STUDIA MATHEMATICA 199 (1) (2010)

Distances to spaces of affine Baire-one functions

by

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Abstract. Let E be a Banach space and let $\mathcal{B}_1(B_{E^*})$ and $\mathfrak{A}_1(B_{E^*})$ denote the space of all Baire-one and affine Baire-one functions on the dual unit ball B_{E^*}, respectively. We show that there exists a separable L_1-predual E such that there is no quantitative relation between $\operatorname{dist}(f, \mathcal{B}_1(B_{E^*}))$ and $\operatorname{dist}(f, \mathfrak{A}_1(B_{E^*}))$, where f is an affine function on B_{E^*}. If the Banach space E satisfies some additional assumption, we prove the existence of some such dependence.

1. Introduction. If K is a compact (Hausdorff) space, we write $\mathcal{C}(K)$ for the space of all real-valued continuous functions on K and $\mathcal{M}(K)$ for the space of all signed Radon measures on K. (By a *Radon measure* we mean a complete measure that is inner regular with respect to compact sets and is defined on a σ-algebra including all Borel subsets of K. A signed measure is Radon if the total variation $|\mu|$ of μ is a Radon measure. We refer the reader to [9, Section 416] for more information on Radon measures.) Let $\mathcal{M}^1(K)$ denote the set of all Radon probability measures on K. We always consider $\mathcal{M}(K)$ endowed with the weak* topology. We say that a function $f : K \to \mathbb{R}$ is *universally measurable* if f is μ-measurable for every $\mu \in \mathcal{M}^1(K)$. We denote the space of all bounded universally measurable functions on K by $\mathcal{U}^b(K)$.

If X is a compact convex subset of a real locally convex space, let $\mathfrak{A}^b(X)$ and $\mathfrak{A}^c(X)$ denote the spaces of all bounded affine functions on X and continuous affine functions on X, respectively. Any $\mu \in \mathcal{M}^1(X)$ has its unique *barycenter* $r(\mu) \in X$, i.e., the point $x \in X$ satisfying $f(x) = \mu(f)$ for any $f \in \mathfrak{A}^c(X)$ (see [1, Proposition I.2.1]). We sometimes say that μ *represents* x. A function $f : X \to \mathbb{R}$ is *strongly affine* (or satisfies the *barycentric formula*) if f is universally measurable, $\mu(f)$ exists and $f(r(\mu)) = \mu(f)$ for any $\mu \in \mathcal{M}^1(X)$. We write $\mathfrak{A}_{\mathrm{bf}}(X)$ for the space of all strongly affine functions on X (i.e. functions satisfying the barycentric formula) and recall that

2010 *Mathematics Subject Classification*: Primary 46B99; Secondary 52A07.
Key words and phrases: L_1-predual, affine function, Baire-one function, fragmented function, simplex.

DOI: 10.4064/sm199-1-3

it is easy to see that any strongly affine function is affine and bounded (see the proof of [13, Satz 2.1(c)]).

By a result of B. Cascales, W. Marciszewski and M. Raja [5, Proposition 4.1], $\operatorname{dist}(f, \mathcal{C}(X)) = \operatorname{dist}(f, \mathfrak{A}^c(X))$ for any $f \in \mathfrak{A}^b(X)$. If E is a Banach space, its dual unit ball B_{E^*} endowed with the weak* topology is an example of a compact convex set. Given an element $x^{**} \in E^{**}$, let f denote its restriction to B_{E^*}. By the fact above, $\operatorname{dist}(f, \mathcal{C}(B_{E^*})) = \operatorname{dist}(f, \mathfrak{A}^c(B_{E^*}))$ (see [5, Corollary 4.2]).

As a further step, a paper [2] by C. Angosto, B. Cascales and I. Namioka investigates how to measure distance of a function to the space of Baire-one functions. Let us recall that, given two topological spaces K and E, the space $\mathcal{B}_1(K, E)$ consists of all mappings $f : K \to E$ that can be obtained as the pointwise limit of a sequence of continuous mappings from K to E. If $E = \mathbb{R}$, we write $\mathcal{B}_1(K)$ for $\mathcal{B}_1(K, \mathbb{R})$. If $f : K \to E$ is a mapping from a topological space K to a metric space E, f is said to be *ε-fragmented* if for any closed set $F \subset K$ there exists a relatively open nonempty subset U of F such that $\operatorname{diam} f(U) < \varepsilon$ (see [2, p. 105]). Then $\operatorname{frag}(f)$ is defined as

$$\operatorname{frag}(f) = \inf\{\varepsilon > 0\colon f \text{ is } \varepsilon\text{-fragmented}\}$$

if such an $\varepsilon > 0$ exists, and $\operatorname{frag}(f) = \infty$ otherwise. If $f : K \to \mathbb{R}$ is a function on a metrizable compact space, it follows from [2, Corollary 2.6] that $\operatorname{dist}(f, \mathcal{B}_1(K)) = \frac{1}{2} \operatorname{frag}(f)$.

If X is a compact convex set, let $\mathfrak{A}_1(X)$ stand for the space of all pointwise limits of sequences of functions from $\mathfrak{A}^c(X)$. By [20, Théorème 80] (see also [7, p. 611]), $\mathcal{B}_1(X) \cap \mathfrak{A}^b(X) = \mathfrak{A}_1(X)$, and any function in $\mathfrak{A}_1(X)$ is a pointwise limit of a bounded sequence in $\mathfrak{A}^c(X)$. If $f \in \mathfrak{A}^b(X)$, following the result on continuous functions we might ask whether $\operatorname{dist}(f, \mathcal{B}_1(X)) = \operatorname{dist}(f, \mathfrak{A}_1(X))$. The aim of our paper is to present an example that disproves this. (We recall that a Banach space is an *L_1-predual* if its dual is isometric to a space $L^1(\mu)$ for a suitable measure μ; see [7, p. 625].)

THEOREM 1.1. *There exists a separable L_1-predual E with the following property: for any $\varepsilon > 0$ there exists $x^{**} \in B_{E^{**}}$ such that the function $f = x^{**}|_{B_{E^*}}$ satisfies*

- *f is strongly affine,*
- *$\operatorname{dist}(f, \mathcal{B}_1(B_{E^*})) < \varepsilon$,*
- *$\operatorname{dist}(f, \mathfrak{A}_1(B_{E^*})) \geq 1/2$.*

If an L_1-predual E satisfies an additional topological condition imposed on the set $\operatorname{ext} B_{E^*}$ of all extreme points of its dual unit ball B_{E^*}, we obtain a quantitative relation between the distance to Baire-one functions and the distance to affine Baire-one functions. We recall that a subset H of a topological space K is said to be an *H-set* (or a *resolvable* set) if the characteristic function χ_H satisfies $\operatorname{frag}(\chi_H)=0$ (see [14, §12]). We recall that a mapping $f : K \to E$ between two topological spaces is *Baire measurable* if $f^{-1}(U)$ is a Baire subset of K for any $U \subset E$ open.

THEOREM 1.2. *Let E be an L_1-predual such that the set of extreme points of the dual unit ball is a Lindelöf H-set in the weak* topology. Let $x^{**} \in E^{**}$ and $f = x^{**}|_{B_{E^*}}$. If*

- *E is separable, or*
- *f is Baire measurable,*

then $\operatorname{dist}(f, \mathfrak{A}_1(B_{E^*})) \leq 5 \operatorname{dist}(f, \mathcal{B}_1(B_{E^*}))$.

We remark that, for a separable space E, the topological condition imposed on $\operatorname{ext} B_{E^*}$ is equivalent to $\operatorname{ext} B_{E^*}$ being of type F_σ. This can be seen from the following two facts: a subset of a compact metrizable space is an H-set if and only if it is both of type F_σ and G_δ (use [14, §26, X] and the Baire category theorem); the set of extreme points in a metrizable compact convex set is of type G_δ (see [1, Corollary I.4.4]).

We also point out that the topological assumption in Theorem 1.2 is satisfied when $\operatorname{ext} B_{E^*}$ is an F_σ set. To see this, we first notice that $\operatorname{ext} B_{E^*}$ is then a Lindelöf space. Second, we need to check that $\operatorname{ext} B_{E^*}$ is an H-set in B_{E^*}. To this end, assume that $F \subset B_{E^*}$ is a nonempty closed set such that both $F \cap \operatorname{ext} B_{E^*}$ and $F \setminus \operatorname{ext} B_{E^*}$ are dense in F. By [25, Théorème 2], we can write

$$\operatorname{ext} B_{E^*} = \bigcap_{n=1}^{\infty} (H_n \cup V_n),$$

where $H_n \subset B_{E^*}$ is closed and $V_n \subset B_{E^*}$ is open, $n \in \mathbb{N}$. Thus both $F \setminus \operatorname{ext} B_{E^*}$ and $F \cap \operatorname{ext} B_{E^*}$ are comeager disjoint sets in F, contradicting the Baire category theorem. Hence $\operatorname{ext} B_{E^*}$ is an H-set.

The following result presents a condition of a different type that still yields a conclusion similar to that of Theorem 1.2.

THEOREM 1.3. *Let E be a Banach space not containing ℓ^1, $x^{**} \in E^{**}$ and let $f = x^{**}|_{B_{E^*}}$. Then* $\operatorname{dist}(f, \mathfrak{A}_1(B_{E^*})) \leq 2 \operatorname{dist}(f, \mathcal{B}_1(B_{E^*}))$.

If E above is assumed to be separable, any element $x^{**} \in E^{**}$ is in $\mathfrak{A}_1(B_{E^*})$ when restricted to B_{E^*} (see [18] or [3, Theorem II.1.3]) and thus the inequality is vacuously satisfied (the author would like to thank M. Raja for this important remark).

We also present a variant of [2, Theorem 2.5] for nonmetrizable compact spaces needed for our purposes.

THEOREM 1.4. *Let K be a compact space and $f : K \to E$ be a function from K to a Banach space E.*

(a) $\frac{1}{2}\operatorname{frag}(f) \le \operatorname{dist}(f, \mathcal{B}_1(K,E))$.
(b) *If f is Baire measurable, then* $\operatorname{dist}(f, \mathcal{B}_1(K,E)) \le \operatorname{frag}(f)$.
(c) *If f is Baire measurable and $E = \mathbb{R}$, then* $\operatorname{dist}(f, \mathcal{B}_1(K)) = \frac{1}{2}\operatorname{frag}(f)$.

Our construction of a separable L_1-predual in Theorem 1.1 is based upon the notion of a *simplicial function space*. We recall that, given a compact space K, a *function space* $\mathcal{H}$ is a subspace of $\mathcal{C}(K)$ that contains constants and separates points of K. We use the construction from [23] to get the desired example of Theorem 1.1.

Throughout, we follow the notation and definitions from [23].

We just recall that, given a function space $\mathcal{H}$ on a compact space K, the *state space* $\mathbf{S}(\mathcal{H})$ of $\mathcal{H}$ is defined as

$$\mathbf{S}(\mathcal{H}) = \{s \in \mathcal{H}^* \colon \|s\| = s(1) = 1\}.$$

If $\mathbf{S}(\mathcal{H})$ is endowed with the weak* topology, it is a compact convex set. The space K is homeomorphically embedded into $\mathbf{S}(\mathcal{H})$ via the evaluation mapping $\phi : K \to \mathbf{S}(\mathcal{H})$ defined by

$$\phi(x)(h) = h(x), \quad h \in \mathcal{H},\ x \in K.$$

We denote

$$\mathcal{U}^b(K) \cap \mathcal{H}^{\perp\perp} = \{f \in \mathcal{U}^b(K) \colon \mu(f) = 0 \text{ for all } \mu \in \mathcal{H}^{\perp}\}$$

and $X = \mathbf{S}(\mathcal{H})$. Let $\pi : \mathcal{M}^1(K) \to X$ denote the restriction mapping. It follows from [23, Theorem 2.5] that the formula

$$If(s) = \mu(f), \qquad \pi(\mu) = s,\ s \in X, \qquad f \in \mathcal{U}^b(K) \cap \mathcal{H}^{\perp\perp}, \tag{1.1}$$

defines an isometric isomorphism $I \colon \mathcal{U}^b(\mathcal{K}) \cap \mathcal{H}^{\perp\perp} \to \mathfrak{A}_{\mathrm{bf}}(X)$. Moreover,

$$I(\mathcal{B}^b_\alpha(K) \cap \mathcal{H}^{\perp\perp}) = \mathcal{B}^b_\alpha(X) \cap \mathfrak{A}_{\mathrm{bf}}(X), \quad \alpha \in [0, \omega_1).$$

To illuminate relations between compact convex sets and Banach spaces, let us recall the following facts. If X is a compact convex set and $E = \mathfrak{A}^c(X)$, the state space $\mathbf{S}(\mathfrak{A}^c(X))$ is affinely homeomorphic to X via the evaluation mapping ϕ. The dual unit ball B_{E^*} equals $\operatorname{co}(\phi(X) \cup -\phi(X))$ and the weak topology on E coincides with the topology of pointwise convergence on $\mathfrak{A}^c(X)$. Any function $f \in \mathfrak{A}^b(X)$ has a unique extension to $E^* = \operatorname{span}\phi(X)$. This provides an identification of E^{**} with $\mathfrak{A}^b(X)$. Moreover, the weak* topology on E^{**} coincides with the topology of pointwise convergence on $\mathfrak{A}^b(X)$.

A compact convex set X is a *simplex* if $\mathfrak{A}^c(X)$ is an L_1-predual (for more information on simplices, see [1, Chapter II, §3], [4, Section 2.7], [6, Chapter 6, §28], [7, Section 3], [15, Chapter 7, §20], [19, Chapter 10] or [21, Chapter 6, §23]).

If E is a Banach space, a function $f : B_{E^*} \to \mathbb{R}$ is the restriction of an element of E if and only if $f \in \mathfrak{A}^c(B_{E^*})$ and $f(0) = 0$.

2. Construction of function spaces. We use the construction of a function space from [23, Section 5]. For a fixed natural number $m > 1$, let $\mathcal{H}_0, \dots, \mathcal{H}_m$ be the simplicial function spaces on the metrizable compact spaces $K_0, \dots, K_m$ constructed in [23, Inductive construction 5.2]. Let $\{F_s\colon s \in \mathbb{N}^{<\mathbb{N}}\}$ and $\mathcal{F}_n = \{F_n(k)\colon k \in \mathbb{N}\}$, $n = 0, \dots, m$, be the objects defined there.

We recall from [23, Lemma 3.3] that $\{F_s\colon s \in \mathbb{N}^{<\mathbb{N}}\}$ is a family of sets in $K_0 = [0,1]$ with the following properties:

(a) $F_\emptyset = K_0$,
(b) $\{F_{s^\wedge n} : n \in \mathbb{N}\}$ is a disjoint family of nonempty nowhere dense perfect subsets of F_s,
(c) $\bigcup\{F_{s^\wedge n} : n \in \mathbb{N}\}$ is dense in F_s,
(d) $\operatorname{diam} F_s < 2^{-(s_1+\dots+s_{|s|})}$, $s \in \mathbb{N}^{<\mathbb{N}}$.

We further demand that the family $\{F_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ satisfies the following stronger version of (c):

(e) both $\{F_{s^\wedge n} : n \text{ odd}\}$ and $\{F_{s^\wedge n} : n \text{ even}\}$ are dense in F_s, $s \in \mathbb{N}^{<\mathbb{N}}$.

This family is used in [23, Inductive construction 5.2] for an inductive construction of function spaces with increasing complexity. Roughly speaking, the construction proceeds as follows.

We take the sets of the first level in K_0, i.e., $\{F_n\colon n \in \mathbb{N}\}$, and create a new compact space K_1 by adding to K_0 two copies of each F_n. We imagine each point of F_n to be the average of two points "above" and "below" and encode it in the definition of the new function space $\mathcal{H}_1$ on K_1. For each set F_n, we consider the sets of the second level $\{F_{n^\wedge k}\colon k \in \mathbb{N}\}$ and transfer them into the just created copies of F_n. These new sets form the family $\mathcal{F}_1$.

The second step splits up the sets from $\mathcal{F}_1$ and transfers the sets $\{F_s\colon |s| = 3\}$ of the third level into them. Proceeding inductively, we create function spaces $\mathcal{H}_m$ on compact spaces K_m, $m = 0, 1, \dots$.

Lemma 2.1. *The following asertions hold:*

(a) *$\mathcal{H}_m$ is a simplicial function space with $\mathcal{A}_c(\mathcal{H}_m) = \mathcal{H}_m$,*
(b) *$K_m \setminus \operatorname{Ch}_{\mathcal{H}_m} K_m = \bigcup_{n=0}^{m-1} \bigcup \mathcal{F}_n$.*

Proof. Assertion (a) follows by inductive use of [23, Lemma 5.1(c),(e)], and (b) from [23, Lemma 5.1(d)]. ∎

Let $\delta = (2m+1)^{-1}$.

Definition 2.2. We define inductively a function $f_m\colon K_m \to [-1,1]$ such that f_m is constant on each element of $\mathcal{F}_n$ for every $n \in \{0, \dots, m\}$. The definition is as follows:

- For $x \in K_0$, we set
$$f_m(x) = \begin{cases} \delta, & x \in F_s, s \in \mathbb{N} \text{ is odd}, \\ -\delta, & x \in K_0 \setminus \bigcup\{F_s : s \in \mathbb{N} \text{ is odd}\}. \end{cases}$$
- Assume that f_m is defined on each $K_0, \dots, K_n$ for some $n \in \{0, \dots, m-1\}$. Let $\mathcal{F}_n = \{F_n(k) \colon k \in \mathbb{N}\}$ be the enumeration of the family $\mathcal{F}_n$ and let $a_n(k)$ be the value of f_m on $F_n(k)$. Let
$$F(s,k,+), F(s,k,-), \qquad k \in \mathbb{N},\ s \in \mathbb{N}^{n+2},$$
be as in equations (8) of [23, Inductive construction 5.2]. Let $\mathbb{N}^{n+2}_{\text{odd}}$ denote the set of all sequences $s \in \mathbb{N}^{n+2}$ with s_{n+2} odd. Then we define the function f_m for
$$x \in K_{n+1} \setminus K_n = \bigcup_{k=1}^{\infty} (F_n(k) \times \{1/k\}) \cup (F_n(k) \times \{-1/k\})$$
as
$$f_m(x) = \begin{cases} a_n(k) + 2\delta, & x \in \bigcup\{F(s,k,+) \colon s \in \mathbb{N}^{n+2}_{\text{odd}}\}, \\ a_n(k), & x \in (F_n(k) \times \{1/k\}) \setminus \bigcup\{F(s,k,+) \colon s \in \mathbb{N}^{n+2}_{\text{odd}}\}, \\ a_n(k) - 2\delta, & x \in \bigcup\{F(s,k,-) \colon s \in \mathbb{N}^{n+2}_{\text{odd}}\}, \\ a_n(k), & x \in (F_n(k) \times \{-1/k\}) \setminus \bigcup\{F(s,k,-) \colon s \in \mathbb{N}^{n+2}_{\text{odd}}\}. \end{cases}$$

Lemma 2.3. *The function f_m from Definition 2.2 has the following properties:*

(a) $f_m(K_m) \subset [-1,1]$,
(b) $\operatorname{frag}(f_m) = 2\delta$,
(c) $f_m \in \mathcal{B}^b_2(K_m) \cap \mathcal{H}^{\perp\perp}_m$,
(d) $\operatorname{dist}(f_m, \mathcal{B}^b_1(K_m) \cap \mathcal{H}^{\perp\perp}_m) \geq 1/2$.

Proof. To verify (a), we notice that Definition 2.2 yields the following fact: The greatest value of f_m is $\delta + 2\delta m = 1$ and the least value of f_m is $-\delta - 2\delta m = -1$.

To prove (b), we note that K_m can be written as
$$K_m = K_0 \cup \bigcup_{n=1}^{m} (K_n \setminus K_{n-1}). \tag{2.1}$$
We show that $f_m|_{K_0}$ and $f_m|_{K_n \setminus K_{n-1}}$, $n = 1, \dots, m$, are 2δ-fragmented. Obviously, $f_m|_{K_0}$ is 2δ-fragmented. If $n \in \{1, \dots, m\}$, $K_n \setminus K_{n-1}$ can be written as a countable union of clopen subsets of K_n such that the restriction of f_m to each of them is 2δ-fragmented. Let $F \subset K_m$ be a closed set and $\varepsilon > 2\delta$. If $F \subset K_0$, it is easy to find a relatively open subset of F with $\operatorname{diam} f_m(U) < \varepsilon$. Otherwise we find the greatest index $n \in \{1, \dots, m\}$ such

that $F \cap (K_n \setminus K_{n-1}) \neq \emptyset$. Since $K_n \setminus K_{n-1}$ is an open subset of K_{n-1}, there exists a relatively open subset U of F with $\operatorname{diam} f_m(U) < \varepsilon$. Hence assertion (b) follows.

We start the proof of (c) by observing that it is enough to show that the restriction of f_m to any member of the partition from (2.1) is a Baire-two function. This is easy on K_0 and, as above, we find that every $K_n \setminus K_{n-1}$, $n = 1, \dots, m$, is a countable union of clopen subsets of K_n and that the restriction of f_m to each member of this family is a Baire-two function.

For the second part of (c), the function f_m is in $\mathcal{A}(\mathcal{H}_m)$ by inductive use of [23, Lemma 5.1(f)]. Indeed, $f_m|_{K_0} \in \mathcal{A}(\mathcal{H}_0)$ obviously. By Definition 2.2, $f_m|_{K_1}$ satisfies equations (6) of [23, Key step 5.1], and thus $f_m|_{K_1} \in \mathcal{A}(\mathcal{H}_1)$ by [23, Lemma 5.1(f)]. Proceeding inductively, we verify that $f_m|_{K_n} \in \mathcal{A}(\mathcal{H}_n)$ for every $n \in \{0, \dots, m\}$. Since $\mathcal{H}_m$ is simplicial, [23, Theorem 2.6(b2)] yields

$$f \in (\mathcal{A}_c(\mathcal{H}_m))^{\perp\perp} = \mathcal{H}_m^{\perp\perp}.$$

To show (d), let $g \in \mathcal{B}_1^b(K_m) \cap \mathcal{H}_m^{\perp\perp}$ be arbitrary. We fix $\varepsilon \in (0, \delta)$ and inductively find $F_n \in \mathcal{F}_n$, $n = 0, \dots, m$, such that

- $|g - f_m| > (n+1)\delta - 3^n\varepsilon$ on F_n, $n = 0, \dots, m$.

For $n = 0$, we find $x \in K_0$ such that $g|_{K_0}$ is continuous at x_0 (see [14, §27, X]). Let $U \subset K_0$ be a neighborhood of x such that $\operatorname{diam} g(U) < \varepsilon$. It follows from properties (d) and (e) of the system $\mathcal{F}_0$ and from Definition 2.2 that there exist $F, F' \in \mathcal{F}_0$ such that $F \cup F' \subset U$ and $f_m = \delta$ on F and $f_m = -\delta$ on F'. Hence it follows, by distinguishing the cases $g(x) \leq 0$ and $g(x) \geq 0$, that there exists $F_0 \in \mathcal{F}_0$ such that $|g - f_m| > \delta - \varepsilon$ on F_0.

Assume now that the construction has been completed up to the nth step for some $n \in \{0, \dots, m-1\}$. Hence we have $F_n \in \mathcal{F}_n$ such that

$$|g - f_m| > (n+1)\delta - 3^n\varepsilon \quad \text{on } F_n. \tag{2.2}$$

Let $k \in \mathbb{N}$ be the index of F_n in $\mathcal{F}_n$; that is, $F_n = F_n(k)$. Let $a_n(k)$ denote the value of f_m on $F_n(k)$. Then

$$(F_n(k) \times \{1/k\}) \cup (F_n(k) \times \{-1/k\}) \subset K_{n+1} \subset K_m.$$

We find $x = (x, 0) \in F_n(k)$ such that $(x, 1/k)$ is a point of continuity of the function $g|_{F_n(k) \times \{1/k\}}$.

Case 1. Assume first that

$$g(x, 1/k) \in (-\infty, a_n(k) - n\delta) \cup (a_n(k) + (n+2)\delta, \infty).$$

If $g(x, 1/k) \in (-\infty, a_n(k) - n\delta)$, we find a neighborhood U of x in $F_n(k)$ such that the same holds for all elements of $U \times \{1/k\}$. By Definition 2.2 and properties of $\mathcal{F}_{n+1}$ described above, there exists a set $F_{n+1} \in \mathcal{F}_{n+1}$ such that $F_{n+1} \subset U \times \{1/k\}$ and

$$f_m - g = a_n(k) + 2\delta - g > a_n(k) + 2\delta - (a_n(k) - n\delta) = (n+2)\delta \quad \text{on } F_{n+1}.$$

Analogously, if $g(x, 1/k) \in (a_n(k)+(n+2)\delta, \infty)$, again we find a neighborhood U of x in $F_n(k)$ such that the same holds for all elements of $U \times \{1/k\}$. By Definition 2.2 and properties (d) and (e) of $\mathcal{F}_{n+1}$, there exists a set $F_{n+1} \in \mathcal{F}_{n+1}$ such that $F_{n+1} \subset U \times \{1/k\}$ and

$$g - f_m = g - a_n(k) > a_n(k) + (n+2)\delta - a_n(k) = (n+2)\delta \quad \text{on } F_{n+1}.$$

This finishes the inductive step in this case.

Case 2. Assume now that

$$g(x, 1/k) \in [a_n(k) - n\delta, a_n(k) + (n+2)\delta].$$

Let U be a neighborhood of x in $F_n(k)$ such that

$$-3^n\varepsilon + a_n(k) - n\delta < g < a_n(k) + (n+2)\delta + 3^n\varepsilon \quad \text{on } U \times \{1/k\}.$$

Let $y = (y, 0) \in U$ be such that $(y, -1/k)$ is a point of continuity of $g|_{F_n(k)\times\{-1/k\}}$. We see from (2.2) that

$$|g(y, 0) - f_m(y, 0)| > (n+1)\delta - 3^n\varepsilon.$$

Case 2a. Assume first that

$$g(y, 0) < f_m(y, 0) - (n+1)\delta + 3^n\varepsilon = a_n(k) - (n+1)\delta + 3^n\varepsilon.$$

Since g is $\mathcal{H}_m$-affine, [23, equations (6) in Key step 5.1] yield

$$\begin{aligned} g(y, -1/k) &= 2g(y, 0) - g(y, 1/k) \\ &< 2a_n(k) - 2(n+1)\delta + 2\cdot 3^n\varepsilon - (-3^n\varepsilon + a_n(k) - n\delta) \\ &= a_n(k) - (n+2)\delta + 3^{n+1}\varepsilon. \end{aligned}$$

By the continuity of $g|_{F_n(k)\times\{-1/k\}}$ at $(y, -1/k)$, there exists a neighborhood V of y in $F_n(k)$ such that $V \subset U$ and

$$g < a_n(k) - (n+2)\delta + 3^{n+1}\varepsilon \quad \text{on } V \times \{-1/k\}.$$

By properties of $\mathcal{F}_{n+1}$ and Definition 2.2, there exists $F_{n+1} \in \mathcal{F}_{n+1}$ such that $F_{n+1} \subset V \times \{-1/k\}$ and

$$g < a_n(k) - (n+2)\delta + 3^{n+1}\varepsilon = f_m - (n+2)\delta + 3^{n+1}\varepsilon \quad \text{on } F_{n+1}.$$

This finishes the inductive step in this case.

Case 2b. If

$$g(y, 0) > f_m(y, 0) + (n+1)\delta - 3^n\varepsilon = a_n(k) + (n+1)\delta - 3^n\varepsilon,$$

[23, equations (6) in Key step 5.1] give

$$\begin{aligned} g(y, -1/k) &= 2g(y, 0) - g(y, 1/k) \\ &> 2a_n(k) + 2(n+1)\delta - 2\cdot 3^n\varepsilon - (a_n(k) + (n+2)\delta + 3^n\varepsilon) \\ &= a_n(k) + n\delta - 3^{n+1}\varepsilon. \end{aligned}$$

By the continuity of $g|_{F_n(k)\times\{-1/k\}}$ at $(y,-1/k)$, there exists a neighborhood V of y in $F_n(k)$ such that $V\subset U$ and

$$g > a_n(k) + n\delta - 3^{n+1}\varepsilon \quad \text{ on } V\times\{-1/k\}.$$

By properties of $\mathcal{F}_{n+1}$ and Definition 2.2, there exists $F_{n+1}\in\mathcal{F}_{n+1}$ such that $F_{n+1}\subset V\times\{-1/k\}$ and

$$g > a_n(k) + n\delta - 3^{n+1}\varepsilon = f_m + (n+2)\delta - 3^{n+1}\varepsilon \quad \text{ on } F_{n+1}.$$

The inductive step is finished also in this case.

After the mth step of the construction we obtain a set $F_m\in\mathcal{F}_m$ such that

$$|g-f_m| > (m+1)\delta - 3^m\varepsilon \quad \text{ on } F_m.$$

Thus

$$\|g-f_m\| > (m+1)\delta - 3^m\varepsilon = \frac{m+1}{2m+1} - 3^m\varepsilon \ge \frac{1}{2} - 3^m\varepsilon.$$

Since $\varepsilon\in(0,\delta)$ is arbitrary, $\|g-f_m\|\ge 1/2$. Hence $\operatorname{dist}(f_m,\mathcal{B}_1^b(K_m)\cap\mathcal{H}_m^{\perp\perp}) \ge 1/2$. ∎

3. Auxiliary results

LEMMA 3.1. *Let $\varphi : X\to Y$ be a continuous surjection of a compact space X onto a compact space Y and let $g: Y\to Z$ be a function from Y to a metric space (Z,ρ). Then* $\operatorname{frag}(g)=\operatorname{frag}(g\circ\varphi)$.

Proof. If $\operatorname{frag}(g)=\infty$, then $\operatorname{frag}(g\circ\varphi)\le\operatorname{frag}(g)$. Assume that $\operatorname{frag}(g)<\infty$ and let $\varepsilon>0$ be such that g is ε-fragmented. If $F\subset X$ is a nonempty closed set, let $W\subset Y$ be an open set intersecting $\varphi(F)$ such that $\operatorname{diam} g(W\cap\varphi(F))<\varepsilon$. Then $\operatorname{diam}(g\circ\varphi)(F\cap\varphi^{-1}(W))<\varepsilon$, and thus $\operatorname{frag}(g\circ\varphi)\le\operatorname{frag}(g)$.

To prove the opposite inequality, assume that $\operatorname{frag}(g\circ\varphi)<\infty$. Let $\varepsilon>0$ be such that $g\circ\varphi$ is ε-fragmented and let $H\subset Y$ be a nonempty closed set. Using compactness and Zorn's lemma, we find a closed set $F\subset X$ such that $\varphi(F)=H$ and F is a closed set which is a minimal set (with respect to inclusion) with this property. Let $U\subset X$ be an open set intersecting F with $\operatorname{diam}(g\circ\varphi)(U\cap F)<\varepsilon$. Then $H\setminus\varphi(F\setminus U)$ is a nonempty relatively open subset of H (it is nonempty by the minimality of F) satisfying

$$\operatorname{diam} g(H\setminus\varphi(F\setminus U))<\varepsilon.$$

Hence $\operatorname{frag}(g)\le\operatorname{frag}(g\circ\varphi)$, which concludes the proof. ∎

LEMMA 3.2. *Let K be a metrizable compact space and let $f\in\mathcal{U}^b(K)$. If $\widehat{f}: B_{\mathcal{M}(K)}\to\mathbb{R}$ is defined as $\widehat{f}(\mu)=\mu(f)$, $\mu\in B_{\mathcal{M}(K)}$, then* $\operatorname{frag}(f)=\operatorname{frag}(\widehat{f})$.

Proof. Let $\varepsilon > \frac{1}{2}\operatorname{frag}(f)$ be arbitrary. Using [2, Corollary 2.6], we find a function $g \in \mathcal{B}_1(K)$ such that $\|f - g\| < \varepsilon$. Without loss of generality we may assume that $\|g\| = \|f\|$. If $\widehat{g} : B_{\mathcal{M}(K)} \to \mathbb{R}$ is defined as

$$\widehat{g}(\mu) = \mu(g), \qquad \mu \in B_{\mathcal{M}(K)},$$

then $\|\widehat{f} - \widehat{g}\| < \varepsilon$. Hence $\operatorname{dist}(\widehat{f}, \mathcal{B}_1(B_{\mathcal{M}(K)})) < \varepsilon$, and thus $\frac{1}{2}\operatorname{frag}(\widehat{f}) < \varepsilon$. It follows that $\operatorname{frag}(\widehat{f}) \le \operatorname{frag}(f)$. Since the opposite inequality is obvious, the proof is complete. ∎

LEMMA 3.3. *If $\mathcal{H}$ is a function space on a metrizable compact space K and $f \in \mathcal{U}^b(K) \cap \mathcal{H}^{\perp\perp}$, then* $\operatorname{frag}(f) = \operatorname{frag}(If)$.

Proof. Let $\varepsilon > \operatorname{frag}(f)$ be arbitrary. If $\widehat{f} : \mathcal{M}^1(K) \to \mathbb{R}$ is defined as

$$\widehat{f}(\mu) = \mu(f), \qquad \mu \in \mathcal{M}^1(K),$$

then $\operatorname{frag}(\widehat{f}) < \varepsilon$ by Lemma 3.2. Since $\pi : \mathcal{M}^1(K) \to \mathbf{S}(\mathcal{H})$ is a continuous surjection, Lemma 3.1 gives $\operatorname{frag}(If) < \varepsilon$. Since $\varepsilon > \operatorname{frag}(f)$ is arbitrary, $\operatorname{frag}(If) \le \operatorname{frag}(f)$.

The opposite inequality follows from the fact that $If \circ \phi = f$. ∎

The following fact is a variant of the argument in [19, p. 88].

LEMMA 3.4. *Let $f : X \to \mathbb{R}$ be a convex function on a compact convex set X such that* $\operatorname{frag}(f) < \infty$. *Then f is lower bounded.*

Proof. Without loss of generality we may assume that $0 \in X$. Assume that there exists a sequence $\{x_n\}$ of points in X such that $f(x_n) \to -\infty$. We consider the set

$$S = \Big\{\lambda \in \ell^1 : \sum_{n=1}^{\infty} \lambda(n) \le 1,\ \lambda(n) \ge 0 \text{ for each } n \in \mathbb{N}\Big\}$$

with the weak* topology (as usual, the space ℓ^1 is identified with the dual space of c_0) and a mapping $\varphi : S \to X$ defined by

$$\varphi(\lambda) = \sum_{n=1}^{\infty} \lambda(n) x_n, \qquad \lambda \in S.$$

Then φ is a continuous affine mapping and, by Lemma 3.1,

$$\operatorname{frag}(f \circ \varphi) = \operatorname{frag}(f|_{\varphi(S)}) \le \operatorname{frag}(f) = \eta < \infty.$$

Since S is metrizable, [2, Corollary 2.6] yields the existence of a function $g \in \mathcal{B}_1(S)$ with $\|f \circ \varphi - g\| < \eta + 1$.

By [14, §27, X], g has a point of continuity, and thus there exist a nonempty open set $U \subset S$ and $C \in \mathbb{R}$ such that $g > C$ on U. We pick $\lambda \in U$ and find $t \in (0,1)$ with $t\lambda \in U$. If e_n, $n \in \mathbb{N}$, denote the standard basic vectors in ℓ^1, then $e_n \to 0$, and thus $t\lambda + (1-t)e_n \in U$ for all but finitely many $n \in \mathbb{N}$. For these indices, we obtain

$$\begin{aligned} C - \eta - 1 \le g(t\lambda + (1-t)e_n) - \eta - 1 &\le (f \circ \varphi)(t\lambda + (1-t)e_n) \\ &\le t f(\varphi(\lambda)) + (1-t) f(x_n). \end{aligned}$$

This contradiction finishes the proof. ∎

We will need the following quantitative version of [7, Proposition 2.19]. We recall that $\int^*$ and $\int_*$ denote the *upper* and *lower integral*, respectively (see [8, 133I]).

LEMMA 3.5. *Let $f : X \to \mathbb{R}$ be an affine function on a compact convex set X and $\mu \in \mathcal{M}^1(X)$. Then*

$$f(r(\mu)) - \operatorname{frag}(f) \le \int_* f\, d\mu \le \int^* f\, d\mu \le f(r(\mu)) + \operatorname{frag}(f).$$

Proof. If $\operatorname{frag}(f) = \infty$, the inequalities obviously hold. Otherwise we may assume by Lemma 3.4 that f is bounded. Let x denote the barycenter of μ. We start the proof by fixing $\eta > \operatorname{frag}(f)$. We define

$$\begin{aligned} (3.1)\qquad \mathcal{U} = \Big\{ U \subset X : U &\text{ is open and there are compact convex sets} \\ &K_n \subset X \text{ such that } \mu\Big(U \setminus \bigcup_{n=1}^{\infty} K_n\Big) = 0 \text{ and } \operatorname{diam} f(K_n) < \eta \Big\}. \end{aligned}$$

Then $V = \bigcup\{U : U \in \mathcal{U}\} \in \mathcal{U}$. Indeed, V is obviously open. Since μ is inner regular with respect to compact sets, there exists a sequence $\{H_k\}$ of compact sets such that $\mu(H_k) \nearrow \mu(V)$. By compactness, we can cover each H_k by a finite family $\{U_1, \ldots, U_{n_k}\}$ of sets contained in $\mathcal{U}$. For every $k \in \mathbb{N}$ and U_i, $i = 1, \ldots, n_k$, we find a countable family of compact convex sets guaranteed by (3.1). Putting together all these families, we obtain a countable family $\mathcal{L}$ of compact convex sets which covers μ-almost all of V and $\operatorname{diam} f(K) < \eta$ for each $K \in \mathcal{L}$.

Our aim is to prove that $X \in \mathcal{U}$. To this end, let $\mathcal{K}$ be the family of all closed convex subsets of X whose complement in X is contained in $\mathcal{U}$. Let Z be the intersection of $\mathcal{K}$. By the argument above, Z is the smallest element of $\mathcal{K}$. Set

$$Y = \{x \in Z : \operatorname{osc}_Z f(x) \ge \eta\}.$$

(Here $\operatorname{osc}_Z f(x)$ denotes the oscillation of the function $f|_Z$ at the point x.) Then Y is a closed convex subset of Z. If $x \in Z \setminus Y$, then there exists an open convex neighborhood U of x such that $\overline{U} \cap Y = \emptyset$ and $\operatorname{diam} f(\overline{U} \cap Z) < \eta$. Since $U \setminus Z \in \mathcal{U}$ and $\overline{U} \cap Z$ contains $U \cap Z$, we observe that $U \in \mathcal{U}$. By the properties of $\mathcal{U}$, Y is a closed convex subset of Z whose complement in X is contained in $\mathcal{U}$. By the minimality of Z, we have $Y = Z$.

Then there is no open set $W \subset X$ intersecting Z with $\operatorname{diam} f(W \cap Z) < \eta$. Since $\eta > \operatorname{frag}(f)$, this implies that $Z = \emptyset$. Hence $X \in \mathcal{U}$.

To finish the proof, we choose $\varepsilon > 0$. Let $\{K_n\}$ be a sequence of compact convex subsets of X such that

$$\operatorname{diam} f(K_n) < \eta \quad \text{and} \quad \mu\Big(X \setminus \bigcup_{n=1}^{\infty} K_n\Big) = 0.$$

Let $k \in \mathbb{N}$ be such that

$$\mu(X \setminus (K_1 \cup \cdots \cup K_k)) < \varepsilon \tag{3.2}$$

and let

$$E_n = K_n \setminus \bigcup_{i=1}^{n-1} K_i, \quad n = 1, \ldots, k, \qquad E_0 = X \setminus \bigcup_{n=1}^{k} K_n,$$
$$\lambda_n = \mu(E_n), \qquad n = 0, \ldots, k.$$

Without loss of generality we may assume that $\lambda_n > 0$ for $n = 1, \ldots, k$. We define probability measures μ_n, $n = 0, \ldots, k$, by

$$\mu_n = \begin{cases} \frac{1}{\lambda_n} \mu|_{E_n} & \text{if } \lambda_n > 0, \\ \varepsilon_x & \text{if } \lambda_n = 0. \end{cases}$$

Let x_n be the barycenter of μ_n, $n = 0, \ldots, k$. Then $x_n \in \overline{\operatorname{co}}\, E_n \subset K_n$, $n = 1, \ldots, k$. Obviously,

$$\sum_{n=0}^{N} \lambda_n = 1, \qquad \sum_{n=0}^{k} \lambda_n x_n = x, \qquad \sum_{n=0}^{k} \lambda_n \mu_n = \mu, \tag{3.3}$$

and

$$f(x_0) - 2\|f\| \le \int_* f \, d\mu_0 \le \int^* f \, d\mu_0 \le f(x_0) + 2\|f\|. \tag{3.4}$$

Since $\operatorname{diam} f(K_n) < \eta$, from (3.2)–(3.4) we obtain

$$\begin{aligned}
\int^* f \, d\mu &= \lambda_0 \int^* f \, d\mu_0 + \sum_{n=1}^{k} \lambda_n \int^* f \, d\mu_n \\
&\le \lambda_0 (f(x_0) + 2\|f\|) + \sum_{n=1}^{k} \int^*_{E_n} f \, d\mu \\
&\le \lambda_0 (f(x_0) + 2\|f\|) + \sum_{n=1}^{k} \int_{E_n} (f(x_n) + \eta) \, d\mu \\
&= \lambda_0 (f(x_0) + 2\|f\|) + \sum_{n=1}^{k} \lambda_n (f(x_n) + \eta) \\
&\le f\Big(\sum_{n=0}^{k} \lambda_n x_n\Big) + \eta + \varepsilon 2\|f\| = f(x) + \eta + \varepsilon 2\|f\|.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain $\int^* f\,d\mu \le f(x) + \eta$. Since η is arbitrary, $\int^* f\,d\mu \le f(x) + \mathrm{frag}(f)$. Analogously we obtain the reverse inequality $f(x) - \mathrm{frag}(f) \le \int_* f\,d\mu$, which concludes the proof. ∎

4. Proofs of the main results

Proof of Theorem 1.4. Let $f : K \to E$ be a mapping. To verify (a), we notice that the proof of the inequality

$$\tfrac{1}{2}\,\sigma\text{-}\,\mathrm{frag}_c(f) \le \mathrm{dist}(f, \mathcal{B}_1(X, E))$$

in [2, Theorem 2.5] does not require any assumption on X (here σ-$\mathrm{frag}_c(f)$ is the index of σ-fragmentability defined in [2, Definition 1]). By [2, Theorem 2.1], σ-$\mathrm{frag}_c(f) = \mathrm{frag}(f)$ for hereditarily Baire spaces and thus assertion (a) follows.

For the proof of (b), assume that f is Baire measurable. We use [10, Theorem 1] to deduce that the range $f(K)$ is K-analytic, and thus separable. Hence there exists $\alpha \in (0, \omega_1)$ such that f is $\Sigma_{\alpha+1}(\mathrm{Baire}(K))$-measurable. By [22, Corollary 5.5], f is a mapping of Baire class α (i.e., $f \in \mathcal{C}_\alpha(K, E)$). It follows that there exists a countable family $\mathcal{F} = \{f_n : n \in \mathbb{N}\} \subset \mathcal{C}(K, E)$ such that $f \in \mathcal{F}_\alpha$.

(We recall the following notation from [22, Definition 2.4]. If $\mathcal{F}$ is a family of mappings from a set X to a topological space Y, we inductively define Baire classes generated by $\mathcal{F}$ as follows: Let $\mathcal{F}_0 = \mathcal{F}$ and for each countable ordinal $\alpha \in (0, \omega_1)$, let $\mathcal{F}_\alpha$ be the family of all pointwise limits of sequences from $\bigcup_{\beta<\alpha} \mathcal{F}_\beta$.)

Let $\varphi : K \to E^{\mathbb{N}}$ be defined by

$$\varphi(x) = \{f_n(x)\}_{n\in\mathbb{N}}, \qquad x \in K.$$

Then $L = \varphi(K)$ is a compact metrizable space. Since, for $x_1, x_2 \in K$, $f_n(x_1) = f_n(x_2)$ for each $n \in \mathbb{N}$ implies $f(x_1) = f(x_2)$, there exists a mapping $g : L \to E$ such that $f = g \circ \varphi$. By Lemma 3.1, $\mathrm{frag}(f) = \mathrm{frag}(g)$. By [2, Theorem 2.5], for every $\eta > \mathrm{frag}(g)$, there exists a function $h \in \mathcal{B}_1(L, E)$ such that $\|g - h\| < \eta$. Then $\|f - h \circ \varphi\| < \eta$, and hence $\mathrm{dist}(f, \mathcal{B}_1(K, E)) < \eta$. Since $\eta > \mathrm{frag}(f)$ is arbitrary, $\mathrm{dist}(f, \mathcal{B}_1(K, E)) \le \mathrm{frag}(f)$.

If f is Baire measurable and $E = \mathbb{R}$ then we proceed as in the proof of (b) and obtain, from Lemma 3.1 and [2, Theorem 2.5],

$$\tfrac{1}{2}\,\mathrm{frag}(f) = \tfrac{1}{2}\,\mathrm{frag}(g) = \mathrm{dist}(g, \mathcal{B}_1(Y)) \ge \mathrm{dist}(f, \mathcal{B}_1(X)).$$

This concludes the proof. ∎

Theorem 4.1. *There exists a metrizable simplex X with the following property: for any $\varepsilon > 0$ there exists a strongly affine function $f : X \to [-1, 1]$ such that* $\mathrm{frag}(f) < \varepsilon$ *and* $\mathrm{dist}(f, \mathfrak{A}_1(X)) \ge 1/2$.

Proof. For each natural number $m > 1$, let $(K_m, \mathcal{H}_m)$ be the function space from Section 2 and let $f_m : K_m \to [-1,1]$ be the function from Definition 2.2. Let $K = \bigcup_{m=2}^{\infty} K_m \cup \{x_\infty\}$ be the one-point compactification of the topological union of the spaces K_m and let

$$\mathcal{H} = \{f \in \mathcal{C}(K)\colon f|_{K_m} \in \mathcal{H}_m,\ m > 1\}.$$

It is easy to verify that $(K, \mathcal{H})$ is a simplicial function space with $\operatorname{Ch}_{\mathcal{H}} K = \{x_\infty\} \cup \bigcup_{m=2}^{\infty} \operatorname{Ch}_{\mathcal{H}_m} K_m$ and $\mathcal{A}_c(\mathcal{H}) = \mathcal{H}$. If X denotes the state space of $\mathcal{H}$, we obtain a metrizable simplex (see [23, Theorem 2.6(a)]). We claim that X has the required property.

To see this, we fix $\varepsilon > 0$. Let $m > 1$ be a natural number satisfying $2(2m+1)^{-1} < \varepsilon$. If $f_m : K_m \to [-1,1]$ is as in Definition 2.2, we define $f : K \to [-1,1]$ as

$$f(x) = \begin{cases} f_m(x), & x \in K_m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Let $I : \mathcal{U}^b(K) \cap \mathcal{H}^{\perp\perp} \to \mathfrak{A}_{\mathrm{bf}}(X)$ be the identification from (1.1). Since $\operatorname{frag}(f) = 2(2m+1)^{-1}$, $\operatorname{frag}(If) < \varepsilon$ by Lemma 3.3. If g is any function in $\mathfrak{A}_1(X)$, it follows from [23, Theorem 2.5(f)] that $I^{-1}g \in \mathcal{B}_1^b(K) \cap \mathcal{H}^{\perp\perp}$. Then $I^{-1}g|_{K_m} \in \mathcal{B}_1^b(K_m) \cap \mathcal{H}_m^{\perp\perp}$, and thus $\|f - I^{-1}g\| \geq 1/2$ by Lemma 2.3(d). Hence $\|If - g\| \geq 1/2$, and the proof is complete. ∎

Proof of Theorem 1.1. Let $(K, \mathcal{H})$ be the simplicial function space constructed in the proof of Theorem 4.1 and let X be the state space of $\mathcal{H}$. Then $\mathcal{H}$ is isometrically isomorphic to $\mathfrak{A}^c(X)$ via the mapping I, and thus it is a separable L_1-predual (see [7, Proposition 3.23]). Given $\varepsilon > 0$, let $m > 1$ be a natural number with $(2m+1)^{-1} < \varepsilon$ and let $f : K \to [-1,1]$ be the function from (4.1). If $\pi : B_{\mathcal{M}(K)} \to B_{\mathcal{H}^*}$ is the restriction mapping, let $\widehat{f} : B_{\mathcal{H}^*} \to [-1,1]$ be defined as

$$\widehat{f}(s) = \mu(f), \qquad \pi(\mu) = s, \qquad s \in B_{\mathcal{H}^*}.$$

Obviously, $\widehat{f}$ is a restriction of an element from $\mathcal{H}^{**}$ to $B_{\mathcal{H}^*}$. By Lemmas 3.2 and 3.1,

$$\operatorname{dist}(\widehat{f}, \mathcal{B}_1(B_{E^*})) = \tfrac{1}{2}\operatorname{frag}(\widehat{f}) = (2m+1)^{-1} < \varepsilon.$$

By [23, Theorem 2.5(a)], $\widehat{f} \in \mathfrak{A}_{\mathrm{bf}}(B_{\mathcal{H}^*})$. Finally,

$$\tfrac{1}{2} \leq \operatorname{dist}(If, \mathfrak{A}_1(\mathbf{S}(\mathcal{H}))) \leq \operatorname{dist}(\widehat{f}, \mathfrak{A}_1(B_{\mathcal{H}^*})).$$

This concludes the proof. ∎

Proof of Theorem 1.2. Let E be an L_1-predual such that $\operatorname{ext} B_{E^*}$ is a Lindelöf H-set and let $f : B_{E^*} \to \mathbb{R}$ be the restriction of an element $x^{**} \in E^{**}$. By [17, Theorem], there exists a simplex X, an isometric embedding $j : E \to \mathfrak{A}^c(X)$ and a projection $P : \mathfrak{A}^c(X) \to j(E)$ of norm 1. Moreover, if E is separable, X can be chosen to be metrizable. Further, it is proved in [17, Corollary III] that there exists an affine continuous surjection $\varphi : X \to B_{E^*}$ such that

(1) $\varphi(\operatorname{ext} X) = \operatorname{ext} B_{E^*} \cup \{0\}$ and $\varphi^{-1}(\operatorname{ext} B_{E^*}) \subset \operatorname{ext} X$,
(2) $\varphi|_{\operatorname{ext} X}$ is injective,
(3) $\operatorname{ext} X \setminus \varphi^{-1}(\operatorname{ext} B_{E^*})$ is a singleton,
(4) $j(e)(x) = (e|_{B_{E^*}} \circ \varphi)(x)$, $e \in E$, $x \in X$.

(In the notation of [17], the embedding j is denoted by T and φ is denoted by q. Conditions (1), (2) and (3) are explicitly stated in [17, Corollary III], condition (4) follows from the definitions of T on p. 175 and q on p. 176.)

We claim that $\operatorname{ext} X$ is a Lindelöf H-set. To show this, we first observe that $\operatorname{ext} X$ differs from the H-set $\varphi^{-1}(\operatorname{ext} B_{E^*})$ by a singleton (see (1) and (3)), and thus it is an H-set. Second, let $F \subset X \setminus \operatorname{ext} X$ be a compact set. By (1), $\varphi(F)$ is disjoint from $\operatorname{ext} B_{E^*}$. Since $\operatorname{ext} B_{E^*}$ is Lindelöf, [24, Lemma 14] provides an F_σ set A with

$$\operatorname{ext} B_{E^*} \subset A \subset \operatorname{ext} B_{E^*} \setminus \varphi(F).$$

If $x_0 \in X$ is the singleton $\operatorname{ext} X \setminus \varphi^{-1}(\operatorname{ext} B_{E^*})$, then $\varphi^{-1}(A)$ is an F_σ set in X satisfying

$$\operatorname{ext} X \subset \varphi^{-1}(A) \cup \{x_0\} \subset X \setminus F.$$

By [24, Lemma 15], $\operatorname{ext} X$ is a Lindelöf space.

If f is a Baire measurable function on B_{E^*}, then $f \circ \varphi$ is Baire measurable on X. If E is separable, X is metrizable. In both cases, Lemma 3.1 and Theorem 1.4 give

$$\operatorname{dist}(f \circ \varphi, \mathcal{B}_1(X)) = \tfrac{1}{2} \operatorname{frag}(f \circ \varphi) = \tfrac{1}{2} \operatorname{frag}(f) = \operatorname{dist}(f, \mathcal{B}_1(B_{E^*})).$$

We fix $\eta > \operatorname{dist}(f, \mathcal{B}_1(B_{E^*}))$. Let $g \in \mathcal{B}_1(X)$ satisfy $\|f \circ \varphi - g\| < \eta$. Without loss of generality we may assume that $\|f \circ \varphi\| = \|g\|$. By [24, Theorem 1], there exists a function $h \in \mathfrak{A}_1(X)$ such that $h = g$ on $\operatorname{ext} X$ and $\|h\| = \|g\|$.

We claim that $\|h - f \circ \varphi\| \leq 3\eta$. To this end, let $x \in X$ be given. We find a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$ (see [1, Proposition I.2.1]). If f is Baire measurable, the set

$$F = \{x \in X : |h(x) - (f \circ \varphi)(x)| \leq \eta\}$$

is a Baire set in X containing $\operatorname{ext} X$. By [1, Corollary I.4.12 and the subsequent Remark], $\mu(X \setminus F) = 0$. Hence, by Lemma 3.5,

$$h(x) = \mu(h) = \int_F h \, d\mu \leq \int_F^* f \circ \varphi \, d\mu + \eta \leq (f \circ \varphi)(x) + \eta + 2\eta.$$

If E is separable, X is metrizable, and thus $\mu(X \setminus \operatorname{ext} X) = 0$ (see [1, Corollary I.4.12]). As above we obtain

$$\mu(h) = \int_{\operatorname{ext} X} h \, d\mu \le \int_{\operatorname{ext} X}^{*} f \circ \varphi \, d\mu \le (f \circ \varphi)(x) + 3\eta.$$

Analogously,

$$h(x) \ge (f \circ \varphi)(x) - 3\eta.$$

Thus $\|h - f \circ \varphi\| \le 3\eta$.

Since $P : \mathfrak{A}^c(X) \to j(E)$ is a projection of norm 1, to any $x \in X$ we can assign a measure $\mu_x \in B_{\mathcal{M}(X)}$ such that

$$Pf(x) = \mu_x(f), \qquad f \in \mathfrak{A}^c(X). \tag{4.2}$$

Since P is identity on $j(E)$, we obtain

$$\mu_x(e|_{B_{E^*}} \circ \varphi) = (e|_{B_{E^*}} \circ \varphi)(x), \qquad x \in X,\ e \in E.$$

We use equality (4.2) to extend the domain of P to any bounded universally measurable function on X.

We claim that

$$|\mu_x(h) - f(\varphi(x))| \le 5\eta, \qquad x \in X. \tag{4.3}$$

To verify this, let $x \in X$ be given. We write

$$\mu_x = a_1\mu_1 - a_2\mu_2, \qquad a_1, a_2 \ge 0 \text{ with } a_1 + a_2 \le 1,\ \mu_1, \mu_2 \in \mathcal{M}^1(X),$$

and let $x_1, x_2 \in X$ be the barycenters of μ_1, μ_2, respectively. Then

$$\varphi(x) = a_1\varphi(x_1) - a_2\varphi(x_2). \tag{4.4}$$

Indeed, if $e \in E$ is arbitrary, let $\widehat{e}$ denote its restriction to B_{E^*}. Let $\varphi_\sharp : \mathcal{M}^1(X) \to \mathcal{M}^1(B_{E^*})$ denote the mapping induced by $\varphi : X \to B_{E^*}$ (see [9, Theorems 418I and 418L]). Then

$$\begin{aligned}
\widehat{e}(\varphi(x)) &= \mu_x(\widehat{e} \circ \varphi) = a_1\mu_1(\widehat{e} \circ \varphi) - a_2\mu_2(\widehat{e} \circ \varphi) \\
&= a_1(\varphi_\sharp\mu_1)(\widehat{e}) - a_2(\varphi_\sharp\mu_2)(\widehat{e}) = a_1\mu_1(\widehat{e} \circ \varphi) - a_2\mu_2(\widehat{e} \circ \varphi) \\
&= a_1\widehat{e}(\varphi(x_1)) - a_2\widehat{e}(\varphi(x_2)) = \widehat{e}(a_1\varphi(x_1) - a_2\varphi(x_2)).
\end{aligned}$$

Hence (4.4) holds.

Further, by Lemma 3.5,

$$\begin{aligned}
\mu_1(h) &\le \int_X^{*} f \circ \varphi \, d\mu_1 + 3\eta = \int_{B_{E^*}}^{*} f \, d(\varphi_\sharp\mu_1) + 3\eta \\
&\le f(r(\varphi_\sharp\mu_1)) + 3\eta + 2\eta = f(\varphi(x_1)) + 5\eta.
\end{aligned}$$

Analogously,

$$\mu_1(h) \ge f(\varphi(x_1)) - 5\eta.$$

Hence

$$|\mu_1(h) - f(\varphi(x_1))| \le 5\eta.$$

Similarly we obtain

$$|\mu_2(h) - f(\varphi(x_2))| \le 5\eta.$$

By combining these inequalities and (4.4) we have

$$\begin{aligned}|\mu_x(h) - f(\varphi(x))| &= |a_1\mu_1(h) - a_2\mu_2(h) - f(a_1\varphi(x_1) - a_2\varphi(x_2))|\\ &= |a_1(\mu_1(h) - f(\varphi(x_1))) - a_2(\mu_2(h) - f(\varphi(x_2)))|\\ &\le 5\eta(a_1 + a_2) = 5\eta.\end{aligned}$$

This gives (4.3).

If $\{h_n\}$ is a bounded sequence in $\mathfrak{A}^c(X)$ pointwise converging to h, the Lebesgue bounded convergence theorem implies that $Ph_n \to Ph$. Since $Ph_n \in j(E)$, there exist elements $e_n \in E$, $n \in \mathbb{N}$, such that

$$Ph_n = e_n|_{B_{E^*}} \circ \varphi, \qquad n \in \mathbb{N}.$$

Then $\{e_n|_{B_{E^*}}\}$ converges to a function $e \in \mathfrak{A}_1(B_{E^*})$. It follows that $Ph = e|_{B_{E^*}} \circ \varphi$ and, by (4.3),

$$\|e|_{B_{E^*}} - f\| = \|e|_{B_{E^*}} \circ \varphi - f \circ \varphi\| = \|Ph - f \circ \varphi\| \le 5\eta.$$

Hence $\operatorname{dist}(f, \mathfrak{A}_1(B_{E^*})) \le 5\eta$. Since $\eta > \operatorname{dist}(f, \mathcal{B}_1(B_{E^*}))$ is arbitrary, we obtain

$$\operatorname{dist}(f, \mathfrak{A}_1(B_{E^*})) \le 5 \operatorname{dist}(f, \mathcal{B}_1(B_{E^*})). \ \blacksquare$$

THEOREM 4.2. *Let X be a compact convex set such that $\mathfrak{A}^c(X)$ does not contain ℓ^1 and $f : X \to \mathbb{R}$ be an affine function. Then* $\operatorname{dist}(f, \mathfrak{A}_1(X)) \le 2 \operatorname{dist}(f, \mathcal{B}_1(X))$.

Proof. If $\operatorname{dist}(f, \mathcal{B}_1(X)) = \infty$, the assertion obviously holds. We assume that $\operatorname{dist}(f, \mathcal{B}_1(X)) < \infty$ and fix $\eta > \operatorname{dist}(f, \mathcal{B}_1(X))$. By Theorem 1.4 and Lemma 3.4, f is bounded. We find a function $g \in \mathcal{B}_1(X)$ such that $\|f - g\| < \eta$. Without loss of generality we may assume that $\|g\| = \|f\|$. It is easy to find (see e.g. [16, Exercise 3.G.1]) sequences $\{u_n\}$ and $\{l_n\}$ of functions on X such that every u_n is upper semicontinuous, every l_n is lower semicontinuous and

$$-\|g\| \le u_n \nearrow g, \qquad \|g\| \ge l_n \searrow g.$$

We fix $n \in \mathbb{N}$ and $x \in X$. By [1, Corollary I.3.6], there exist measures $\mu_1, \mu_2 \in \mathcal{M}^1(X)$ representing x such that

$$(u_n - \eta)^*(x) = \mu_1(u_n - \eta) \quad \text{and} \quad (l_n + \eta)_*(x) = \mu_2(l_n + \eta).$$

(We recall that f^* and f_* are the upper and lower envelopes of a function f, respectively; see [1, p. 4].) By [11, Theorem 4.2], f is universally measurable and $\mu(f) = f(r(\mu))$ for every $\mu \in \mathcal{M}^1(X)$. (Here we use the identification of $\mathfrak{A}^b(X)$ with $(\mathfrak{A}^c(X))^{**}$.) Hence

$$(u_n-\eta)^*(x)=\mu_1(u_n-\eta)<\int f\,d\mu_1=f(x),$$
$$(l_n+\eta)_*(x)=\mu_2(l_n+\eta)>\int f\,d\mu_2=f(x).$$

Since the upper envelope is an upper semicontinuous concave function and the lower envelope is a lower semicontinuous convex function (see [1, p. 4]), the Hahn–Banach theorem provides a function $h_n\in\mathfrak{A}^c(X)$ such that

$$(u_n-\eta)^*<h_n<(l_n+\eta)_*.$$

Since $\mathfrak{A}^c(X)$ does not contain ℓ^1, Rosenthal's theorem (see [12, p. 18]) provides a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ that converges pointwise to a function h. Then $h\in\mathfrak{A}_1(X)$ and

$$g-\eta=\lim_{k\to\infty}u_{n_k}-\eta\le h\le\lim_{k\to\infty}l_{n_k}+\eta=g+\eta.$$

Since $\|g-f\|<\eta$, we obtain

$$\|f-h\|<2\eta.$$

Since $\eta>\operatorname{dist}(f,\mathcal{B}_1(X))$ is arbitrary, the proof is finished. ∎

Proof of Theorem 1.3. This follows from Theorem 4.2. ∎

Acknowledgements. The author would like to thank his colleagues at the Faculty of Mathematics and Physics for a substantial help during the work on the paper. He would also like to thank the referee for several helpful comments.

The author was supported in part by the grant GAAV IAA 100190901 and in part by the Research Project MSM 0021620839 from the Czech Ministry of Education.

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Received December 14, 2009
Revised version January 29, 2010 (6769)