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Homomorphisms on algebras of Lipschitz functions

by

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Abstract. We characterize a class of $*$-homomorphisms on $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$, a non-commutative Banach $*$-algebra of Lipschitz functions on a compact metric space and with values in $\mathcal{B}(\mathcal{H})$. We show that the zero map is the only multiplicative $*$-preserving linear functional on $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$. We also establish the algebraic reflexivity property of a class of $*$-isomorphisms on $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$.

1. Introduction. We consider a compact metric space (X, d) and

$$(1.1)\qquad \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) = \left\{ f : X \to \mathcal{B}(\mathcal{H}) \;\middle|\; \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)} < \infty \right\}$$

with the norm

$$\|f\|_* = \|f\|_\infty + \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)}.$$

The space $\mathcal{B}(\mathcal{H})$ represents the bounded operators on a separable complex Hilbert space $\mathcal{H}$. We set $f^*(x) = [f(x)]^*$ for all $x \in X$. Since $\|f^*\|_* = \|f\|_*$ and $\|fg\|_* \le \|f\|_* \|g\|_*$, $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ is a non-commutative Banach $*$-algebra with identity.

We denote by $\mathrm{Const}_*(X, \mathcal{B}(\mathcal{H}))$ the $*$-subalgebra of all constant operator valued functions on X. This subalgebra is a C^*-algebra. In this paper, we derive a characterization of a class of $*$-algebra homomorphisms from $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ into $\mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$, with X and Y compact metric spaces. We recall that a $*$-*homomorphism*, ψ, on $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ is an algebra homomorphism that satisfies $\psi(f^*) = \psi(f)^*$ for all f.

Since it is not known whether $*$-homomorphisms in this setting are continuous we introduce a weaker continuity hypothesis, called "ps-continuity".

DEFINITION 1.1. We say that a sequence of functions in $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$, $\{f_n\}$, is *ps-convergent* to f if $\|(f_n(x) - f(x))\mathbf{v}\|_{\mathcal{H}} \to 0$ for every $x \in X$ and

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$\mathbf{v} \in \mathcal{H}$. We then write

$$f_n \rightharpoonup f.$$

We say that ψ is *ps-continuous* at $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ if the sequence $\{\psi(f_n)\}$ is ps-convergent to $\psi(f)$, for every sequence $\{f_n\}$ which is ps-convergent to f:

$$f_n \rightharpoonup f \;\Rightarrow\; \psi(f_n) \rightharpoonup \psi(f).$$

Furthermore, if ψ is a $*$-isomorphism, we say that ψ is a *ps-homeomorphism* if both ψ and ψ^{-1} are ps-continuous.

We remark that if $f_n \to f$ relative to the norm $\|\cdot\|_*$, then

$$\|f_n - f\|_\infty \to 0 \quad \text{and} \quad f_n \rightharpoonup f.$$

Our main theorem characterizes all ps-continuous $*$-homomorphisms between Lipschitz algebras of the type described in (1.1).

We first introduce some additional notation and terminology. We denote by id_X the function defined on X and everywhere equal to $\mathrm{Id}_{\mathcal{H}}$, the identity on $\mathcal{H}$. We say that a $*$-homomorphism $\psi : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$ *preserves constant functions* if $\psi(\mathrm{Id}_X) = \mathrm{Id}_Y$ and

$$\psi(\mathrm{Const}_*(X, \mathcal{B}(\mathcal{H}))) \subseteq \mathrm{Const}_*(Y, \mathcal{B}(\mathcal{H})).$$

We also say that ψ *fixes constant functions* if $\psi(\tilde{A})(y) = A$ for every $y \in Y$ and $A \in \mathcal{B}(\mathcal{H})$, with $\tilde{A}$ the constant function everywhere equal to A.

A family of partial isometries on $\mathcal{H}$, say $\{U_\alpha\}_{\alpha\in\Lambda}$, with $\mathcal{H}_\alpha$ denoting the range of U_α, is said to be *orthogonally ranged* if $\mathcal{H}_\alpha$ is orthogonal to $\mathcal{H}_\beta$ for $\alpha \neq \beta$, and for each $u \in \mathcal{H}$, $u = \sum_\alpha u_\alpha$ with $u_\alpha \in \mathcal{H}_\alpha$ (cf. [13] and [15]). We also observe that the adjoint of U_α, $U_\alpha^* : \mathcal{H}_\alpha \to \mathcal{H}$, satisfies $U_\alpha U_\alpha^* = \mathrm{Id}_{\mathcal{H}_\alpha}$ and $U_\alpha^* U_\alpha = \mathrm{Id}_{\mathcal{H}}$.

We denote by $\langle\ ,\ \rangle$ the inner product in $\mathcal{H}$. We now state our main result.

THEOREM 1.2. *Let X and Y be compact metric spaces. If $\psi : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$ is a ps-continuous $*$-algebra homomorphism that preserves constant functions then there exist a Lipschitz function $\varphi : Y \to X$ and a countable orthogonally ranged family of partial isometries $U_n : \mathcal{H} \to \mathcal{H}$, $n = 1, 2, \ldots,$ such that*

$$\psi(f)(y) = \sum_n U_n f(\varphi(y)) U_n^* \quad \text{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \text{ and } y \in Y. \tag{1.2}$$

The proof of this theorem uses the following characterization of $*$-homomorphisms on $\mathcal{B}(\mathcal{H})$, due to Molnár.

THEOREM 1.3 (cf. [14]). *Let $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a continuous $*$-homomorphism. If ψ is spectrum non-increasing, then there is a countable orthogonally ranged family of partial isometries $U_n : \mathcal{H} \to \mathcal{H}$ $(n = 1, 2, \ldots)$ with* *range $\mathcal{H}_n$ such that ψ is of the form*

$$\psi(A) = \sum_n U_n A U_n^* \qquad (A \in \mathcal{B}(\mathcal{H})). \tag{1.3}$$

We also use the characterization of algebra homomorphisms on scalar valued Lipschitz spaces due to Sherbert. Although we use Sherbert's theorem, our proof techniques are necessarily different to account for the noncommutative setting.

THEOREM 1.4 (cf. [22]). *Let X and Y be compact metric spaces. If $\psi : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$ is an algebra homomorphism then there exists a unique Lipschitz function $\varphi : Y \to X$ such that*

$$\psi(f)(y) = f(\varphi(y)) \quad \text{for all } f \in \mathrm{Lip}_*(X) \text{ and } y \in Y. \tag{1.4}$$

If $\psi : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$ is an algebra isomorphism then φ is a lipeomorphism.

2. Proof of the main theorem. In this section we prove Theorem 1.2. We first prove that a ps-continuous $*$-homomorphism that fixes constant functions is a composition operator. For simplicity of notation we denote a constant function everywhere equal to $A \in \mathcal{B}(\mathcal{H})$ simply by A. We also denote the resolvent set of an operator A by $\rho(A)$, i.e.

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \, \mathrm{Id}_{\mathcal{H}} \text{ is invertible}\}.$$

PROPOSITION 2.1. *Let X and Y be compact metric spaces. Let $\psi : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$ be a mapping that fixes constant functions. If ψ is a ps-continuous $*$-homomorphism then there exists a unique Lipschitz function $\varphi : Y \to X$ such that*

$$\psi(f)(y) = f(\varphi(y)) \quad \text{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \text{ and } y \in Y. \tag{2.1}$$

If $\psi : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$ is a ps-homeomorphism and a $*$-isomorphism then φ is a lipeomorphism.

Proof. Given $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ and $y \in Y$ we set

$$A_{f,y} = \{x \in X : 0 \notin \rho(f(x) - \psi(f)(y))\}.$$

(i) $A_{f,y} \neq \emptyset$.

Suppose that $A_{f,y} = \emptyset$. For every $x \in X$, $0 \in \rho(f(x) - \psi(f)(y))$. Equivalently, $f(x) - \psi(f)(y)$ is invertible. We define $h : X \to \mathcal{B}(\mathcal{H})$ by $h(x) = [f(x) - \psi(f)(y)]^{-1}$. It is easy to see that $h \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$. Firstly, the continuity of h is a consequence of Theorem 10.11 in [17]. Secondly,

$$\frac{\|h(x_0) - h(x_1)\|}{d(x_0, x_1)} \leq \|h\|_\infty^2 \sup_{x_0 \neq x_1} \frac{\|f(x_0) - f(x_1)\|}{d(x_0, x_1)} < \infty.$$

We set $g(x) = f(x) - \psi(f)(y)$. The function $g \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ and $hg = \mathrm{Id}_X$ (with $\mathrm{Id}_X(x) = \mathrm{Id}_{\mathcal{H}}$, the identity operator on $\mathcal{H}$). Therefore $\psi(hg) = \psi(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{H}}$, but $\psi(g)(y) = 0$. This contradiction establishes (i).

(ii) *The family* $\{A_{f,y} : f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))\}$ *has the finite intersection property.*

Given a finite set of functions in $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$, say $\{f_1, \ldots, f_k\}$, we set

$$g(x) = \sum_{i=1}^{k} [f_i(x) - \psi(f_i)(y)]^* [f_i(x) - \psi(f_i)(y)].$$

It is clear that $g \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$. Claim (i) asserts that $A_{g,y} \neq \emptyset$. Therefore there exists x_0 such that $0 \notin \rho(g(x_0) - \psi(g)(y))$. Since $\psi(g)(y) = 0$ we have $0 \notin \rho(g(x_0))$. Furthermore, since g has range in the space of hermitian operators, there must exist a sequence $\{v_n\}$ of unit vectors such that $\|g(x_0)v_n\| \to 0$. We have

$$\sum_{i=1}^{k} \|(f_i(x_0) - \psi(f_i)(y))v_n\|^2$$
$$= \Big\langle \sum_{i=1}^{k} [f_i(x_0) - \psi(f_i)(y)]^* [f_i(x_0) - \psi(f_i)(y)] v_n, v_n \Big\rangle \leq \|g(x_0)v_n\|.$$

This implies that for every i, $\|(f_i(x_0) - \psi(f_i)(y))v_n\| \to 0$ and $0 \in \rho(f_i(x_0) - \psi(f_i)(y))$, that is, $x_0 \in \bigcap_{i=1}^{k} A_{f_i,y}$. This proves (ii).

(iii) $\bigcap_{f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))} A_{f,y}$ *is a singleton.*

The compactness of X and the finite intersection property of $\{A_{f,y} : f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))\}$ imply that $A_y = \bigcap_{f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))} A_{f,y}$ is non-empty. Now we suppose that there exist x_1 and x_2 in A_y. We define $f(x) = d(x, x_1)\,\mathrm{Id}_X$. We find that $0 \notin \rho(f(x_1) - \psi(f)(y)) = \rho(-\psi(f)(y))$ and $0 \notin \rho(f(x_2) - \psi(f)(y))$. Then there exists a sequence of unit vectors, say $\{v_n\}$, such that $\|\psi(f)(y)v_n\| \to 0$. We fix $w \in \mathcal{H}$, a vector of norm 1. For each n, let V_n be a unitary operator such that $V_n v_n = w$. Thus $\psi(f)V_n = V_n\psi(f)$. Consequently,

$$\|\psi(f)(y)w\| = \|\psi(f)(y)V_n v_n\| = \|V_n\psi(f)(y)v_n\| = \|\psi(f)(y)v_n\| \to 0.$$

This implies that $\psi(f)(y)$ is the zero function. Hence $0 \notin \rho(f(x_2))$ and $x_1 = x_2$. This proves (iii).

We denote by x_y the only element in $\bigcap_{f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))} A_{f,y}$. We define $\varphi : Y \to X$ by $\varphi(y) = x_y$.

Let $\mathcal{E} = \{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and $P_i = e_i \otimes e_i$ a rank one projection with range the span of e_i. We now consider a real valued function λ in $\mathrm{Lip}_*(X)$ and set $f(x) = \lambda(x)P_i$. Since $P_i f = fP_i = f$ we have $P_i\psi(f)(y) = \psi(f)(y)P_i = \psi(f)(y)$. Moreover $\psi(f)(y)e_k = \sum_j \alpha_j^k e_j$ with $\alpha_j^k = \langle \psi(f)(y)e_k, e_j\rangle$. Therefore $P_i\psi(f)(y)e_k = \alpha_i^k e_i = \sum_j \alpha_j^k e_j$. Thus $\alpha_j^k = 0$ for $j \neq i$. Since $\psi(f)(y)e_k = \alpha_i^k e_i$, we also have $\psi(f)(y)P_i e_k = 0$ for $i \neq k$. Hence $\alpha_i^k = 0$ for $i \neq k$. This implies that $\psi(f)(y) = \alpha_i(y)P_i$.

For a fixed i, the map ψ induces an algebra homomorphism

$$\tau_i^{\mathcal{E}} : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y), \qquad \lambda \mapsto \tau_i^{\mathcal{E}}(\lambda),$$

given by $\tau_i^{\mathcal{E}}(\lambda)(y) = \langle \psi(\lambda P_i)(y)e_i, e_i\rangle$. We use the superscript $\mathcal{E}$ to emphasize the dependence on the orthonormal basis. Theorem 1.4 asserts that there exists a Lipschitz function $\varphi_i^{\mathcal{E}} : Y \to X$ such that

$$\tau_i^{\mathcal{E}}(\lambda)(y) = \lambda(\varphi_i^{\mathcal{E}}(y)).$$

On the other hand, given $g(x) = \lambda(x)\,\mathrm{Id}_X$, $0 \notin \rho(g(\varphi(y)) - \psi(g)(y))$. This implies the existence of a sequence $\{v_n\}$ of unit vectors such that

$$\lim_n \|(g(\varphi(y)) - \psi(g)(y))v_n\| = 0.$$

Similar techniques to those employed in the proof of claim (iii) allow us to conclude that $g(\varphi(y)) = \psi(g)(y)$. We set $g_n(x) = \lambda(x)\sum_{i=1}^n P_i$; then $g_n \rightharpoonup g$. The continuity assumption on ψ implies that $\psi(g_n) \rightharpoonup \psi(g)$. Therefore we have

$$\psi(g_n)(y) = \sum_{i=1}^{n} \lambda(\varphi_i^{\mathcal{E}}(y))P_i \rightharpoonup \lambda(\varphi(y))\,\mathrm{Id}_{\mathcal{H}}\,. \tag{2.2}$$

We claim that

($\star$) $\varphi_i^{\mathcal{E}}(y) = \varphi(y)$ for all i and $y \in Y$.

Suppose that there exist i_0 and y_0 such that $\varphi_{i_0}^{\mathcal{E}}(y_0) \neq \varphi(y_0)$. Let $\lambda(x) = d(x, \varphi_{i_0}^{\mathcal{E}}(y_0))$. Equation (2.2) implies that

$$\|\psi(g_n)(y_0)e_{i_0}\| \to \lambda(\varphi(y_0)) \neq 0.$$

This contradiction establishes claim ($\star$). Moreover, this also implies that φ is Lipschitz and $\tau_i^{\mathcal{E}}$ is independent of the orthonormal basis. We conclude that given a function of the form λP with λ a scalar valued function in $\mathrm{Lip}_*(X)$ and P a projection on $\mathcal{H}$, we have

$$\psi(\lambda P)(y) = \lambda(\varphi(y))P.$$

We extend this representation to functions of the form λA with λ a real valued Lipschitz function and A a hermitian operator in $\mathcal{B}(\mathcal{H})$. We use the spectral representation for hermitian operators (see [6]) to set

$$A = \int_m^{M+\epsilon} \lambda\, dE(\lambda)$$

with $m = \inf_{\|v\|=1} \|Av\|$, $M = \sup_{\|v\|=1} \|Av\|$, $0 < \epsilon < 1$ and $\{E(\lambda)\}$ a spectral family of projections associated with A. We define $E(\triangle_k) = E(\lambda_k) - E(\lambda_{k-1})$, with $m = \lambda_0 < \lambda_1 < \cdots < \lambda_n = M + \epsilon$. The sequence $\{\sum_{k=1}^{n} \lambda_k \, E(\triangle_k)\}_n$ converges uniformly to A. The continuity assumption on ψ and the representation previously derived imply that $\psi(f)(y) = \lambda(\varphi(y))A$. An arbitrary operator A has the representation

$$A = \frac{A + A^*}{2} + i \, \frac{A - A^*}{2i},$$

as the linear combination of two hermitian operators. Therefore given a complex valued Lipschitz function λ and a bounded operator A on $\mathcal{H}$, we have

$$\psi(\lambda A)(y) = \lambda(\varphi(y))A.$$

Now given a function $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$, f is clearly continuous relative to the $\|\cdot\|_\infty$. It is shown in [2, p. 224] (see also [11, Theorem 1.13, p. 9]) that the tensor product space $\mathcal{C}(X) \otimes \mathcal{B}(\mathcal{H})$, with the least crossnorm, is dense in $\mathcal{C}(X, \mathcal{B}(\mathcal{H}))$, the space of all continuous $\mathcal{B}(\mathcal{H})$ valued functions equipped with the $\|\cdot\|_\infty$. Therefore, there exists a sequence $\{F_n\}$ in $\mathcal{C}(X) \otimes \mathcal{B}(\mathcal{H})$ that converges uniformly to f. We identify the space $\mathcal{C}(X) \otimes \mathcal{B}(\mathcal{H})$ with all the functions of the form $\sum_{i=1}^{m} \lambda_i A_i$ with $\lambda_i \in \mathcal{C}(X)$. Each function F_n is represented as follows:

$$F_n(x) = \sum_{i=1}^{k_n} \lambda_i^n(x) A_i^n$$

with $\lambda_i^n \in \mathcal{C}(X)$ and $A_i^n \in \mathcal{B}(\mathcal{H})$. Without loss of generality we may assume that λ_i^n are Lipschitz functions (see [8, Theorem 6.8]). Once more, the ps-continuity of ψ allows us to conclude that

$$\psi(f)(y) = f(\varphi(y))$$

for every $y \in Y$ and $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$.

It is easy to show that φ is unique. This concludes the proof of the first statement in the proposition.

If ψ is an isomorphism then ψ and ψ^{-1} are both composition operators of the form

$$\psi(f)(y) = f(\varphi(y)) \quad \text{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \text{ and } y \in Y,$$
$$\psi^{-1}(g)(x) = g(\lambda(x)) \quad \text{for all } g \in \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H})) \text{ and } x \in X,$$

where $\varphi : Y \to X$ and $\lambda : X \to Y$ are Lipschitz functions. Hence $\varphi \circ \lambda = \mathrm{Id}$ and $\lambda \circ \varphi = \mathrm{Id}$. Therefore φ is a lipeomorphism. ∎

In the proof of Theorem 1.2 we use the characterization of a class of $*$-homomorphisms of $\mathcal{B}(\mathcal{H})$ due to Molnár (see Theorem 1.3). Moreover, we apply a basic result on $*$-homomorphisms between C^*-algebras that we state first.

THEOREM 2.2 (cf. [10, Theorems 4.1.8 and 4.1.9]). *Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are C^*-algebras and ϕ is a $*$-homomorphism from $\mathcal{C}_1$ into $\mathcal{C}_2$. Then for each $A \in \mathcal{C}_1$, $\mathrm{sp}(\phi(A)) \subseteq \mathrm{sp}(A)$, $\|\phi(A)\| \le \|A\|$ and $\phi(\mathcal{C}_1)$ is a C^*-subalgebra of $\mathcal{C}_2$.*

Proof of Theorem 1.2. We fix $y \in Y$ and define $T_y : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $T_y(A) = \psi(A)(y)$. It is clear that T_y is a $*$-homomorphism on $\mathcal{B}(\mathcal{H})$. In fact, T_y is independent of y, since ψ preserves constants. Theorem 2.2 implies that T_y satisfies the hypotheses of Theorem 1.3. Consequently, there exists an orthogonally ranged family of partial isometries U_n $(n = 1, 2, \ldots)$ such that $T_y(A) = \sum_n U_n A U_n^*$ for all $y \in Y$. We now define $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ as follows: $\tau(f)(y) = \sum_n U_n^* \psi(f)(y) U_n$. The mapping τ is a $*$-homomorphism that leaves invariant all the constant functions. Proposition 2.1 applies and so $\tau(f)(y) = f(\varphi(y))$ for some Lipschitz function $\varphi : Y \to X$. Ergo, $\psi(f)(y) = \sum_n U_n f(\varphi(y)) U_n^*$. ∎

REMARK 2.3. If ψ is a $*$-isomorphism and a ps-homeomorphism, then φ is a lipeomorphism. Furthermore $\psi(\mathrm{Id}_X) = \mathrm{Id}_Y$. In fact, for every $A \in \mathcal{B}(\mathcal{H})$, we denote by $\tilde{A}$ the Lipschitz function everywhere equal to A. There exists $f_A \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ such that $\psi(f_A) = \tilde{A}$. Therefore $\psi(f_A \,\mathrm{Id}_X) = A\psi(\mathrm{Id}_X) = \psi(\mathrm{Id}_X)A = A$. For every $y \in Y$ we have $A\psi(\mathrm{Id}_X)(y) = \psi(\mathrm{Id}_X)(y)A = A$. This implies that $\psi(\mathrm{Id}_X)(y) = \mathrm{Id}_{\mathcal{H}}$.

COROLLARY 2.4. *Let X and Y be compact metric spaces. If $\psi : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$ is a $*$-algebra isomorphism that preserves constant functions and ψ is a ps-homeomorphism, then there exist a lipeomorphism $\varphi : Y \to X$ and a unitary $U : \mathcal{H} \to \mathcal{H}$ such that*

$$\psi(f)(y) = Uf(\varphi(y))U^* \quad \textit{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \textit{ and } y \in Y. \tag{2.3}$$

Proof. The map ψ induces a $*$-isomorphism $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. Corollary 5.42 on page 143 in [3] asserts the existence of a unitary U such that $T(A) = UAU^*$. We define τ as follows: $\tau(f)(y) = U^*\psi(f)(y)U$. The map τ fixes constant functions and then Proposition 2.1 applies. This completes the proof. ∎

COROLLARY 2.5. *Let X and Y be compact metric spaces. If $\psi : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ is a ps-continuous $*$-algebra homomorphism that preserves constant functions then ψ is continuous.*

Proof. Theorem 1.2 asserts the existence of a Lipschitz function $\varphi : Y \to X$ and a countable orthogonally ranged family of partial isometries

$U_n : \mathcal{H} \to \mathcal{H}_n$ such that

$$\psi(f)(y) = \sum_n U_n f(\varphi(y)) U_n^* \quad \text{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \text{ and } y \in Y.$$

Hence we have

$$\|\psi(f)(y)\| = \sup_{\|v\| \le 1} \Big\| \sum_n (U_n f(\varphi(y)) U_n^*)(v) \Big\|.$$

Given $v \in \mathcal{H}$, $v = \sum_n v_n$, with $v_n \in \mathcal{H}_n$, we have

$$\begin{aligned}
&\Big\| \sum_n (U_n f(\varphi(y)) U_n^*)(v) \Big\|^2 \\
&\qquad = \Big\langle \sum_n U_n f(\varphi(y)) U_n^*(v_n), \sum_k U_k f(\varphi(y)) U_k^*(v_k) \Big\rangle \\
&\qquad = \sum_{n,k} \langle U_n f(\varphi(y)) U_n^*(v_n), U_k f(\varphi(y)) U_k^*(v_k) \rangle \\
&\qquad = \sum_n \langle f(\varphi(y)) U_n^*(v_n), f(\varphi(y)) U_n^*(v_n) \rangle \le \|f(\varphi(y))\|^2 \|v\|^2.
\end{aligned}$$

This implies $\|\psi(f)\|_\infty \le \|f\|_\infty$. Similarly we show that $L(\psi(f)) \le L(f)L(\varphi)$. We conclude that $\|\psi(f)\|_* \le \max\{1, L(\varphi)\}\|f\|_*$. ∎

Remark 2.6. We observe that whenever φ is a contraction then ψ is also a contraction, as for $*$-homomorphisms between C^*-algebras (cf. Theorem 2.2).

Corollary 2.7. *If X is a compact metric space then every ps-homeomorphism and $*$-isomorphism on* $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ *is continuous.*

3. Multiplicative linear functionals on $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$. In this section we show that the $*$-algebra $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ has no nontrivial multiplicative linear functionals. We denote by δ_ξ the point evaluation defined by $\delta_\xi(f) = f(\xi)$.

We first prove that a $*$-homomorphism $F : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H})$ such that $F(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{H}}$ is a point evaluation.

Theorem 3.1. *Let X be a compact metric space. For every $*$-homomorphism $F : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H})$ that maps Id_X to $\mathrm{Id}_{\mathcal{H}}$, there exist a unique $\xi \in X$ and a family of orthogonally ranged partial isometries $U_n : \mathcal{H} \to \mathcal{H}$, $n = 1, 2, \ldots$, such that $F(f) = \sum_n U_n \delta_\xi(f) U_n^*$ for all $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$.*

Proof. We consider a metric space Y equal to a single point, say $Y = \{y\}$. We identify $\mathcal{B}(\mathcal{H})$ with $\mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$. The homomorphism F is identified with $\tilde{F} : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}_*(Y, \mathcal{B}(\mathcal{H}))$ such that $\tilde{F}(f)(y) = F(f)$. The function $\tilde{F}$ is a $*$-homomorphism that satisfies the hypotheses of Theorem 1.2. Therefore

$$\tilde{F}(f)(y) = \sum_n U_n f(\varphi(y)) U_n^* \quad \text{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})),$$

with $\varphi : \{y\} \to X$. We set $\xi = \varphi(y)$. ∎

Theorem 3.2. *The only multiplicative linear functional on the $*$-algebra $\mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$ is the zero functional.*

Proof. Let $F : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathbb{C}$ be a $*$-homomorphism. Then

$$F(\mathrm{Id}_X) = 0 \quad \text{or} \quad F(\mathrm{Id}_X) = 1.$$

If $F(\mathrm{Id}_X) = 1$, we define $\tilde{F} : \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H})$ by $\tilde{F}(f) = F(f)\,\mathrm{Id}_{\mathcal{H}}$. Theorem 3.1 implies the existence of $\xi \in X$ and a family of orthogonally ranged partial isometries $U_n : \mathcal{H} \to \mathcal{H}_n$, $n = 1, 2, \ldots$, such that

$$\tilde{F}(f) = \sum_n U_n \delta_\xi(f) U_n^* \tag{3.1}$$

for all $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$. For every k we have

$$U_k^* \sum_n (U_n \delta_\xi(f) U_n^*) U_k = U_k^* U_k \delta_\xi(f) U_k^* U_k = F(f)\,\mathrm{Id}_{\mathcal{H}}.$$

Therefore $\delta_\xi(f) = F(f)\,\mathrm{Id}_{\mathcal{H}}$ for every $f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H}))$. This shows that $F(\mathrm{Id}_X) = 0$, and completes the proof. ∎

4. Some remarks on algebraic reflexivity of classes of $*$-isomorphisms on $\mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$. Let X be a compact metric space which supports an injective mapping into the complex numbers. We consider the class of continuous $*$-isomorphisms on $\mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ such that each isomorphism preserves the C^*-subalgebra of constant functions. We denote this class by $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$. We say that a ps-continuous $*$-homomorphism ψ on $\mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ is *locally in* $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$ if for every $f \in \mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ there exists an isomorphism T_f in $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$ such that

$$\psi(f) = T_f(f).$$

Furthermore we say that $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$ is *algebraically reflexive* if every ps-continuous $*$-homomorphism on $\mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ that is locally in $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$, is in $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$. For background on algebraic reflexivity we refer the reader to [9], [14], [16], [19], [20] and [21].

Proposition 4.1. *Let X be a compact metric space. If there exists an injective mapping $\lambda : X \to \mathbb{C}$, then $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$ is algebraically reflexive.*

Proof. Let ψ be a ps-continuous $*$-homomorphism on $\mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ that preserves constant functions. We assume that for every $f \in \mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$

there exists $T_f \in \mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$ such that

$$\psi(f) = T_f(f).$$

Theorem 1.2 implies the existence of a Lipschitz function $\varphi : Y \to X$ and a countable orthogonally ranged family of partial isometries $U_n : \mathcal{H} \to \mathcal{H}$, $n \in \Lambda$ (a countable subset of the natural numbers), such that

$$\psi(f)(x) = \sum_{n\in\Lambda} U_n f(\varphi(x)) U_n^*, \qquad f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})),\ x \in X. \tag{4.1}$$

For each f, Corollary 2.4 implies that there exist a lipeomorphism $\varphi_f : X \to X$ and a unitary $V_f : \mathcal{H} \to \mathcal{H}$ such that

$$T_f(f)(x) = V_f f(\varphi_f(x)) V_f^*, \qquad f \in \mathrm{Lip}_*(X, \mathcal{B}(\mathcal{H})),\ x \in X. \tag{4.2}$$

Let g be given by $g(x) = \lambda(x)\,\mathrm{Id}_{\mathcal{H}}$. We have

$$\sum_{n\in\Lambda} U_n g(\varphi(x)) U_n^* = V_g g(\varphi_g(x)) V_g^*.$$

This implies that $\varphi(x) = \varphi_g(x)$ for every $x \in X$. Therefore φ is a lipeomorphism. Now we show that the family of orthogonally ranged partial isometries must consist of a single element, U. Consequently, U must be unitary.

We consider an orthonormal basis for $\mathcal{H}$, $\{e_i\}_{i\in\mathbb{N}}$. We define the constant function $f = e_1 \otimes e_1$, where $e_1 \otimes e_1$ represents the rank one operator $(e_1 \otimes e_1)(v) = \langle v, e_1\rangle e_1$.

Previous considerations imply that

$$e_1 \otimes e_1 = U_k^* V_f e_1 \otimes e_1 V_f^* U_k \qquad \text{for } k \in \Lambda. \tag{4.3}$$

We set $W_k = V_f^* U_k$. Given e_j (with $j \neq 1$) we have

$$(e_1 \otimes e_1)(e_j) = (U_k^* V_f e_1 \otimes e_1 V_f^* U_k)(e_j).$$

Therefore $0 = \langle W_k e_j, e_1\rangle W_k^* e_1$. This implies that either $W_k^* e_1 = 0$ or $\langle e_j, W_k^* e_1\rangle = 0$ for every $j \neq 1$.

We start by observing that $W_k^* e_1 \neq 0$. Otherwise, (4.3) would imply that $e_1 = \langle W_k e_1, e_1\rangle W_k^* e_1 = 0$. Therefore, for every $k \in \Lambda$, $W_k^* e_1 = \alpha_k e_1$ for some scalar α_k. Since

$$e_1 = \langle e_1, \alpha_k e_1\rangle \alpha_k e_1,$$

it follows that $|\alpha_k| = 1$.

We have shown that $U_k^* V_f e_1 = \alpha_k e_1$ for every k. This implies that $V_f e_1 = \sum_{k\in\Lambda} \alpha_k U_k e_1$. Therefore,

$$1 = \langle V_f e_1, V_f e_1 \rangle = \Big\langle \sum_{k\in\Lambda} \alpha_k U_k e_1, \sum_{j\in\Lambda} \alpha_j U_j e_1 \Big\rangle$$
$$= \sum_{k,j\in\Lambda} \alpha_k \overline{\alpha_j} \langle U_k e_1, U_j e_1 \rangle = \sum_{k\in\Lambda} \langle U_k e_1, U_k e_1 \rangle.$$

This previous equality is only possible if Λ reduces to a single point and the family $\{U_k\}_{k\in\Lambda}$ consists of a single unitary operator, U. Thus we have $\psi(f)(x) = Uf(\varphi(x))U^*$ with φ a lipeomorphism and U unitary. This implies that ψ is in $\mathrm{CI}(\mathrm{Lip}(X, \mathcal{B}(\mathcal{H})))$. ∎

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