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Proper holomorphic liftings and new formulas for the Bergman and Szegő kernels

by

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Abstract. We consider a large class of convex circular domains in $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ which contains the oval domains and minimal balls. We compute their Bergman and Szegő kernels. Our approach relies on the analysis of some proper holomorphic liftings of our domains to some suitable manifolds.

1. Introduction. The use of the Bergman projection plays an important role in the study of proper holomorphic mappings. See Bell [B] or Ligocka [L]. In this paper we shall conversely make use of proper holomorphic mappings to compute Bergman and Szegő kernels. We consider a large class of circular domains in $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ which contains the generalized oval domains considered in [D'A1], [D'A2], [FH1] and [FH2] and the minimal ball introduced in [HP]. We compute their Bergman and Szegő kernels. Our method consists in associating to a domain in our class an appropriate proper holomorphic lifting in which good analysis can be developed. Then we use a suitable operator to deduce the Bergman and Szegő kernels of the domain from those of its proper holomorphic lifting.

If p and q are two positive integers we denote by $M_{p,q}(\mathbb{C})$ the pq-dimensional complex vector space of all $(p\times q)$-matrices with complex coefficients. If $Z = (z_{jk})_{1\le j\le p;1\le k\le q}$ is an element of $M_{p,q}(\mathbb{C})$, we set

$$|Z| := \Big(\sum_{j=1}^{p}\sum_{k=1}^{q}|z_{jk}|^2\Big)^{1/2},$$

$$\|Z\|_* := \Big(\sum_{j=1}^{p}\Big(\sum_{k=1}^{q}|z_{jk}|^2 + \Big|\sum_{k=1}^{q} z_{jk}^2\Big|\Big)\Big)^{1/2}.$$

Let d be a positive integer and let $m = (m_1,\ldots,m_d)$ and $n = (n_1,\ldots,n_d)$ be two d-tuples of integers such that $m_j \ge 1$ and $n_j \ge 1$ for all $j = 1,\ldots,d$. Let $a = (a_1,\ldots,a_d) \in \mathbb{R}^d$ where $a_j \ge 1$, $j = 1,\ldots,d$, and consider the

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function $\varrho(Z) = \varrho_{a,m,n}(Z)$ defined for $Z = (Z(1), \ldots, Z(d)) \in M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ by

$$\varrho(Z) := \sum_{j=1}^{d} \|Z(j)\|_*^{2a_j},$$

and set $\Omega = \Omega_{a,m,n} := \{Z \in M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C}) : \varrho(Z) < 1\}$.

Note that for $n = (1, \ldots, 1)$ the domain Ω describes, as a and m vary, the class of generalized oval domains considered in [D'A1], [D'A2], [FH1] and [FH2].

Note also that for $d = a = m = 1$, Ω is just the minimal ball $\mathbb{R}_*$ in $\mathbb{C}^n$ introduced by Hahn and Pflug [HP].

In what follows, we denote by j_0 the number of those n_j's that are equal to 1. If $j_0 \geq 1$, we may assume without loss of generality that $n_1 = \ldots = n_{j_0} = 1$ and $n_j \geq 2$ for $j = j_0 + 1, \ldots, d$.

For each $s > -1$, we set $dv_s(Z) := (1 - \varrho(Z))^s dv(Z)$, where v denotes the normalized Lebesgue measure on Ω. Let $\mathcal{A}_s^2(\Omega)$ be the Hilbert space of all holomorphic functions on Ω which are square integrable with respect to the measure $dv_s(Z)$ and denote by $\mathcal{K}_{s,\Omega}(Z, W)$ its reproducing kernel.

If $\mathbb{S}$ is an arbitrary non-empty set, we let $\mathbb{S}^m$ denote the set of all finite multi-sequences $t = (t_{jl})_{l=1,\ldots,m_j;j=1,\ldots,d}$ where the entries t_{jl} are elements of $\mathbb{S}$. Let $\mathbb{N}_0$ denote the set of all non-negative integers. We use the notation

$$t^k := \prod_{j=1}^{d} \prod_{l=1}^{m_j} t_{jl}^{k_{jl}} \tag{1.1}$$

for $k = (k_{jl})_{1\leq l\leq m_j;1\leq j\leq d} \in \mathbb{N}_0^m$ and $t = (t_{jl})_{1\leq l\leq m_j;1\leq j\leq d} \in \mathbb{C}^m$. We also consider the action of $\mathbb{C}^m$ on itself given for $t = (t_{jl})$ and $u = (u_{jl})$, $l = 1, \ldots, m_j$, $j = 1, \ldots, d$, by

$$tu := (t_{jl}u_{jl})_{l=1,\ldots,m_j;j=1,\ldots,d}, \quad \text{and set} \quad t^2 := tt. \tag{1.2}$$

Consider the differential operator $\mathcal{D} = \mathcal{D}(m, n, a)$ acting on functions $f(t)$ for $t = (t_{jl})_{l=1,\ldots,m_j;j=1,\ldots,d}$ in some region of $\mathbb{C}^m$ by

$$(\mathcal{D}f)(t) := a_1 \ldots a_d(\mathcal{D}_{1,1} \ldots \mathcal{D}_{1,m_1}) \ldots (\mathcal{D}_{d,1} \ldots \mathcal{D}_{d,m_d})(f(t)), \tag{1.3}$$

where

$$(\mathcal{D}_{j,l}f)(t) := \begin{cases} \dfrac{2}{(n_j - 1)!}\left(2t_{jl}\dfrac{df}{dt_{jl}} + (n_j - 1)f(t)\right) & \text{if } j > j_0, \\ 4f(t) & \text{if } j \leq j_0. \end{cases}$$

We also consider the differential operator $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}(m, n, a)$ acting on functions

$g(x)$ for $x = (x_1, \ldots, x_d)$, defined by

$$(1.4)\qquad (\widetilde{\mathcal{D}}g)(x) := \widetilde{\mathcal{D}}_1 \ldots \widetilde{\mathcal{D}}_d\Big(\prod_{j=1}^{j_0} x_j^{m_j-1} \prod_{j=j_0+1}^{d} x_j^{m_j n_j - m_j - 1} g(x)\Big),$$

where

$$\widetilde{\mathcal{D}}_j := \begin{cases} \dfrac{d^{m_j-1}}{dx_j^{m_j-1}} & \text{if } j \le j_0, \\[2ex] \dfrac{d^{m_j n_j - m_j - 1}}{dx_j^{m_j n_j - m_j - 1}} & \text{if } j > j_0. \end{cases}$$

Next, consider the product group $\Lambda := \{-1,1\}^{m_{j_0+1}} \times \ldots \times \{-1,1\}^{m_d}$. We shall extend each element $\varepsilon = (\varepsilon_{jl})$, $l = 1, \ldots, m_j$, $j = j_0+1, \ldots, d$, of Λ to an element $\widetilde{\varepsilon}$ of $\{-1,1\}^m$ be setting

$$\widetilde{\varepsilon}_{jl} := \begin{cases} 1 & \text{if } j \le j_0, \\ \varepsilon_{jl} & \text{if } j > j_0. \end{cases}$$

If $f(t)$ is a smooth function defined on the ball $\sum_{j=1}^{d} \sum_{l=1}^{m_j} |t_{jl}| < 1$, we consider the function $\widetilde{f}$ on the region $\sum_{j=1}^{d} \sum_{l=1}^{m_j} (|t_{jl}| + |u_{jl}|) < 1$ in $\mathbb{C}^m \times \mathbb{C}^m$ by setting

$$\widetilde{f}(t,u) := \sum_{\varepsilon \in \Lambda} \frac{f(t + \widetilde{\varepsilon} u)}{\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} \varepsilon_{jl} u_{jl}}.$$

Then for each t the partial function $\widetilde{f}(t,\cdot)$ is invariant under the group Λ so that there is a unique function $\mathcal{L}f$ that satisfies

$$\mathcal{L}f(t,u^2) = \widetilde{f}(t,u).$$

Moreover, the mapping $\mathcal{L} : f \mapsto \mathcal{L}f$ is linear.

Remark 1.1. The operator $\mathcal{L}$ maps polynomials to polynomials and rational functions to rational functions. This can be checked directly by induction.

For $s \ge -1$, consider the series defined for $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$ by

$$G_s(x) := \widetilde{\mathcal{D}}\bigg(\sum_{h \in \mathbb{N}_0^d} \frac{\Gamma\big(s + 1 + \sum_{j=1}^{j_0} \frac{m_j + h_j}{a_j} + \sum_{j=j_0+1}^{d} \frac{m_j n_j - m_j + h_j}{a_j}\big)}{\prod_{j=1}^{j_0} \Gamma\big(\frac{m_j + h_j}{a_j}\big) \prod_{j=j_0+1}^{d} \Gamma\big(\frac{m_j n_j - m_j + h_j}{a_j}\big)} x^h\bigg).$$

We shall see that the series $G_s(x)$ converges for $|x| < 1$. Finally, we set

$$(1.5)\qquad R_s(t) := G_s\Big(\sum_{l=1}^{m_1} t_{1l}, \ldots, \sum_{l=1}^{m_d} t_{dl}\Big), \qquad H_s(t,u) := (\mathcal{L}(\mathcal{D}R_s))(t,u).$$

It follows from Lemma 2.2 below that the function $H_s(t,u)$ is well defined provided the variables t and u satisfy $\sum_{j=1}^{d} \sum_{l=1}^{m_j} (|t_{jl}| + \sqrt{|u_{jl}|}) < 1$.

Throughout the paper we use the notation $x \bullet y := \sum_{k=1}^{p} x_k y_k$ if p is a positive integer and $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ are two vectors in $\mathbb{C}^p$. We extend the operation $\bullet$ to a $\mathbb{C}^m$-valued bilinear mapping defined for $Z, W \in M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ by setting

$$Z \bullet W := (Z_l(j) \bullet W_l(j))_{l=1,\ldots,m_j; j=1,\ldots,d}$$

if $Z = (Z(1), \ldots, Z(d)), W = (W(1), \ldots, W(d))$ and

$$Z(j) = \begin{pmatrix} Z_1(j) \\ \vdots \\ Z_{m_j}(j) \end{pmatrix}, \quad W(j) = \begin{pmatrix} W_1(j) \\ \vdots \\ W_{m_j}(j) \end{pmatrix}.$$

We shall establish the following

Theorem A. *For each $s > -1$, the weighted Bergman kernel of $\Omega = \Omega_{a,m,n}$ is given by the formula*

$$\mathcal{K}_{s,\Omega}(Z, W) = \frac{1}{v_s(\Omega)} \cdot \frac{H_s(t, u)}{H_s(0, 0)}, \qquad Z, W \in \Omega,$$

where $t = Z \bullet \overline{W}$ and $u = (Z \bullet Z)(\overline{W \bullet W})$. The latter product is understood in the sense of (1.2).

Remark 1.2. 1) As a consequence of Theorem A, we obtain an explicit formula for the weighted Bergman kernel for oval domains (see [D'A1]) and the minimal ball (see [OPY] and [MY]).

2) Using the same argument as in [FH1] and [FH2], we can rewrite (1.5) in terms of the Appel hypergeometric function in several variables so that by Remark 1.1, the Bergman kernel $\mathcal{K}_{0,\Omega}(Z, W)$ is a rational fraction if $1/a_1, \ldots, 1/a_d$ are positive integers.

Let $\partial\Omega$ denote the boundary of Ω. Each $Z \in M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$, $Z \neq 0$, can be written uniquely in the form

$$Z = Z(r, W) := (r^{1/a_1} W(1), \ldots, r^{1/a_d} W(d)) \tag{1.6}$$

where r is a positive number and $W = (W(1), \ldots, W(d)) \in \partial\Omega$. This parametrization of $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ induces a boundary measure σ on $\partial\Omega$ given by the formula

$$\int\limits_{M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})} f(Z)\, dv(Z) = \int\limits_0^\infty r^{-1+2\sum_{j=1}^{d} m_j n_j / a_j} \int\limits_{\partial\Omega} f(Z(r, W))\, d\sigma(W)\, dr$$

for all compactly supported continuous functions f on $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$.

We denote by $\mathcal{A}(\Omega)$ the space of all continuous functions f on the closure $\overline{\Omega}$ of Ω such that f is holomorphic on Ω. We define the *Szegő kernel* of Ω to be the kernel function $\mathcal{S}_\Omega : \Omega \times \overline{\Omega} \to \mathbb{C}$ that satisfies the following:

1) $\mathcal{S}_\Omega(\cdot, W)$ is holomorphic on Ω for all $W \in \overline{\Omega}$,
2) $\mathcal{S}_\Omega(Z,W) = \overline{\mathcal{S}_\Omega(W,Z)}$ for all $Z, W \in \Omega$,
3) for all $f \in \mathcal{A}(\Omega)$ we have

$$f(Z) = \int\limits_{\partial\Omega} \mathcal{S}_\Omega(Z,W) f(W)\, d\sigma(W).$$

One of the main purposes of this paper is to compute the Szegő kernel $\mathcal{S}_\Omega(Z,W)$. More precisely, we shall prove the following

Theorem B. *The Szegő kernel of Ω is given by the formula*

$$\mathcal{S}_\Omega(Z,W) = \frac{1}{\sigma(\partial\Omega)} \cdot \frac{H_{-1}(t,u)}{H_{-1}(0,0)}, \qquad Z, W \in \Omega,$$

where $t = Z \bullet \overline{W}$ and $u = (Z \bullet Z)(\overline{W \bullet W})$. The latter product is understood in the sense of (1.2).

Corollary C. *The Szegő kernel of $\mathbb{B}_*$ is given by the formula*

$$\mathcal{S}_{\mathbb{B}_*}(z,w) = \frac{1}{2n^3\sigma(\partial\mathbb{B}_*)} \times \frac{\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^{n-2k-2} y^k [2nx - (n-1-2k)(x^2-y)]}{(x^2-y)^n}$$

where $x = 1 - z \bullet \overline{w}$, $y = z \bullet z\ \overline{w \bullet w}$ and $[(n-1)/2]$ is the greatest integer smaller than or equal to $(n-1)/2$.

For each $s > -1$ and $p \geq 1$, let $L^p_s(\Omega)$ the Banach space of all functions on Ω which are L^p-integrable with respect to the measure $\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} |Z_l(j) \bullet Z_l(j)|^{(p-2)/2} dv_s(Z)$. We denote by $\mathcal{A}^p_s(\Omega)$ the space of all holomorphic functions on Ω which are in the space $L^p_s(\Omega)$, and $\mathcal{H}^p(\Omega)$ the Hardy space with respect to the measure σ on the boundary of Ω. We shall show that the spaces $\mathcal{A}^p_s(\Omega)$ and $\mathcal{H}^p(\Omega)$ furnished respectively with the norms of $L^p_s(\Omega)$ and $L^p(\partial\Omega, \sigma)$ are Banach spaces.

2. Some integral and summation formulas. We use the notations of Section 1. We let $m = (m_1, \ldots, m_d), n = (n_1, \ldots, n_d)$ and $a = (a_1, \ldots, a_d)$ be as in the introduction. We set

$$\mathbb{E}_{a,m} := \Big\{ t \in]0,\infty[^m : \sum_{j=1}^{d} \Big[\sum_{l=1}^{m_j} t_{jl}^2 \Big]^{a_j} < 1 \Big\},$$

$$\partial\mathbb{E}_{a,m} := \Big\{t \in \,]0,\infty[^m : \sum_{j=1}^{d}\Big[\sum_{l=1}^{m_j} t_{jl}^2\Big]^{a_j} = 1\Big\}.$$

Each $t = (t_{jl}) \in \,]0,\infty[^m$ can be written in the form $t = t(r,u) := (r^{1/a_j}u_{jl})$ where $r \in \,]0,\infty[$ and $u = (u_{jl}) \in \partial\mathbb{E}_{a,m}$. Thus the canonical volume form $dt = \wedge_{j=1}^{d} \wedge_{l=1}^{m_j} dt_{jl}$ on $\mathbb{R}^m$ induces a boundary volume form ϕ on $\partial\mathbb{E}_{a,m}$ given by the formula

$$(2.1)\qquad \int\limits_{]0,\infty[^n} f(t)\,dt = \int\limits_{]0,\infty[\times\partial\mathbb{E}_{a,m}} r^{-1+\sum_{j=1}^{d} m_j/a_j} f(t(r,u))\,dr \wedge \phi(u).$$

If $k = (k_{jl}) \in \mathbb{N}_0^m$ and $j \in \{1,\ldots,d\}$, set

$$k(j) := (k_{j1},\ldots,k_{jm_j}),\qquad |k(j)| := \sum_{l=1}^{m_j} k_{jl},\qquad |k| := \sum_{j=1}^{d} |k(j)|.$$

For $s > -1$, let $\chi_s(t) := (1-\sum_{j=1}^{d}[\sum_{l=1}^{m_j} t_{jl}^2]^{a_j})^s$. Then we have the following

LEMMA 2.1. *For each $s > -1$, and $k \in \mathbb{N}_0^m$, we have*

$$\int\limits_{\mathbb{E}_{a,m}} \chi_s(t) \prod_{j=1}^{j_0}\prod_{l=1}^{m_j} t_{jl}^{1+2k_{jl}} \prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j} t_{jl}^{2n_j-3+2k_{jl}}\,dt$$
$$= \frac{\Gamma(s+1)\prod_{j=1}^{j_0}\Gamma\big(\frac{m_j+|k(j)|}{a_j}\big)\prod_{j=j_0+1}^{d}\Gamma\big(\frac{m_jn_j-m_j+|k(j)|}{a_j}\big)}{2^{|m|}a_1\ldots a_d\Gamma\big(s+1+\sum_{j=1}^{j_0}\frac{m_j+|k(j)|}{a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j+|k(j)|}{a_j}\big)}$$
$$\times \prod_{j=1}^{j_0}\frac{k(j)!}{\Gamma(m_j+|k(j)|)}\prod_{j=j_0+1}^{d}\frac{\prod_{l=1}^{m_j}\Gamma(n_j-1+k_{jl})}{\Gamma(m_jn_j-m_j+|k(j)|)},$$

and

$$\int\limits_{\partial\mathbb{E}_{a,m}} \prod_{j=1}^{j_0}\prod_{l=1}^{m_j} t_{jl}^{1+2k_{jl}} \prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j} t_{jl}^{2n_j-3+2k_{jl}}\,\phi(t)$$
$$= \frac{\prod_{j=1}^{j_0}\Gamma\big(\frac{m_j+|k(j)|}{a_j}\big)\prod_{j=j_0+1}^{d}\Gamma\big(\frac{m_jn_j-m_j+|k(j)|}{a_j}\big)}{2^{|m|-1}a_1\ldots a_d\Gamma\big(\sum_{j=1}^{j_0}\frac{m_j+|k(j)|}{a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j+|k(j)|}{a_j}\big)}$$
$$\times \prod_{j=1}^{j_0}\frac{k(j)!}{\Gamma(m_j+|k(j)|)}\prod_{j=j_0+1}^{d}\frac{\prod_{l=1}^{m_j}\Gamma(n_j-1+k_{jl})}{\Gamma(m_jn_j-m_j+|k(j)|)}.$$

Proof. To prove the first part of the lemma we use integration in polar coordinates in several variables and Lemma 1 of [D'A2]. The second part follows from the first one and identity (2.1). ∎

If p is a non-negative integer and q is a positive integer, let

$$N(p,q) := \frac{(2p+q-1)(p+q-2)!}{p!(q-1)!},$$

with the understanding that $N(0,1) := 2$.

LEMMA 2.2. *Let R_s be the function given by* (1.5) *and let*

$$C(k,s) := \frac{\prod_{j=1}^{d}\prod_{l=1}^{m_j} N(k_{jl},n_j)}{\int_{\mathbb{E}_{a,m}} \chi_s(t) \prod_{j=1}^{j_0}\prod_{l=1}^{m_j} t_{jl}^{1+2k_{jl}} \prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j} t_{jl}^{2n_j-3+2k_{jl}}\,dt}$$

$$D(k) := \frac{\prod_{j=1}^{d}\prod_{l=1}^{m_j} N(k_{jl},n_j)}{\int_{\partial\mathbb{E}_{a,m}} \prod_{j=1}^{j_0}\prod_{l=1}^{m_j} t_{jl}^{1+2k_{jl}} \prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j} t_{jl}^{2n_j-3+2k_{jl}}\,\phi(t)}.$$

Then

$$\sum_{k\in\mathbb{N}_0^m} C(k,s)t^k = \frac{(\mathcal{D}R_s)(t)}{\Gamma(s+1)}, \qquad \sum_{k\in\mathbb{N}_0^m} D(k)t^k = \frac{(\mathcal{D}R_{-1})(t)}{2}.$$

Proof. First observe that in view of Lemma 2.1 we have

$$C(k,s) = \frac{2^{|m|}a_1\ldots a_d \Gamma\big(s+1+\sum_{j=1}^{j_0}\frac{m_j+|k(j)|}{a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j+|k(j)|}{a_j}\big)}{\Gamma(s+1)\prod_{j=1}^{j_0}\Gamma\big(\frac{m_j+|k(j)|}{a_j}\big)\prod_{j=j_0+1}^{d}\Gamma\big(\frac{m_jn_j-m_j+|k(j)|}{a_j}\big)}$$
$$\times \prod_{j=1}^{j_0} 2^{m_j}\frac{\Gamma(m_j+|k(j)|)}{k(j)!}$$
$$\times \prod_{j=j_0+1}^{d}\bigg(\frac{\Gamma(m_jn_j-m_j+|k(j)|)}{((n_j-1)!)^{m_j}k(j)!}\prod_{l=1}^{m_j}(2k_{jl}+n_j-1)\bigg),$$

Therefore,

$$\Gamma(s+1)\sum_{k\in\mathbb{N}_0^m} C(k,s)t^k = \sum_{h\in\mathbb{N}_0^d} C'(h,s)\sum_{|k(j)|=h_j} C''(h,k,s)t^k,$$

where

$$C'(h,s) = \frac{\Gamma\big(s+1+\sum_{j=1}^{j_0}\frac{m_j+h_j}{a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j+h_j}{a_j}\big)}{\prod_{j=1}^{j_0}\Gamma\big(\frac{m_j+h_j}{a_j}\big)\prod_{j=j_0+1}^{d}\Gamma\big(\frac{m_jn_j-m_j+h_j}{a_j}\big)}$$
$$\times \prod_{j=1}^{j_0}\frac{\Gamma(m_j+h_j)}{h_j!}\prod_{j=j_0+1}^{d}\frac{\Gamma(m_jn_j-m_j+h_j)}{h_j!},$$

$$C''(h,k,s) = 2^{|m|}\prod_{j=1}^{j_0}\frac{a_j 2^{m_j} h_j!}{k(j)!} \times \prod_{j=j_0+1}^{d}\biggl(\frac{a_j h_j!}{((n_j-1)!)^{m_j}k(j)!}\prod_{l=1}^{m_j}(2k_{jl}+n_j-1)\biggr).$$

This fact, combined with the multinomial theorem, yields

$$\sum_{|k(j)|=h_j} C''(h,k,s)t^k = \mathcal{D}\Bigl(\prod_{j=1}^{d}\Bigl(\sum_{l=1}^{m_j} t_{jl}\Bigr)^{h_j}\Bigr)$$

and thus

$$\sum_{k\in\mathbb{N}_0^m} C(k,s)t^k = \frac{1}{\Gamma(s+1)}\mathcal{D}\Bigl(u\Bigl(\sum_{l=1}^{m_1} t_{1l},\ldots,\sum_{l=1}^{m_d} t_{dl}\Bigr)\Bigr),$$

where u is the function in d variables $x=(x_1,\ldots,x_d)$ defined by

$$u(x) := \sum_{h\in\mathbb{N}_0^d} C'(h,s)x^h.$$

Now setting

$$v(x) := \sum_{h\in\mathbb{N}_0^d}\frac{\Gamma\bigl(s+1+\sum_{j=1}^{j_0}\frac{m_j+h_j}{a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j+h_j}{a_j}\bigr)}{\prod_{j=1}^{j_0}\Gamma\bigl(\frac{m_j+h_j}{a_j}\bigr)\prod_{j=j_0+1}^{d}\Gamma\bigl(\frac{m_jn_j-m_j+h_j}{a_j}\bigr)}x^h,$$

we see that $u(x) = (\widetilde{\mathcal{D}}v)(x) = G_s(x)$. This proves the first equality in the lemma. The proof of the second equality follows in an analogous manner. ∎

3. Preparatory results. First assume that $d=1$ and let $m=m_1$ and $n=n_1$ be positive integers. In this case, for $z,\xi\in\mathbb{C}^{n+1}$, we have $z\bullet\xi = \sum_{l=1}^{n+1} z_l\xi_l$. Let

$$\mathbb{H}_n := \{z\in\mathbb{C}^{n+1}\setminus\{0\} : z\bullet z = 0\} \quad \text{for } n\ge 2,$$
$$\mathbb{H}_1 := \{(z,iz) : z\in\mathbb{C}\setminus\{0\}\}.$$

We set $\Gamma_n := \{z\in\mathbb{H}_n : |z|=1\}$. Then $O(n+1,\mathbb{R})$ acts transitively on Γ_n and thus there is a unique $O(n+1,\mathbb{R})$-invariant probability measure μ_n on Γ_n induced by the Haar measure of $O(n+1,\mathbb{R})$.

If k and l are two non-negative integers with $k\ne l$ and if f and g are holomorphic polynomials on $\mathbb{C}^{n+1}$ such that f is k-homogeneous and g is l-homogeneous, then for all $\xi\in\mathbb{H}_n$ we have the identities

$$
(3.1) \qquad \begin{cases} \int\limits_{\Gamma_n} f(w)(\xi \bullet \overline{w})^k \, d\mu_n(w) = \dfrac{f(\xi)}{N(k,n)}, \\ \int\limits_{\Gamma_n} f(w)\overline{g(w)} \, d\mu_n(w) = 0. \end{cases}
$$

The latter formulas were established [I], [W] and [MY] for $n \geq 2$. For $n = 1$, they can be checked by direct computation since in this case we have $\Gamma_1 = \{(\frac{\sqrt{2}}{2}\zeta, i\frac{\sqrt{2}}{2}\zeta) : \zeta \in \mathbb{S}^1\}$ and the measure μ_1 is given by

$$
(3.2) \qquad d\mu_1\left(\frac{\sqrt{2}}{2}\,\zeta, i\,\frac{\sqrt{2}}{2}\,\zeta\right) = \frac{dt}{2\pi} \quad \text{for } \zeta = e^{it},\ t \in [0, 2\pi].
$$

As observed in [OPY], when $n \geq 2$, there is a unique (up to a multiplicative constant) $SO(n+1, \mathbb{C})$-invariant holomorphic form α_n on $\mathbb{H}_n$ given in the open charts $\mathcal{U}_j = \{z \in \mathbb{H}_n : z_j \neq 0\}$ by

$$
\alpha_n(z) = (n+1)\,\frac{(-1)^{j-1}}{z_j}\, dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_{n+1}.
$$

We set $\alpha_1(z) := dz_1$. It was proved in [MY] that the form $\alpha_n(z) \wedge \overline{\alpha}_n(z)$ contracted with the vector field $z \mapsto z$ induces an $SO(n+1, \mathbb{R})$-invariant $(2n-1)$-volume form ω_n on Γ_n defined by

$$
\omega_n(z)(X) := i_z(\alpha_n \wedge \overline{\alpha}_n)(X) = (\alpha_n(z) \wedge \overline{\alpha}_n(z))(z, X)
$$

for all elements X of the $(2n-1)$-fold tangent space to Γ_n at z. Furthermore, a little calculation shows that both forms $\alpha_n \wedge \overline{\alpha}_n$ and ω_n are also S^1-invariant. In addition, if $\omega_n(\Gamma_n) := \int_{\Gamma_n} \omega_n(\xi)$, then the measure μ_n and the forms α_n and ω_n are related by the formula

$$
(3.3) \qquad \int\limits_{\mathbb{H}_n} f(z)\alpha_n(z) \wedge \overline{\alpha}_n(z) = \int\limits_{]0,\infty[\times\Gamma_n} t^{2n-3} f(t\xi)\, dt \wedge \omega_n(\xi)
$$
$$
= \omega_n(\Gamma_n) \int\limits_0^\infty t^{2n-3} \int\limits_{\Gamma_n} f(t\xi)\, d\mu_n(\xi)\, dt,
$$

which holds for all compactly supported C^∞-functions f on $\mathbb{H}_n$. This formula, combined with the fact that the forms $\alpha_n \wedge \overline{\alpha}_n$ and ω_n are S^1-invariant, implies that the measure μ_n is also S^1-invariant, and hence $S^1 \cdot SO(n+1, \mathbb{R})$-invariant.

LEMMA 3.1. *Let $k, l \in \mathbb{Z}$ be such that $k \neq l$. Assume that f and g are two holomorphic functions on $\mathbb{H}_n$ such that f is k-homogeneous and g is l-homogeneous. Then*

$$
\int\limits_{\Gamma_n} f(w)\overline{g(w)}\, d\mu_n(w) = 0.
$$

Proof. We distinguish two cases.

First assume $n \geq 2$. Notice that in this case the complex space $\mathbb{X} := \mathbb{H}_n \cup \{0\}$ is normal. In fact, 0 is the only singular point of $\mathbb{X}$, and this singularity is normal in view of [Wh, Theorem 1B(d), p. 251]. Therefore, both f and g are holomorphic on $\mathbb{X}$ so that by homogeneity $f\overline{g}$ vanishes identically for $k < 0$ or $l < 0$. This, combined with (3.1), shows that the lemma holds for $n \geq 2$.

It remains to prove the lemma for $n = 1$. In this case $\mathbb{H}_1 = \{(z, iz) : z \in \mathbb{C} \setminus \{0\}\}$. Since the function $z \mapsto f(z, iz)$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and k-homogeneous, it has the form cz^k where c is a constant. Similarly, the function $z \mapsto g(z, iz)$ must be of the form $c'z^l$ where c' is some constant. Using this fact and (3.2), a little computing shows that the lemma holds also for $n = 1$. ■

Now consider the general case $d \geq 1$ and let $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d)$ be as in the introduction. If $Z = (Z(1), \ldots, Z(d)) \in M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C})$ and $t = (t_{jl}) \in \mathbb{C}^m$, then for each $j = 1, \ldots, d$ and each $l = 1, \ldots, m_j$ set $(t \centerdot Z)_l(j) := t_{jl} Z_l(j)$, where $Z_l(j)$ is the lth row of $Z(j)$. Then we define $(t \centerdot Z)(j)$ to be the $m_j \times (n_j + 1)$-matrix whose lth row is $(t \centerdot Z)_l(j)$ and put

$$t \centerdot Z := ((t \centerdot Z)(1), \ldots, (t \centerdot Z)(d)).$$

If $k \in \mathbb{Z}_m$, then a function $f : M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C}) \to \mathbb{C}$ is said to be *k-homogeneous* if $f(t \centerdot Z) = t^k f(Z)$ for all $t \in (\mathbb{C} \setminus \{0\})^m$. Here the power t^k is understood in the sense of (1.1).

We denote by $\mathbb{H}_{m,n}$ the $\sum_{j=1}^d m_j n_j$-dimensional complex submanifold consisting of all $Z = (Z(1), \ldots, Z(d)) \in M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C})$ such that for each $j = 1, \ldots, d$ all the rows $Z_l(j)$, $l = 1, \ldots, m_j$, of the matrix $Z(j)$ are elements of $\mathbb{H}_{n_j}$. We also denote by $\Gamma_{m,n}$ the $(2\sum_{j=1}^d m_j n_j - |m|)$-dimensional real submanifold of $\mathbb{H}_{m,n}$ consisting of all $Z = (Z(1), \ldots, Z(d)) \in \mathbb{H}_{m,n}$ such that for each $j = 1, \ldots, d$ all the rows $Z_l(j), l = 1, \ldots, m_j$, of the matrix $Z(j)$ are elements of Γ_{n_j}. Then the mapping $(t, W) \mapsto Z = t \centerdot W$ is a diffeomorphism from $]0, \infty[^m \times \Gamma_{m,n}$ onto $\mathbb{H}_{m,n}$.

The form

$$\Theta(Z) := \wedge_{j=1}^d \wedge_{l=1}^{m_j} \alpha_{n_j}(Z_l(j)) \wedge \overline{\alpha_{n_j}(Z_l(j))}$$

is a volume form on $\mathbb{H}_{m,n}$ which is $(S^1)^m$-invariant under the action of $(S^1)^m$ on $\mathbb{H}_{m,n}$ given by the mapping $(t, Z) \mapsto t \centerdot Z$ from $(t, Z) \in (S^1)^m \times \mathbb{H}_{m,n}$ onto $\mathbb{H}_{m,n}$.

Using the coordinates $Z = t \centerdot W$, where $t \in]0, \infty[^m$ with $t_{jl} := |Z_l(j)|$ and $W \in \Gamma_{m,n}$, and applying (3.3) we see that

$$\Theta(Z) = \prod_{j=1}^{j_0} \prod_{l=1}^{m_j} t_{jl} \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} t_{jl}^{2n_j-3} dt \wedge \omega_{m,n}(W) \tag{3.4}$$

where $dt = \wedge_{j=1}^{d} \wedge_{l=1}^{m_j} dt_{jl}$ is the canonical volume form on $\mathbb{R}^m$ and $\omega_{m,n}(W)$ is the volume form on $\Gamma_{m,n}$ given by

$$\omega_{m,n}(W) = \wedge_{j=1}^{d} \wedge_{l=1}^{m_j} \omega_{n_j}(W_l(j)).$$

Now fix $a = (a_1, \ldots, a_d)$ with $a_1 \geq 1, \ldots, a_d \geq 1$, and set

$$\mathbb{M} := \Big\{ Z = (Z(1), \ldots, Z(d)) \in \mathbb{H}_{m,n} : \sum_{j=1}^{d} |Z(j)|^{2a_j} < 1 \Big\},$$

$$\partial\mathbb{M} := \Big\{ Z = (Z(1), \ldots, Z(d)) \in \mathbb{H}_{m,n} : \sum_{j=1}^{d} |Z(j)|^{2a_j} = 1 \Big\}.$$

Then the mapping $(t, W) \mapsto Z = t\,\boldsymbol{.}\,W$ is a diffeomorphism from $\mathbb{E}_{a,m} \times \Gamma_{m,n}$ onto $\mathbb{M}$ and from $\partial\mathbb{E}_{a,m} \times \Gamma_{m,n}$ onto $\partial\mathbb{M}$.

Let $\phi(t)$ denote the volume form on $\partial\mathbb{E}_{a,m}$ given by (2.1) and consider the volume form ϑ on $\partial\mathbb{M}$ given in the coordinates $Z = t\,\boldsymbol{.}\,W$ by

$$\vartheta(Z) := \prod_{j=1}^{j_0} \prod_{l=1}^{m_j} t_{jl} \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} t_{jl}^{2n_j-3} \phi(t) \wedge \omega_{m,n}(W). \tag{3.5}$$

The form ϑ induces a probability measure μ on $\partial\mathbb{M}$ given for all continuous functions f on $\partial\mathbb{M}$ by

$$\int_{\partial\mathbb{M}} f(W)\, d\mu(W) = \frac{1}{\vartheta(\partial\mathbb{M})} \int_{\partial\mathbb{M}} f(W) \vartheta(W), \tag{3.6}$$

where

$$\vartheta(\partial\mathbb{M}) := \int_{\partial\mathbb{M}} \vartheta.$$

LEMMA 3.2. *For any C^∞-function f on $\mathbb{H}_{m,n}$ we have*

$$\begin{aligned}\int_{\mathbb{H}_{m,n}} f(Z)\Theta(Z) &= \int_{]0,\infty[\times\partial\mathbb{M}} \{ r^{-1+2[\sum_{j=1}^{j_0} \frac{m_j}{a_j} + \sum_{j=j_0+1}^{d} \frac{m_j n_j - m_j}{a_j}]} \\ &\qquad \times f(r^{1/a_1} W(1), \ldots, r^{1/a_d} W(d)) \}\, dr \wedge \vartheta(W) \\ &= \vartheta(\partial\mathbb{M}) \int_{]0,\infty[\times\partial\mathbb{M}} \{ r^{-1+2[\sum_{j=1}^{j_0} \frac{m_j}{a_j} + \sum_{j=j_0+1}^{d} \frac{m_j n_j - m_j}{a_j}]} \\ &\qquad \times f(r^{1/a_1} W(1), \ldots, r^{1/a_d} W(d)) \}\, d\mu(W)\, dr,\end{aligned}$$

provided that the integrals make sense.

Proof. Follows from (2.1), (3.4), (3.5) and (3.6). ■

If $k \in \mathbb{N}_0^m$, let $\mathcal{P}^k(\mathbb{H}_{m,n})$ be the space of all restrictions to $\mathbb{H}_{m,n}$ of k-homogeneous holomorphic polynomials on $M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C})$.

Lemma 3.3. *Let $k = (k_{jl})$ and $k' = (k'_{jl})$ be two elements of $\mathbb{Z}_m$. Suppose that f and g are holomorphic on $\mathbb{H}_{m,n}$, f is k-homogeneous and g is k'-homogeneous. Let $s > -1$.*

(1) *If $k \neq k'$, then*

$$\int_{\mathbb{M}} f(W)\overline{g(W)}\, d\Theta_s(W) = \int_{\partial\mathbb{M}} f(W)\overline{g(W)}\, d\vartheta(W) = 0.$$

(2) *If $k \in \mathbb{N}_0^m$ and $f \in \mathcal{P}^k(\mathbb{H}_{m,n})$, then for all $Z \in \mathbb{H}_{m,n}$ we have*

$$\begin{aligned} f(Z) &= C(k,s) \int_{\mathbb{M}} f(W)(Z \bullet \overline{W})^k \Big(1 - \sum_{j=1}^{d} |W(j)|^{2a_j}\Big)^s \Theta(W) \\ &= D(k) \int_{\partial\mathbb{M}} f(W)(Z \bullet \overline{W})^k\, d\vartheta(W), \end{aligned}$$

where $C(k,s)$ and $D(k)$ are given in Lemma 2.2.

Proof. We may assume that $s = 0$. For $k \in \mathbb{Z}_m$, let Λ_k denote the set of all $\beta = (\beta_{jl})$ with $\beta_{jl} \in \mathbb{Z}^{n_j}$ and $|\beta_{jl}| = k_{jl}$ for all $j = 1, \dots, d$ and $l = 1, \dots, m_j$. Observe that the space $\mathcal{P}^k(\mathbb{H}_{m,n})$ is the linear span of the monomials

$$f_\beta(Z) := \prod_{j=1}^{d} \prod_{l=1}^{m_j} Z_l(j)^{\beta_{jl}}, \qquad \text{where } \beta \in \Lambda_k.$$

Therefore, it suffices to prove the lemma for $f = f_\beta$ and $g = g_{\beta'}$ where $\beta \in \Lambda_k, \beta' \in \Lambda_{k'}$.

Since $k \neq k'$ there are $j \in \{1, \dots, d\}$ and $l \in \{1, \dots, m_j\}$ such that $\beta_{jl} \neq \beta'_{jl}$ so that by Lemma 3.1 we see that

$$\int_{\Gamma_{n_j}} W_l(j)^{\beta_{jl}}\, \overline{W_l(j)^{\beta'_{jl}}}\, \omega_{n_j}(W_l(j)) = 0.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{M}} f_\beta(W)\, \overline{g_{\beta'}(W)}\, \Theta(W) &= \int_{\mathbb{E}_{a,m}} t^{k+k'} \prod_{j=1}^{j_0} t_{jl} \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} t_{jl}^{2n_j-3} dt \\ &\quad \times \prod_{j=1}^{d} \prod_{l=1}^{m_j} \int_{\Gamma_{n_j}} W_l(j)^{\beta_{jl}}\, \overline{W_l(j)^{\beta'_{jl}}}\, \omega_{n_j}(W_l(j)) \\ &= 0. \end{aligned}$$

Now by Lemma 3.2 we see that

$$\int\limits_{\partial\mathbb{M}} f_\beta(W)\,\overline{g_{\beta'}(W)}\,\vartheta(W)$$
$$= \frac{\int_{\mathbb{M}} f_\beta(W)\,\overline{g_{\beta'}(W)}\,\Theta(W)}{\int_0^1 r^{-1+2[\sum_{j=1}^{j_0}\frac{m_j}{a_j}+\frac{|k(j)|+|k'(j)|}{2a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j}{a_j}+\frac{|k(j)|+|k'(j)|}{2a_j}]}dr} = 0.$$

To prove part (2), observe by (3.1) and (3.3) that if $\beta \in \Lambda_k$, then

$$\int\limits_{\Gamma_{n_j}} W_l(j)^{\beta_{jl}}(Z_l(j)\bullet\overline{W_l(j)})^{k_{jl}}\omega_{n_j}(W_l(j)) = Z_l(j)^{\beta_{jl}}\,\frac{\int_{\Gamma_{n_j}}\omega_{n_j}}{N(k_{jl},n_j)}.$$

This shows that

$$\int\limits_{\Gamma_{m,n}} f_\beta(W)(Z\bullet\overline{W})^k\omega_{m,n}(W) = f_\beta(Z)\,\frac{\int_{\Gamma_{m,n}}\omega_{m,n}}{\prod_{j=1}^d\prod_{l=1}^{m_j}N(k_{jl},n_j)}.$$

This fact, combined with (3.4), implies that

$$\int\limits_{\mathbb{M}} f_\beta(W)\Big(1-\sum_{j=1}^d|W(j)|^{2a_j}\Big)^s(Z\bullet\overline{W})^k\Theta(W)$$
$$= \int\limits_{\mathbb{E}_{a,m}}\chi_s(t)\prod_{j=1}^{j_0}\prod_{l=1}^{m_j}t_{jl}^{1+2k_{jl}}\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}t_{jl}^{2n_j-3+2k_{jl}}\,dt$$
$$\times\int\limits_{\Gamma_{m,n}} f_\beta(W)(Z\bullet\overline{W})^k\omega_{m,n}(W)$$
$$= \frac{f_\beta(Z)}{C(k,s)}.$$

Thus by Lemma 3.2 we have

$$\int\limits_{\partial\mathbb{M}} f_\beta(W)(Z\bullet\overline{W})^k\vartheta(W)$$
$$= \frac{\int_{\mathbb{M}} f_\beta(W)(1-\sum_{j=1}^d|W(j)|^{2a_j})^s(Z\bullet\overline{W})^k\Theta(W)}{\int_0^1 r^{-1+2[\sum_{j=1}^{j_0}\frac{m_j+|k(j)|}{a_j}+\sum_{j=j_0+1}^{d}\frac{m_jn_j-m_j+|k(j)|}{a_j}]}(1-r^2)^s\,dr}$$
$$= \frac{f_\beta(Z)}{D(k)}. \ ■$$

4. The Bergman and Szegő kernels of $\mathbb{M}$. For $s > -1$ set

$$\Theta_s(Z) := \Big(1-\sum_{j=1}^d|Z(j)|^{2a_j}\Big)^s\Theta(Z).$$

We denote by $L^p_s(\mathbb{M})$ the space of holomorphic functions that satisfy

$$\|f\|^p_{L^p_s(\mathbb{M})} = \Big(\int\limits_{\mathbb{M}} |f(Z)|^p \Theta_s(Z)\Big)^{1/p} < \infty,$$

and by $\mathcal{A}^p_s(\mathbb{M})$ the subspace of $L^p_s(\mathbb{M})$ consisting of holomorphic functions on $\mathbb{M}$. Using local coordinates in $\mathbb{M}$, it is easy to see that for each compact set $\mathbb{A}$ in $\mathbb{M}$, there is a positive constant $C = C(s,p,n)$ such that

$$\sup_{Z\in\mathbb{A}} |g(Z)| \le C\|g\|_{\mathcal{A}^p_s(\mathbb{M})} \tag{4.1}$$

for all $g \in \mathcal{A}^p_s(\mathbb{M})$. This shows that $\mathcal{A}^p_s(\mathbb{M})$ is a Banach space.

For $p \ge 1$, denote by $\mathcal{H}^p(\mathbb{M})$ the Hardy space of $\mathbb{M}$. This is the space of all those holomorphic functions f on $\mathbb{M}$ that satisfy

$$\|f\|_{\mathcal{H}^p} := \sup_{0<r<1} \Big\{ \int\limits_{\partial\mathbb{M}} |f(r^{1/a_1}W(1),\dots,r^{1/a_d}W(d))|^p \, d\mu(W)\Big\}^{1/p} < \infty.$$

We shall see below that $\mathcal{H}^p(\mathbb{M})$ can be identified with a closed subspace of $L^p(\partial\mathbb{M},\mu)$ and thus is a Banach space.

Proposition 4.1. *If $f \in \mathcal{A}^2_s(\mathbb{M})$ then there exists a multi-sequence $\{f_k\}_{k\in\mathbb{N}^m}$ of holomorphic polynomials such that $f_k \in \mathcal{P}^k(\mathbb{M})$ and*

$$f(Z) = \sum_{k\in\mathbb{N}^m_0} f_k(Z)$$

where the series is convergent uniformly on compact sets of the domain $\mathbb{B}_{m,n,a}$ in $M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C})$ given by

$$\mathbb{B}_{m,n,a} := \Big\{ Z \in M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C}) : \sum_{j=1}^{d} |Z(j)|^{2a_j} < 1 \Big\}.$$

Proof. Step 1. First note that if $n_1 \ge 2, \dots, n_d \ge 2$, then the conclusion of the proposition holds for any holomorphic function f on $\mathbb{M}$. Indeed, set

$$\mathbb{X} = \bigcap_{j=1}^{d} \bigcap_{l=1}^{m_j} \{Z \in \mathbb{B}_{m,n,a} : Z_l(j) \bullet Z_l(j) = 0\}.$$

Then $\mathbb{X}$ is a complex space which is the zero set of the $\sum_{j=1}^d m_j$ holomorphic functions $g_{jl}(Z) = Z_l(j) \bullet Z_l(j)$, $l = 1,\dots,m_j$, $j = 1,\dots,d$, in $\mathbb{B}_{m,n,a}$. Moreover, $\mathbb{X}$ is of constant dimension $\sum_{j=1}^d m_j n_j$ and its set of singular points is

$$\mathbb{X}^{\times} = \bigcap_{j=1}^{d} \bigcap_{l=1}^{m_j} \{(Z(1),\dots,Z(d)) \in \mathbb{B}_{m,n,a} : Z_l(j) = 0\}$$

and for each point $Z \in \mathbb{X}^\times$ we have $\dim_Z \mathbb{X}^\times \le -2 + \sum_{j=1}^d m_j n_j$ so that by [Wh, Theorem 1B(d′), p. 251] we see that Z is a normal singular point in $\mathbb{X}$. On the other hand, the set of regular points of $\mathbb{X}$ is precisely the manifold $\mathbb{M}$ and of course consists of normal points. It follows that $\mathbb{X}$ is a normal complex space. By the second Riemann removable singularity theorem (see [KK, pp. 307]) every holomorphic function f on $\mathbb{M}$ can be extended uniquely to a holomorphic function $\widetilde{f}$ in $\mathbb{X}$. Since $\mathbb{B}_{m,n,a}$ is a domain of holomorphy the proposition follows from the Oka–Cartan Theorem B.

Step 2. Consider now the situation where $j_0 \ge 1$. If $Z \in \mathbb{H}_{m,n}$ is fixed and if $r = (r_{jl})_{l=1,\dots,m_j;j=1,\dots,d}$ is a finite sequence of positive numbers we set

$$\varDelta_{m,n} := \{\lambda = (\lambda_{jl}) \in \mathbb{C}^m : 0 < |\lambda_{jl}| \le r_{jl}\}.$$

We choose the r_{jl}'s sufficiently small so that for all $\lambda \in \varDelta_{m,n}$ we have $\lambda_\bullet Z \in \mathbb{M}$. Then $\lambda \mapsto \lambda_\bullet Z$ maps continuously $\varDelta_{m,n}$ into $\mathbb{M}$ and is holomorphic on the interior of $\varDelta_{m,n}$. Consider the function

$$\varphi_Z(\lambda) := f(\lambda \centerdot Z), \qquad \lambda \in \varDelta_{m,n}.$$

The function φ_Z has a Laurent series expansion of the form

$$\varphi_Z(\lambda) = \sum_{k\in\mathbb{Z}_m} f_k(Z)\lambda^k$$

where $f_k : \mathbb{H}_{m,n} \to \mathbb{C}$ is a k-homogeneous holomorphic function which is independent of the choice of $r = (r_{jl})_{l=1,\dots,m_j;j=1,\dots,d}$. Indeed, f_k is given only in terms of f by

$$f_k(Z) = \frac{1}{(2i\pi)^{|m|}} \int\limits_{\{|\lambda_{jl}|=r_{jl}:l=1,\dots,m_j;\, j=1,\dots,d\}} f(\lambda \centerdot Z) \prod_{j,l} \frac{d\lambda_{jl}}{\lambda_{jl}^{k_{jl}+1}}. \tag{4.3}$$

In particular, if $Z \in \mathbb{M}$, then we can choose $r_{jl} = 1$ so that by (4.3) we have

$$|f_k(Z)| \le \frac{1}{(2i\pi)^{|m|}} \int\limits_{(S^1)^m} |f(\lambda \centerdot Z)| \prod_{j=1}^{d} \prod_{l=1}^{m_j} \frac{d\lambda_{jl}}{\lambda_{jl}}.$$

This implies that if $f \in \mathcal{A}_s^2(\mathbb{M})$, then

$$\begin{aligned} \int\limits_{\mathbb{M}} |f_k(Z)|^2 \varTheta_s(Z) &\le \frac{1}{(2i\pi)^{|m|}} \int\limits_{(S^1)^m} \int\limits_{\mathbb{M}} |f(\lambda \centerdot Z)|^2 \varTheta_s(Z) \prod_{j=1}^{d} \prod_{l=1}^{m_j} \frac{d\lambda_{jl}}{\lambda_{jl}} \\ &= \int\limits_{\mathbb{M}} |f(Z)|^2 \varTheta_s(Z). \end{aligned}$$

where the latter equality holds because $\varTheta_s$ is $(S^1)^m$-invariant. If we choose $r_{jl} \in]0,1[$ sufficiently small so that $\lambda_\bullet Z \in \mathbb{M}$ for all $\lambda \in \varDelta_{m,n}$ and $Z \in \varGamma_{m,n}$, then we have

$$\prod_{j=1}^{j_0}\prod_{l=1}^{m_j}\int_0^{r_{jl}} u_{jl}^{2k_{jl}+1}\,du_{jl}\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}\int_0^{r_{jl}} u_{jl}^{2|n_j|+2k_{jl}-3}\,du_{jl}\int_{\Gamma_{m,n}}|f_k(W)|^2\omega_{m,n}(W)$$
$$=\int_{\{\lambda\boldsymbol{.}W:\lambda\in\Delta_{m,n},Z\in\Gamma_{m,n}\}}|f_k(Z)|^2\Theta(Z)\le(1+2^{|s|})\int_{\mathbb{M}}|f(Z)|^2\Theta_s(Z)<\infty.$$

This shows that if f_k does not vanish identically on $\mathbb{H}_{m,n}$, then

$$\prod_{j=1}^{j_0}\prod_{l=1}^{m_j}\int_0^{r_{jl}} u_{jl}^{2k_{jl}+1}\,du_{jl}<\infty,$$

which in turn implies that $k_{jl}\ge 0$ provided $n_j=1$. If we fix j_0 variables $Z(1),\dots,Z(j_0)\in\mathbb{H}_1$, then the function $g_k:(Z(j_0+1),\dots,Z(d))\mapsto f_k(Z(1),\dots,Z(d))$ is holomorphic on the manifold $\mathbb{H}_{n_{j_0+1}}\times\dots\times\mathbb{H}_{n_d}$. Now the same reasoning as in Step 1 implies that g_k is the restriction of a holomorphic function in the domain in $M_{m_{j_0+1},n_{j_0+1}+1}(\mathbb{C})\times\dots\times M_{m_d,n_d+1}(\mathbb{C})$ consisting of all elements $(Z(j_0+1),\dots,Z(d))$ that satisfy

$$\sum_{j=j_0+1}^{d}|Z(j)|^{2a_j}<\Big(1-\sum_{j=1}^{j_0}|Z(j)|^{2a_j}\Big)^{1/2}.$$

Now by homogeneity of g_k we see that g_k vanishes identically provided that $k_{jl}<0$ for some j,l. This proves that f_k vanishes identically for $k\in\mathbb{Z}_m\setminus\mathbb{N}_0^m$. By the Parseval equality we see that if $Z\in\mathbb{M}$, then

$$\sum_{k\in\mathbb{N}^m}|f_k(Z)|^2=\frac{1}{(2i\pi)^{|m|}}\int_{(S^1)^m}|f(\lambda\boldsymbol{.}Z)|^2\prod_{j=1}^{d}\prod_{l=1}^{m_j}\frac{d\lambda_{jl}}{\lambda_{jl}}.$$

The homogeneity of the polynomials f_k and the Cauchy–Schwarz inequality now show that the series $\sum_{k\in\mathbb{N}^m}|f_k(Z)|$ converges uniformly on compact subsets of $\mathbb{M}$, which completes the proof of the proposition. ∎

THEOREM 4.2. *The weighted Bergman kernel of $\mathcal{A}_s^2(\mathbb{M})$ is given by the formula*

$$\mathcal{K}_{s,\mathbb{M}}(Z,W)=\frac{(\mathcal{D}R_s)(Z\bullet\overline{W})}{\Gamma(s+1)},$$

where R_s is the function defined in (1.5).

Proof. Putting together Proposition 4.1 and the identities in Lemma 3.3, we obtain

$$\mathcal{A}_s^2(\mathbb{M})=\bigoplus_{k\in\mathbb{N}_0^d}\mathcal{P}^k(\mathbb{M})$$

where the direct sum is orthogonal with respect to the inner product of $\mathcal{A}_s^2(\mathbb{M})$. Thus each $f \in \mathcal{A}_s^2(\mathbb{M})$ is the sum of a series of holomorphic polynomials $f_k \in \mathcal{P}^k(\mathbb{M})$, $k \in \mathbb{N}_0^m$. Let $C(s,k)$ be the coefficients appearing in Lemma 2.2. By Lemma 3.3 we have

$$\begin{aligned} f(Z) = \sum_{k\in\mathbb{N}_0^m} f_k(Z) &= \sum_{k\in\mathbb{N}_0^m} C(s,k) \int_{\mathbb{M}} (Z \bullet \overline{W})^k f_k(W) \Theta_s(W) \\ &= \int_{\mathbb{M}} \Big(\sum_{k\in\mathbb{N}_0^m} C(s,k)(Z \bullet \overline{W})^k \Big) f(W) \Theta_s(W). \end{aligned}$$

Thus

$$\mathcal{K}_{s,\mathbb{M}}(Z,W) = \sum_{k\in\mathbb{N}_0^m} C(s,k)(Z \bullet \overline{W})^k = \frac{1}{\Gamma(s+1)} (\mathcal{D}R_s)(Z \bullet \overline{W})$$

where the latter equality holds in view of Lemma 2.2. ∎

Theorem 4.3. *The Szegő kernel of* $\mathbb{M}$ *is given by the formula*

$$\mathcal{S}_{\mathbb{M}}(Z,W) = \frac{1}{2} (\mathcal{D}R_{-1})(Z \bullet \overline{W}).$$

Proof. We use the same argument as in the proof of Theorem 4.2. ∎

5. An operator between function spaces on $\mathbb{M}$ and function spaces on Ω. If $Z = (Z(1), \ldots, Z(d)) \in M_{m_1,n_1+1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d+1}(\mathbb{C})$, let $F(Z) = (W(1), \ldots, W(d))$ be the element of $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ where $W(j)$ is the $(m_j \times n_j)$-matrix obtained from $Z(j)$ by deleting the (n_j+1)th column. Then

$$\varrho(F(Z)) = \sum_{j=}^{d} |Z(j)|^{2a_j}.$$

Set

$$\mathbb{V}_0 := \bigcup_{j=1}^{d} \bigcup_{l=1}^{m_j} \{W \in M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C}) : W_l(j) = 0\}.$$

Then F is a proper holomorphic mapping of degree $2^{|m|}$ which induces a proper mapping (denoted again by F) from $\mathbb{M}$ onto $\Omega \setminus \mathbb{V}_0$ and from $\partial\mathbb{M}$ onto $\partial\Omega \setminus \mathbb{V}_0$. In addition, the branching locus $\mathbb{W}$ of F is given by

$$\mathbb{W} = \bigcup_{j=1}^{d} \bigcup_{l=1}^{m_j} \{Z \in \mathbb{H}_{m,n} : Q_{jl}(Z) = 0\},$$

where $Q_{jl}(Z)$ is the (n_j+1)th component of the row $Z_l(j)$ of the matrix

$Z(j)$. In addition, the image $F(\mathbb{W})$ is given by

$$F(\mathbb{W}) = \bigcup_{j=1}^{d} \bigcup_{l=1}^{m_j} \{W \in \Omega \setminus \mathbb{V}_0 : W_l(j) \bullet W_l(j) = 0\}.$$

We set $\mathbb{V} := F(\mathbb{W}) \cup \mathbb{V}_0 = \bigcup_{j=1}^{d} \bigcup_{l=1}^{m_j} \{W \in \Omega : W_l(j) \bullet W_l(j) = 0\}$.

The mapping F has card(Λ) local inverses $(U_\varepsilon)_{\varepsilon \in \Lambda}$, where the local inverse mapping U_ε, $\varepsilon = (\varepsilon_{jl}) \in \Lambda$, is defined locally for $Z = (Z(1), \ldots, Z(d)) \in \Omega \setminus \mathbb{V}$ by

$$U_\varepsilon(Z) := (W(1), \ldots, W(d))$$

where $W(j)$ is the $(m_j \times (n_j + 1))$-matrix whose rows are given in terms of the rows $Z_l(j)$ and ε by

$$W_l(j) := \begin{cases} (Z_l(j), i\varepsilon_{jl}\sqrt{Z_l(j) \bullet Z_l(j)}) & \text{if } j > j_0, \\ (Z_l(j), iZ_l(j)) & \text{if } j \le j_0. \end{cases}$$

Let $dV(z) := dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_{n_j} \wedge d\bar{z}_{n_j}$ be the canonical volume form on $\mathbb{C}^{n_j}$. Then the canonical volume form on $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ is given by

$$dV(Z) := \wedge_{j=1}^{d} \wedge_{l=1}^{m_j} dV(Z_l(j)).$$

In addition, a little computing shows that

$$U_\varepsilon^*(\Theta) = \frac{\prod_{j=j_0+1}^{d} (1 + n_j)^{2m_j}}{\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} |Z_l(j) \bullet Z_l(j)|} \, dV(Z). \tag{5.1}$$

Let Σ be the set of all boundary points $Z = (Z(1), \ldots, Z(d)) \in \partial\Omega$ with

$$Z(j) = \begin{pmatrix} Z_1(j) \\ \vdots \\ Z_{m_j}(j) \end{pmatrix}$$

such that $Z_l(j) \bullet Z_l(j) \neq 0$ for all $l = 1, \ldots, m_j$, $j = 1, \ldots, d$. A little computing shows that Σ is a smooth submanifold of $M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$. Using the parametrization (1.5) we see that the form $dV(Z)$ induces a volume form η on Σ by the formula

$$dV(Z) = r^{-1+2\sum_{j=1}^{d} m_j n_j / a_j} dr \wedge \eta. \tag{5.2}$$

In addition, we have

$$\int_{\Sigma} f(W)\eta(W) = \int_{\partial\Omega} f(W) \, d\sigma(W)$$

for all compactly supported continuous functions f on $\partial\Omega$.

LEMMA 5.1. *For each $\varepsilon \in \Lambda$ we have*

$$U_\varepsilon^*(\vartheta) = \frac{\prod_{j=j_0+1}^d (1+n_j)^{2m_j}}{\prod_{j=j_0+1}^d \prod_{l=1}^{m_j} |Z_l(j) \bullet Z_l(j)|}\, \eta.$$

Proof. We use the polar decompositions of $dV(Z)$ and Θ given respectively by (5.2) and Lemma 3.2. ∎

If $f : \Omega \to \mathbb{C}$ is a measurable function and if $Z \in \mathbb{M}$, we define the operator T by

$$(Tf)(Z) := \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} Q_{jl}(Z_l)(f \circ F)(Z). \tag{5.3}$$

LEMMA 5.2. *If f is an integrable compactly supported function on $\Omega \backslash \mathbb{V}$, then for $p \geq 1$ the operator*

$$T_p := \Big(\operatorname{card}(\Lambda) \prod_{j=j_0+1}^{d} (n_j+1)^{2m_j}\Big)^{-1/p} T$$

satisfies the identities

$$\int_{\mathbb{M}} |(T_p f)(Z)|^p \Theta(Z) = \int_{\Omega} |f(W)|^p \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} |W_l(j) \bullet W_l(j)|^{(p-2)/2} dv(W)$$

$$\int_{\partial\mathbb{M}} |(T_p g)(Z)|^p \, d\mu(Z) = \int_{\partial\Omega} |g(W)|^p \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} |W_l(j) \bullet W_l(j)|^{(p-2)/2} d\sigma(W),$$

for all f in $L^p(\Omega, \prod_{j=j_0+1}^d \prod_{l=1}^{m_j} |Z_l(j) \bullet Z_l(j)|^{(p-2)/2} dv(Z)$ and all g in $L^p(\partial\Omega, \prod_{j=j_0+1}^d \prod_{l=1}^{m_j} |W_l(j) \bullet W_l(j)|^{(p-2)/2} d\sigma(W)$.

Proof. Using a partition of unity we may assume that f is compactly supported in $\Omega \setminus \mathbb{V}$ and all the local inverses $(U_\varepsilon)_{\varepsilon\in\Lambda}$ of F are defined on a neighborhood of the support of f. Therefore,

$$\begin{aligned}
\int_{\mathbb{M}} |(Tf)(Z)|^p \Theta(Z) &= \int_{\mathbb{M}} \Big| \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} Q_{jl}(Z)(f \circ F)(Z)\Big|^p \Theta(Z) \\
&= \int_{\mathbb{M}\backslash\mathbb{W}} \Big| \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} Q_{jl}(Z)(f \circ F)(Z)\Big|^p \Theta(Z) \\
&= \sum_{\varepsilon\in\Lambda} \int_{\Omega\backslash\mathbb{V}} \Big| \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} i\varepsilon_{jl} \sqrt{W_l(j) \bullet W_l(j)}\, f(W)\Big|^p U_\varepsilon^*(\Theta)(W)
\end{aligned}$$

$$= \operatorname{card}(\Lambda)\prod_{j=1}^{d}(n_j+1)^{2m_j}\int_{\Omega\setminus\mathbb{V}}\frac{|f(W)|^p}{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}|W_l(j)\bullet W_l(j)|^{(p-2)/2}}\,dv(W)$$

by (5.1). The second equality can be proved in an analogous manner. ∎

LEMMA 5.3. *For each $p\ge 1$ and $s>-1$, let $\mathcal{E}_s^p(\mathbb{M})$ and $\mathcal{R}^p(\mathbb{M})$ denote respectively the images of $\mathcal{A}_s^p(\Omega)$ and $\mathcal{H}^p(\Omega)$ under the operator T. Then*

(1) *$\mathcal{E}_s^p(\mathbb{M})$ is a closed subspace of $\mathcal{A}_s^p(\mathbb{M})$ and T_p is a unitary operator from $\mathcal{A}_s^p(\Omega)$ onto $\mathcal{E}_s^p(\mathbb{M})$. In particular, $\mathcal{A}_s^p(\Omega)$ is a Banach space. Moreover, T is surjective if and only if $n_1=\ldots=n_d=1$.*

(2) *$\mathcal{R}^p(\mathbb{M})$ is a closed subspace of $\mathcal{H}^p(\mathbb{M})$ and T_p is a unitary operator from $\mathcal{H}^p(\Omega)$ onto $\mathcal{R}^p(\mathbb{M})$. In particular, $\mathcal{H}^p(\Omega)$ is a Banach space. Moreover, T is surjective if and only if $n_1=\ldots=n_d=1$.*

Proof. To establish (1), observe by Lemma 5.2 that T_p is a unitary operator from $\mathcal{A}_s^p(\Omega)$ onto $\mathcal{E}_s^p(\mathbb{M})$. We now show that $\mathcal{E}_s^p(\mathbb{M})$ is a closed subspace of $\mathcal{A}_s^p(\mathbb{M})$. If D is a compact set in $\Omega\setminus\mathbb{V}$, then $E=F^{-1}(D)$ is a compact subset of $\mathbb{M}$ since F is proper. Therefore by (4.1) there is a positive constant C' such that

$$\sup_{Z\in D}|f(Z)|\le C'\|Tf\|_{\mathcal{A}_s^p(\mathbb{M})} \tag{5.4}$$

for all $f\in\mathcal{A}_s^p(\Omega)$. These estimates imply that if g is in the closure of $\mathcal{E}_s^p(\mathbb{M})$ then g is holomorphic in $\mathbb{M}$ and there exists $f\in L_s^p(\Omega)$ such that f is holomorphic on $\Omega\setminus\mathbb{V}$ and $g=Tf$. Notice that $\Omega\cap\mathbb{V}$ is an analytic set in Ω. Since g is holomorphic in $\mathbb{M}$ it follows from (5.4) that $f\in L^2_{\mathrm{loc}}(\Omega)$. Therefore, f can be extended holomorphically to Ω, and thus $g=Tf\in\mathcal{E}_s^p(\mathbb{M})$. This proves that $\mathcal{E}_s^p(\mathbb{M})$ is closed in $L_s^p(\mathbb{M})$. Finally, observe that $\mathcal{A}_s^p(\mathbb{M})$ is the closure of all the polynomials. However, $\mathcal{E}_s^p(\mathbb{M})$ contains all the polynomials if and only if $n_1=\ldots=n_d=1$. This shows that $\mathcal{E}_s^p(\mathbb{M})=\mathcal{A}_s^p(\mathbb{M})$ if and only if $n_1=\ldots=n_d=1$ and thus the proof of (1) is complete.

To prove (2), let $g\in\mathcal{R}^p(\mathbb{M})$. Then there is a sequence $\{f_q\}_{q\in\mathbb{N}_0}\subset\mathcal{H}^p(\Omega)$ such that $(T_pf_q)_q$ converges to g in $\mathcal{H}^p(\mathbb{M})$. By Lemma 3.2 we see that $(T_pf_q)_q$ converges to g in $\mathcal{A}_0^p(\mathbb{M})$. By (1), there is $\widetilde{f}\in\mathcal{A}_0^p(\Omega)$ such that $g=Tf$. In virtue of (5.4) we see that f_q converges to f pointwise on $\Omega\setminus\mathbb{V}$. On the other hand, by Lemmas 3.2 and 5.2 we see that $\{f_q\}$ is bounded in $\mathcal{H}^p(\Omega)$ and thus $f\in\mathcal{H}^p(\Omega)$. This proves that $\mathcal{R}^p(\mathbb{M})$ is a closed subspace of $\mathcal{H}^p(\mathbb{M})$. Furthermore, these spaces are equal if and only if $\mathcal{R}^p(\mathbb{M})$ contains all the polynomials, but this occurs if and only if $n_1=\ldots=n_d=1$. ∎

LEMMA 5.4. *Let $B_{s,\mathbb{M}}:L_s^2(\mathbb{M})\to\mathcal{A}_s^2(\mathbb{M})$ be the weighted Bergman projection with respect to the volume form $(1-\sum_{j=1}^d|Z(j)|^{2a_j})^s\Theta(Z)$. Then*

$$B_{s,\mathbb{M}}\circ T=T\circ B_{s,\Omega} \tag{5.5}$$

where $B_{s,\Omega} : L^2_s(\Omega) \to \mathcal{A}^2_s(\Omega)$ is the weighted Bergman projection with respect to the measure $dv_s(W)$.

Proof. We first observe that the lemma holds for holomorphic functions. Indeed, if $f \in \mathcal{A}^2_s(\Omega)$ then $Tf \in \mathcal{A}^2_s(\mathbb{M})$ and thus (5.5) holds at f. Next, we show that (5.5) also holds when f is orthogonal to holomorphic functions.

If $W = (W(1), \ldots, W(d)) \in M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C})$ let w_{jlp} be the entry of the matrix $W(j)$ corresponding to the lth row and pth column. Let $\partial_{jlp} := \partial/\partial w_{jlp}$ be the holomorphic derivative with respect to the variable w_{jlp} and set

$$\mathcal{F}_s = \{(1-\varrho^2)^{-s}\partial_{jlp} g : g \in C_0^\infty(\Omega \setminus \mathbb{V}),\\ j = 1, \ldots, d;\ l = 1, \ldots, m_j;\ p = 1, \ldots, n_j\}$$

where $C_0^\infty(\Omega \setminus \mathbb{V})$ denotes the space of all C^∞-functions with compact support in $\Omega \setminus \mathbb{V}$.

Now let $h \in L^2_s(\mathbb{M})$ be a holomorphic function and let $g \in C_0^\infty(\Omega \setminus \mathbb{V})$ be such that all the local inverses (U_ε) are defined on a neighborhood of the support of g. Then we have

$$\tag{5.6} \int_{\mathbb{M}} h(Z)\, \overline{T\bigg(\frac{1}{(1-\varrho^2)^s}\partial_{jlp} g\bigg)}\, \Theta_s(Z) \\ = \sum_{\varepsilon \in \Lambda} \int_{\Omega \setminus \mathbb{V}} \frac{(f \circ U_\varepsilon)(W)}{\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} (Q_{jl} \circ U_\varepsilon)(W)}\, \overline{\partial_{jlp} g(W)}\, dv_s(W) = 0$$

where the latter equality holds by integration by parts. Therefore,

$$\bigg\langle h, T\bigg(\frac{1}{(1-\varrho^2)^s}\partial_{jlp} g\bigg)\bigg\rangle_{L^2_s(\mathbb{M})} = 0$$

and thus

$$\tag{5.7} (B_{s,\mathbb{M}} \circ T)\bigg(\frac{1}{(1-\varrho(s)^2)^s} \cdot \frac{\partial g}{\partial w_{jlp}}\bigg) = 0.$$

If now g is an arbitrary C^∞-function with compact support in $\Omega \setminus \mathbb{V}$, then using a partition of unity we see that (5.7) holds at g. This shows that the space $\mathcal{F}_s$ is contained in the orthogonal complement $\mathcal{A}^2_s(\Omega)^\perp$ of $\mathcal{A}^2_s(\Omega)$ in $L^2_s(\Omega)$. It remains to show that $\mathcal{F}_s$ is dense in $\mathcal{A}^2_s(\Omega)^\perp$. Let $h \in \mathcal{A}^2_s(\Omega)^\perp$ be orthogonal to $\mathcal{F}_s$ in $L^2_s(\Omega)$; then for any $g \in C_0^\infty(\Omega \setminus \mathbb{V})$,

$$\int_{\Omega} h(W)\, \overline{(\partial_{jlp} g)(W)}\, dv(W) = \int_{\Omega} h(W)\, \overline{(1-\varrho^2)^{-s}(\partial_{jlp} g)(W)}\, dv_s(W) = 0.$$

Thus h satisfies the Cauchy–Riemann equations on $\Omega \setminus \mathbb{V}$ (in the sense of distributions). Therefore h is holomorphic in $\Omega \setminus \mathbb{V}$. Since $h \in L^2_s(\Omega)$, it is also locally in $L^2(\Omega)$. It follows from [Ra, E.3.2, p. 40] that h extends holomorphically across $\mathbb{V}$ in Ω. Hence $h \equiv 0$. ∎

LEMMA 5.5. *Let $S_{\mathbb{M}} : L^2(\partial\mathbb{M}) \to \mathcal{H}^2(\partial\mathbb{M})$ be the Szegő projection with respect to the measure μ. Then*

$$S_{\mathbb{M}} \circ T = T \circ S_{\Omega} \tag{5.8}$$

where $S_{\Omega} : L^2(\partial\Omega) \to \mathcal{H}^2(\Omega)$ is the Szegő projection with respect to the measure σ.

Proof. Clearly, if $f \in \mathcal{H}^2(\Omega)$, then (5.8) holds at f. Suppose now that f is orthogonal to $\mathcal{H}^2(\Omega)$ with respect to the inner product of $L^2(\partial\Omega)$. Consider the function

$$\widetilde{f}(W) := f\left(\frac{W(1)}{\varrho^{1/a_1}(W)}, \dots, \frac{W(d)}{\varrho^{1/a_d}(W)}\right), \quad W \in \Omega \setminus \mathbb{V}.$$

Then by Lemmas 3.2 and 5.2 we see that $\widetilde{f} \in L^2(\Omega)$ and $\widetilde{f}$ is orthogonal to $\mathcal{A}^2(\Omega)$. By Lemma 5.4, $B_{0,\mathbb{M}}(T\widetilde{f}) = (T \circ B_{0,\Omega})(\widetilde{f}) = 0$. This shows that

$$\int_{\mathbb{M}} h(Z)\,\overline{(T\widetilde{f})(Z)}\,\Theta(Z) = 0$$

for all bounded holomorphic functions h on $\mathbb{M}$. In particular, if $k \in \mathbb{N}_0^m$, then for any $h \in \mathcal{P}^k(\mathbb{M})$ we have

$$\int_{\partial\mathbb{M}} h(W)\,\overline{(Tf)(W)}\,d\mu(W) = \frac{\int_{\mathbb{M}} h(Z)\,\overline{(T\widetilde{f})(Z)}\,\Theta(Z)}{\sum_{j=1}^{d} \frac{|k(j)|}{a_j} + 2\sum_{j=1}^{j_0} \frac{m_j}{a_j} + 2\sum_{j=j_0+1}^{d} \frac{m_j n_j}{a_j}} = 0,$$

showing that Tf is orthogonal to $\mathcal{H}^2(\mathbb{M})$ and hence (5.8) holds at f. ∎

For $k \in \mathbb{N}_0^m$, let $\Pi_{k,\mathbb{M}}$ (resp. $\Pi_{k,\Omega}$) denote the orthogonal projection from $\mathcal{H}^2(\mathbb{M})$ (resp. $\mathcal{H}^2(\Omega)$) onto $\mathcal{P}^k(\mathbb{M})$ (resp. $\mathcal{P}^k(\Omega)$). We denote by $\mathbb{I}$ the element k of $\mathbb{N}_0^m$ such that all the components k_{jl}, $j \ge j_0 + 1$, of k are equal to 1 and the remaining components are equal to 0.

LEMMA 5.6. *The following diagram commutes*:

$$\begin{tikzcd}
L^2(\partial\Omega) \arrow[r, "S_\Omega"] \arrow[d, "T"'] & \mathcal{H}^2(\Omega) \arrow[r, "\Pi_{k,\Omega}"] \arrow[d, "T"'] & \mathcal{P}^k(\Omega) \arrow[d, "T"'] \\
L^2(\partial\mathbb{M}) \arrow[r, "S_\mathbb{M}"] & \mathcal{H}^2(\mathbb{M}) \arrow[r, "\Pi_{k+\mathbb{I},\mathbb{M}}"] & \mathcal{P}^{k+\mathbb{I}}(\mathbb{M})
\end{tikzcd}$$

Proof. In view of Lemma 5.5 it is sufficient to prove that the diagram

$$\begin{tikzcd}
\mathcal{H}^2(\Omega) \arrow[r, "T"] \arrow[d, "\Pi_{k,\Omega}"'] & \mathcal{H}^2(\mathbb{M}) \arrow[d, "\Pi_{k+\mathbb{I},\mathbb{M}}"'] \\
\mathcal{P}^k(\Omega) \arrow[r, "T"] & \mathcal{P}^{k+\mathbb{I}}(\mathbb{M})
\end{tikzcd}$$

commutes. To do so, note that if $f \in \mathcal{P}^k(\Omega)$, then $Tf \in \mathcal{P}^{k+\mathbb{I}}(\mathbb{M})$ and thus

$$\Pi_{k,\Omega} f = f \quad \text{and} \quad \Pi_{k+\mathbb{I},\mathbb{M}} Tf = Tf,$$

showing that $(T \circ \Pi_{k,\Omega})(f) = (\Pi_{k+\mathbb{I},\mathbb{M}} \circ T)(f)$. On the other hand, if $f \in \mathcal{H}^2(\Omega)$ is orthogonal to $\mathcal{P}^k(\Omega)$ with respect to the inner product of $\mathcal{H}^2(\Omega)$, then $\Pi_{k,\Omega} f = 0$ and thus $(T \circ \Pi_{k,\Omega})(f) = 0$. Moreover, if we expand f in the form $f = \sum_{l \in \mathbb{N}_0^m \setminus \{k\}} f_l$, where $f_l \in \mathcal{P}^l(\Omega)$, then we have $Tf = \sum_{l \in \mathbb{N}_0^m \setminus \{k\}} Tf_l$. Since for all $l \in \mathbb{N}_0^m \setminus \{k\}$ the polynomial Tf_l is orthogonal to $\mathcal{P}^{k+\mathbb{I}}(\mathbb{M})$ with respect to the inner product of $\mathcal{H}^2(\mathbb{M})$ we see that $\Pi_{k+\mathbb{I},\mathbb{M}} Tf = 0$. This completes the proof of the lemma. ■

Proof of Theorem A. Let $W \in \Omega \setminus \mathbb{V}$ and choose an open neighborhood $\mathcal{O}_W$ of W so that $\mathcal{O}_W \subset \Omega \setminus \mathbb{V}$ and all the local inverses $\{U_\varepsilon\}_{\varepsilon \in \Lambda}$ are well defined in $\mathcal{O}_W$. In view of Remark 6.1.4 in [JP], there is a C^∞-function $\varphi : M_{m_1,n_1}(\mathbb{C}) \times \ldots \times M_{m_d,n_d}(\mathbb{C}) \to [0,\infty[$ such that $\operatorname{supp} \varphi \subset \mathcal{O}_W$ and

$$f(W) = \int_\Omega f(Z)\varphi(Z)\,dv(Z) = \int_\Omega (1-\varrho^2(Z))^{-s} f(Z)\varphi(Z)\,dv_s(Z) \tag{5.9}$$

for any holomorphic function f in $\mathcal{O}_W$. Therefore,

$$\mathcal{K}_{s,\Omega}(\cdot, W) = B_{s,\Omega}\left(\frac{\varphi}{(1-\varrho^2)^s}\right).$$

Let $U := U_{\mathbb{I}}$ be the local inverse of F corresponding to $\varepsilon = \mathbb{I}$. If $Z \in \Omega \setminus \mathbb{V}$, then in view of (5.1), for $Z \in \mathbb{M}$,

$$\begin{aligned}
\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} (Q_{jl} \circ U)(Z)\mathcal{K}_{s,\Omega}(Z,W) &= (T \circ B_{s,\Omega})\left(\frac{\varphi}{(1-\varrho^2)^s}\right)(U(Z)) \\
&= (B_{s,\mathbb{M}} \circ T)\left(\frac{\varphi}{(1-\varrho^2)^s}\right)(U(Z)) \\
&= \int_{\mathbb{M}} \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} (Q_{jl}X)\, \frac{(\varphi \circ F)(X)}{(1-\sum_{j=1}^d |X(j)|^{2a_j})^s}\, \mathcal{K}_{s,\mathbb{M}}(U(Z),X)\Theta_s(X) \\
&= \prod_{j=j_0+1}^{d} (n_j+1)^{2m_j} \sum_{\varepsilon \in \Lambda} \int_{\Omega \setminus \mathbb{V}} \varphi(Y)\, \frac{\mathcal{K}_{s,\mathbb{M}}(U(Z),U_\varepsilon(Y))}{\overline{\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} (Q_{jl} \circ U_\varepsilon)(Y)}}\, dv(Y) \\
&= \prod_{j=j_0+1}^{d} (n_j+1)^{2m_j} \sum_{\varepsilon \in \Lambda} \frac{\mathcal{K}_{s,\mathbb{M}}(U(Z),U_\varepsilon(W))}{\overline{\prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} (Q_{jl} \circ U_\varepsilon)(W)}},
\end{aligned}$$

where the latter equality holds because of (5.9). This shows that

$$\mathcal{K}_{s,\Omega}(Z,W) = \prod_{j=j_0+1}^{d}(n_j+1)^{2m_j}\sum_{\varepsilon\in\Lambda}\frac{\mathcal{K}_{s,\mathbb{M}}(U(Z),U_\varepsilon(W))}{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}[(Q_{jl}\circ U)(Z)\,\overline{(Q_{jl}\circ U_\varepsilon)(W)}]}.$$

Setting $t := Z \bullet \overline{W}$ and $u := (Q_{jl}\circ U)(Z)\,\overline{(Q_{jl}\circ U)(W)}$ and applying Theorem 4.2 we see that for each $\varepsilon\in\Lambda$ we have

$$\frac{\mathcal{K}_{s,\mathbb{M}}(U(Z),U_\varepsilon(W))}{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}[(Q_{jl}\circ U)(Z)\,\overline{(Q_{jl}\circ U_\varepsilon)(W)}]} = \frac{R_s(t+\varepsilon u)}{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}\varepsilon_{jl}u_{jl}},$$

from which the theorem follows because $u^2 = (Z\bullet Z)\,\overline{(W\bullet W)}$. ■

Proof of Theorem B. We use the same notations as in the proof of Theorem B. Let $W\in\partial\Omega\setminus\mathbb{V}$. Using the coordinates (1.5) we choose $r_0>0$ sufficiently small so that the subset

$$\mathcal{O}_W := \{Z(r,X) : r\in\,]1-r_0,1+r_0[,\ |W-X|<r_0,\ X\in\partial\Omega\}$$

is contained in $M_{m_1,n_1}(\mathbb{C})\times\ldots\times M_{m_d,n_d}(\mathbb{C})\setminus\mathbb{V}$ and the local inverses $\{U_\varepsilon\}_{\varepsilon\in\Lambda}$ are well defined in $\mathcal{O}_W$. As in the proof of Theorem A there is a C^∞-function $\varphi : M_{m_1,n_1}(\mathbb{C})\times\ldots\times M_{m_d,n_d}(\mathbb{C})\to[0,\infty[$ with support in $\mathcal{O}_W$ and

$$f(W) = \int_{\mathcal{O}_W} f(Z)\varphi(Z)\,dv(Z)$$

for any holomorphic function f in $\mathcal{O}_W$. If $k\in\mathbb{N}_0^m$, we set

$$\psi_k(X) := \int_{1-r_0}^{1+r_0} r^{-1+\sum_{j=1}^{d}|k(j)|/a_j+2\sum_{j=1}^{d}m_jn_j/a_j}\varphi(Z(r,X))\,dr,\qquad X\in\partial\Omega.$$

Then $\operatorname{supp}\psi_k\subset\mathcal{O}_W\cap\partial\Omega$. In addition, by integration in polar coordinates we see that

$$f(W) = \int_{\partial\Omega} f(X)\psi_k(X)\,d\sigma(X) \tag{5.10}$$

for all $f\in\mathcal{P}^k(M_{m_1,n_1}(\mathbb{C})\times\ldots\times M_{m_d,n_d}(\mathbb{C}))$. Therefore,

$$\Pi_{k,\Omega}(\mathcal{S}_\Omega(\cdot,W)) = (\Pi_{k,\Omega}\circ S_\Omega)(\psi_k).$$

Recalling that U is the local inverse of F corresponding to $\varepsilon=\mathbb{I}$ and applying Lemma 5.6 we see that if $Z\in\Omega\setminus\mathbb{V}$ then

$$\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}(Q_{jl}\circ U)(Z)\Pi_{k,\Omega}(S_\Omega(\cdot,W))(Z) = (T\circ\Pi_{k,\Omega}\circ S_\Omega)(\psi_k)(U(Z))$$

$$= (\Pi_{k+\mathbb{I},\mathbb{M}} \circ S_{\mathbb{M}} \circ T)(\psi_k)(U(Z))$$
$$= \int_{\partial\mathbb{M}} \prod_{j=j_0+1}^{d} \prod_{l=1}^{m_j} (Q_{jl}(X)(\psi_k \circ F)(X)\mathcal{S}_{k+\mathbb{I},\mathbb{M}}(U(Z), X)\, d\mu(X)$$
$$= \prod_{j=j_0+1}^{d} (n_j+1)^{2m_j} \sum_{\varepsilon\in\Lambda} \int_{\partial\Omega\setminus\mathbb{V}} \psi_k(Y) \frac{\mathcal{S}_{k+\mathbb{I},\mathbb{M}}(U(Z), U_\varepsilon(Y))}{\overline{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}(Q_{jl}\circ U_\varepsilon)(Y)}}\, d\sigma(Y)$$
$$= \prod_{j=j_0+1}^{d} (n_j+1)^{2m_j} \sum_{\varepsilon\in\Lambda} \frac{\mathcal{S}_{k+\mathbb{I},\mathbb{M}}(U(Z), U_\varepsilon(W))}{\overline{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}(Q_{jl}\circ U_\varepsilon)(W)}},$$

where the latter equality holds because of (5.10). This shows that

$$\mathcal{S}_{k,\Omega}(Z,W)$$
$$= \prod_{j=j_0+1}^{d} (n_j+1)^{2m_j} \sum_{\varepsilon\in\Lambda} \frac{\mathcal{S}_{k+\mathbb{I},\mathbb{M}}(U(Z), U_\varepsilon(W))}{\prod_{j=j_0+1}^{d}\prod_{l=1}^{m_j}[(Q_{jl}\circ U)(Z)\,\overline{(Q_{jl}\circ U_\varepsilon)(W)}]}.$$

This completes the proof. ∎

Finally, Corollary C can be proved using Theorem B and the same calculation as in the proof of [OPY, Theorem, pp. 222–223].

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