STUDIA MATHEMATICA 147 (2) (2001)

The alternative Dunford–Pettis Property in the predual of a von Neumann algebra

by

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Abstract. Let A be a type II von Neumann algebra with predual A_*. We prove that A_* does not have the alternative Dunford–Pettis property introduced by W. Freedman [7], i.e., there is a sequence (φ_n) converging weakly to φ in A_* with $\|\varphi_n\| = \|\varphi\| = 1$ for all $n \in \mathbb{N}$ and a weakly null sequence (x_n) in A such that $\varphi_n(x_n) \nrightarrow 0$. This answers a question posed in [7].

1. Introduction. A Banach space X is said to have the *Dunford–Pettis property* (*DP*) if $f_n(x_n) \to 0$ for any weakly null sequences (x_n) in X and (f_n) in X^*. It is a classical result [6, 8] that the spaces $C(K)$ and $L_1(\mu)$ have DP. We refer to [5] as a good survey on DP.

In the 90's, the Dunford–Pettis property was studied in the setting of some algebraic structures. The von Neumann algebras having DP were characterized by C. H. Chu and B. Iochum [3] as the finite direct sums of type I_n von Neumann algebras. In [2], L. Bunce showed that the predual A_* of a von Neumann algebra A has DP if, and only if, A is of type I finite.

Another property which has been studied in the setting of preduals of von Neumann algebras is the so-called *Kadec–Klee property* (*KKP* in what follows). Recall that a Banach space has KKP if any sequence in the unit sphere whose weak limit is also in the unit sphere, is norm convergent. It is known [7, Theorem 3.4] that a C*-algebra has KKP if, and only if, it is finite-dimensional. In the case of the predual of a semifinite von Neumann algebra A, G. Dell'Antonio [4] showed that A_* satisfies KKP if, and only if, A is of type I with atomic center (see also [7, Remark 2.3]).

Recently, W. Freedman [7] has introduced a new property weaker than DP and KKP, called DP1. A Banach space X has *DP1* if $f_n(x_n) \to 0$ for

2000 *Mathematics Subject Classification*: 46L05, 46L10.

First author partially supported by Spanish MCYT project no. BFM 2000-1467, and Junta de Andalucía grant FQM-0185.

Second author partially supported by Spanish DGESIC project no. PB98-1371, and Junta de Andalucía grant FQM-0199.

any weakly convergent sequences $x_n \to x$ in X and $f_n \to 0$ in X^* such that $\|x_n\| = \|x\| = 1$. Of course, the condition $\|x_n\| = \|x\| = 1$ can be replaced by $\|x_n\| \to \|x\|$. This property has also been studied under some algebraic assumptions. For instance, W. Freedman also proved that DP1 is equivalent to DP for von Neumann algebras [7, Theorem 3.5], but it is strictly weaker than DP and KKP for preduals of von Neumann algebras [7, Example 2.4]. In a more general setting, it is shown in [1, Corollary 2] that a JBW*-triple satisfies DP1 if, and only if, it satisfies KKP or DP.

In [7], it is asked if every predual of a von Neumann algebra satisfies DP1. The aim of this paper is to show that this is not the case. Indeed, we prove that the predual of every type II von Neumann algebra fails to have DP1 (Theorem 3).

2. The results. Let A be a von Neumann algebra. In what follows, A_{sa} will denote the set of all hermitian elements in A. We recall that a *spin system* in A is a set S of at least two symmetries not equal to ± 1 satisfying $st + ts = 0$ whenever $s \neq t$ in S. By a *normal state* on A we mean a weak*-continuous, norm-one, positive, linear functional on A. The next proposition gives a sufficient condition for the predual of a von Neumann algebra to fail DP1.

PROPOSITION 1. *Let A be a von Neumann algebra. Suppose that there exist a countable spin system $\{s_n\}_{n\in\mathbb{N}}$ in A_{sa} and a normal state ϱ on A such that $\varrho(s_n x) = \varrho(x s_n)$ for every $x \in A$ and $n \in \mathbb{N}$. Then A_* does not satisfy DP1.*

Proof. For $n \in \mathbb{N}$, let ϱ_n be the element of A_* defined by

$$\varrho_n(x) = \varrho((1 + s_n)x) \quad (x \in A).$$

Since for every positive element $x \in A$, we have

$$0 \leq \varrho((1 + s_n)x(1 + s_n)) = \varrho((1 + s_n)^2 x) = 2\varrho((1 + s_n)x),$$

we deduce that each ϱ_n is a positive linear functional in A_*. Therefore,

$$\|\varrho_n\| = \varrho_n(1) = 1 + \varrho(s_n) \tag{1}$$

for every $n \in \mathbb{N}$. It is well known (see [9, p. 135]) that the real Banach subspace V of A_{sa} generated by $\{s_n\}$ (the so-called *spin factor*) is isomorphic to a real Hilbert space containing $\{s_n\}$ as an orthonormal system. Thus, the sequence (s_n) is weakly null in V and hence in A. In particular, it follows from (1) that

$$\|\varrho_n\| \to 1 = \|\varrho\|.$$

On the other hand, since ϱ is positive, the mapping

$$(a, b) \mapsto (a|b)_\varrho := \varrho(ab^*) \quad (a, b \in A)$$

is a positive sesquilinear form on A. If we write $N_\varrho = \{a \in A : \varrho(aa^*) = 0\}$, then the quotient A/N_ϱ can be completed to a Hilbert space denoted by H_ϱ. The natural quotient map from A to H_ϱ will be denoted by J_ϱ. It is easy to check that $(s_n|s_m)_\varrho = \delta_{nm}$ for $n, m \in \mathbb{N}$, and therefore $\{J_\varrho(s_n)\}$ is an orthonormal sequence in H_ϱ. Now, for every $x \in A$, we have

$$\|x\|^2 \geq \|J_\varrho x\|_\varrho^2 = (x|x)_\varrho \geq \sum_{n\in\mathbb{N}} |(x|s_n)_\varrho|^2 = \sum_{n\in\mathbb{N}} |\varrho(xs_n)|^2.$$

Therefore $\varrho(s_n x) \to 0$, and hence (ϱ_n) converges weakly to ϱ in A_*. Finally, since $(\|\varrho_n\|)$ goes to $\|\varrho\|$, (s_n) is weakly null, and

$$\varrho_n(s_n) = 1 + \varrho(s_n) \not\to 0,$$

we conclude that A_* does not satisfy DP1. ∎

In [4, Lemma 4], G. Dell'Antonio established that the predual of a von Neumann algebra A does not satisfy KKP if there exists a projection p in A such that the predual of pAp does not satisfy KKP. The result is still true for DP1.

Remark 2. It is easy to check that the property DP1 is inherited by complemented subspaces. Now, let p be a projection in a von Neumann algebra A. Since the product on A is separately weak*-continuous, it follows that the map $x \mapsto pxp$ is a weak*-continuous projection from A onto pAp. This implies that $(pAp)_*$ is complemented in A_*. Hence, A_* fails DP1 whenever $(pAp)_*$ does.

Now, we can prove the existence of von Neumann algebras whose preduals do not satisfy DP1.

Theorem 3. *Let A be a type II von Neumann algebra. Then A_* does not satisfy DP1.*

Proof. It is well known (see [10, Theorem V.1.19]) that there exists a projection p on A such that pAp is of type II_1. Therefore, using the above remark, we can assume that A is a type II_1 von Neumann algebra.

Now, by [11, §3] (see also [9, §6]), A contains a countable infinite spin system $\{s_n\}$, consisting of nontrivial symmetries s_n in A_{sa} satisfying $s_n s_m + s_m s_n = 0$ whenever $n \neq m$. On the other hand, the (unique) normal trace ϱ on A is a normal state on A which satisfies $\varrho(s_n x) = \varrho(x s_n)$ for every $x \in A$ and every $n \in \mathbb{N}$ (see [10, §V.2]). To finish the proof, just apply Proposition 1. ∎

Remark 4. We do not know if the predual of every type I von Neumann algebra has DP1.

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Received December 29, 2000
Revised version March 19, 2001 (4663)