On unitary equivalence of quasi-free Hilbert modules

by

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Abstract. We characterize unitary equivalence of quasi-free Hilbert modules, which complements Douglas and Misra's earlier work [New York J. Math. 11 (2005)]. We first confine our arguments to the classical setting of reproducing Hilbert spaces and then relate our result to equivalence of Hermitian vector bundles.

1. Introduction. In a 1978 Acta paper ([2]), Cowen and Douglas established the Cowen–Douglas theory, making a connection between operator theory and complex geometry. Since then, various generalizations and applications have been found, and many ideas and methods have been extended to investigation of Hilbert modules ([1], [4], [5], [6]).

The theory of Hilbert modules, introduced by Douglas and Paulsen ([6]), provides a powerful tool to study multivariate operator theory. For a bounded domain Ω in $\mathbb{C}^m$, let $A(\Omega)$ be the function algebra consisting of functions that are holomorphic in some neighbourhood of $\overline{\Omega}$ with supremum norm. A Hilbert space R is said to be a *contractive Hilbert module* over $A(\Omega)$ if there exists a module action $A(\Omega) \times R \to R$ satisfying

$$\|\varphi \cdot f\|_R \leq \|\varphi\|_{A(\Omega)} \|f\|_R$$

for φ in $A(\Omega)$ and f in R. Two Hilbert modules R and R' are said to be *isomorphic* if there exists a unitary module map from R onto R'.

Quasi-free Hilbert modules were introduced by Douglas and Misra in the study of resolutions of general Hilbert modules ([4]). It was shown that a large class of Hilbert modules admit resolutions with quasi-free Hilbert modules, therefore quasi-free Hilbert modules can be regarded as building blocks of general Hilbert modules.

A quasi-free Hilbert module can be naturally regarded as a vector-valued reproducing Hilbert space to which one can associate a Hermitian vector bundle. Geometric methods, including curvatures and the Rigidity Theorem,

play a significant role in obtaining unitary invariants for quasi-free Hilbert modules (see [1], [2], [5]).

To describe unitary equivalence of quasi-free Hilbert modules, Douglas and Misra ([5]) introduced a modulus function $\mu_{(R,R')}$ for a pair of quasi-free Hilbert modules R and R'. It is closely related to the geometry of the Hermitian vector bundles associated to the modules; its definition will be given in the next section. Let $\mathrm{Hol}(\Omega)$ be the set of holomorphic matrix-valued functions on Ω. Douglas and Misra gave the following necessary condition for two modules to be unitarily equivalent:

THEOREM 1.1 ([5]). *If finite rank quasi-free Hilbert modules R and R' over $A(\Omega)$ are unitarily equivalent, then the modulus function $\mu_{(R,R')}$ is the absolute value of a function Ψ in $\mathrm{Hol}(\Omega)$.*

We will mainly consider the converse of Theorem 1.1. In [5], the converse of Theorem 1.1 was proved under the additional hypothesis that the bounded domain Ω is strongly pseudo-convex. In this paper, by studying the reproducing kernels of the Hilbert modules via calculating the Gramians of certain frames, we are able to show that the condition of Theorem 1.1 is sufficient for any bounded domain Ω. Our main result is as follows:

THEOREM 1.2. *If the modulus function $\mu_{(R,R')}$ for the two finite rank quasi-free Hilbert modules R and R' over $A(\Omega)$ is the absolute value of a function Ψ in $\mathrm{Hol}(\Omega)$, then R and R' are unitarily equivalent.*

The reproducing kernel approach to Cowen–Douglas theory was first adopted by Curto and Salinas ([3]). It enables us to argue without using the language of complex geometry, as will be seen in Section 3.

In Section 4, we provide a geometric characterization of unitary equivalence. The notion of *spectral sheaf* will be involved and the proof relies on the calculations made in Section 3. This characterization gives a geometric interpretation of the main theorem.

2. Preliminaries. In this section, we will make a quick review of quasi-free Hilbert modules. Throughout the rest of this paper, we use $\{e_1, \ldots, e_n\}$ to denote the standard basis of $\mathbb{C}^n$.

First we recall some basic elements on localization of Hilbert modules, introduced in [6]. Fix z in Ω. We can construct a one-dimensional Hilbert module $\mathbb{C}_z$, which is the one-dimensional Hilbert space $\mathbb{C}$ with module action $\varphi \cdot \xi = \varphi(z)\xi$, for φ in $A(\Omega)$ and ξ in $\mathbb{C}$. We denote the complex number 1 by 1_z in the Hilbert module $\mathbb{C}_z$.

Given a Hilbert module R over $A(\Omega)$, the Hilbert module tensor product $R \otimes_A \mathbb{C}_z$ is called the *localization of R at z*, which is canonically isomorphic to the quotient module R/R_z (which in turn is isomorphic to $R_z^\perp$ with

compressed module action), where

$$R_z = [A(\Omega)_z R]^-, \quad A(\Omega)_z = \{\varphi \in A(\Omega) \mid \varphi(z) = 0\}.$$

The isomorphism is given by

$$R \otimes_A \mathbb{C}_z \to R_z^\perp : f \otimes_A 1_z \mapsto P_z f,$$

where P_z denotes the orthogonal projection from R onto $R_z^\perp$. It is easy to see, by definition of tensor product of Hilbert modules ([6]), that the module action of a function φ in $A(\Omega)$ on $R \otimes_A \mathbb{C}_z$ is just multiplication by $\varphi(z)$.

Recall that a Hilbert module R is said to be *finitely generated* if R admits a finite generating set E in the sense that the finite linear sums

$$\sum_{i=1}^n \varphi_i \cdot f_i, \quad f_i \in E, \, \varphi_i \in A(\Omega),$$

are dense in R. The minimal cardinality of a generating set is called the *rank* of R.

Now we give the definition of quasi-free Hilbert modules introduced by Douglas and Misra.

DEFINITION 2.1 ([4]). A finitely generated Hilbert module R of rank n over $A(\Omega)$ is said to be *quasi-free* relative to the generating set $\{f_1, \ldots, f_n\}$ if

    (i) $f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z$ forms a basis for $R \otimes_A \mathbb{C}_z$ for every z in Ω,
    (ii) the linear map $X_z : R \otimes_A \mathbb{C}_z \to \mathbb{C}^n$ defined by

$$X_z(f_i \otimes_A 1_z) = e_i$$

       is locally uniformly bounded,
    (iii) $f \otimes_A 1_z = 0$ for every z in Ω if and only if $f = 0$.

By (i), X_z is a well-defined invertible linear map. If we denote the vector $X_z(f \otimes_A 1_z)$ by $f(z)$, condition (iii) just says that $f(z) = 0$ for every z if and only if $f = 0$ in R. In other words, R is a subspace of $\mathbb{C}^n$-valued functions on Ω. It is easy to see that

$$f(z) = 0 \iff X_z(f \otimes_A 1_z) = 0 \iff f \otimes_A 1_z = 0 \iff f \in R_z.$$

Therefore, $R_z = \{f \in R \mid f(z) = 0\}$, and condition (iii) is equivalent to

$$R = \bigvee_{z \in \Omega} R_z^\perp.$$

Condition (ii) ensures that the evaluation $f \mapsto f(z)$ is bounded, i.e., R is a reproducing Hilbert space. Moreover, the locally uniformly bounded assumption implies that R consists of holomorphic functions.

In fact, for any finite linear combination $\sum_{i=1}^{k} \varphi_i \cdot f_i$, $\varphi_i \in A(\Omega)$, we have

$$\Big(\sum_{i=1}^{k} \varphi_i \cdot f_i\Big)(z) = \sum_{i=1}^{k} X_z(\varphi_i \cdot f_i \otimes_A 1_z) = \sum_{i=1}^{k} \varphi_i(z) X_z(f_i \otimes_A 1_z) = \sum_{i=1}^{k} \varphi_i(z) e_i,$$

so $\sum_{i=1}^{k} \varphi_i \cdot f_i$ is holomorphic. On the other hand, any function f in R can be approximated in norm by such linear combinations. By (ii), $f(z)$ is the limit function of a sequence of locally uniformly convergent holomorphic functions, hence must be holomorphic.

We summarize these facts as a proposition:

PROPOSITION 2.2. *Any finite rank quasi-free Hilbert module R over $A(\Omega)$ with generating set $\{f_1, \ldots, f_n\}$ is a reproducing Hilbert space consisting of $\mathbb{C}^n$-valued holomorphic functions over Ω, with evaluation given by*

$$f(z) = X_z(f \otimes_A 1_z),$$

where X_z is given by Definition 2.1(ii). In particular, the generating function f_i takes the constant value e_i.

Let R and R' be two quasi-free Hilbert modules relative to generators $\{f_1, \ldots, f_n\}$ and $\{f_1', \ldots, f_n'\}$ respectively. For any z in Ω, we can define an invertible operator $\delta_z : R \otimes_A \mathbb{C}_z \to R' \otimes_A \mathbb{C}_z$ by setting $\delta_z(f_i \otimes_A 1_z) = f_i' \otimes_A 1_z$. The *modulus function* $\mu_{(R,R')}$ is defined by $\mu_{(R,R')}(z) = (\delta_z^* \delta_z)^{1/2}$, which is a positive operator on $R \otimes_A \mathbb{C}_z$ for any z in Ω.

We say a function Ψ belongs to $\mathrm{Hol}(\Omega)$ if for any z, $\Psi(z)$ is a linear operator on $R \otimes_A \mathbb{C}_z$ such that its matrix relative to the basis $f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z$ varies holomorphically in z. In other words, $\mathrm{Hol}(\Omega)$ can be identified with the space of holomorphic $n \times n$ matrix-valued functions once we fix the frame.

In the rank one case, the Laplacian of the logarithm of the modulus function as a complex two-form is the difference of the curvatures of the line bundles associated with R and R' ([5]). Hence by Theorem 1.1, the curvatures of the two associated line bundles coincide if the two modules are unitarily equivalent.

3. Unitary equivalence of quasi-free Hilbert modules. Recall that a Hilbert space H of functions from Ω to $\mathbb{C}^n$ is a *reproducing Hilbert space* if the evaluation functional $\mathrm{ev}_z : f \mapsto f(z)$ is bounded for every z. In this case, H admits an $n \times n$ matrix-valued kernel function $K(\cdot, \cdot)$, satisfying

$$\langle f, K(z, \cdot)\xi \rangle_R = \langle f(z), \xi \rangle_{\mathbb{C}^n}$$

for any ξ in $\mathbb{C}^n$ and z in Ω. Here $K(z, \cdot)\xi = \mathrm{ev}_z^* \xi$ and $K(z, w) = \mathrm{ev}_w \mathrm{ev}_z^*$. For any $\{z_i\}$ in Ω, and $\{\xi_i\}, \{\eta_i\}$ in $\mathbb{C}^n$, we have the following well-known

identity:

$$\Big\langle \sum_{i=1}^{k} K(z_i, \cdot)\xi_i, \sum_{i=1}^{k} K(z_i, \cdot)\eta_i \Big\rangle_R = \sum_{1 \le i,j \le k} \langle K(z_i, z_j)\xi_i, \eta_j \rangle_{\mathbb{C}^n}.$$

Moreover, the finite linear sums $\sum_{i=1}^{k} K(z_i, \cdot)\xi_i$ form a dense subspace of H. If H consists of holomorphic functions, the kernel function $K(z, w)$ is holomorphic in w and anti-holomorphic in z.

Using the *dual set of kernel functions* introduced in [5], the kernel function $K(z, w)$ of a quasi-free Hilbert module R can be easily calculated.

As before, let $\{f_1, \ldots, f_n\}$ be the generating set of R. By the Riesz Theorem, for every z in Ω and $1 \le i \le n$, there exist $k_z^1, \ldots, k_z^n$ in R satisfying the following duality:

$$(3.1) \qquad \langle f(z), e_i \rangle_{\mathbb{C}^n} = \langle f, k_z^i \rangle_R, \quad f \in R.$$

In other words,

$$k_z^i = \mathrm{ev}_z^* e_i$$

for z in Ω and $1 \le i \le n$. Consequently, $f \in R_z$ if and only if $f \perp k_z^i$ for every $1 \le i \le n$, thus $R_z^{\perp} = \mathrm{span}\{k_z^1, \ldots, k_z^n\}$. Moreover, regarding vectors in $\mathbb{C}^n$ as columns of complex numbers, we have

$$K(z, w) = [\langle \mathrm{ev}_w \, \mathrm{ev}_z^* e_j, e_i \rangle]_{1 \le i,j \le n} = [\langle \mathrm{ev}_z^* e_j, \mathrm{ev}_w^* e_i \rangle]_{1 \le i,j \le n}$$
$$= [\langle k_z^j, k_w^i \rangle]_{1 \le i,j \le n}.$$

In particular,

$$(3.2) \qquad K(z, z) = [\langle k_z^j, k_z^i \rangle]_{1 \le i,j \le n},$$

which is the transpose of the Gram matrix of the frame $\{k_z^i\}$.

Under mild conditions, operators between two reproducing Hilbert spaces commuting with coordinate operators M_{z_i} have been completely characterized in terms of the reproducing kernels by Curto and Salinas ([3]). Following their ideas, we can give a corresponding result characterizing unitary equivalence of quasi-free Hilbert modules. We begin with a lemma on module maps.

LEMMA 3.1. *Let R and R' be two quasi-free Hilbert modules, and T a unitary operator from R to R'. The following are equivalent*:

  (i) *T is a module map.*
  (ii) *$TR_z \subset R'_z$ for every $z \in \Omega$.*
  (iii) *$TR_z^{\perp} \subset R_z'^{\perp}$ for every $z \in \Omega$.*

In any case, $TR_z = R'_z$ and $TR_z^{\perp} = R_z'^{\perp}$.

Proof. (i)⇒(ii). This is obvious from the definition of R_z and R'_z.

(ii)$\Rightarrow$(i). We have to show that $TM_\varphi = M_\varphi T$, or equivalently, $M_\varphi^* T^* = T^* M_\varphi^*$. Since $R' = \bigvee_{z\in\Omega} R_z'^\perp$, it suffices to show that for any $z \in \Omega$ and $f \in R_z'^\perp$, $M_\varphi^* T^* f = T^* M_\varphi^* f$.

The condition $TR_z \subset R_z'$ implies $T^* R_z'^\perp \subset R_z^\perp$, and it is easy to check that the action of M_φ^* on $R_z^\perp$ and $R_z'^\perp$ is scalar multiplication by $\overline{\varphi(z)}$. Therefore,

$$M_\varphi^* T^* f = \overline{\varphi(z)}\, T^* f = T^* \,\overline{\varphi(z)}\, f = T^* M_\varphi^* f,$$

as desired.

Note that (i)$\Leftrightarrow$(ii) holds for any bounded operator T.

(i)$\Rightarrow$(iii). Recall that $A(\Omega)_z = \{\varphi \in A(\Omega) \mid \varphi(z) = 0\}$ and $R_z' = [A(\Omega)_z R']^-$. It suffices to show that for any $f \in R_z^\perp$, $\varphi \in A(\Omega)_z$ and $g \in R'$,

$$\langle Tf, M_\varphi g\rangle = 0.$$

Since T is a unitary module map, it commutes with M_φ and M_φ^*, hence

$$\langle Tf, M_\varphi g\rangle = \langle M_\varphi^* Tf, g\rangle = \langle T M_\varphi^* f, g\rangle = \overline{\varphi(z)}\langle Tf, g\rangle = 0.$$

(iii)$\Rightarrow$(i). Similar arguments to those in (ii)$\Rightarrow$(i) imply that $TM_\varphi^* = M_\varphi^* T$. Since T is unitary, $TM_\varphi = M_\varphi T$. The rest is obvious. ∎

The following theorem characterizes unitary equivalence of quasi-free Hilbert modules in terms of reproducing kernels. While our setting is slightly different from [3], the proof is essentially the same, and we only include the detailed proof of the sufficiency part for completeness.

THEOREM 3.2. *Let R and R' be two quasi-free Hilbert modules over $A(\Omega)$ with kernel functions $K(z,w)$ and $K'(z,w)$. R and R' are unitarily equivalent if and only if there exists a matrix-valued function Φ which is invertible at every z in Ω such that $K'(z,w) = \Phi^*(w)K(z,w)\Phi(z)$. In this case, the unitary module map is given by $M_{\Phi^*} : f(\cdot) \mapsto \Phi^*(\cdot)f(\cdot)$.*

Proof. Sufficiency: We will show that the multiplication operator M_{Φ^*} is a unitary module map from R to R'. Since R_z and R_z' consist of functions vanishing at z, the operator M_{Φ^*} obviously satisfies the second condition of Lemma 3.1, so we only need to show that M_{Φ^*} is a *well defined* unitary operator.

The identity $K'(z,w) = \Phi^*(w)K(z,w)\Phi(z)$ implies that for any finite set $\{z_i\} \subseteq \Omega$ and $\{\xi_i\} \subseteq \mathbb{C}^n$, $1 \leq i \leq k$,

$$M_{\Phi^*} \sum_{i=1}^k K(z_i,\cdot)\xi_i = \Phi^*(\cdot)\sum_{i=1}^k K(z_i,\cdot)\xi_i = \sum_{i=1}^k K'(z_i,\cdot)\Phi^{-1}(z_i)\xi_i.$$

Note that the right-hand side of the equation above lies in R', so M_{Φ^*} is a well defined operator from the subspace of finite linear combina-

tions $\sum_{i=1}^{k} K(z_i, \cdot)\xi_i$ *onto* the subspace of finite linear combinations $\sum_{i=1}^{k} K'(z_i, \cdot)\eta_i$, which are dense in R and R' respectively.

Moreover,

$$\left\| \sum_{i=1}^{k} K'(z_i, \cdot)\Phi^{-1}(z_i)\xi_i \right\|^2 = \sum_{1 \le i,j \le k} \langle K'(z_i, z_j)\Phi^{-1}(z_i)\xi_i, \Phi^{-1}(z_j)\xi_j \rangle$$

$$= \sum_{1 \le i,j \le k} \langle \Phi^{*-1}(z_j)K'(z_i, z_j)\Phi^{-1}(z_i)\xi_i, \xi_j \rangle$$

$$= \sum_{1 \le i,j \le k} \langle K(z_i, z_j)\xi_i, \xi_j \rangle = \left\| \sum_{i=1}^{k} K(z_i, \cdot)\xi_i \right\|^2.$$

Now the operator $M_{\Phi^*} : f(\cdot) \to \Phi^*(\cdot)f(\cdot)$ is an isometry from a dense subspace of R onto a dense subspace of R', thus admits a unique unitary extension. By continuity, the extension operator is also given by M_{Φ^*}. ∎

To prove our main theorem, we need to establish an elementary lemma in linear algebra. The transpose and conjugate of a matrix A will be denoted by A^T and $\overline{A}$ respectively.

LEMMA 3.3. *Let H be a Hilbert space and M be an n-dimensional subspace such that $\{g_1, \ldots, g_n\}$ is a base for M. If $h_1, \ldots, h_n$ are n vectors in H such that $\langle h_i, g_j \rangle = \delta_{ij}$ (the Kronecker symbol), then*

$$\text{Gr}(Ph_1, \ldots, Ph_n) = \text{Gr}^{*-1}(g_1, \ldots, g_n),$$

where P denotes the orthogonal projection onto M and where

$$\text{Gr}(g_1, \ldots, g_n) = [\langle g_i, g_j \rangle]_{1 \le i,j \le n}$$

and

$$\text{Gr}(Ph_1, \ldots, Ph_n) = [\langle Ph_i, Ph_j \rangle]_{1 \le i,j \le n}$$

are the Gramians.

Proof. We first prove the following projection formula:

$$(3.3) \qquad (Ph_1, \ldots, Ph_n) = (g_1, \ldots, g_n)\,\text{Gr}^{T-1}(g_1, \ldots, g_n).$$

To this end, we denote the right-hand side of the above equation by $(g'_1, \ldots, g'_n)$, which is a row of vectors in M. We have to check that $h_i - g'_i$ is orthogonal to M, or

$$\langle h_i - g'_i, g_j \rangle = 0$$

for any $1 \le i, j \le n$. Equivalently,

$$[\langle g'_i, g_j \rangle]_{1 \le i,j \le n} = I.$$

For simplicity, we introduce the following notation: for two matrices $A = [a_{ij}]_{m' \times m}$ and $B = [b_{ij}]_{m \times m''}$ with entries a_{ij}, b_{ij} in H, define

$$\langle A, B \rangle = \Big[\sum_k \langle a_{ik}, b_{kj} \rangle \Big]_{m' \times m''},$$

which is a numerical matrix. It is easy to see that if C and D are both numerical matrices,

$$\langle CA, BD \rangle = C \langle A, B \rangle \overline{D}.$$

Given vectors $f_1, \ldots, f_n$ and $f'_1, \ldots, f'_n$ in H, we have

$$[\langle f_i, f'_j \rangle]_{1 \le i,j \le n} = \langle (f_1, \ldots, f_n)^T, (f'_1, \ldots, f'_n) \rangle.$$

In particular,

$$\mathrm{Gr}(f_1, \ldots, f_n) = \langle (f_1, \ldots, f_n)^T, (f_1, \ldots, f_n) \rangle.$$

Now we have

$$\begin{aligned}
[\langle g'_i, g_j \rangle]_{1 \le i,j \le n} &= \langle (g'_1, \ldots, g'_n)^T, (g_1, \ldots, g_n) \rangle \\
&= \langle (\mathrm{Gr}^{T-1}(g_1, \ldots, g_n))^T (g_1, \ldots, g_n)^T, (g_1, \ldots, g_n) \rangle \\
&= \mathrm{Gr}^{-1}(g_1, \ldots, g_n) \langle (g_1, \ldots, g_n)^T, (g_1, \ldots, g_n) \rangle \\
&= \mathrm{Gr}^{-1}(g_1, \ldots, g_n) \mathrm{Gr}(g_1, \ldots, g_n) = I,
\end{aligned}$$

hence the projection formula (3.3) follows.

By this formula,

$$\begin{aligned}
\mathrm{Gr}(Ph_1, \ldots, Ph_n) &= \langle (Ph_1, \ldots, Ph_n)^T, (Ph_1, \ldots, Ph_n) \rangle \\
&= (\mathrm{Gr}^{T-1}(g_1, \ldots, g_n))^T \mathrm{Gr}(g_1, \ldots, g_n) \overline{\mathrm{Gr}^{T-1}(g_1, \ldots, g_n)} \\
&= \mathrm{Gr}^{*-1}(g_1, \ldots, g_n). \ \blacksquare
\end{aligned}$$

COROLLARY 3.4. *For a quasi-free Hilbert module R relative to the generating set $\{f_1, \ldots, f_n\}$, we have*

$$\mathrm{Gr}(f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z) = \mathrm{Gr}^{*-1}(k_z^1, \ldots, k_z^n),$$

where $\{k_z^i\}$ is the dual set of kernel functions at z.

Proof. Since the map $R \otimes_A \mathbb{C}_z \to R_z^\perp : f \otimes_A 1_z \mapsto P_z f$ is unitary, we have

$$\mathrm{Gr}(f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z) = \mathrm{Gr}(P_z f_1, \ldots, P_z f_n).$$

Recall that $R_z^\perp = \mathrm{span}\{k_z^1, \ldots, k_z^n\}$, and by (3.1) we have

$$\langle f_i, k_z^j \rangle_R = \langle f_i(z), e_j \rangle_{\mathbb{C}^n} = \langle e_i, e_j \rangle_{\mathbb{C}^n} = \delta_{ij},$$

hence Lemma 3.3 can be applied and we are done. $\blacksquare$

Now we are ready to give the proof of our main theorem.

Proof of Theorem 1.2. Let $\{f_1, \ldots, f_n\}$ and $\{f'_1, \ldots, f'_n\}$ be the generating sets relative to which R and R' are quasi-free. By the hypothesis of the

theorem, we have

$$\delta_z^* \delta_z = \Psi(z)^* \Psi(z)$$

for a function $\Psi(z)$ in $\mathrm{Hol}(\Omega)$. Since δ_z is invertible, so is $\Psi(z)$, hence we have

$$I = [\delta_z \Psi(z)^{-1}]^* [\delta_z \Psi(z)^{-1}].$$

Thus $\delta_z \Psi(z)^{-1}$ is unitary in the following diagram:

$$
\begin{array}{ccc}
R \otimes_A \mathbb{C}_z & \xrightarrow{\Psi^{-1}(z)} & R \otimes_A \mathbb{C}_z \\
& \searrow{\scriptstyle \delta_z \Psi^{-1}(z)} \quad \swarrow{\scriptstyle \delta_z} & \\
& R' \otimes_A \mathbb{C}_z &
\end{array}
$$

with matrix representation

$$\delta_z \Psi(z)^{-1} : R \otimes_A \mathbb{C}_z \to R' \otimes_A \mathbb{C}_z,$$
$$(f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z) \mapsto (f_1' \otimes_A 1_z, \ldots, f_n' \otimes_A 1_z)\Psi(z)^{-1}.$$

Hence we have the following relation of Gramians:

$$\mathrm{Gr}(f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z) = \Psi(z)^{T-1} \, \mathrm{Gr}(f_1' \otimes_A 1_z, \ldots, f_n' \otimes_A 1_z)\overline{\Psi(z)^{-1}},$$

where $\Psi(z)$ is a holomorphic matrix-valued function such that $\Psi(z)$ is invertible for every z. Appealing to Corollary 3.4 we get

$$\mathrm{Gr}^{*-1}(k_z^1, \ldots, k_z^n) = \Psi(z)^{T-1}\mathrm{Gr}^{*-1}(k_z'^1, \ldots, k_z'^n)\overline{\Psi(z)^{-1}}.$$

Equivalently,

$$(3.4) \qquad \mathrm{Gr}(k_z^1, \ldots, k_z^n) = \overline{\Psi(z)} \, \mathrm{Gr}(k_z'^1, \ldots, k_z'^n)\Psi(z)^T.$$

Combining this with (3.2), we come to the desired identity:

$$(3.5) \qquad K(z,z) = \Psi(z)K'(z,z)\Psi^*(z).$$

We claim the following stronger identity:

$$(3.6) \qquad K(z,w) = \Psi(w)K'(z,w)\Psi^*(z).$$

In fact, since $\Psi(z)$ is holomorphic, the matrix-valued function $K(z,w) - \Psi(w)K'(z,w)\Psi^*(z)$ is holomorphic in w and anti-holomorphic in z, and (3.5) implies that this function vanishes on the diagonal. Using Taylor expansion, it is not hard to see that such a function must vanish identically ([7, Proposition 1]), hence (3.6) holds.

Finally, we rewrite (3.6) as

$$K'(z,w) = \Psi(w)^{-1}K(z,w)\Psi^{-1*}(z),$$

so that Theorem 3.2 can be applied by taking $\Phi(z) = \Psi^{-1*}(z)$, completing the proof. In light of Theorem 3.2, the unitary map is given by $M_{\Psi^{-1}} : f(\cdot) \mapsto \Psi^{-1}(\cdot)f(\cdot)$. ∎

4. Equivalence of associated Hermitian vector bundles. As a complement of our main result, we characterize unitary equivalence of quasi-free Hilbert modules in terms of Hermitian vector bundles. We say that two holomorphic (resp. anti-holomorphic) Hermitian vector bundles are equivalent if there exists a holomorphic (resp. anti-holomorphic) *isometric* bundle map between them.

Given a quasi-free Hilbert module R with generator $\{f_1, \ldots, f_n\}$ and dual kernel $\{k_z^1, \ldots, k_z^n\}$, we can construct a natural bundle $\bigcup_{z \in \Omega} R_z^{\perp}$. As shown in [5], for any z in Ω and $1 \leq i \leq n$, the map $z \mapsto k_z^i$ is anti-holomorphic in z, thus $\bigcup_{z \in \Omega} R_z^{\perp}$ is an anti-holomorphic Hermitian vector bundle such that $\{k_z^i\}$ forms an anti-holomorphic frame. This bundle, obtained as the pull-back of the anti-holomorphic map $z \mapsto R_z^{\perp}$ from Ω to the Grassmann manifold of n-dimensional subspaces of R, is closely related to unitary invariants of quasi-free Hilbert modules.

Using the Rigidity Theorem established in [2], one can show that two finite rank quasi-free Hilbert modules R and R' are unitarily equivalent if and only if $\bigcup_{z \in \Omega} R_z^{\perp}$ and $\bigcup_{z \in \Omega} R_z'^{\perp}$ are equivalent.

On the other hand, there is another vector bundle $\mathrm{Sp}(R)$ that can be associated to a quasi-free Hilbert module R, which is holomorphic with fibre $R \otimes_A \mathbb{C}_z$ at each z in Ω. This bundle $\mathrm{Sp}(R) = \bigcup_{z \in \Omega} R \otimes_A \mathbb{C}_z$ is called the *spectral sheaf* of R. Douglas and Misra proved the following result:

PROPOSITION 4.1 ([5]). *For a finite rank quasi-free Hilbert module over $A(\Omega)$, there is a unique well-defined holomorphic structure on $\mathrm{Sp}(R)$ relative to which the functions $z \mapsto k \otimes_A 1_z$ are holomorphic sections for each k in R.*

Since $\{f_i \otimes 1_z\}$ spans $R \otimes_A \mathbb{C}_z$, it is a holomorphic frame for $\mathrm{Sp}(R)$. The metric of the fibre $R \otimes_A \mathbb{C}_z$ is induced by the metric on $R_z^{\perp}$ in R via the unitary map

$$R \otimes_A \mathbb{C}_z \to R_z^{\perp} : f \otimes_A 1_z \mapsto P_z f.$$

Douglas and Misra showed in [5] that this induced metric makes $\mathrm{Sp}(R)$ a holomorphic Hermitian vector bundle over Ω.

Finally, we give the following geometric characterization.

THEOREM 4.2. *For two finite rank quasi-free Hilbert modules R and R', the following are equivalent*:

(i) *R and R' and unitarily equivalent.*
(ii) *The anti-holomorphic Hermitian vector bundles $\bigcup_{z \in \Omega} R_z^{\perp}$ and $\bigcup_{z \in \Omega} R_z'^{\perp}$ are equivalent.*
(iii) *The holomorphic Hermitian vector bundles $\mathrm{Sp}(R)$ and $\mathrm{Sp}(R')$ are equivalent.*

Proof. (i)⇔(ii). Combine the Rigidity Theorem ([2]) with Lemma 3.1.

(ii)⇒(iii). By definition, there exists an anti-holomorphic matrix-valued function $\Theta(z)$ such that for any z in Ω, the fibre map

$$\Theta(z) : R_z^{\perp} \to R_z'^{\perp},$$
$$(k_z^1, \ldots, k_z^n) \mapsto (k_z'^1, \ldots, k_z'^n)\Theta(z),$$

is isometric. Hence we have

$$\mathrm{Gr}(k_z^1, \ldots, k_z^n) = \Theta(z)^T \,\mathrm{Gr}(k_z'^1, \ldots, k_z'^n)\overline{\Theta(z)}.$$

Applying Corollary 3.4, we get

$$\mathrm{Gr}(f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z) = \overline{\Theta(z)^{-1}} \,\mathrm{Gr}(f_1' \otimes_A 1_z, \ldots, f_n' \otimes_A 1_z)\Theta^{T-1}(z).$$

Therefore, the fibre map defined by

$$\widehat{\Theta}(z) : R \otimes_A \mathbb{C}_z \to R' \otimes_A \mathbb{C}_z,$$
$$(f_1 \otimes_A 1_z, \ldots, f_n \otimes_A 1_z) \mapsto (f_1' \otimes_A 1_z, \ldots, f_n' \otimes_A 1_z)\Theta^{*-1}(z),$$

is isometric. Noting that $\{f_i \otimes 1_z\}$ is a holomorphic frame, and $\Theta^{*-1}(z)$ is holomorphic since $\Theta(z)$ is anti-holomorphic, we see that the fibre maps $\widehat{\Theta}(z)$ glue to an isometric holomorphic bundle map from $\mathrm{Sp}(R)$ to $\mathrm{Sp}(R')$, hence $\mathrm{Sp}(R)$ and $\mathrm{Sp}(R')$ are equivalent.

(iii)⇒(ii) follows in the same way. ∎

REMARK 4.3. It is easy to see that if the modulus function $\mu_{(R,R')}$ is the absolute value of a function Ψ in $\mathrm{Hol}(\Omega)$, then $\mathrm{Sp}(R)$ and $\mathrm{Sp}(R')$ are equivalent via the fibre map given by $\Psi(z)^{-1}$ with respect to the holomorphic frames, therefore Theorem 1.2 can be regarded as a consequence of Theorem 4.2, and this essentially relies on the Rigidity Theorem.

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