STUDIA MATHEMATICA 147 (2) (2001)

Local dual spaces of a Banach space

by

Manuel González (Santander)
and Antonio Martínez-Abejón (Oviedo)

Abstract. We study the *local dual spaces* of a Banach space X, which can be described as the subspaces of X^* that have the properties that the principle of local reflexivity attributes to X as a subspace of X^{**}.

We give several characterizations of local dual spaces, which allow us to show many examples. Moreover, every separable space X has a separable local dual Z, and we can choose Z with the metric approximation property if X has it. We also show that a separable space containing no copies of ℓ_1 admits a smallest local dual.

1. Introduction. The principle of local reflexivity [14] shows that there is a close relation between a Banach space X and its second dual X^{**} from a finite-dimensional point of view: X^{**} is finitely dual representable in X with ε-isometries that fix points (see Definition 2.1). This means that X can be considered as a "local" dual of X^*.

In [9] the authors introduced the polar property as a test to check if X^* is finitely dual representable in its subspaces. Here we consider a smaller class of subspaces Z of X^* (Definition 2.1) that satisfy the principle of local reflexivity in full force: X^* is finitely dual representable in Z with ε-isometries that fix points. So we can properly refer to these subspaces Z as *local dual spaces* of X. We give several characterizations of such spaces, and we describe examples of local dual spaces for some classical spaces like $C[0,1]$, $L_1[0,1]$, and for some families of Banach spaces, like $\ell_1(X^*)$, $\ell_\infty(X)$, $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$ in the case that Y^* has the metric approximation property (M.A.P., for short). We show that for μ a finite positive measure, $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$, improving a result of Díaz [3]. We also prove that every separable space with the M.A.P. has a separable local dual space with the M.A.P.

2000 *Mathematics Subject Classification*: Primary 46B10, 46B20; Secondary 46B04, 46B08.

Key words and phrases: local dual space, local reflexivity, norming subspace.

Research supported in part by DGICYT Grant PB 97–0349 (Spain).

The relation between a Banach space X and its local dual spaces is symmetric, in the sense that any local dual Z of X has a local dual isometric to X. This fact can also be seen as an extension of the local reflexivity principle.

We prove that every subspace L of X^* is contained in a local dual Z of X with $\mathrm{dens}(Z) = \max\{\mathrm{dens}(L), \mathrm{dens}(X)\}$. Using this fact and some results of Godefroy and Kalton [7], we show that a separable space X containing no copies of ℓ_1 admits a smallest local dual Z_d which is also separable. This result provides an answer to a question in [7]. We also give a partial answer to another question in [7] by showing that a space X isometric to a dual space has a smallest local dual Z_d if and only if it admits a smallest norming subspace Z_n, and in this case $Z_d = Z_n$.

In the paper X and Y are Banach spaces, B_X the closed unit ball of X, S_X the unit sphere of X, and X^* the dual of X. We identify X with a subspace of X^{**}. A *subspace* is always a closed subspace. For $A \subset X$ we consider the sets

$$A^\circ := \{f \in X^* : |\langle f, x\rangle| \le 1 \text{ for every } x \in A\},$$
$$A^\perp := \{f \in X^* : \langle f, x\rangle = 0 \text{ for every } x \in A\}.$$

Analogously, for $C \subset X^*$, we define the subsets $C_\circ$ and $C_\perp$ of X. We denote by $\mathcal{B}(X,Y)$ the space of all (bounded linear) operators from X into Y, and by $\mathcal{K}(X,Y)$ the subspace of all compact operators. Given $T \in \mathcal{B}(X,Y)$, $N(T)$ and $R(T)$ are the range and the kernel of T, and T^* is the conjugate operator of T.

Given a number $0 < \varepsilon < 1$, an operator $T \in \mathcal{B}(X,Y)$ is an *ε-isometry* if it satisfies $(1+\varepsilon)^{-1} < \|Tx\| < 1+\varepsilon$ for all $x \in S_X$. A space X is said to be *finitely representable in* Y if for each $\varepsilon > 0$ and each finite-dimensional subspace M of X there is an ε-isometry $T : M \to Y$. We denote by $\mathbb{N}$ the set of all positive integers.

2. Local dual spaces. Recall that a subspace Z of X^* is *norming* if

$$\|x\| = \sup\{|\langle f, x\rangle| : f \in B_Z\} \quad \text{for every } x \in X.$$

Moreover, X^* is *finitely dual representable* (f.d.r., for short) *in* Z [9, Definition 1] if for every couple of finite-dimensional subspaces F of X^* and G of X, and for every $0 < \varepsilon < 1$, there is an ε-isometry $L : F \to Z$ such that $\langle Lf, x\rangle = \langle f, x\rangle$ for all $x \in G$ and all $f \in F$.

Clearly, if X^* is f.d.r. in Z, then Z is norming. However, the converse implication does not hold [9, Remark after Theorem 4].

Now we introduce a concept which is strictly stronger than finite dual representability (see Example 2.11).

DEFINITION 2.1. Let Z be a subspace of X^*. We say that Z is a *local dual space* of X if for every couple of finite-dimensional subspaces F of X^* and G of X, and every number $0 < \varepsilon < 1$, there is an ε-isometry $L : F \to Z$ satisfying the following conditions:

(a) $\langle Lf, x\rangle = \langle f, x\rangle$ for all $x \in G$ and all $f \in F$, and
(b) $Lf = f$ for all $f \in F \cap Z$.

REMARK 2.2. (a) Obviously, X^* is a local dual of X.

(b) A local dual of X provides an almost-isometric local representation of X^*. This could be useful when we do not have a description of X^*, but it is possible to find a local dual. This happens for the ultrapowers $X_{\mathfrak{U}}$ of X and for $L_\infty(\mu, X)$ (see (c) and Corollary 2.7).

(c) The principle of local reflexivity [14] establishes that a Banach space X (as well as every isometric predual of X^*) is a local dual of X^*, and the principle of local reflexivity for ultrapowers [12, Theorem 7.3] establishes that $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.

(d) There are spaces X so that X^* contains no proper norming subspaces. Hence X^* is the only local dual of X. This is the case when X is an M-ideal in its bidual [11, Corollary III.2.16], or more generally, when X is Hahn–Banach smooth. This means that every $x^* \in X^*$ admits only one Hahn–Banach extension to X^{**} [21].

For the structure of Banach spaces admitting no proper norming subspaces, we refer to [6], specially Theorem 8.3 where a characterization of these spaces is given, and [2].

The following technical result is a direct application of the Hahn–Banach Theorem and the principle of local reflexivity.

LEMMA 2.3. *Let Z be a norming subspace of X^*. Then for every couple of finite-dimensional subspaces E of $Z^{\perp\perp}$ and F of X, and every $0 < \varepsilon < 1$, there is an ε-isometry $L : E \to Z$ such that $Le = e$ for all $e \in E \cap Z$, and $\langle Le, x\rangle = \langle e, x\rangle$ for all $e \in E$ and all $x \in F$.*

Proof. The canonical isometry $T : Z^{\perp\perp} \to Z^{**}$ maps $f \in Z^{\perp\perp}$ to the functional $\widehat{f} \in Z^{**}$ defined by $\langle \widehat{f}, \zeta\rangle := \langle f, \zeta_e\rangle$, where $\zeta_e \in (X^*)^*$ is any Hahn–Banach extension of $\zeta \in Z^*$. Note that $\langle T^{-1}\widehat{f}, g\rangle := \langle \widehat{f}, g|_Z\rangle$ for every $g \in X^{**}$.

Let $E_1 := T(E)$. Since Z is norming, the map that sends $x \in F$ to the functional $\widehat{x} \in Z^*$ given by $\langle \widehat{x}, z\rangle := \langle z, x\rangle$ is an isometry. So we can apply the principle of local reflexivity to get an ε-isometry $\Lambda : E_1 \to Z$ such that $\langle a, \widehat{x}\rangle = \langle \Lambda a, x\rangle$ for all $a \in E_1$ and $x \in F$, and $\Lambda a = a$ for all $a \in E_1 \cap Z$. The ε-isometry $L := \Lambda T|_E$ satisfies the requirements of our statement. ■

DEFINITION 2.4. Given a couple of subspaces Z of X^* and G of Z^*, an operator $L : G \to X^{**}$ is said to be an *extension operator* if $Lf|_Z = f$ for every $f \in G$.

The following result will be very useful to find examples of local dual spaces of a Banach space X. The proof uses an ultrapower version of the Lindenstrauss compactness principle and some ideas of [15]. These ideas have also been used in [8, Proposition 3.6] in order to give a local characterization of subspaces of a Banach space which are unconditional ideals.

THEOREM 2.5. *For a subspace Z of X^*, the following statements are equivalent*:

(1) *Z is a local dual of X*;

(2) *for every couple of finite-dimensional subspaces F of X^* and G of X, and every $0 < \varepsilon < 1$, there is an ε-isometry $L : F \to Z$ such that*

(a′) *$|\langle Lf, x\rangle - \langle f, x\rangle| < \varepsilon\|f\| \cdot \|x\|$ for all $x \in G$ and all $f \in F$, and*
(b′) *$\|Lf - f\| \le \varepsilon\|f\|$ for all $f \in F \cap Z$*;

(3) *there is an isometric extension operator $L : Z^* \to X^{**}$ so that $R(L) \supset X$*;

(4) *there exists a norm-one projection $P : X^{**} \to X^{**}$ such that $N(P) = Z^\perp$ and $R(P) \supset X$*;

(5) *there exists a norm-one projection $Q : X^{***} \to X^{***}$ such that $R(Q) = Z^{\perp\perp}$ and $N(Q) \subset X^\perp$ (where $X^{***} = X^* \oplus X^\perp$).*

Proof. We denote by ι the natural inclusion operator from Z into X^*. Observe that $\iota^* : X^{**} \to Z^*$ is the restriction operator: $\iota^*(F) = F|_Z$.

(1)⇒(2). This is trivial.

(2)⇒(3). First, for every compact operator $T : Z \to Y$ we obtain a compact extension $\widetilde{T} : X^* \to Y$ with $\|\widetilde{T}\| = \|T\|$, as follows:

Let $\mathcal{A}$ be the family of all pairs $\alpha = (E_\alpha, F_\alpha)$ of finite-dimensional subspaces $E_\alpha \subset X^*$ and $F_\alpha \subset X$. We define $|\alpha| := \dim E_\alpha + \dim F_\alpha$. For every $\alpha \in \mathcal{A}$ we select an $|\alpha|^{-1}$-isometry $L_\alpha : E_\alpha \to Z$ such that $|\langle L_\alpha e, x\rangle - \langle e, x\rangle| < \varepsilon\|e\| \cdot \|x\|$ for all $e \in E_\alpha$ and all $x \in F_\alpha$, and $\|L_\alpha z - z\| \le |\alpha|^{-1}\|z\|$ for all $z \in E_\alpha \cap Z$.

We fix an ultrafilter $\mathfrak{U}$ on $\mathcal{A}$ refining the order filter associated to the order inclusion. Taking $L_\alpha g = 0$ for $g \notin E_\alpha$, we can define the operator

$$\widetilde{T}g := \lim_{\alpha \to \mathfrak{U}} TL_\alpha g, \qquad g \in X^*.$$

Note that $(TL_\alpha g)_{\alpha \in \mathcal{A}}$ is contained in the compact set $2\|g\| \cdot \overline{TB_Z}$. Since $L_\alpha|_{E_\alpha \cap Z}$ converges to the identity map, $\widetilde{T}$ is an extension of T; i.e., $\widetilde{T}\iota = T$. In particular, $\|T\| \le \|\widetilde{T}\|$. Moreover, $\|\widetilde{T}\| = \lim_{\alpha \to \mathfrak{U}} \|TL_\alpha\| \le \|T\|$.

Now, for every finite-dimensional subspace G of Z^*, we consider the quotient map $q_G : Z \to Z/G_\perp$, and denote by ι_G the inclusion operator from G into Z^*. Let $Q_G : X^* \to Z/G_\perp$ be the extension of q_G built as in the first part of the proof. Note that $\iota^* Q_G^* = \iota_G$, so $Q_G^* : G \to X^{**}$ is an isometric extension operator.

Let $\mathfrak{V}$ be an ultrafilter on the set of all the finite-dimensional subspaces of Z, refining the filter associated to the order inclusion. The w^*-compactness of $B_{X^{**}}$ allows us to define $L : Z^* \to X^{**}$ as $Lh := w^*\text{-}\lim_{G\to\mathfrak{V}} Q_G^* h$. Clearly, $\|L\| = 1$ and $\iota^* L$ is the identity, so L is an extension operator.

It only remains to see that $R(L) \supset X$. Since every $x \in X$ belongs to X^{**}, we can consider its restriction $x|_Z \in Z^*$. Let G be any finite-dimensional subspace of Z^* containing $x|_Z$. Then for every $f \in X^*$, we have

$$\begin{aligned}\langle Q_G^*(x|_Z), f\rangle &= \langle x|_Z, Q_G f\rangle = \lim_{\alpha\to\mathfrak{U}} \langle x|_Z, q_G L_\alpha f\rangle \\ &= \lim_{\alpha\to\mathfrak{U}} \langle \iota_G(x|_Z), L_\alpha f\rangle = \lim_{\alpha\to\mathfrak{U}} \langle L_\alpha f, x\rangle = \langle f, x\rangle,\end{aligned}$$

hence $Q_G^*(x|_Z) = x$, so $x = L(x|_Z)$, concluding the proof.

(3)⇒(4). The operator $P := L\iota^*$ defines a projection on X^{**}, because $\iota^* L$ is the identity on Z^*, and $\|P\| = 1$. Also, $R(P) \supset X$ since ι^* is surjective and $R(L) \supset X$. Finally, $N(P) = N(\iota^*) = Z^\perp$.

(4)⇒(5). It is enough to take $Q = P^*$.

(5)⇒(1). Let Q be a norm-one projection on X^{***} such that $R(Q) = Z^{\perp\perp}$ and $N(Q) \subset X^\perp$. First, considering the natural embedding of X^* in X^{***}, we show that the restriction $Q|_{X^*}$ is an isometry. Indeed, given $f \in X^*$ and $0 < \varepsilon < 1$, we select $x \in X$ such that $\|x\| = 1$ and $\langle f, x\rangle > \|f\| - \varepsilon$. Since $R(I - Q) \subset X^\perp$, we have $\langle f, x\rangle = \langle Qf, x\rangle$; hence $\|Qf\| = \|f\|$.

Fix $F \in Z^\perp$ and $x \in X$. We choose $f \in X^*$ so that $\|f\| = 1$ and $\langle f, x\rangle = \|x\|$. Since $R(Q) = Z^{\perp\perp}$ and $N(Q) \subset X^\perp$, we have

$$\|F - x\| \geq |\langle F - x, Qf\rangle| = |\langle x, Qf\rangle| = \|x\|.$$

By [5, Lemma I.1], we conclude that Z is norming.

Now, in order to show that Z is a local dual of X, we take a number $0 < \varepsilon < 1$ and finite-dimensional subspaces $E \subset X^*$ and $F \subset X$, and set $E_1 := Q(E) \subset Z^{\perp\perp}$.

Applying Lemma 2.3, we get an ε-isometry $L : E_1 \to Z$ so that

$$\begin{aligned}\langle Le, x\rangle &= \langle e, x\rangle \quad &&\text{for all } e \in E_1,\ x \in F, \\ Le &= e &&\text{for all } e \in E_1 \cap Z.\end{aligned}$$

Thus $LQ : E \to Z$ is an ε-isometry. Moreover, given $e \in E \cap Z$, we have $Le = e$ and $Qe = e$, so $LQe = e$. In addition, given $e \in E$, $x \in F$, we see that

$$\langle LQe, x\rangle = \langle Qe, x\rangle = \langle e, x\rangle,$$

and the proof is complete. ∎

Remark 2.6. Since X is weak*-dense in X^{**}, the projection P in Theorem 2.5(4) cannot be weak*-continuous.

A direct application of Theorem 2.5 gives the following improvement of a result of Díaz [3, Theorem 2.1].

Corollary 2.7. *Let μ be a finite positive measure. Then $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$.*

Proof. It is enough to observe that, in the case that μ is a probability measure, [3, Theorem 2.1] establishes that $L_1(\mu, X^*)$ satisfies condition (2) in Theorem 2.5. ■

Next we apply Theorem 2.5 to show examples of local dual spaces for some classical Banach spaces.

Let λ be a positive Borel measure on a metrizable compact space K. We denote by $B(K)$ the Banach algebra of all scalar, Borel-measurable bounded functions on K, endowed with the supremum norm. We can identify $L_\infty(\lambda)$ with the quotient $B(K)/\mathcal{J}_0$, where $\mathcal{J}_0 := \{f \in B(K) : \int |f|\,d\lambda = 0\}$. We denote by

$$\pi_\lambda : B(K) \to L_\infty(\lambda)$$

the canonical quotient map. Using the continuum hypothesis, it was proved in [18, Theorem 3], in the case $\mathrm{supp}(\lambda) = K$, that π_λ admits a *strong Borel lifting*; i.e., there exists an algebra homomorphism

$$\varrho_\lambda : L_\infty(\lambda) \to B(K)$$

so that for every $f \in L_\infty(\lambda)$ we have $\varrho_\lambda(f)(t) = f(t)$ for λ-almost all $t \in K$, and $\varrho_\lambda(f) = f$ for every $f \in C(K)$.

It follows from these properties that ϱ_λ is a right inverse of π_λ that satisfies

$$\|\varrho_\lambda(f)\| = \|f\|_\infty \quad \text{for every } f \in L_\infty(\lambda).$$

For m the Lebesgue measure on $[0,1]$, the existence of ϱ_m can be derived from the results of von Neumann and Stone in [19]. In this case we write $L_\infty[0,1]$ rather than $L_\infty(m)$. Recall that $L_1[0,1]^* = L_\infty[0,1]$ and $C[0,1]^* = M[0,1]$, the space of all regular Borel measures on $[0,1]$. For every positive $\lambda \in M[0,1]$, the space $L_1(\lambda)$ is embedded in $M[0,1]$ through the map $f \mapsto \lambda_f$, where $\lambda_f(U) = \int_U f\,d\lambda$.

Proposition 2.8. *Assume the continuum hypothesis $2^\omega = \omega_1$.*

(a) *The natural copy of $C[0,1]$ in $L_\infty[0,1]$ is a local dual of $L_1[0,1]$.*
(b) *The natural copy of $L_1[0,1]$ in $M[0,1]$ is a local dual of $C[0,1]$.*

Proof. (a) We consider the map $L : M[0,1] \to L_\infty[0,1]^*$ given by

$$\langle L\mu, f\rangle := \int_0^1 \varrho_m(f)(t)\,d\mu(t).$$

This map is well defined because $\varrho_m(f)$ is Borel measurable for every f in $L_\infty[0,1]$. Since $\varrho_m(g) = g$ for every $g \in C[0,1]$, L is an isometric extension operator from $C[0,1]^*$ into $L_1[0,1]^{**}$. Moreover, $\varrho_m(f)(t) = f(t)$ a.e. for every f implies that $L(h) = h$ for every $h \in L_1[0,1]$; hence $L(M[0,1]) \supset L_1[0,1]$.

(b) The map $L_m : L_\infty[0,1] \to M[0,1]^*$ defined by

$$\langle L_m f, \mu\rangle := \int_0^1 \varrho_m(f)(t)\, d\mu(t)$$

is an isometric extension operator from $L_1[0,1]^*$ into $C[0,1]^{**}$, because $M[0,1] \supset L_1[0,1]$ and $\varrho_m(f)(t) = f(t)$ a.e. for every f. Moreover, $\varrho_m(f) = f$ for every $f \in C[0,1]$ implies $L_m(L_\infty[0,1]) \supset C[0,1]$. ■

REMARK 2.9. (a) In the proof of Proposition 2.8, we have applied the continuum hypothesis in order to select a Borel function for every $f \in L_\infty$, so that we can define an isometric extension operator $L : C[0,1]^* \to L_\infty[0,1]^*$. However, in our opinion it should be possible to find a proof in which the continuum hypothesis is not necessary.

(b) The Radon–Nikodym theorem allows us to write

$$C[0,1]^* = L_1[0,1] \oplus_1 M_{\rm sing}[0,1].$$

So if Q is the projection with range $L_1[0,1]$ and kernel $M_{\rm sing}[0,1]$, then Q^* is a norm-one projection on $C[0,1]^{**}$ with $N(Q^*) = L_1[0,1]^\perp$. However, $R(Q^*) = M_{\rm sing}[0,1]^\perp \not\supset C[0,1]$. Thus, we cannot apply part (4) of Theorem 2.5 to derive that $L_1[0,1]$ is a local dual of $C[0,1]$.

(c) Suppose that $\lambda \in M[0,1]$ is a positive Borel measure with support equal to $[0,1]$ and satisfying $m \perp \lambda$. Then using an argument similar to that in Proposition 2.8, we can prove that $L_1(\lambda)$ is a local dual of $C[0,1]$. Hence, $C[0,1]$ admits two local dual spaces with intersection $\{0\}$.

As an example of a measure λ so that $L_1(\lambda) \cap L_1[0,1] = \{0\}$, we can consider the discrete measure associated to a dense sequence in $[0,1]$.

Proposition 2.8 and the principle of local reflexivity suggest that the relation of "being a local dual" is symmetric. Next we prove it.

Let Z be a local dual of X. Denoting by $\widehat{x}$ the vector $x \in X$ viewed as an element of X^{**}, we consider the following natural map:

$$\Upsilon : x \in X \mapsto \widehat{x}|_Z \in Z^*.$$

Note that Υ is an isometry, because Z is norming.

PROPOSITION 2.10. *Let Z be a local dual of X and let $L : Z^* \to X^{**}$ be an isometric extension such that $L(Z^*) \supset X$. Then*

(a) *$L\Upsilon$ is the natural embedding from X into X^{**}.*

(b) *$\Upsilon(X)$ is a local dual of Z isometric to X.*

Proof. (a) Let J and ι denote the embedding of X in X^{**} and the embedding of Z in X^*, respectively. Then $\Upsilon = \iota^* J$. Moreover, for every

$z \in Z$ and $z^* \in Z^*$,

$$\langle \iota^* L z^*, z \rangle = \langle L z^*, \iota z \rangle = \langle z^*, z \rangle;$$

hence $\iota^* L$ is the identity on Z^*. Thus, $L\iota^*$ is a projection on X^{**} with $R(L\iota^*) = R(L) \supset X$; hence $L\Upsilon = L\iota^* J = J$.

(b) We define $\Lambda : \Upsilon(X)^* \to Z^{**}$ by

$$\langle \Lambda f, g \rangle := \langle Lg, f \circ \Upsilon \rangle, \qquad f \in \Upsilon(X)^*, \ g \in Z^*.$$

Clearly $\|\Lambda\| \le 1$. Moreover, for every $f \in \Upsilon(X)^*$ and every $\Upsilon x \in \Upsilon(X)$,

$$\langle \Lambda f, \Upsilon x \rangle = \langle L\Upsilon x, f \circ \Upsilon \rangle = \langle f \circ \Upsilon, x \rangle = \langle f, \Upsilon x \rangle.$$

Thus Λ is an isometric extension operator. Moreover, for every $y \in Z$ we can write $y = f \circ \Upsilon$ with $f \in \Upsilon(X)^*$. Then

$$\langle \Lambda f, g \rangle = \langle Lg, f \circ \Upsilon \rangle = \langle Lg, y \rangle = \langle g, y \rangle$$

for every $g \in Z^*$; hence $\Lambda(\Upsilon(X)^*) \supset Z$. ∎

Now we show that X^* f.d.r. in Z does not imply that Z is a local dual of X. In order to do that, observe that

$$Z_1 \subset Z_2 \subset X^* \text{ and } X^* \text{ f.d.r. in } Z_1 \ \Rightarrow \ X^* \text{ f.d.r. in } Z_2.$$

The following example shows that this implication is not valid for local dual spaces.

Example 2.11. The principle of local reflexivity implies that ℓ_∞ is f.d.r. in c_0.

On the other hand, since the quotient map $q : \ell_\infty \to \ell_\infty / c_0$ is not weakly compact, ℓ_∞ / c_0 contains a complemented subspace isomorphic to ℓ_∞ [17, Proposition 2.f.4]. Therefore, there is a subspace N of ℓ_∞ / c_0 such that the quotient space $(\ell_\infty / c_0)/N$ is isomorphic to ℓ_2. We take $M := q^{-1}(N)$.

Claim. *$\ell_1^* = \ell_\infty$ is f.d.r. in M, but M is not a local dual of ℓ_1.*

Note that ℓ_∞ / M is isomorphic to $(\ell_\infty / c_0)/N$, so $M^\perp$ is isomorphic to ℓ_2. Since $c_0 \subset M$, we see that ℓ_∞ is f.d.r. in M. But $M^\perp$ is non-complemented in ℓ_∞^* because ℓ_∞^* has the Dunford–Pettis property. Therefore, by Theorem 2.5, M is not a local dual of ℓ_1. ∎

Proposition 2.12. *Let X be a Banach space. Then*

(a) *$\ell_1(X^*)$ is a local dual of $\ell_\infty(X)$, and*
(b) *$\ell_\infty(X)$ is a local dual of $\ell_1(X^*)$.*

Proof. (a) For every couple $\alpha := (E, F)$ of finite-dimensional subspaces of $\ell_1(X^*)$, $\ell_\infty(X^{**})$, we select a pair of sequences (E_n), (F_n) of finite-dimensional subspaces of X^* and X^{**} respectively so that $E \subset \ell_1(E_n)$ and $F \subset \ell_\infty(F_n)$. We define $|\alpha| := \dim(E) + \dim(F)$.

The principle of local reflexivity allows us to find, for every n, an $|\alpha|^{-1}$-isometry $S_n^\alpha : F_n \to X$ so that $\langle S_n^\alpha f, e\rangle = \langle e, f\rangle$ for every $e \in E_n$ and $f \in F_n$, and $S_n^\alpha(f) = f$ for every $f \in F_n \cap X$. We consider the (non-linear) map $S^\alpha : \ell_\infty(X^{**}) \to \ell_\infty(X)$ given by $S^\alpha(z_n) := (S_n^\alpha(z_n))$ if $(z_n) \in F$, and $S^\alpha(z_n) := 0$ otherwise.

Let $\mathfrak{U}$ be an ultrafilter in the set of all couples $\alpha = (E, F)$ of finite-dimensional subspaces E of $\ell_1(X^*)$ and F of $\ell_\infty(X^{**})$ refining the order filter. We define an operator $\Lambda : \ell_1(X^*)^* = \ell_\infty(X^{**}) \to \ell_\infty(X)^{**}$ by

$$\Lambda((z_n)) := w^*\text{-}\lim_{\alpha\to\mathfrak{U}} S^\alpha(z_n), \qquad (z_n) \in \ell_\infty(X^{**}).$$

Note that Λ is an isometry and $\Lambda(y_n) = (y_n)$ for every $(y_n) \in \ell_\infty(X^{**})$. Therefore, Λ is an isometric extension operator. Moreover, $\Lambda((x_n)) = (x_n)$ if $(x_n) \in \ell_\infty(X)$. In particular $\Lambda(\ell_\infty(X^{**})) \supset \ell_\infty(X)$.

(b) It is enough to observe that the operator $\Upsilon : \ell_\infty(X) \to \ell_1(X^*)^*$, introduced in Proposition 2.10, is the natural inclusion. ∎

Recall that a Banach space X has the *metric approximation property* (M.A.P., for short) if for every $\varepsilon > 0$ and every compact set K in X, there is a finite rank operator T on X such that $\|T\| \le 1$ and $\|Tx - x\| \le \varepsilon$ for every $x \in K$. Note that if X^* has the M.A.P., then so does X [4, Corollary VIII.3.9]. However, the converse implication is not valid [17, Theorem 1.e.7].

The following result is proved using some ideas of [13].

Proposition 2.13. *Assume that X^* or Y^* has the M.A.P. Then*

(a) *$X^* \otimes_\varepsilon Y^*$ is a local dual of $X \otimes_\pi Y$, and*
(b) *$X^* \otimes_\pi Y^*$ is a local dual of $X \otimes_\varepsilon Y$.*

Proof. We assume that Y^* has the M.A.P.

(a) The dual space $(X\otimes_\pi Y)^*$ can be identified with $\mathcal{B}(X, Y^*)$. Moreover, since Y^* has the M.A.P., $X^* \otimes_\varepsilon Y^*$ can be identified with $\mathcal{K}(X, Y^*)$, and there exists a net (A_α) of finite rank operators on Y^* with $\|A_\alpha\| \le 1$ so that $\lim_\alpha \|A_\alpha g - g\| = 0$ for every $g \in Y^*$. We can assume that (A_α) is $\sigma(\mathcal{K}(Y^*)^{**}, \mathcal{K}(Y^*)^*)$-convergent.

For $T \in \mathcal{B}(X, Y^*)$ and $\Phi \in \mathcal{K}(X, Y^*)^*$, the expression $\Phi_T(A) := \Phi(AT)$ defines $\Phi_T \in \mathcal{K}(Y^*)^*$. Then we define $\Lambda : \mathcal{K}(X, Y^*)^* \to \mathcal{B}(X, Y^*)^*$ by

$$\langle \Lambda\Phi, T\rangle := \lim_\alpha \langle \Phi, A_\alpha T\rangle = \lim_\alpha \langle A_\alpha, \Phi_T\rangle.$$

Note that for every $f \otimes g \in X^* \otimes_\varepsilon Y^*$ we have

$$\langle \Lambda\Phi, f \otimes g\rangle = \lim_\alpha \langle \Phi, A_\alpha(g) \cdot f\rangle = \langle \Phi, f \otimes g\rangle.$$

So Λ is an isometric extension operator. Analogously, we can check that for every $x \otimes y \in X \otimes_\pi Y \subset \mathcal{B}(X, Y^*)^*$, we have $\Lambda(x \otimes y|_{\mathcal{K}(X,Y^*)}) = x \otimes y$. Thus $X \otimes_\pi Y \subset \Lambda(\mathcal{K}(X, Y^*)^*)$, and it is enough to apply Theorem 2.5.

(b) The proof is analogous, identifying $(X\otimes_\varepsilon Y)^*$ with the space $\mathcal{I}(X,Y^*)$ of all integral operators from X into Y^*. ∎

Remark 2.14. (a) If we assume in Proposition 2.13 that Y^* has the metric compact approximation property (defined as the M.A.P., but using compact operators instead of finite rank operators), then we find that $\mathcal{K}(X,Y^*)$ is a local dual of $X\otimes_\pi Y$.

(b) It follows from the results of Lima [16, Theorem 13] that if Y^* has the Radon–Nikodym property and $Y^{**}\otimes_\varepsilon Y^*$ is a local dual of $Y^*\otimes_\pi Y$, then Y^* has the M.A.P. So it is not enough to assume in Proposition 2.13 that X or Y has the M.A.P.

(c) Let μ be a finite positive measure and let K be a compact space. Since the spaces $L_1(\mu)^*\equiv L_\infty(\mu)$ and $C(K)^*\equiv M(K)$ have the M.A.P., it follows from Proposition 2.13 that $X^*\otimes_\varepsilon L_\infty(\mu)$ is a local dual of $L_1(\mu,X)=X\otimes_\pi L_1(\mu)$, and that $X^*\otimes_\pi M(K)$ is a local dual of $C(K,X)=X\otimes_\varepsilon C(K)$.

(d) The tensor product $X^*\otimes_\varepsilon L_\infty(\mu)$ in part (c) can be identified with a (proper, in general) subspace of $L_\infty(\mu,X^*)$.

It has been proved in [10] that $L_\infty(\mu,X^*)$ is also a local dual of $L_1(\mu,X)$.

Casazza and Kalton [1] proved that for every separable Banach space X with the M.A.P., we can find a sequence (T_n) of finite rank operators on X such that

(a) $\lim_{n\to\infty}\|T_nx-x\|=0$ for all $x\in X$,
(b) $\lim_{n\to\infty}\|T_n\|=1$ and
(c) $T_nT_k=T_kT_n=T_{\min\{k,n\}}$;

i.e., X admits a *commuting* 1-*approximating sequence* (T_n). Using this fact we show in the following result that a separable Banach space with the M.A.P. admits a local dual of X with the M.A.P. Its proof is similar to the proof of [7, Lemma II.2].

Theorem 2.15. *Let X be a separable Banach space with the M.A.P., and let (T_n) be a commuting 1-approximating sequence on X. Then $\overline{\bigcup_{n=1}^\infty R(T_n^*)}$ is a local dual of X, and has the M.A.P.*

Proof. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$. We define a map P on X^{**} by

$$Pz:=w^*\text{-}\lim_{k\to\mathfrak{U}}T_k^{**}z,\qquad z\in X^{**}.$$

From $T_n^{**}T_k^{**}=T_k^{**}T_n^{**}=T_{\min\{k,n\}}^{**}$ and the weak*-continuity of the operators T_n^{**}, it follows that for every $n\in\mathbb{N}$ and every $z\in X^{**}$, we have

$$T_n^{**}Pz=PT_n^{**}z=T_n^{**}z. \tag{1}$$

Hence $P^2z=w^*\text{-}\lim_{n\to\mathfrak{U}}T_n^{**}Pz=Pz$. Since $\lim_{n\to\infty}\|T_n\|=1$, P is a norm-one projection. Also, it follows from formula (1) that $N(T_n^{**})\supset N(P)$

for every $n \in \mathbb{N}$. Since the intersection of the kernels $N(T_n^{**})$ is clearly contained in $N(P)$, we get

$$N(P) = \bigcap_{n=1}^{\infty} N(T_n^{**}).$$

As a consequence, $N(P)$ is weak*-closed. And clearly $P(X^{**}) \supset X$.

Note that $T_n T_k = T_k T_n = T_{\min\{k,n\}}$ implies $N(T_n^{**})_\perp = R(T_n^*) \subset R(T_{n+1}^*)$ for every n. Therefore

$$N(P)_\perp = \overline{\bigcup_{n=1}^{\infty} R(T_n^*)},$$

and it follows from Theorem 2.5 that $\overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$ is a local dual of X.

Moreover, since $T_n^* f$ is weak*-convergent for every $f \in X^*$, and compact operators take weak*-convergent sequences to norm-convergent sequences, by formula (1) we have $\lim_{k\to\infty} \|T_k^* f - f\| = \lim_{k\to\infty} \|T_n^*(T_k^* g - g)\| = 0$ for every $f = T_n^* g \in R(T_n^*)$. Since (T_k^*) is bounded, we get $\lim_{k\to\infty} \|T_k^* f - f\| = 0$ for every $f \in \overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$; hence $\overline{\bigcup_{n=1}^{\infty} R(T_n^*)}$ has the M.A.P. ∎

REMARK 2.16. If X has a monotone Schauder basis, then the local dual of X provided by Theorem 2.15 is the subspace generated by the coefficient functionals of the basis.

As an application of Theorem 2.15, we give another example of a local dual space of $L_1[0,1]$.

EXAMPLE 2.17. *The subspace Z of $L_\infty[0,1]$ generated by the characteristic functions $\chi_{n,i}$ of the dyadic intervals*

$$\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right], \qquad n = 0,1,2,\ldots;\ i = 1,\ldots 2^n,$$

is a local dual of $L_1[0,1]$ isometric to $C(\Delta)$, where Δ denotes the Cantor set.

It is enough to check that the sequence (P_n) of projections defined by

$$P_n f := \sum_{i=1}^{2^n} \langle 2^n \chi_{n,i}, f\rangle \chi_{n,i}$$

is a commuting 1-approximating sequence in $L_1[0,1]$, and that $\bigcup_n R(P_n^*)$ is the subspace generated by the functions $\chi_{n,i}$.

In relation to the necessity of the continuum hypothesis in Proposition 2.8, note that $C(\Delta)$ is isomorphic, but not isometric to $C[0,1]$.

We have seen in part (a) of the previous example that there are local dual spaces Z_1 and Z_2 of $C[0,1]$ so that $Z_1 \cap Z_2$ is finite-dimensional. Now we will show that this cannot happen for spaces that contain no copies of ℓ_1.

Godefroy and Kalton [7] considered the family $\mathcal{P}_X$ of all the subspaces Y of X^{**} for which there is a norm-one projection on X^{**} such that $Y = N(P)$ and $R(P) \supset X$. The following result is an application of [7, Proposition V.1] and our previous results.

Proposition 2.18. *If X contains no copies of ℓ_1, then it admits a smallest local dual*; *i.e., there exists a local dual Z_d contained in every local dual of X.*

Proof. If X contains no copies of ℓ_1, then $\mathcal{P}_X$ consists of weak*-closed subspaces of X^{**} and has a largest element L [7, Proposition V.1]. By Theorem 2.5, the local dual spaces of X are precisely the subspaces Z of X^* such that $Z^\perp \in \mathcal{P}_X$. Thus $Z_d := L_\perp$ is the smallest local dual of X. ∎

The following result was obtained by Sims and Yost [20] (see [11, Lemmas III.4.3 and III.4.4]). Here, $\text{dens}(X)$ stands for the *density character* of X, defined as the smallest cardinal κ for which X has a dense subset of cardinality κ.

Proposition 2.19. *Let L be a subspace of Y, and let F be a subspace of Y^* with* $\text{dens}(F) \le \text{dens}(L)$. *Then there exists a subspace M of Y with* $\text{dens}(M) = \text{dens}(L)$ *and $M \supset L$ for which there exists an isometric extension operator $T : M^* \to Y^*$ such that $T(M^*) \supset F$.*

We now prove our next result about the existence of local dual spaces.

Proposition 2.20. *Every subspace L of X^* is contained in a local dual Z_L of X with* $\text{dens}(Z_L) = \max\{\text{dens}(L), \text{dens}(X)\}$.

Proof. Given a subspace L of X^*, it is easy to find a subspace L_0 of X^* so that $L \subset L_0$ and $\text{dens}(L_0) = \max\{\text{dens}(L), \text{dens}(X)\}$. If we apply Proposition 2.19 to L_0 as a subspace of X^* and X as a subspace of X^{**} we get a subspace Z_L of X^* with $Z_L \supset L$ and $\text{dens}(Z_L) = \max\{\text{dens}(L), \text{dens}(X)\}$ for which there exists an isometric extension operator $T : Z_L^* \to X^{**}$ such that $T(Z_L^*) \supset X$. By Theorem 2.5, this is the desired local dual of X. ∎

Remark 2.21. (a) Assume that X is separable and contains no copies of ℓ_1, and that X^* is not separable. By Proposition 2.20, the smallest local dual space Z_d provided by Proposition 2.18 is separable; in particular, $Z_d \neq X^*$. This fact gives an affirmative answer to a question of Godefroy and Kalton in [7, Remarks V.3].

(b) Assume that X contains no copies of ℓ_1. In this case, apart from the smallest local dual Z_d there also exists a smallest norming subspace $Z_n \subset X^*$ [5, Lemma I.2 and Theorem II.3]. Clearly Z_n is contained in Z_d. However, we do not know whether or not $Z_n = Z_d$.

Question [7, Remarks V.3]. Assume that both Z_n and Z_d exist for X. Is $Z_n = Z_d$?

We can only give an affirmative answer for dual spaces.

PROPOSITION 2.22. *Assume that X is isometric to a dual space. Then X admits a smallest local dual Z_d if and only if it admits a smallest norming subspace Z_n. In this case $Z_d = Z_n$, and this space is the unique isometric predual of X.*

Proof. By [5, Lemma I.2], the smallest norming subspace Z_n exists if and only if

$$Z_n^{\perp} = \{z \in X^{**} : \|z - x\| \geq \|x\| \text{ for every } x \in X\}.$$

In this case $X^{**} = X \oplus Z_n^{\perp}$ and Z_n is the unique predual of X [5, Theorem II.1].

Clearly, the projection P on X^{**} with kernel $Z_n^{\perp}$ and range X satisfies $\|P\| = 1$ and the remaining conditions in Theorem 2.5. Hence Z_n is a local dual of X, and it is the smallest one, because every local dual is norming.

Conversely, assume that the smallest local dual Z_d exists, and let P be the associated projection. If X_* is a predual of X, then $X^{**} = X \oplus X_*^{\perp} = P(X) \oplus Z_d^{\perp}$. Since $X \subset P(X)$ and $Z_d \subset X_*$ (hence $X_*^{\perp} \subset Z_d^{\perp}$), we conclude that $Z_d = X_*$ and $X_*^{\perp} = Z_d^{\perp}$. In particular,

$$Z_d^{\perp} = \{z \in X^{**} : \|z - x\| \geq \|x\| \text{ for every } x \in X\};$$

hence Z_d is the smallest norming subspace of X. ■

REMARK 2.23. (a) In Proposition 2.22, we have seen that a dual space admitting a smallest norming subspace has a unique predual. However, this condition is not sufficient, since $L_\infty[0,1]$ has a unique predual but it does not admit a smallest norming subspace [5, Proposition IV.2].

(b) There are spaces X containing no copies of ℓ_1 so that X^* is f.d.r. in a subspace Z which is not a local dual of X.

Indeed, let Y be a separable space such that Y^{**}/Y is isomorphic to c_0, and let $Q : Y^{**} \to Y^{**}/Y$ denote the quotient map. We select a subspace M of c_0 such that $M^{\perp}$ is not complemented in ℓ_1. For example, we can take M so that $M^{\perp}$ is isomorphic to $\ell_1(\ell_2^n)$.

The space $X = Y^*$ contains no copies of ℓ_1, and X^* is f.d.r. in $Z := Q^{-1}(M)$, because Y is contained in Z. However, Z is not a local dual of X because $Z^{\perp} = M^{\perp}$ is not complemented.

Acknowledgements. We are grateful to the referee for several valuable comments.

References

[1] P. G. Casazza and N. J. Kalton, *Notes on approximation properties in separable Banach spaces*, in: London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, 1990, 49–63.

[2] D. Chen and B.-L. Li, *Ball topology of Banach spaces*, Houston J. Math. 22 (1996), 821–833.

[3] S. Díaz, *A local approach to functionals on* $L^\infty(\mu, X)$, Proc. Amer. Math. Soc. 128 (2000), 101–109.

[4] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, 1977.

[5] G. Godefroy, *Existence and uniqueness of isometric preduals*: *a survey*, in: Contemp. Math. 85, Amer. Math. Soc., 1989, 131–193.

[6] G. Godefroy and N. J. Kalton, *The ball topology and its applications*, ibid., 195–237.

[7] —, —, *Approximating sequences and bidual projections*, Quart. J. Math. Oxford 48 (1997), 179–202.

[8] G. Godefroy, N. J. Kalton and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. 104 (1993), 13–59.

[9] M. González and A. Martínez-Abejón, *Local reflexivity of dual Banach spaces*, Pacific J. Math. 56 (1999), 263–278.

[10] —, —, *Local dual spaces of Banach spaces of vector-valued functions*, preprint.

[11] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[12] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.

[13] J. Johnson, *Remarks on Banach spaces of compact operators*, J. Funct. Anal. 32 (1979), 304–311.

[14] W. B. Johnson, H. P. Rosenthal and M. Zippin, *On bases*, *finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. 9 (1971), 488–506.

[15] N. J. Kalton, *Locally complemented subspaces and* $\mathcal{L}_p$*-spaces for* $0 < p < 1$, Math. Nachr. 115 (1984), 71–97.

[16] Å. Lima, *The metric approximation property*, *norm-one projections and intersection properties of balls*, Israel J. Math. 84 (1993), 451–475.

[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, 1977.

[18] S. P. Lloyd, *Two lifting theorems*, Proc. Amer. Math. Soc. 42 (1974), 128–134.

[19] J. von Neumann and M. H. Stone, *The determination of representative elements in the residual classes of a Boolean algebra*, Fund. Math. 25 (1935), 353–378.

[20] B. Sims and D. Yost, *Banach spaces with many projections*, Proc. Centre Math. Anal. Austral. Nat. Univ. 14 (1986), 335–342.

[21] F. Sullivan, *Geometrical properties determined by the higher duals of a Banach space*, Illinois J. Math. 21 (1977), 315–331.

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Cantabria
E-39071 Santander, Spain
E-mail: gonzalem@ccaix3.unican.es

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Oviedo
E-33007 Oviedo, Spain
E-mail: ama@pinon.ccu.uniovi.es

Received September 21, 2000
Revised version March 19, 2001 (4608)