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Monotone extenders for bounded c-valued functions

by

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Abstract. Let c be the Banach space consisting of all convergent sequences of reals with the sup-norm, $C_\infty(A,c)$ the set of all bounded continuous functions $f : A \to c$, and $C_A(X,c)$ the set of all functions $f : X \to c$ which are continuous at each point of $A \subset X$. We show that a Tikhonov subspace A of a topological space X is strong Choquet in X if there exists a monotone extender $u : C_\infty(A,c) \to C_A(X,c)$. This shows that the monotone extension property for bounded c-valued functions can fail in GO-spaces, which provides a negative answer to a question posed by I. Banakh, T. Banakh and K. Yamazaki.

In this paper, vector spaces mean real vector spaces. Let X and Y be topological spaces. Then $C(X,Y)$ stands for the set of all continuous functions $f : X \to Y$. If Y is a topological vector space, the set of all bounded continuous functions $f : X \to Y$ is denoted by $C_\infty(X,Y)$, where $f : X \to Y$ is *bounded* if $f(X)$ is a bounded subset of Y, that is, for each 0-neighborhood U of Y there exists $r \in \mathbb{R}$ such that $f(X) \subset rU$. For $A \subset X$, a map $u : C(A,Y) \to C(X,Y)$ is called an *extender* if $u(f)|A = f$ for each $f \in C(A,Y)$. For a topological vector space Y, an extender $u : C(A,Y) \to C(X,Y)$ is said to be a conv-*extender* (resp. $\overline{\text{conv}}$-*extender*) if $u(f)(X)$ is a subset of the convex hull (resp. the closed convex hull) of $f(A)$ for each $f \in C(A,Y)$. For a topological space Y with a partial order structure $\le$, an extender $u : C(A,Y) \to C(X,Y)$ (or $u : C_\infty(A,Y) \to C(X,Y)$) is said to be *monotone* if $u(f) \le u(g)$ for each $f,g \in C(A,Y)$ (or $f,g \in C_\infty(A,Y)$) with $f \le g$. A vector space Y with a partial order structure $\le$ is called an *ordered vector space* if the following axioms are satisfied:

$(O)_1$ $x \le y$ implies $x+z \le y+z$ for all $x,y,z \in Y$;
$(O)_2$ $x \le y$ implies $\lambda x \le \lambda y$ for all $x,y \in Y$ and all $\lambda > 0$.

A topological vector space Y that is also an ordered vector space is called an *ordered topological vector space* if the positive cone $\{y \in Y : y \ge 0\}$ is

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closed. Note that, for an ordered topological vector space Y, each linear $\overline{\text{conv}}$-extender $u : C(A, Y) \to C(X, Y)$ is monotone. As usual, $C(X)$ and $C_\infty(X)$ stand for $C(X, \mathbb{R})$ and $C_\infty(X, \mathbb{R})$, respectively.

Dugundji's extension theorem [6] states that for a metric space X, a closed subset A of X and a locally convex topological vector space Y, there exists a linear conv-extender $u : C(A, Y) \to C(X, Y)$; this is an improvement of an earlier result by K. Borsuk [4] that for a closed separable subset of a metric space X, there exists a norm-one linear extender $u : C_\infty(A) \to C_\infty(X)$. Now it is known that Dugundji's extension theorem holds in some classes of generalized metric spaces X (C. J. R. Borges [3], I. S. Stares [11]), but does not hold for all GO-spaces X. Indeed, for the Michael line $\mathbb{R}_\mathbb{Q}$, R. W. Heath and D. J. Lutzer [9] show that there exists no linear $\overline{\text{conv}}$-extender $u : C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$; E. K. van Douwen [5] extends it by showing that there is no monotone extender $u : C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ (see also I. S. Stares and J. E. Vaughan [12]). For related results on Dugundji extenders and retracts in GO-spaces, see G. Gruenhage, Y. Hattori and H. Ohta [8].

On extenders for bounded functions, R. W. Heath and D. J. Lutzer [9] establish that for a closed subset A of a GO-space X, there exists a linear $\overline{\text{conv}}$-extender $u : C_\infty(A) \to C_\infty(X)$; van Douwen's result [5] shows that "$\overline{\text{conv}}$-extender" in the above cannot be strengthened to "conv-extender". For normed-space-valued functions, I. Banakh, T. Banakh and K. Yamazaki [1, Theorem 4.1] prove that a normed space Y is reflexive if and only if for every closed subset A of a GO-space X, there exists a linear $\overline{\text{conv}}$-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$. From these viewpoints, a natural further question arises: let Y be a non-reflexive normed space which is an ordered topological vector space; does there exist, for every closed subset A of every GO-space X, a monotone extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$? The answer is "yes" for $Y = l_1$ ([1, Theorem 9.1]), "no" for $Y = c_0$ ([1, Corollary 6.3]), and for $Y = c$ the following is asked in [1, Question 6.4]: *Is there a monotone extender* $u : C_\infty(\mathbb{Q}, c) \to C(\mathbb{R}_\mathbb{Q}, c)$? In this paper, we give a negative answer to this question (Corollary 4). In fact, we show that any subset A which is not strong Choquet in X fails to possess such monotone extenders (Corollary 3).

Let us recall some terminology. The symbol c_0 (resp. c) stands for the Banach space consisting of all sequences of reals that converge to 0 (resp. of all convergent sequences of reals) with the sup-norm and a partial order structure $\le$, where for $x = (x_n)_{n\in\omega}$ and $y = (y_n)_{n\in\omega} \in c_0$ (or c), $x \le y$ if $x_n \le y_n$ for each $n \in \omega$. Recall from [10] and [2] that a Hausdorff space X is a *generalized ordered space* (= *GO-space*) if X has a linear order structure and has a base of the topology consisting of order-convex sets. The *Michael line* $\mathbb{R}_\mathbb{Q}$ is the set $\mathbb{R}$ endowed with the topology $\{U \cup V : U \in \tau,\ V \subset \mathbb{R} \setminus \mathbb{Q}\}$, where

τ is the usual topology of $\mathbb{R}$ and $\mathbb{Q}$ is the set of all rational numbers ([7]). The Michael line $\mathbb{R}_{\mathbb{Q}}$ is a typical example of a GO-space.

As in [1], the relative strong Choquet game $G_r(A,X)$ is played by two players, I and II, for a subset A of a topological space X. Player I starts the game selecting a point $a_0 \in A$ and a neighborhood U_0 of a_0 in X. Player II responds with a neighborhood $V_0 \subset U_0$ of a_0 in X. At the nth inning player I selects a point $a_n \in V_{n-1} \cap A$ and a neighborhood $U_n \subset V_{n-1}$ of a_n in X, while player II responds with a neighborhood $V_n \subset U_n$ of a_n in X. Thus players construct a sequence $\{a_n\}_{n\in\omega}$ of points of A and sequences $(U_n)_{n\in\omega}$ and $(V_n)_{n\in\omega}$ of open sets of X such that $a_n \in V_n \subset U_n \subset V_{n-1}$ for all $n \in \mathbb{N}$. Player I is declared the winner in the game $G_r(A,X)$ if $\emptyset \neq \bigcap_{n\in\omega} U_n \subset X\setminus A$. Otherwise, player II wins. If player II has a winning strategy in the game $G_r(A,X)$, then the set A is said to be *strong Choquet in* X. Note that $\mathbb{Q}$ is not strong Choquet in $\mathbb{R}_{\mathbb{Q}}$ ([1, Corollary 3.7]). Another important example of a subset which is not strong Choquet in the whole space is given in [12] (see [1, Remark 3.8]).

Let Y be a topological space with a partial order structure. Then a continuous function $\gamma : [0,\infty) \to Y$ is an *ω-increasing ray* (resp. *ω-decreasing ray*) if $\gamma(n) \leq \gamma(t)$ (resp. $\gamma(n) \geq \gamma(t)$) for any integer $n \in \omega$ and any real $t \geq n$ ([1]). In order to improve [1, Theorem 6.1], we introduce key notions which are modifications of (almost) upper boundedness of $\gamma(\omega)$ for an ω-increasing ray γ appearing in [1]. For $Y_0 \subset Y$, we say Y_0 has the *ω-decreasing intersection property* in Y if for each increasing sequence $\{y_n\}_{n\in\omega}$ in Y_0 and each decreasing sequence $\{z_n\}_{n\in\omega}$ in Y_0 with $y_n \leq z_n$ for each $n \in \omega$, $\bigcap_{n\in\omega}\{y \in Y : y_n \leq y \leq z_n\} \neq \emptyset$. We also say Y_0 has the *almost ω-decreasing intersection property* in Y if for each ω-increasing ray $\gamma_1 : [0,\infty) \to Y_0$, each ω-decreasing ray $\gamma_2 : [0,\infty) \to Y_0$ with $\gamma_1(r_1) \leq \gamma_2(r_2)$ for each $r_1, r_2 \in [0,\infty)$, and each family $\{G_n^i\}_{n\in\omega}^{i=1,2}$ of G_δ-sets of Y with $\gamma_i(n) \in G_n^i$, $n \in \omega$, $i = 1,2$, it follows that $\bigcap_{n\in\omega}\bigcup_{b_1\in G_n^1,\, b_2\in G_n^2}\{y \in Y : b_1 \leq y \leq b_2\} \neq \emptyset$.

For $A \subset X$, $C_A(X,Y)$ denotes the set of all functions $f : X \to Y$ which are continuous at each point of A.

THEOREM 1. *Let A be a Tikhonov subset of a topological space X, Y a topological space with a partial order structure and $Y_0 \subset Y$. If there exists a monotone extender $u : C(A,Y_0) \to C_A(X,Y)$, then either A is strong Choquet in X or else Y_0 has the almost ω-decreasing intersection property in Y.*

Proof. Let $u : C(A,Y_0) \to C_A(X,Y)$ be a monotone extender. Assume Y_0 does not have the almost ω-decreasing intersection property in Y. Namely, there exist an ω-increasing ray $\gamma_1 : [0,\infty) \to Y_0$, an ω-decreasing ray $\gamma_2 : [0,\infty) \to Y_0$ with $\gamma_1(r_1) \leq \gamma_2(r_2)$ for each $r_1, r_2 \in [0,\infty)$, and a family $\{G^i_n\}^{i=1,2}_{n\in\omega}$ of G_δ-sets of Y with $\gamma_i(n)\in G^i_n$, $n\in\omega$, $i=1,2$, such that

$$\bigcap_{n\in\omega}\ \bigcup_{b_1\in G^1_n, b_2\in G^2_n}\{y\in Y: b_1\le y\le b_2\}=\emptyset. \tag{1}$$

Set $y_n=\gamma_1(n)$ and $z_n=\gamma_2(n)$ for each $n\in\omega$. For each $n\in\omega$, take decreasing sequences $(O_m(y_n))_{m\ge n}$ and $(O_m(z_n))_{m\ge n}$ of open neighborhoods of y_n and z_n, respectively, such that $\bigcap_{m\ge n}O_m(y_n)\subset G^1_n$ and $\bigcap_{m\ge n}O_m(z_n)\subset G^2_n$. Now we describe a winning strategy of player II in the game $G_r(A,X)$. For $a_0\in A$ and a neighborhood U_0 of a_0 in X, player II takes a neighborhood V_0 of a_0 in X with $V_0\subset U_0$ and sets a function $\lambda_0\equiv 0$ on A. In the nth inning ($n>0$), for $a_n\in V_{n-1}\cap A$ and a neighborhood $U_n\subset V_{n-1}$ of a_n, player II takes a continuous function $\lambda_n: A\to[0,1]$ and a neighborhood V_n of a_n in X such that

$$V_n\subset U_n,\qquad a_n\in V_n\cap A\subset\lambda_n^{-1}(1)\subset\lambda_n^{-1}((0,1])\subset U_n, \tag{2}$$

$$u\Big(\gamma_1\circ\sum_{i=0}^k\lambda_i\Big)(V_n)\subset O_n(y_k)\quad\text{and}\quad u\Big(\gamma_2\circ\sum_{i=0}^k\lambda_i\Big)(V_n)\subset O_n(z_k), \tag{3}$$

for each $k\le n$. Indeed, since A is Tikhonov, take a continuous function $\lambda_n: A\to[0,1]$ such that $a_n\in\operatorname{int}_A\lambda_n^{-1}(1)\subset\lambda_n^{-1}((0,1])\subset U_n$. For each i with $1\le i<n$, it follows from $a_n\in U_n\cap A\subset V_i\cap A\subset\lambda_i^{-1}(1)$ that $\lambda_i(a_n)=1$. Hence, $u(\gamma_1\circ\sum_{i=0}^k\lambda_i)(a_n)=\gamma_1(k)=y_k$ and $u(\gamma_2\circ\sum_{i=0}^k\lambda_i)(a_n)=\gamma_2(k)=z_k$ for each $k\le n$. Since $u(\gamma_1\circ\sum_{i=0}^k\lambda_i)$ and $u(\gamma_2\circ\sum_{i=0}^k\lambda_i)$ are in $C_A(X,Y)$, choose a neighborhood V_n of a_n in X such that $u(\gamma_1\circ\sum_{i=0}^k\lambda_i)(V_n)\subset O_n(y_k)$ and $u(\gamma_2\circ\sum_{i=0}^k\lambda_i)(V_n)\subset O_n(z_k)$ for each $k\le n$ and $V_n\cap A\subset\lambda_n^{-1}(1)$.

Then the condition $\emptyset\ne\bigcap_{n\in\omega}U_n\subset X\setminus A$ fails. To show this, assume on the contrary that there exists $c\in\bigcap_{n\in\omega}U_n=\bigcap_{n\in\omega}V_n\subset X\setminus A$. By (3),

$$\begin{aligned}&u\Big(\gamma_1\circ\sum_{i=0}^k\lambda_i\Big)(c)\in\bigcap_{n\ge k}O_n(y_k)\subset G^1_k,\\ &u\Big(\gamma_2\circ\sum_{i=0}^k\lambda_i\Big)(c)\in\bigcap_{n\ge k}O_n(z_k)\subset G^2_k,\end{aligned}\tag{4}$$

for each $k\in\omega$. Define a continuous function $s: A\to[0,\infty)$ by $s(a)=\sum_{i\in\omega}\lambda_i(a)$ for each $a\in A$; this is possible by (2) and the fact that $\bigcap_{n\in\omega}U_n=\bigcap_{n\in\omega}V_n\subset X\setminus A$. Then we show $(\gamma_1\circ\sum_{i=0}^k\lambda_i)(a)\le(\gamma_1\circ s)(a)$ for each $a\in A$. Indeed, if $\sum_{i=0}^k\lambda_i(a)$ is an integer, this follows from γ_1 being ω-increasing. If $\sum_{i=0}^k\lambda_i(a)$ is not an integer, the fact that $\lambda_{n+1}^{-1}((0,1])\subset\lambda_n^{-1}(1)$ for each $n\in\mathbb{N}$ implies that $\sum_{i=0}^k\lambda_i(a)=s(a)$. Similarly, $\gamma_1\circ\sum_{i=0}^k\lambda_i\le\gamma_1\circ s\le\gamma_2\circ s\le\gamma_2\circ\sum_{i=0}^k\lambda_i$ for each $k\in\omega$. Since $\gamma_1\circ\sum_{i=0}^k\lambda_i$, $\gamma_1\circ s$, and $\gamma_2\circ\sum_{i=0}^k\lambda_i$ are in $C(A,Y_0)$ and u is monotone, it follows that

$u(\gamma_1 \circ \sum_{i=0}^k \lambda_i) \le u(\gamma_1 \circ s) \le u(\gamma_2 \circ \sum_{i=0}^k \lambda_i)$ for each $k \in \omega$. By (4), $u(\gamma_1 \circ s)(c) \in \bigcap_{n\in\omega} \bigcup_{b_1 \in G_n^1, b_2 \in G_n^2} \{y \in Y : b_1 \le y \le b_2\}$, a contradiction to (1). Hence A is strong Choquet in X. This completes the proof. ∎

LEMMA 2. *Let Y be a topological vector space and an ordered vector space, and Y_0 a convex subset of Y. If Y_0 has countable pseudo-character in Y, the following conditions are equivalent:*

(1) *Y_0 has the almost ω-decreasing intersection property in Y;*
(2) *Y_0 has the ω-decreasing intersection property in Y.*

Proof. (2)⇒(1) is obvious. To show (1)⇒(2), let $\{y_n\}_{n\in\omega}$ be an increasing sequence in Y_0 and $\{z_n\}_{n\in\omega}$ a decreasing sequence in Y_0 with $y_n \le z_n$ for each $n \in \omega$. Since Y_0 has countable pseudo-character in Y, we set $G_n^1 = \{y_n\}$ and $G_n^2 = \{z_n\}$ for each $n \in \omega$. Define $\gamma_1, \gamma_2 : [0,\infty) \to Y_0$ by $\gamma_1(r) = (n_r + 1 - r)y_{n_r} + (r - n_r)y_{n_r+1}$ and $\gamma_2(r) = (n_r + 1 - r)z_{n_r} + (r - n_r)z_{n_r+1}$, where $n_r \in \omega$ with $r \in [n_r, n_r + 1]$. Then γ_1 and γ_2 are Y_0-valued continuous, because Y_0 is convex and $\{[n, n+1] : n \in \omega\}$ is a locally finite closed collection in $[0,\infty)$. It is easy to see that γ_1 is an ω-increasing ray and γ_2 is an ω-decreasing ray. To show $\gamma_1(r) \le \gamma_2(s)$ for each $r, s \in [0,\infty)$, fix $r, s \in [0,\infty)$ arbitrarily. Set $n_0 = \max\{n_r + 1, n_s + 1\}$. Then it follows from $(O)_1$ and $(O)_2$ that

$$\begin{aligned}\gamma_1(r) &= (n_r+1-r)y_{n_r} + (r-n_r)y_{n_r+1} \le (n_r+1-r)y_{n_r+1} + (r-n_r)y_{n_r+1}\\ &= y_{n_r+1} \le y_{n_0} \le z_{n_0} \le z_{n_s+1} = (n_s+1-s)z_{n_s+1} + (s-n_s)z_{n_s+1}\\ &\le (n_s+1-s)z_{n_s} + (s-n_s)z_{n_s+1} = \gamma_2(s).\end{aligned}$$

Since $\gamma_1(n) = y_n$ and $\gamma_2(n) = z_n$ for each $n \in \omega$, it follows from (1) that $\bigcap_{n\in\omega}\{y \in Y : y_n \le y \le z_n\} \ne \emptyset$. ∎

COROLLARY 3. *A Tikhonov subspace A of a topological space X is strong Choquet in X if there exists a monotone extender $u : C_\infty(A, c) \to C_A(X, c)$.*

Proof. Let $Y = c$ and $Y_0 = \{y \in c : 0 \le y \le 1\}$, where $0 = (0, 0, \ldots)$, $1 = (1, 1, \ldots) \in c$. Then Y_0 is convex and bounded in Y. Define $y_n, z_n \in c$, $n \in \omega$, by $y_n(m) = 1$ if $m = 2j + 1$, $j \le n - 1$, $j \in \omega$; $y_n(m) = 0$ otherwise; $z_n(m) = 0$ if $m = 2j$, $j \le n - 1$, $j \in \omega$; $z_n(m) = 1$ otherwise. Then $\{y_n\}_{n\in\omega} \subset Y_0$ is increasing, $\{z_n\}_{n\in\omega} \subset Y_0$ is decreasing with $y_n \le z_n$ for each $n \in \omega$, and $\bigcap_{n\in\omega}\{y \in c : y_n \le y \le z_n\} = \emptyset$. By Lemma 2, Y_0 does not have the almost ω-decreasing intersection property in c. Since $C(A, Y_0) \subset C_\infty(A, c)$, Theorem 1 completes the proof of Corollary 3. ∎

COROLLARY 4. *There is no monotone extender $u : C_\infty(\mathbb{Q}, c) \to C(\mathbb{R}_{\mathbb{Q}}, c)$.*

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