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On the limiting empirical measure of eigenvalues of the sum of rank one matrices with log-concave distribution

by

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Abstract. We consider $n \times n$ real symmetric and hermitian random matrices H_n that are sums of a non-random matrix $H_n^{(0)}$ and of m_n rank-one matrices determined by i.i.d. isotropic random vectors with log-concave probability law and real amplitudes. This is an analog of the setting of Marchenko and Pastur [Mat. Sb. 72 (1967)]. We prove that if $m_n/n \to c \in [0, \infty)$ as $n \to \infty$, and the distribution of eigenvalues of $H_n^{(0)}$ and the distribution of amplitudes converge weakly, then the distribution of eigenvalues of H_n converges weakly in probability to the non-random limit, found by Marchenko and Pastur.

1. Introduction. Let $\{Y_\alpha\}_{\alpha=1}^m$ be i.i.d. random vectors of $\mathbb{R}^n$ (or $\mathbb{C}^n$), and $\{\tau_\alpha\}_{\alpha=1}^m$ be a collection of real numbers. Consider the random matrix

$$M_n = \sum_{\alpha=1}^{m} \tau_\alpha L_{Y_\alpha}, \tag{1.1}$$

where L_Y is the rank-one matrix corresponding to $Y \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and defined as

$$L_Y X = (X, Y)Y, \quad \forall X \in \mathbb{R}^n \text{ (resp. } X \in \mathbb{C}^n\text{)}, \tag{1.2}$$

with $(\ ,\ )$ denoting the standard euclidian (or hermitian) scalar product in $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Let also $H_n^{(0)}$ be a real symmetric (or hermitian) $n \times n$ deterministic matrix. We then consider the real symmetric (or hermitian) $n \times n$ random matrix

$$H_n = H_n^{(0)} + M_n. \tag{1.3}$$

Denote by

$$-\infty < \lambda_1 \le \cdots \le \lambda_n < \infty \tag{1.4}$$

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the eigenvalues of H_n and introduce their normalized counting (or empirical) measure N_n, setting for any interval $\Delta \subset \mathbb{R}^n$,

$$N_n(\Delta) = \mathrm{Card}\{l \in [1,n] : \lambda_l \in \Delta\}/n. \tag{1.5}$$

Likewise, define the normalized counting measures $N_n^{(0)}$ of the eigenvalues $\{\lambda_l^{(0)}\}_{l=1}^n$ of $H_n^{(0)}$,

$$N_n^{(0)}(\Delta) = \mathrm{Card}\{l \in [1,n] : \lambda_l^{(0)} \in \Delta\}/n, \tag{1.6}$$

and the normalized counting measures σ_m of $\{\tau_\alpha\}_{\alpha=1}^m$:

$$\sigma_m = \mathrm{Card}\{\alpha \in [1,m] : \tau_\alpha \in \Delta\}/m. \tag{1.7}$$

Assume that the sequences $\{N_n^{(0)}\}$ and $\{\sigma_m\}$ converge weakly to probability measures $N^{(0)}$ and σ:

$$\lim_{n\to\infty} N_n^{(0)} = N^{(0)}, \tag{1.8}$$

$$\lim_{m\to\infty} \sigma_m = \sigma. \tag{1.9}$$

It was shown in [20] that if the i.i.d. $\{Y_\alpha\}_{\alpha=1}^m$ satisfy certain conditions, valid in particular for vectors with independent components and vectors uniformly distributed over the unit sphere of $\mathbb{R}^n$ (or $\mathbb{C}^n$), and $\{m_n\}$ is a sequence such that

$$c_n := m_n/n \to c \in [0,\infty), \qquad n \to \infty, \tag{1.10}$$

then there exists a non-random probability measure N ($N(\mathbb{R}) = 1$) such that for any interval $\Delta \subset \mathbb{R}$ we have the convergence in probability

$$\lim_{n\to\infty} N_n(\Delta) = N(\Delta). \tag{1.11}$$

The limiting non-random measure N can be found as follows. Introduce the Stieltjes transform

$$f^{(0)}(z) = \int_{\mathbb{R}} \frac{N^{(0)}(d\lambda)}{\lambda - z}, \qquad \Im z \neq 0, \tag{1.12}$$

of the measure $N^{(0)}$ of (1.8) and the Stieltjes transform

$$f(z) = \int_{\mathbb{R}} \frac{N(d\lambda)}{\lambda - z}, \qquad \Im z \neq 0, \tag{1.13}$$

of the measure N of (1.11). Then f is uniquely determined by the functional equation

$$f(z) = f^{(0)}\Big(z - c\int_{\mathbb{R}} \frac{\tau\,\sigma(d\tau)}{1+\tau f(z)}\Big), \tag{1.14}$$

considered in the class of functions analytic in $\mathbb{C} \setminus \mathbb{R}$ and such that $\Im f(z)\Im z \geq 0$ for $\Im z \neq 0$. Since the Stieltjes transform determines N uniquely by the formula

$$\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\varphi(\lambda)\Im f(\lambda+i\varepsilon)\,d\lambda=\int_{\mathbb{R}}\varphi(\lambda)\,N(d\lambda), \tag{1.15}$$

valid for any continuous function φ of compact support, (1.14) determines N uniquely.

The same result is valid if the components $\{Y_{\alpha j}\}_{j=1}^{n}$ of Y_α, $\alpha=1,\dots,m$, are i.i.d. random variables of zero mean and unit variance (see [4, 16, 20, 24, 26] and references therein). A particular case of this random matrix for $H_n^{(0)}=0$, $\tau_\alpha=1$, $\alpha=1,\dots,m$, and i.i.d. Gaussian $\{Y_{\alpha j}\}_{j=1}^{n}$ has been known since the 30s in statistics as the (null) *Wishart matrix* (see e.g. [22]). The same random matrix also appears in the local theory of Banach spaces or so-called asymptotic convex geometry (see e.g. [10, 27]). An important problem that enters these frameworks is the study of some geometric parameters associated to i.i.d. random points Y_α, $\alpha=1,\dots,m$, uniformly distributed over a convex body in $\mathbb{R}^n$ and the asymptotic geometry of the random convex polytope generated by these points (see e.g. [3, 7, 14, 15, 19]).

In this paper we prove (1.11) and (1.14) for the case where the common probability law of the i.i.d. vectors $\{Y_\alpha\}_{\alpha=1}^{m}$ is isotropic and log-concave (see the next section for the corresponding definitions). A preliminary unpublished result obtained in 2004 by the authors is on the vectors that are uniformly distributed in the euclidian unit ball of $\mathbb{R}^n$. This case was also obtained by a different approach in [2].

The paper is organized as follows. In Section 2 we present necessary spectral and probabilistic facts and recent results on isotropic random vectors with log-concave distribution. Section 3 contains the proof of our main result (Theorem 3.3) which combines the method of [20] and the later papers [16, 24, 26].

2. Necessary spectral and probabilistic facts. We will begin by recalling several facts on the resolvent of real symmetric (hermitian) matrices. Here and below, $|\dots|$ denotes the euclidian (or hermitian) norm of vectors and matrices.

LEMMA 2.1. *Let A be a real symmetric (hermitian) matrix and*

$$G_A(z)=(A-z)^{-1},\qquad \Im z\neq 0, \tag{2.1}$$

be its resolvent.

(i) *We have*

$$|G_A(z)|\le|\Im z|^{-1}. \tag{2.2}$$

(ii) *If A_1 and A_2 are hermitian matrices, then*

$$G_{A_2}(z)=G_{A_1}(z)-G_{A_2}(z)(A_2-A_1)G_{A_1}(z). \tag{2.3}$$

(iii) *If for $Y \in \mathbb{R}^n$ or ($Y \in \mathbb{C}^n$), L_Y is the corresponding rank-one matrix* (1.2) *and $\tau \in \mathbb{R}$, then*

$$G_{A+\tau L_Y}(z) = G_A(z) - \tau G_A(z) L_Y G_A(z)(1+\tau(G_A(z)Y,Y))^{-1}, \quad \Im z \neq 0. \tag{2.4}$$

We also need the following simple fact, a version of the min-max principle of linear algebra (see e.g. [8, Section I.4]).

Lemma 2.2. *Let A_1 and A_2 be hermitian matrices and $N_n^{(1)}$ and $N_n^{(2)}$ be their normalized counting measures. Then for any interval $\Delta \subset \mathbb{R}$,*

$$|N_n^{(1)}(\Delta) - N_n^{(2)}(\Delta)| \le \operatorname{rank}(A_1 - A_2)/n. \tag{2.5}$$

Next we give a version of the martingale-difference bounds for the variance of a Borelian function of independent random variables (see e.g. [11]). The technique of martingale differences was used in studies of random matrices by Girko (see e.g. [16] for results and references).

Lemma 2.3. *Let $\{Y_\alpha\}_{\alpha=1}^m$ be a collection of i.i.d. random vectors of $\mathbb{R}^n$ (or $\mathbb{C}^n$) with a common probability law F, and $\Phi : \mathbb{R}^{nm} \to \mathbb{C}$ (resp. $\Phi : \mathbb{C}^{nm} \to \mathbb{C}$) be a bounded Borelian function, satisfying the inequalities*

$$\sup_{X_1,\dots,X_m \in \mathbb{R}^n \ (resp.\, \mathbb{C}^n)} |\Phi - \Phi|_{X_\alpha=0}| \le C < \infty, \quad \alpha = 1,\dots,m. \tag{2.6}$$

Then

$$\mathbf{Var}\{\Phi(Y_1,\dots,Y_m)\} \le 4C^2 m. \tag{2.7}$$

We denoted above by $\Phi|_{X_\alpha=0}$ the function Φ composed with the projection $(X_1,\dots,X_m) \mapsto (X_1,\dots,X_{\alpha-1},0,X_{\alpha+1},\dots,X_m)$, and

$$\mathbf{Var}\{\Phi\} = \mathbf{E}\{|\Phi|^2\} - |\mathbf{E}\{\Phi\}|^2.$$

We now discuss isotropic random vectors with a log-concave probability law.

Definition 2.4. A random real vector $Y \in \mathbb{R}^n$ is called *isotropic* if

$$\mathbf{E}\{(Y,X)\} = 0, \quad \mathbf{E}\{(Y,X)^2\} = n^{-1}|X|^2, \quad \forall X \in \mathbb{R}^n, \tag{2.8}$$

(where $(\cdot,\cdot)$ denotes the euclidian scalar product in $\mathbb{R}^n$), or, in terms of components $\{y_j\}_{j=1}^n$ of Y,

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_j y_k\} = n^{-1}\delta_{jk}, \quad j,k \in [1,n].$$

A random complex vector $Y \in \mathbb{C}^n$ is called $\mathbb{R}$-*isotropic* or simply *isotropic* if $(\Re Y, \Im Y) \in \mathbb{R}^{2n}$ is isotropic. In terms of components $\{y_j\}_{j=1}^n$ of Y, this is equivalent to

$$\mathbf{E}\{y_j\} = \mathbf{E}\{y_j y_k\} = 0, \quad \mathbf{E}\{y_j \overline{y}_k\} = 2n^{-1}\delta_{jk}, \quad j,k = 1,\dots,n.$$

Observe that if $Y \in \mathbb{C}^n$ is isotropic then so are $\Re Y$ and $\Im Y$ as real vectors.

We are going to find the limiting normalized counting measure of the eigenvalues of (1.3), not attempting to obtain bounds on the corresponding rate of convergence that depends on a number of factors. It is shown below that in this situation it suffices to consider random vectors described by

Definition 2.5. A random isotropic vector $Y = \{y_j\}_{j=1}^n$ of $\mathbb{R}^n$ is called *good* if for any $n \times n$ complex matrix A such that its operator norm satisfies $|A| \leq 1$ we have

$$\mathbf{Var}\{(AY, Y)\} \leq \delta_n, \qquad \delta_n = o(1), \qquad n \to \infty. \tag{2.9}$$

Remark 2.6. A complex version of the above definition is

$$\mathbf{Var}\{(AY, Y)\} \leq \delta_n, \qquad \delta_n = o(1), \qquad n \to \infty, \tag{2.10}$$

where now $(\cdot, \cdot)$ is the hermitian scalar product in $\mathbb{C}^n$.

It is easy to check that if $\{\xi_j\}_{j=1}^n$ are i.i.d. random variables of zero mean and unit variance, then the vector $Y = \{n^{-1/2}\xi_j\}_{j=1}^n$ is good and

$$\delta_n = 2\mathbf{Var}\{\xi_1^2\}/n. \tag{2.11}$$

Likewise, the vector uniformly distributed over the unit sphere in $\mathbb{R}^n$ and the vector uniformly distributed over the ball of radius $\sqrt{(n+2)/n}$ in $\mathbb{R}^n$ are good and have $\delta_n = 2/n$. A class of random vectors for which (2.9) is valid was considered in [20].

On the other hand, the vector assuming the values $\pm e_j$, $j = 1, \dots, n$, with probability $(2n)^{-1}$, where $\{e_j\}_{j=1}^n$ is the canonical basis in $\mathbb{R}^n$, is not good. Indeed, in this case we have (2.8), but not (2.9), since

$$\mathbf{Var}\{(AY, Y)\} = n^{-1} \sum_{j=1}^n |A_{jj}|^2 - \Big| n^{-1} \sum_{j=1}^n A_{jj} \Big|^2,$$

and if $n = 2m$, A is diagonal, $A_{jj} = 1$, $j = 1, \dots, m$, and $A_{jj} = -1$, $j = m+1, \dots, 2m$, then $\mathbf{Var}\{(AY, Y)\} = 1$.

The fact that for the vector uniformly distributed over the unit sphere in $\mathbb{R}^n$ and the vector uniformly distributed over the ball of radius $\sqrt{(n+1)/n}$ we have the same δ_n in (2.9) is a simple manifestation of a rather general concentration phenomenon, according to which the vector uniformly distributed over a symmetric convex body in $\mathbb{R}^n$ is concentrated for large n within a very thin shell adjacent to its surface. It follows from Lemma 2.7 below that because of the concentration phenomenon, in particular, by important recent results of [17, 18] and also [12], such random vectors are good, as also are the vectors whose probability law is log-concave. It is shown in Theorem 3.3 (whose proof is different from that of [20]) that the results of [20] are valid for matrices (1.3), where $\{Y_\alpha\}$ are good vectors.

Recall that a measure m on $\mathbb{R}^n$ (or $\mathbb{C}^n$) is called *log-concave* if for any measurable subsets A, B of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and any $\theta \in [0,1]$,

$$m(\theta A + (1-\theta)B) \geq m(A)^\theta m(B)^{1-\theta}$$

whenever the set

$$\theta A + (1-\theta)B = \{\theta X_1 + (1-\theta)X_2 : X_1 \in A,\ X_2 \in B\}$$

is measurable.

If Y is a random vector with a log-concave distribution, then any affine image of Y has a log-concave distribution. In particular, the projection of a log-concave measure is log-concave. If Y_1 and Y_2 are independent random vector with log-concave distributions, then (Y_1, Y_2) and $Y_1 + Y_2$ have log-concave distributions as well (see [5, 9, 25]). The Brunn–Minkowski inequality provides examples of log-concave measures, namely the uniform Lebesgue measures on compact convex subsets of $\mathbb{R}^n$ as well as their marginals. More generally, Borell's theorem [6] characterizes the log-concave measures that are not supported by any hyperplane as the absolutely continuous measures (with respect to Lebesgue measure) with a log-concave density f, i.e.,

$$f(\theta X_1 + (1-\theta)X_2) \geq f^\theta(X_1) f^{1-\theta}((1-\theta)X_2),\ \forall X_1, X_2 \in \mathbb{R}^n,\ \forall \theta \in [0,1].$$

Note that the distribution of an isotropic vector is not supported by any hyperplane.

In recent years considerable progress has been achieved in understanding the properties of isotropic and log-concave distributed random vectors, which prove to be fairly similar to those of vectors uniformly distributed over the euclidian ball. In particular, it follows from the results of [18] (see Lemma 2.8 below) that if Y is an isotropic random vector with a log-concave probability law, then

$$\mathbf{Var}\{\,|Y|^2\} \leq \varphi_n, \tag{2.12}$$

where

$$\varphi_n = C_1 n^{-\beta_1} \tag{2.13}$$

for some $C_1 < \infty$ and $\beta_1 > 0$. It was also shown in [12, Theorem 1] that if Y is uniformly distributed over a convex body, then

$$\varphi_n = C_2 (\log n)^{-\beta_2} \tag{2.14}$$

for some $C_2 < \infty$ and $\beta_2 > 0$.

We now show that isotropic and log-concave distributed random vectors are good under a rather mild condition.

Lemma 2.7. *Let Y be a random isotropic vector with log-concave distribution in $\mathbb{R}^n$ (or $\mathbb{C}^n$) satisfying* (2.12) *in which*

$$\lim_{n\to\infty} \varphi_n = 0, \qquad n^2\varphi_n \text{ is nondecreasing.} \tag{2.15}$$

Then (2.9) *is valid with*

$$\delta_n = C\varphi_n, \tag{2.16}$$

where C is an absolute constant.

Proof. Writing $A = R + iI$, where R and I are hermitian, we have

$$\mathbf{Var}\{(AY, Y)\} = \mathbf{Var}\{(RY, Y)\} + \mathbf{Var}\{(IY, Y)\}.$$

Hence up to a factor 2, the proof reduces to the case where A is hermitian. If V is an isometry in $\mathbb{R}^n$ (respectively $\mathbb{C}^n$), then VY is also an isotropic (respectively $\mathbb{R}$-isotropic) random vector in $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) with log-concave distribution.

Hence, we can assume that A is diagonal, i.e., $A = \operatorname{diag}\{a_j\}_{j=1}^n$ with $|a_j| \le 1$ for all $1 \le j \le n$, and set

$$q(a_1, \dots, a_n) = \mathbf{Var}\{(AY, Y)\} = \mathbf{E}\Big\{\Big|\sum_{j=1}^n a_j |y_j|^2\Big|^2\Big\} - \Big|n^{-1}\sum_{j=1}^n a_j\Big|^2,$$

where $Y = \{y_j\}_{j=1}^n$. Since q is a positive quadratic form, its maximum on the cube $[-1, 1]^n$ is attained at one of its vertices $\{-1, 1\}^n$. In order to estimate q at a vertex, let J be a subset of $\{1, \dots, n\}$. Then

$$\begin{aligned}
\mathbf{Var}\Big\{\sum_{j\in J}|y_j|^2 - \sum_{j\notin J}|y_j|^2\Big\} &\le \Big(\mathbf{Var}^{1/2}\Big\{\sum_{j\in J}|y_j|^2\Big\} + \mathbf{Var}^{1/2}\Big\{\sum_{j\notin J}|y_j|^2\Big\}\Big)^2, \\
&\le \Big(\mathbf{Var}^{1/2}\Big\{\sum_{j\in J}|\Re y_j|^2\Big\} + \mathbf{Var}^{1/2}\Big\{\sum_{j\in J}|\Im y_j|^2\Big\} \\
&\quad + \mathbf{Var}^{1/2}\Big\{\sum_{j\notin J}|\Re y_j|^2\Big\} + \mathbf{Var}^{1/2}\Big\{\sum_{j\notin J}|\Im y_j|^2\Big\}\Big)^2.
\end{aligned}$$

Since in the complex case $\Re Y$ and $\Im Y$ are isotropic, up to a factor 16 we may reduce the complex case to the case $Y \in \mathbb{R}^n$. Now observe that if $Y_J = \{y_i\}_{j\in J}$ is the projection of Y onto $\mathbb{R}^J$, then $n^{1/2}\operatorname{Card}\{J\}^{-1/2}Y_J$ is also an isotropic random vector and by [9] it has a log-concave distribution. Thus, (2.15) implies

$$\mathbf{Var}\{|Y_J|^2\} \le n^{-2}\operatorname{Card}\{J\}^2\varphi_{\operatorname{Card}\{J\}} \le \varphi_n.$$

Combining all numerical constants in the reduction, we conclude that in the complex case

$$\mathbf{Var}\{(AY, Y)\} \le 32\varphi_n$$

and $\mathbf{Var}\{(AY, Y)\} \le 8\varphi_n$ when $Y \in \mathbb{R}^n$. ∎

We now derive (2.13) from the results of [17, 18].

LEMMA 2.8. *Let $Y \in \mathbb{R}^n$ be a random isotropic vector with log-concave distribution. Then* (2.12)–(2.13) *is valid.*

Proof. According to [17, Theorem 1.3], if X is an isotropic random vector with log-concave distribution, zero mean and unit covariance matrix, then there exist numerical constants $C < \infty$ and $\kappa > 0$ such that

$$P\left\{\left|\frac{|X|}{n^{1/2}} - 1\right| \geq n^{-\kappa}\right\} \leq Ce^{-n^{\kappa}}.$$

Defining $X/n^{1/2} = Y$, $\eta = |Y - 1|$, and $l(y) = y^2(2+y)^2$, we can write

$$\mathbf{Var}\{|Y|^2\} := \mathbf{E}\{(|Y|^2 - 1)^2\} \leq \mathbf{E}\{l(\eta)\} = \int_0^\infty l'(y)\mathbf{P}\{\eta \geq y\}\,dy$$
$$= \int_0^{n^{-\kappa}} + \int_{n^{-\kappa}}^b + \int_b^\infty = I_1 + I_2 + I_3$$

for some $b \geq 1$. We have obviously

$$I_1 \leq n^{-\kappa} l'(n^{-\kappa}) \leq Bn^{-2\kappa},$$

where B is an absolute constant.

Next, it follows from (2.13) that

$$I_2 \leq Cl(b)e^{-n^{\kappa}}.$$

To estimate I_3 we use a version of Borell's theorem (see e.g. [21, Appendix III, Theorem III.3]), according to which for any random vector Y with log-concave distribution,

$$\mathbf{P}\{|Y| \geq t\mathbf{E}\{|Y|\}\} \leq e^{-dt}, \qquad t \geq D,$$

for some numerical constants $D < \infty$ and $d > 0$. Since in our case $\mathbf{E}\{|Y|\} \leq \mathbf{E}^{1/2}\{|Y|^2\} \leq 1$, for $y \geq \max\{1, D\}$ we obtain

$$\mathbf{P}\{\eta \geq y\} := \mathbf{P}\{|\,|Y| - 1| \geq y\} = \mathbf{P}\{|Y| \geq 1 + y\} \leq D_1 e^{-dy}$$

for some numerical constant D_1. This leads to the bound

$$I_3 \leq l_1(b)e^{-db},$$

where l_1 is a polynomial of degree 3 with non-negative numerical coefficients. We obtain

$$\mathbf{Var}\{|Y|^2\} \leq Bn^{-2\kappa} + Cl(b)e^{-n^{\kappa}} + l_1(b)e^{-db},$$

and it now suffices to choose b such that the last two terms on the r.h.s. do not exceed $Bn^{-2\kappa}$. We conclude that the bound (2.13) is valid with $C_1 = 3B$ and $\beta_1 = 2\kappa$. ∎

3. Proof of the main result. We first prove certain auxiliary facts.

PROPOSITION 3.1. *Let N_n be the normalized counting measure* (1.5) *of the eigenvalues of* (1.3), *in which $\{Y_\alpha\}_{\alpha=1}^{m_n}$ are i.i.d. random vectors* (*not* *necessarily isotropic and/or with log-concave distribution*), $\{\tau_\alpha\}_{\alpha=1}^{m_n}$ *are real numbers*, *and let*

$$g_n(z) = \int_{\mathbb{R}} \frac{N_n(d\lambda)}{\lambda - z}, \qquad \Im z \neq 0, \tag{3.1}$$

be the Stieltjes transform of N_n. *Then*

$$\mathbf{Var}\{N_n(\mathit{\Delta})\} \leq 4c_n/n \tag{3.2}$$

for any interval $\mathit{\Delta} \subset \mathbb{R}$, *with the* c_n *of* (1.10), *and*

$$\mathbf{Var}\{g_n(z)\} \leq 4c_n/(n|\Im z|^2) \tag{3.3}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. To prove (3.2) we use Lemma 2.3 with $\mathit{\Phi} = nN_n(\mathit{\Delta})$, the number of eigenvalues of H_n in $\mathit{\Delta}$. Since

$$H_n - H_n|_{Y_\alpha = 0} = \tau_\alpha L_{Y_\alpha}$$

is a rank 1 matrix, by (2.5) we have

$$|nN_n - nN_n|_{Y_\alpha=0}| \leq 1, \tag{3.4}$$

i.e. the constant C in (2.6) is 1 in this case. This and (2.7) lead to (3.2).

In the case of g_n we choose $\mathit{\Phi} = ng_n := \mathrm{Tr}\,(H_n - z)^{-1}$. To prove an analog of (2.5) for ng_n we use an argument similar to that in the proof of Lemma 2 of [20] (see also Lemma 2.6 of [26]). According to (1.5), (3.1), and the spectral theorem for real symmetric (hermitian) matrices,

$$ng_n(z) = \mathrm{Tr}\,(H_n - z)^{-1} =: \mathrm{Tr}\, G(z).$$

Then (2.4) implies

$$|ng_n(z) - ng_n(z)|_{Y_\alpha=0}| \leq \frac{|\tau_\alpha|\,|(G_\alpha^2 Y_\alpha, Y_\alpha)|}{|1 + \tau_\alpha (G_\alpha Y_\alpha, Y_\alpha)|}, \tag{3.5}$$

where

$$G_\alpha = G|_{Y_\alpha=0}. \tag{3.6}$$

By the spectral theorem for real symmetric (hermitian) matrices there exists a non-negative measure m_α such that for any integer l,

$$(G_\alpha^l Y_\alpha, Y_\alpha) = \int_{\mathbb{R}} \frac{m_\alpha(d\lambda)}{(\lambda - z)^l}.$$

Thus

$$|\tau_\alpha|\,|(G_\alpha^2 Y_\alpha, Y_\alpha)| \leq |\tau_\alpha| \int_{\mathbb{R}} \frac{m_\alpha(d\lambda)}{|\lambda - z|^2},$$

and

$$|1 + \tau_\alpha (G_\alpha Y_\alpha, Y_\alpha)| \geq |\tau_\alpha|\,|\Im (G_\alpha Y_\alpha, Y_\alpha)| = |\tau_\alpha|\,|\Im z| \int_{\mathbb{R}} \frac{m_\alpha(d\lambda)}{|\lambda - z|^2}.$$

This and (3.5) imply the bound

$$|ng_n(z) - ng_n(z)|_{Y_\alpha=0}| \le |\Im z|^{-1}. \tag{3.7}$$

Thus we can choose $|\Im z|^{-1}$ as C in (2.7) and obtain (3.3) from (2.7). ∎

PROPOSITION 3.2. *Let $N^{(0)}$ and σ be probability measures on $\mathbb{R}$, and $f^{(0)}$ be the Stieltjes transform* (1.12) *of $N^{(0)}$. Consider a probability measure N on $\mathbb{R}$ and assume that its Stieltjes transform f* (1.13) *satisfies* (1.14)*. Then f is uniquely determined by* (1.14)*.*

This is proved in Section 5 of [20] and Lemma 5.1 of [26]. Now we are ready to prove our main result.

THEOREM 3.3. *Let $\{m_n\}$ be a sequence of positive integers satisfying* (1.10)*, $\{Y_\alpha\}_{\alpha=1}^{m_n}$ be i.i.d. good random vectors of $\mathbb{R}^n$ (or $\mathbb{C}^n$) in the sense of Definition* 2.5*, and $\{\tau_\alpha\}_{\alpha=1}^{m_n}$ be a collection of real numbers satisfying* (1.9)*. Consider the random matrices H_n of* (1.1)–(1.3) *and assume* (1.8)*. Then there exists a non-random measure N ($N(\mathbb{R}) = 1$) such that for any interval $\Delta \subset \mathbb{R}$ we have in probability*

$$\lim_{n\to\infty,\, m_n/n\to c\in[0,\infty)} N_n(\Delta) = N(\Delta) \tag{3.8}$$

for the normalized counting measure N_n of (1.5) *of the eigenvalues of H_n.*

The limiting non-random measure N is uniquely determined by equation (1.14) *for its Stieltjes transform* (1.13)*.*

Proof. In view of (3.2) and Proposition 3.2 it suffices to prove that the expectations

$$\overline{N}_n = \mathbf{E}\{N_n\} \tag{3.9}$$

of the normalized counting measure (1.5) of the eigenvalues of H_n converge weakly to a probability measure whose Stieltjes transform solves (1.14).

Given a positive integer p, define

$$\tau_\alpha^{(p)} = \begin{cases} \tau_\alpha, & |\tau_\alpha| \le p, \\ 0, & |\tau_\alpha| > p, \end{cases} \tag{3.10}$$

and set

$$H_n^{(p)} = H_n^{(0)} + \sum_{\alpha=1}^{m_n} \tau_\alpha^{(p)} L_{Y_\alpha}. \tag{3.11}$$

Let $N_n^{(p)}$ be the normalized counting measure of $H_n^{(p)}$, and

$$\overline{N}_n^{(p)} = \mathbf{E}\{N_n^{(p)}\}.$$

Since $\overline{N}_n^{(p)}$ is a probability measure for any p and n, there exists a subsequence $\{N_{n_j}^{(p)}\}_{n_j \geq 1}$ and a non-negative measure such that

$$\lim_{n_j \to \infty} \overline{N}_{n_j}^{(p)}(\varDelta) = N^{(p)}(\varDelta), \quad \forall \varDelta \subset \mathbb{R}, |\varDelta| < \infty. \tag{3.12}$$

Let us show that the proof of the theorem reduces to the proof that for any positive integer p there exists $C_p \in [2p, \infty)$ such that the Stieltjes transform $f^{(p)}$ of $N^{(p)}$ satisfies the equation

$$f^{(p)}(z) = f^{(0)}\left(z - c\int_{-p}^{p} \frac{\tau\,\sigma(d\tau)}{1 + \tau f^{(p)}(z)}\right), \quad |\Im z| \geq C_p \geq 2p, \tag{3.13}$$

Indeed, it is an easy consequence of Theorem 3 of [1, Section 69] (see also [13]) that if s is the Stieltjes transform of a non-negative measure m, then

$$\lim_{y \to \infty} y|s(iy)| = m(\mathbb{R}). \tag{3.14}$$

Being the Stieltjes transform of a non-negative measure whose total mass does not exceed 1, $f^{(p)}$ admits the bound $|f^{(p)}(z)| \leq |\Im z|^{-1}$. Thus, the second term in the argument on the r.h.s. of (3.13) is bounded by $2cp$, and by (3.14) and (1.8) we have

$$\lim_{y \to \infty} y|f^{(p)}(iy)| = \lim_{y \to \infty} y|f^{(0)}(iy)| = N^{(0)}(\mathbb{R}) = 1.$$

Thus, $N^{(p)}$ is a probability measure.

Furthermore, it follows from (1.3), (3.11), (2.5), and (3.10) that

$$\operatorname{rank}(H_n - H_n^{(p)}) \leq \operatorname{Card}\{\alpha \in [1, m] : |\tau_\alpha| \geq p\}.$$

This and the min-max principle (2.5) imply

$$|\overline{N}_n(\varDelta) - \overline{N}_n^{(p)}(\varDelta)| \leq c_n \sigma_{m_n}(\mathbb{R} \setminus [-p, p]), \quad \forall \varDelta \in \mathbb{R}, \tag{3.15}$$

where c_n is defined by (1.10).

Let N be a vague (i.e. for all finite intervals) limit point of $\{N^{(p)}\}$, i.e.,

$$\lim_{j \to \infty} N^{(p_j)}(\varDelta) = N(\varDelta), \quad \forall \varDelta \subset \mathbb{R}, |\varDelta| < \infty, \tag{3.16}$$

for a certain sequence $\{p_j\}_{j \geq 1}$. Then, writing

$$\begin{aligned} |\overline{N}_n(\varDelta) - N(\varDelta)| \leq |\overline{N}_n(\varDelta) - \overline{N}_n^{(p)}(\varDelta)| + |\overline{N}_n^{(p)}(\varDelta) - N^{(p)}(\varDelta)| \\ + |N^{(p)}(\varDelta) - N(\varDelta)|, \end{aligned}$$

using (3.15), (1.9), and the subsequent limits $n_j \to \infty$ and $p_j \to \infty$, we find that N is a vague limit point of $\{\overline{N}_n\}$. Now, passing to the limit $n_j \to \infty$ in (3.15) with $\varDelta = \varDelta_q := [-q, q]$ and using (1.9), we obtain the bound

$$|N(\varDelta_q) - N^{(p)}(\varDelta_q)| \leq c\sigma(\mathbb{R} \setminus [-p, p]). \tag{3.17}$$

Letting here $q \to \infty$ and then $p_j \to \infty$, and recalling that $N^{(p)}(\mathbb{R}) = 1$, we find that

$$N(\mathbb{R}) = 1.$$

We conclude that N is a weak limit point of $\{N^{(p)}\}$.

Let us show that N is the weak limit of $\{\overline{N}_n\}$ and that the Stieltjes transform f of N satisfies (1.14) provided that $f^{(p)}$ satisfies (3.13). Note first that since both sides of (3.13) are analytic in $\mathbb{C} \setminus \mathbb{R}$, the equation is valid everywhere in $\mathbb{C} \setminus \mathbb{R}$. Moreover, due to the uniqueness property of analytic functions it suffices to consider the equation for $z = iy$, $0 < \eta_0 \le y \le \eta_1 < \infty$. We then have

$$\Im f^{(p)}(iy) = \int_{\mathbb{R}} \frac{y\, N^{(p)}(d\lambda)}{\lambda^2 + y^2} \ge \int_{-y}^{y} \frac{y\, N^{(p)}(d\lambda)}{\lambda^2 + y^2} \ge (2\eta_1)^{-1} N^{(p)}([-\eta_0, \eta_0]).$$

This and the weak convergence of $\{N^{(p_j)}\}$ to N imply that $\Im f^{(p_j)}(iy)$ is bounded away from zero uniformly in $p_j \to \infty$ and $y \in [\eta_0, \eta_1]$. Now the bound

$$|\tau(1 + \tau f^{(p_j)}(iy))^{-1}| \le (\Im f^{(p_j)}(iy))^{-1}$$

allows us to let $p_j \to \infty$ in (3.13) to obtain (1.14) for $z = iy$, $0 < \eta_0 \le y \le \eta_1 < \infty$, hence everywhere in $\mathbb{C} \setminus \mathbb{R}$. Proposition 3.2 implies that (1.14) is uniquely soluble. Since the Stieltjes transform of a measure determines it uniquely (see (1.15)), the theorem is proved provided that (3.13) is valid for any positive integer p. This is proved in Proposition 3.4 below. ∎

Proposition 3.4. *Let $H_n^{(p)}$ be defined in* (3.11) *and $f^{(p)}$ be the Stieltjes transform of the measure $N^{(p)}$ of* (3.12). *Then $f^{(p)}$ satisfies* (3.13).

Proof. We first outline the idea of the proof, omitting for brevity the superscript p, the subscript n in m_n, and the argument z in corresponding quantities.

We write the resolvent identity (2.3) for $A_1 = H_n^{(0)}$ and $A_2 = H_n$:

$$G = \mathcal{G} - \sum_{\alpha=1}^{m} \tau_\alpha G L_{Y_\alpha} \mathcal{G}, \tag{3.18}$$

where

$$G = (H_n - z)^{-1}, \quad \mathcal{G} = (H_n^{(0)} - z)^{-1}, \quad \Im z \ne 0. \tag{3.19}$$

Hence, if

$$\overline{G} = \mathbf{E}\{G\}, \tag{3.20}$$

then

$$\overline{G} = \mathcal{G} - \sum_{\alpha=1}^{m} \tau_\alpha \boldsymbol{E}\{G L_{Y_\alpha}\} \mathcal{G}. \tag{3.21}$$

It follows from (2.4) that

$$GL_{Y_\alpha} = G_\alpha L_{Y_\alpha}(1+\tau_\alpha(G_\alpha Y_\alpha, Y_\alpha))^{-1}, \tag{3.22}$$

where G_α is given by (3.6), thus

$$\overline{G} = \mathcal{G} - \sum_{\alpha=1}^{m} \mathbf{E}\left\{\frac{\tau_\alpha}{1+\tau_\alpha(G_\alpha Y_\alpha, Y_\alpha)}\, G_\alpha L_{Y_\alpha}\right\}\mathcal{G}. \tag{3.23}$$

Since G_α does not depend on Y_α, from (2.8) we have

$$\mathbf{E}\{G_\alpha L_\alpha\} = n^{-1}\overline{G}_\alpha, \qquad \mathbf{E}\{(G_\alpha Y_\alpha, Y_\alpha)\} = n^{-1}\operatorname{Tr}\overline{G}_\alpha, \tag{3.24}$$

where $\overline{G}_\alpha = \mathbf{E}\{G_\alpha\}$. Moreover, it follows from Lemma 2.7 and (2.2) that the fluctuations of $(G_\alpha Y_\alpha, Y_\alpha)$ and $G_\alpha L_\alpha$ due to Y_α vanish as $n \to \infty$. This allows us to replace these random quantities in (3.23) by their expectations (3.24). In addition, it follows from (2.4) that $n^{-1}(G - G_\alpha) \to 0$ as $n \to \infty$, and we can replace $\overline{G}_\alpha$ by $\overline{G}$ of (3.20), in particular, $n^{-1}\operatorname{Tr}\overline{G}_\alpha$ by

$$f_n := n^{-1}\operatorname{Tr}\overline{G} = \int_{\mathbb{R}} \frac{\overline{N}_n(d\lambda)}{\lambda - z}. \tag{3.25}$$

Thus (3.23) becomes

$$\overline{G} = \mathcal{G} - \sum_{\alpha=1}^{m} \frac{\tau_\alpha}{1+\tau_\alpha f_n}\,\overline{G}\mathcal{G} + o(1), \qquad n \to \infty.$$

Viewing this as an equation for $\overline{G}$ and solving it, we obtain

$$\overline{G}(z) = \widetilde{\mathcal{G}}(z) + o(1), \qquad n \to \infty,$$

where in view of (1.7),

$$\widetilde{\mathcal{G}}(z) = \mathcal{G}(\widetilde{z}_n(z)), \qquad \widetilde{z}_n(z) = z - c_n \int_{\mathbb{R}} \frac{\tau\,\sigma_m(d\tau)}{1+\tau f_n(z)}; \tag{3.26}$$

here and below we write $\lim_{n\to\infty}$ for the limit (1.10) and $o(1)$ for quantities that vanish in this limit uniformly in z such that $|\Im z| \geq \eta > 0$ and η does not depend on n and m. The above implies that

$$f_n(z) = f_n^{(0)}(\widetilde{z}_n(z)) + o(1), \qquad n \to \infty, \tag{3.27}$$

where

$$f_n^{(0)}(z) := n^{-1}\operatorname{Tr}\mathcal{G}(z) = \int_{\mathbb{R}} \frac{N_n^{(0)}(d\lambda)}{\lambda - z}. \tag{3.28}$$

This is a prelimit form of (3.13), and it is not hard now to pass to the limit (1.10) and obtain (1.14).

We will now justify the above scheme. Note first that we obtain the same equation (3.13) if we replace the random part M_n of (1.1) by

$$M_n^{<} = \sum_{\alpha=1}^{m_n} \tau_\alpha \mathbf{1}_{|\,|Y_\alpha|^2-1|\le\delta_n^{1/3}} L_{Y_\alpha}, \tag{3.29}$$

where δ_n is defined in (2.9). Indeed, from (2.5) with $A_1 = H_n^{(0)} + M_n^{<}$ and $A_2 = H_n$ we have

$$\begin{aligned}|N_n(\varDelta) - N_n^{<}(\varDelta)| &\le n^{-1}\mathrm{rank}\Big(\sum_{\alpha=1}^{m_n} \tau_\alpha \mathbf{1}_{|\,|Y_\alpha|^2-1|>\delta_n^{1/3}} L_{Y_\alpha}\Big) \\ &\le n^{-1}\,\mathrm{Card}\{\alpha\in[1,m_n] : |\,|Y_\alpha|^2-1| > \delta_n^{1/3}\}, \quad \forall \varDelta \subset \mathbb{R},\end{aligned}$$

where $N_n^{<}$ is the normalized counting measure of the eigenvalues of $H_n^{(0)} + M_n^{<}$. Thus, if $\overline{N_n^{<}} = \mathbf{E}\{N_n^{<}\}$, then

$$|\overline{N}_n(\varDelta) - \overline{N_n^{<}}(\varDelta)| \le c_n \mathbf{P}\{|\,|Y_1|^2-1| > \delta_n^{1/3}\} = o(1), \quad n\to\infty,$$

since it follows from the Chebyshev inequality, (1.10) and (2.15) that

$$c_n\mathbf{P}\{|\,|Y_1|^2-1| > \delta_n^{1/3}\} \le c_n\delta_n^{-2/3}\mathbf{Var}\{|Y_1|^2\} \le c_n\delta_n^{1/3} = o(1) \tag{3.30}$$

as $n\to\infty$. This and (3.12) imply the assertion.

Thus, replacing in (3.23) τ_α by

$$\upsilon_\alpha = \tau_\alpha \mathbf{1}_{|\,|Y_\alpha|^2-1|\le\delta_n^{1/3}}, \tag{3.31}$$

we obtain an analog of (3.23),

$$\overline{G} = \mathcal{G} - \sum_{\alpha=1}^{m} \mathbf{E}\Big\{\frac{\upsilon_\alpha}{1+\upsilon_\alpha(G_\alpha Y_\alpha, Y_\alpha)}\, G_\alpha L_{Y_\alpha}\Big\}\mathcal{G},$$

where we denote again by G the resolvent of $H_n^{(0)} + M_n^{<}$. Now we write

$$\overline{G} = \mathcal{G} - c_n \int_{-p}^{p} \frac{\tau\,\sigma(d\tau)}{1+\tau f_n(z)}\, \overline{G}\mathcal{G} - \sum_{q=1}^{5} R_q, \tag{3.32}$$

where

$$R_1 = \sum_{\alpha=1}^{m} \mathbf{E}\Big\{\Big(\frac{\upsilon_\alpha}{1+\upsilon_\alpha(G_\alpha Y_\alpha, Y_\alpha)} - \frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n}\Big) G_\alpha L_{Y_\alpha}\Big\}\mathcal{G}, \tag{3.33}$$

$$R_2 = \sum_{\alpha=1}^{m} \mathbf{E}\Big\{\frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n}(G_\alpha L_{Y_\alpha} - n^{-1}G_\alpha)\Big\}\mathcal{G}, \tag{3.34}$$

$$R_3 = \frac{1}{n}\sum_{\alpha=1}^{m} \mathbf{E}\Big\{\frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n}(G_\alpha - G)\Big\}\mathcal{G}, \tag{3.35}$$

$$R_4 = \frac{1}{n}\mathbf{E}\Big\{\sum_{\alpha=1}^{m}\Big(\frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n} - \frac{\tau_\alpha}{1+\tau_\alpha f_n}\Big)G\Big\}\mathcal{G}, \tag{3.36}$$

$$R_5 = c_n\Big(\frac{1}{m}\sum_{\alpha=1}^{m}\frac{\tau_\alpha}{1+\tau_\alpha f_n} - \int_{-p}^{p}\frac{\tau\sigma(d\tau)}{1+\tau f_n(z)}\Big)\mathbf{E}\{G\}\mathcal{G}. \tag{3.37}$$

It follows from (3.32) that if (cf. (3.26))

$$\widehat{\mathcal{G}} = \mathcal{G}(\widehat{z}_n(z)), \quad \widehat{z}_n(z) = z - c_n\int_{-p}^{p}\frac{\tau\,\sigma(d\tau)}{1+\tau f_n(z)}, \tag{3.38}$$

then

$$\overline{G} = \widehat{\mathcal{G}} - \sum_{q=1}^{5}\widehat{R}_q, \tag{3.39}$$

where $\widehat{R}_q$, $q = 1, \dots, 5$, are obtained from R_q, $q = 1, \dots, 5$, by replacing $\mathcal{G}$ by $\widehat{\mathcal{G}}$ in (3.33)–(3.37).

Note that

$$\Im\widehat{z}_n(z) = \Im z + c_n\Im f_n(z)\int_{-p}^{p}\frac{\tau^2\,\sigma(d\tau)}{|1+\tau f_n(z)|^2}, \tag{3.40}$$

and since $\Im f_n(z)\Im z > 0$ and $\Im z \neq 0$ by (3.25), we have

$$|\Im\widehat{z}_n(z)| \geq |\Im z| \tag{3.41}$$

and $\widehat{\mathcal{G}}(z)$ is well defined for $\Im z \neq 0$ in view of (2.2).

Applying to (3.32) the operation $n^{-1}\operatorname{Tr}$ and recalling (1.6), (3.19), (3.25), and the spectral theorem, we obtain

$$f_n(z) = f_n^{(0)}(\widehat{z}_n(z)) - \sum_{q=1}^{5}\widehat{r}_q, \tag{3.42}$$

where now f_n is the Stieltjes transform of $N_n^{<}$, $f_n^{(0)}$ is given by (3.28), and

$$\widehat{r}_q(z) = n^{-1}\operatorname{Tr}\widehat{R}_q. \tag{3.43}$$

We will now prove that if

$$|\Im z| \geq 4p, \tag{3.44}$$

then

$$\widehat{r}_q(z) = o(1), \quad q = 1, \dots, 5. \tag{3.45}$$

Here and below, $o(1)$ is a quantity that tends to zero under conditions (1.10) and (3.44); moreover, we write $O(n^{-p})$ for quantities bounded by $C(z)n^{-p}$, where $C(z)$ does not depend on n and is finite under the same conditions.

Note first that (3.41) and (3.44) imply

$$|\widehat{z}_n(z)| \geq |\Im z| \geq 4p. \tag{3.46}$$

We then have, from (3.33),

$$\widehat{r}_1(z) = -\frac{1}{n}\sum_{\alpha=1}^{m}\mathbf{E}\left\{\frac{\upsilon_\alpha^2((G_\alpha Y_\alpha, Y_\alpha) - f_n)}{(1+\upsilon_\alpha(G_\alpha Y_\alpha, Y_\alpha))(1+\tau_\alpha f_n)}\,(\widehat{\mathcal{G}}GY_\alpha, Y_\alpha)\right\}.$$

According to (2.2) and (3.25) we have $|(G_\alpha Y_\alpha, Y_\alpha)| \leq |\Im z|^{-1}|Y_\alpha|^2$ and $|f_n| \leq |\Im z|^{-1}$. Thus, (3.44), (3.46), and the inequalities $|\upsilon_\alpha| \leq |\tau_\alpha| \leq p$ yield, for $|Y_\alpha|^2 < 1 + \delta_n^{1/3} \leq 2$ and $|\Im z| \geq 4p$,

$$\begin{aligned} |1+\upsilon_\alpha(G_\alpha Y_\alpha, Y_\alpha)| &\geq 1 - 2p/|\Im z| \geq 1/2, \\ |1+\upsilon_\alpha f_n| &\geq 1 - p/|\Im z| \geq 1/2. \end{aligned} \tag{3.47}$$

This, (2.2), and (3.46) lead to the bound

$$|\widehat{r}_1(z)| \leq \frac{4p^2}{|\Im z|^2}\,\frac{1}{n}\sum_{\alpha=1}^{m}\mathbf{E}\{\mathbf{E}_\alpha\{|(G_\alpha Y_\alpha, Y_\alpha) - f_n|\}\}, \tag{3.48}$$

where $\mathbf{E}_\alpha\{...\}$ denotes the expectation only with respect to Y_α. Since Y_α is isotropic and G_α does not depend on Y_α, we have (see (3.24))

$$\mathbf{E}_\alpha\{(G_\alpha Y_\alpha, Y_\alpha)\} = n^{-1}\mathrm{Tr}\, G_\alpha =: g_n^{(\alpha)}. \tag{3.49}$$

Moreover, we have the relation $\mathbf{E}\{g_n\} = f_n$, following from the definitions (1.5), (3.1), (3.20), and (3.25). This and the Schwarz inequality yield

$$\begin{aligned} &\mathbf{E}\{|\mathbf{E}_\alpha\{|(G_\alpha Y_\alpha, Y_\alpha) - f_n|\}\} \\ &\quad \leq \mathbf{E}\{\mathbf{Var}_\alpha^{1/2}\{(G_\alpha Y_\alpha, Y_\alpha)\}\} + \mathbf{E}\{|g_n^{(\alpha)} - g_n|\} + \mathbf{E}\{\mathbf{Var}_\alpha^{1/2}\{g_n\}\}. \end{aligned}$$

Now (2.9) and (2.2) yield a bound $(C\delta_n)^{1/2}/|\Im z| = o(1)$ for the first term on the r.h.s.; (3.6), (3.7), (3.49) yield the bound $1/(n|\Im z|) = O(n^{-1})$ for the second term; and (3.3) yields the bound $2m^{1/2}/(n|\Im z|) = O(n^{-1/2})$ for the third term in view of (1.10). It then follows from (3.48) that $\widehat{r}_1(z) = o(1)$.

Write now

$$\widehat{r}_2(z) = \frac{1}{n}\sum_{\alpha=1}^{m}\mathbf{E}\left\{\frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n}\,((\widehat{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha) - n^{-1}\,\mathrm{Tr}\,\widehat{\mathcal{G}}G_\alpha)\right\},$$

and use the relation $\mathbf{E}_\alpha\{(\widehat{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha)\} = n^{-1}\,\mathrm{Tr}\,\widehat{\mathcal{G}}G_\alpha$, (2.2), and (3.46) to obtain, similarly to the case of $\widehat{r}_1$,

$$|\widehat{r}_2(z)| \leq \frac{2p}{n}\sum_{\alpha=1}^{m}\mathbf{E}\{\mathbf{Var}_\alpha^{1/2}\{(\widehat{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha)\}\} \leq 2C^{1/2}pc_n\delta_n^{1/2}/|\Im z| = o(1).$$

In the case of

$$\widehat{r}_3(z) = \frac{1}{n}\sum_{\alpha=1}^{m}\mathbf{E}\Big\{\frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n}\, n^{-1}\operatorname{Tr}\widehat{\mathcal{G}}(G_\alpha - G)\Big\}$$

we again use (2.4) to write

$$\widehat{r}_3(z) = \frac{1}{n^2}\sum_{\alpha=1}^{m}\mathbf{E}\Big\{\frac{\upsilon_\alpha}{1+\upsilon_\alpha f_n}\cdot\frac{\upsilon_\alpha(G_\alpha\widehat{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha)}{1+\upsilon_\alpha(G_\alpha Y_\alpha, Y_\alpha)}\Big\}.$$

Thus, similarly to the case of $\widehat{r}_1$, we have

$$|\widehat{r}_3(z)| \le \frac{8p^2 c_n}{n|\Im z|^3} = O(n^{-1}).$$

Furthermore, we use (3.31) to write

$$\widehat{r}_4(z) = -\frac{1}{n}\sum_{\alpha=1}^{m}\mathbf{E}\Big\{\frac{\tau_\alpha \mathbf{1}_{|\,|Y_\alpha|^2-1|>\varphi_n^{1/3}}}{(1+\upsilon_\alpha f_n)(1+\tau_\alpha f_n)}\, n^{-1}\operatorname{Tr}\widehat{\mathcal{G}}G\Big\},$$

and using (3.47), (2.2), and the bound

$$n^{-1}|\operatorname{Tr} A| \le |A|, \tag{3.50}$$

valid for any $n\times n$ matrix, we obtain, in view of (3.30),

$$|\widehat{r}_4(z)| \le 4c_n p\mathbf{P}\{|\,|Y_1|^2 - 1| > \delta_n^{1/3}\} = o(1).$$

Finally, from (3.37) and (1.7) we have

$$\widehat{r}_5(z) = c_n\int_{-p}^{p}\frac{\tau}{1+\tau f_n}\,(\sigma_m(d\tau) - \sigma(d\tau))\,\mathbf{E}\{n^{-1}\operatorname{Tr}\widehat{\mathcal{G}}G\}.$$

In view of (3.50), (2.2), and (3.46) the expectation above is bounded by $|\Im z|^{-2}$. Moreover, from (3.47) we have

$$\Big|\frac{\partial}{\partial\tau}\frac{\tau}{1+\tau f_n}\Big| = \frac{1}{|1+\tau f_n|^2} \le 4,$$

and then (1.9) implies that $\widehat{r}_5(z) = o(1)$.

In view of (3.12), the Stieltjes transforms of $\{\overline{N}_{n_j}^{(p)}\}$ converge to the Stieltjes transform $f^{(p)}$ of $N^{(p)}$ uniformly on compact sets $K\subset\mathbb{C}\setminus\mathbb{R}$. Choosing $K\subset\{z\in\mathbb{C} : |\Im z|\ge 4p\}$ and using (1.8) and (3.45) we can pass to the limit in (3.42) along the subsequence $\{n_j\}$ for $z\in K$. This proves (3.13), hence the theorem. ∎

Remark 3.5. (1) An interesting case of the amplitudes $\{\tau_\alpha\}_{\alpha=1}^m$ is where they are the first m terms of an ergodic sequence $\{\tau_\alpha\}_{\alpha=1}^\infty$, independent of $\{Y_\alpha\}_{\alpha=1}^m$, in particular, of a sequence of i.i.d. random variables. In this case it follows from the ergodic theorem (the law of large numbers) that (1.9) is valid with probability 1 in the probability space of $\{\tau_\alpha\}_{\alpha=1}^{\infty}$, and the measure σ is the probability law of τ_1.

(2) Our result can be used for another random matrix, defined via the vectors $\{Y_\alpha\}_{\alpha=1}^{m}$ as

$$\widehat{M}_n = \{(Y_\alpha, Y_\beta)\}_{\alpha,\beta=1}^{m}. \tag{3.51}$$

It can be called the Gram matrix of the collection $\{Y_\alpha\}_{\alpha=1}^{m}$. Writing $Y_\alpha = \{y_{\alpha j}\}_{j=1}^{n}$ we can view $Y = \{y_{\alpha j}\}_{\alpha,j=1}^{m,n}$ as an $m \times n$ random matrix. Then we have the relations $\widehat{M}_n = YY^*$ and $M_n = Y^*Y$. For $n > m$, M_n has $n - m$ zero eigenvalues, whose eigenvectors form a basis of the complement of the span of $\{Y_\alpha\}_{\alpha=1}^{m}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$). All other eigenvalues of M_n coincide with those of $\widehat{M}_n$, and if $\widehat{N}_n$ is the normalized counting measure of the eigenvalues of $\widehat{M}_n$, then

$$\widehat{N}_n = -\frac{n-m}{m}\,\delta_0 + \frac{n}{m}\,N_n.$$

Hence, if N is the limit of N_n, then the limit $\widehat{N}$ of $\widehat{N}_n$ also exists and is equal to $\widehat{N} = -(c^{-1} - 1)\delta_0 + c^{-1}N$. Since in this case $N = (1 - c)\delta_0 + N^*$, where $N^*(d\lambda) = \rho^*(\lambda)d\lambda$, where the support of ρ^* is $[a_-, a_+]$, $a_\pm = (1 \pm \sqrt{c})^2$, and

$$\rho^*(\lambda) = \frac{1}{2\pi\lambda}\sqrt{(a_+ - \lambda)(\lambda - a_-)}, \qquad \lambda \in [a_-, a_+], \tag{3.52}$$

(see [20]), we conclude that $\widehat{N}$ is absolutely continuous and has the density $c^{-1}\rho^*$. A similar argument applies for $m \geq n$.

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