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When unit groups of continuous inverse algebras are regular Lie groups

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Abstract. It is a basic fact in infinite-dimensional Lie theory that the unit group $A^\times$ of a continuous inverse algebra A is a Lie group. We describe criteria ensuring that the Lie group $A^\times$ is regular in Milnor's sense. Notably, $A^\times$ is regular if A is Mackey-complete and locally m-convex.

1. Introduction and statement of the main result. A locally convex, unital, associative topological algebra A over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called a *continuous inverse algebra* if its group $A^\times$ of invertible elements is open and the inversion map $\iota\colon A^\times \to A$, $x \mapsto x^{-1}$, is continuous (cf. [20]). Then ι is $\mathbb{K}$-analytic and hence $A^\times$ is a $\mathbb{K}$-analytic Lie group [6]. Our goal is to describe conditions ensuring that the Lie group $A^\times$ is well-behaved, i.e., it is a regular Lie group in the sense of Milnor [16].

To recall this notion, let G be a Lie group modelled on a locally convex space E, with identity element 1, its tangent bundle TG and the Lie algebra $\mathfrak{g} := T_1G \cong E$. Given $g \in G$ and $v \in T_1G$, let $\lambda_g\colon G \to G$, $x \mapsto gx$ be left translation by g and $gv := T_1(\lambda_g)(v) \in T_gG$. If $\gamma\colon [0,1] \to \mathfrak{g}$ is a continuous map, then there exists at most one C^1-map $\eta\colon [0,1] \to G$ such that

$$\eta'(t) = \eta(t)\gamma(t) \quad \text{for all } t \in [0,1], \quad \text{and} \quad \eta(0) = 1.$$

If such an η exists, it is called the *evolution of* γ. The Lie group G is called *regular* if each $\gamma \in C^\infty([0,1], \mathfrak{g})$ admits an evolution η_γ, and the map $\mathrm{evol}\colon C^\infty([0,1], \mathfrak{g}) \to G$, $\gamma \mapsto \eta_\gamma(1)$, is smooth (see [16] and [17], where also many applications of regularity are described). If G is regular, then its modelling space E is *Mackey-complete* in the sense that every Lipschitz

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curve in E admits a Riemann integral (1) (as shown in [10]). It is a notorious open problem whether, conversely, every Lie group modelled on a Mackey-complete locally convex space is regular ([17, Problem II.2]; cf. [16]).

As a tool for the discussion of $A^\times$, we let $\mu_n\colon A^n \to A$ be the n-linear map defined via $\mu_n(x_1,\ldots,x_n) := x_1\cdots x_n$, for $n\in\mathbb{N}$. Given seminorms $p,q\colon A\to[0,\infty[$, we define $\overline{B}^q_1(0) := \{x\in A : q(x)\le 1\}$ and

$$\|\mu_n\|_{p,q} := \sup\{p(\mu_n(x_1,\ldots,x_n)) : x_1,\ldots,x_n\in\overline{B}^q_1(0)\}\in[0,\infty].$$

Our regularity criterion now reads as follows:

THEOREM 1.1. *Let A be a Mackey-complete continuous inverse algebra such that the following condition is satisfied:*

(∗) *For each continuous seminorm p on A, there exists a continuous seminorm q on A and $r>0$ (which may depend on p and q) such that*
$$\sum_{n=1}^{\infty} r^n\|\mu_n\|_{p,q} < \infty.$$

Then $A^\times$ is a regular Lie group in Milnor's sense.

In fact, $A^\times$ even has certain stronger regularity properties (see Proposition 4.4). Of course, by Hadamard's formula for the radius of convergence of a power series, condition (∗) is equivalent to (2)

$$\limsup_{n\to\infty}\sqrt[n]{\|\mu_n\|_{p,q}}<\infty.$$

It is also equivalent to the existence of $M\in[0,\infty[$ such that $\|\mu_n\|_{p,q}\le M^n$ for all $n\in\mathbb{N}$.

REMARK 1.2. The authors do not know whether condition (∗) can be omitted, i.e., whether $A^\times$ is regular for every Mackey-complete continuous inverse algebra A. Here are some preliminary considerations:

If A is a continuous inverse algebra, then the map $\pi_n\colon A\to A$, $x\mapsto x^n$, is a continuous homogeneous polynomial of degree n, for each $n\in\mathbb{N}_0$. It is known that the analytic inversion map $\iota\colon A^\times\to A$ is given by Neumann's series, $\iota(1-x)=\sum_{n=0}^\infty x^n=\sum_{n=0}^\infty \pi_n(x)$, for x in some 0-neighbourhood of A [6, Lemma 3.3]. Hence, for each continuous seminorm p on A, there exists a continuous seminorm q on A and $s>0$ such that

$$\sum_{n=1}^{\infty} s^n\|\pi_n\|_{p,q}<\infty,$$

(1) See [13] for a detailed discussion of this property.

(2) If $\|\mu_n\|_{p,q}<\infty$, then also $\|\mu_k\|_{p,q}<\infty$ for all $k\in\{1,\ldots,n\}$. In fact, $\|\mu_k\|_{p,q}\le q(1)^{n-k}\|\mu_n\|_{p,q}$ since $\mu_k(x_1,\ldots,x_k)=\mu_n(1,\ldots,1,x_1,\ldots,x_k)$.

where $\|\pi_n\|_{p,q} := \sup\{p(\pi_n(x)) : x \in \overline{B}^q_1(0)\}$ (cf. [2, Proposition 5.1] ([3])). Let S_n be the symmetric group of all permutations of $\{1,\dots,n\}$ and let $\mu_n^{\mathrm{sym}} : A^n \to A$, $(x_1,\dots,x_n) \mapsto (1/n!)\sum_{\sigma\in S_n} x_{\sigma(1)}\cdots x_{\sigma(n)}$ be the symmetrization of μ_n. Then $\pi_n(x) = \mu_n^{\mathrm{sym}}(x,\dots,x)$ and thus $\|\mu_n^{\mathrm{sym}}\|_{p,q} \le (n^n/n!)\|\pi_n\|_{p,q}$ by the Polarization Formula (in the form [11, p. 34, (2)]). Since $\lim_{n\to\infty}(n/\sqrt[n]{n!}) = e$ is Euler's constant (as a consequence of Stirling's Formula), it follows that

$$(1.1)\qquad \sum_{n=1}^{\infty} t^n\|\mu_n^{\mathrm{sym}}\|_{p,q} < \infty \quad \text{for each } t \in]0, s/e[.$$

In general, it is not clear how one could give good estimates for $\|\mu_n\|_{p,q}$ in terms of $\|\mu_n^{\mathrm{sym}}\|_{p,q}$. Hence, it does not seem to be clear in general whether (1.1) implies the existence of some $r > 0$ with $(*)$.

However, $(*)$ is satisfied in some important cases. Following [14], a topological algebra A is called *locally m-convex* if its topology arises from a set of seminorms q which are *submultiplicative*, i.e., $q(xy) \le q(x)q(y)$ for all $x, y \in A$.

COROLLARY 1.3. *Let A be a Mackey-complete continuous inverse algebra. If A is commutative or locally m-convex, then $A^\times$ is a regular Lie group.*

Proof. In fact, if A is commutative, then $\mu_n = \mu_n^{\mathrm{sym}}$, whence $(*)$ is satisfied with any $r \in]0, s/e[$ as in (1.1). Therefore Theorem 1.1 applies ([4]). If A is locally m-convex, then for every continuous seminorm p on A, there is a submultiplicative continuous seminorm q on A such that $p \le q$. Using the submultiplicativity, we see that $\|\mu_n\|_{p,q} \le \|\mu_n\|_{q,q} \le 1$. Thus $(*)$ is satisfied with any $r \in]0, 1[$, and Theorem 1.1 applies. ∎

It can be shown that every Mackey-complete, commutative continuous inverse algebra is locally m-convex (cf. [19]).

REMARK 1.4. We mention that there is a quite direct, alternative proof for the corollary if A is locally m-convex and *complete* ([5]). The easier argu-

([3]) If $\mathbb{K} = \mathbb{R}$, we can apply the proposition to $A_{\mathbb{C}}$, which is a complex continuous inverse algebra (see, e.g., [6, Proposition 3.4]).

([4]) Alternative proof: $(A, +)$ is regular, as it is Mackey-complete [17, Proposition II.5.6]. Since $\exp\colon A \to A^\times$ is a homomorphism of groups (as $A^\times$ is abelian) and a local diffeomorphism (see [6, Theorem 5.6]), it follows that also $A^\times$ is regular [18, Proposition 3].

([5]) Then $A = \varprojlim A_q$ is a projective limit of Banach algebras (where q ranges through the set of all submultiplicative continuous seminorms on A). Being a Banach–Lie group, each $A_q^\times$ is regular [16]. Then $C^\infty([0,1], A) = \varprojlim C^\infty([0,1], A_q)$ and $\mathrm{evol}_{A^\times} = \varprojlim \mathrm{evol}_{A_q^\times}$ is a smooth evolution (cf. [1, Lemma 10.3]).

ments fail however if A is not complete, but merely sequentially complete or Mackey-complete. By contrast, our more elaborate method does not require that A be complete: Mackey-completeness suffices.

Remark 1.5. Our Theorem 1.1 is a variant of the (possibly too optimistic) Theorem IV.1.11 announced in the survey [17], and its proof expands the sketch of proof given there. To avoid the difficulties described in Remark 1.2, we added condition $(*)$.

Remark 1.6. Unit groups of Mackey-complete continuous inverse algebras are so-called *BCH-Lie groups* [6, Theorem 5.6], i.e., they admit an exponential function which is an analytic diffeomorphism around 0 (see [5], [17], [18] for information on such groups). Inspiration for the studies came from an article by Robart [18]. He pursued the (possibly too optimistic) larger goal to show that every BCH-Lie group with Mackey-complete modelling space is regular. However, there seem to be gaps in his arguments ([6]).

Remark 1.7. The following questions are open:

(a) Are there examples of Mackey-complete continuous inverse algebras which satisfy $(*)$ but are not locally m-convex? Or even:
(b) Does every Mackey-complete continuous inverse algebra satisfy $(*)$?

2. Notation and preparatory results

Basic notation. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If X is a set and $n \in \mathbb{N}$, we write $X^n := X \times \cdots \times X$ (with n factors). If $f\colon X \to Y$ is a map, we abbreviate $f^n := f \times \cdots \times f\colon X^n \to Y^n$, $(x_1, \ldots, x_n) \mapsto (f(x_1), \ldots, f(x_n))$. If $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are normed spaces and $\beta\colon E^n \to F$ is a continuous n-linear map, we write $\|\beta\|_{\mathrm{op}}$ for its operator norm, defined as usual as $\sup\{\|\beta(x_1, \ldots, x_n)\|_F : x_1, \ldots, x_k \in E,\ \|x_1\|_E, \ldots, \|x_n\|_E \le 1\}$. If E is a locally convex space, we let $P(E)$ be the set of all continuous seminorms on E. If $p \in P(E)$, we consider the factor space $E_p := E/p^{-1}(0)$ as a normed space with the norm $\|\cdot\|_p$ given by $\|x + p^{-1}(0)\|_p := p(x)$. Then the canonical map $\pi_p\colon E \to E_p$, $x \mapsto x + p^{-1}(0)$, is linear and continuous, with $\|\pi_p(x)\|_p = p(x)$.

Weak integrals. Recall that if E is a locally convex space, $a \le b$ are reals and $\gamma\colon [a, b] \to E$ a continuous map, then the weak integral $\int_a^b \gamma(s)\, ds$ (if it exists) is the unique element of E such that $\lambda(\int_a^b \gamma(s)\, ds) = \int_a^b \lambda(\gamma(s))\, ds$ for each continuous linear functional λ on E. If $\alpha\colon E \to F$ is a continuous

[6] For example, it is unclear whether the limit γ_u constructed in the proof of [18, Proposition 7] takes its values in $\mathrm{Aut}(\mathcal{L})$ (as observed by K.-H. Neeb), and no explanation is given how a smooth curve g in the local group with $\mathrm{Ad}(g) = \gamma_u$ can be obtained.

linear map between locally convex spaces and $\int_a^b \gamma(s)\,ds$ (as before) exists in E, then also $\int_a^b \alpha(\gamma(s))\,ds$ exists in F and is given by

$$\int_a^b \alpha(\gamma(s))\,ds = \alpha\Big(\int_a^b \gamma(s)\,ds\Big) \tag{2.1}$$

(see, e.g., [10] for this observation). If E is sequentially complete, then $\int_a^b \gamma(s)\,ds$ always exists (cf. [2, Lemma 1.1] or [11, 1.2.3]).

C^r-curves. Let $r \in \mathbb{N}_0 \cup \{\infty\}$. As usual, a *$C^r$-curve* in a locally convex space E is a continuous function $\gamma\colon I \to E$ on a non-degenerate interval I such that the derivatives $\gamma^{(k)}\colon I \to E$ of order k exist for all $k \in \mathbb{N}$ with $k \le r$, and are continuous (see, e.g., [10] for more details). The C^∞-curves are also called *smooth curves*.

Smooth maps. If E and F are real locally convex spaces, $U \subseteq E$ is an open subset and $r \in \mathbb{N}_0 \cup \{\infty\}$, then a function $f\colon U \to F$ is called C^r if f is continuous, the iterated directional derivatives $d^{(k)}f(x, y_1, \ldots, y_k) := (D_{y_k} \ldots D_{y_1} f)(x)$ exist for all $k \in \mathbb{N}$ such that $k \le r$, $x \in U$ and $y_1, \ldots, y_k \in E$, and define continuous functions $d^{(k)}f\colon U \times E^k \to F$. If U is not open, but is a convex (or locally convex) subset of E with dense interior U^0, we say that f is C^r if f is continuous, $f|_{U^0}$ is C^r and $d^{(k)}(f|_{U^0})\colon U^0 \times E^k \to F$ has a continuous extension $d^{(k)}f\colon U \times E^k \to F$ for each $k \in \mathbb{N}$ such that $k \le r$. C^∞-maps are also called *smooth*. We abbreviate $df := d^{(1)}f$. It is known that the Chain Rule holds in the form $d(f \circ g)(x, y) = df(g(x), dg(x, y))$, and that compositions of C^r-maps are C^r. Moreover, a C^0-curve $\gamma\colon I \to E$ is a C^r-curve if and only if it is a C^r-map, in which case $\gamma'(t) = d\gamma(t, 1)$ (see [10] for all of these basic facts; cf. also [15], [16], and [4]).

Analytic maps. If E and F are complex locally convex spaces and $n \in \mathbb{N}$, then a function $p\colon E \to F$ is called a *continuous homogeneous polynomial* of degree $n \in \mathbb{N}_0$ if $p(x) = \beta(x, \ldots, x)$ for some continuous n-linear map $\beta\colon E^n \to F$ (if $n = 0$, this means a constant function). A map $f\colon U \to F$ on an open set $U \subseteq E$ is called *complex-analytic* (or $\mathbb{C}$-analytic) if it is continuous and for each $x \in U$, there is a 0-neighbourhood $Y \subseteq E$ with $x + Y \subseteq U$ and continuous homogeneous polynomials $p_n\colon E \to F$ of degree n such that

$$(\forall y \in Y) \qquad f(x + y) = \sum_{n=0}^{\infty} p_n(y)$$

(see [2], [4] and [10] for further information). Following [16], [4] and [10] (but deviating from [2]), given real locally convex spaces E, F, we call a function $f\colon U \to F$ on an open set $U \subseteq E$ *real-analytic* (or $\mathbb{R}$-analytic) if it extends to a complex-analytic map $V \to F_{\mathbb{C}}$, defined on some open subset $V \subseteq E_{\mathbb{C}}$

of the complexification of E, such that $U \subseteq V$. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, it is known that compositions of $\mathbb{K}$-analytic maps are $\mathbb{K}$-analytic. Every $\mathbb{K}$-analytic map is smooth (see, e.g., [10] or [4] for both of these facts).

We shall use the following lemma (proved in Appendix A):

LEMMA 2.1. *Let E and F be complex locally convex spaces, $\widetilde{F}$ be a completion of F such that $F \subseteq \widetilde{F}$ as a dense vector subspace, and $p_n\colon E \to F$ be continuous homogeneous polynomials of degree n for $n \in \mathbb{N}_0$. Assume that*

$$f(x) := \sum_{n\in\mathbb{N}_0} p_n(x)$$

converges in $\widetilde{F}$ for all x in a balanced, open 0-neighbourhood $U \subseteq E$, and $f\colon U \to \widetilde{F}$ is continuous. If F is Mackey-complete, then $f(x) \in F$ for all $x \in U$ and $f\colon U \to F$ is $\mathbb{C}$-analytic.

Function spaces. If E is a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$, let $C^r([0,1], E)$ be the space of all C^r-maps from $[0,1]$ to E. We endow $C^r([0,1], E)$ with the locally convex vector topology defined by the seminorms $\|\cdot\|_{C^k,p}$ given by

$$\|\gamma\|_{C^k,p} := \max_{j=0,\ldots,k}\ \max_{t\in[0,1]} p(\gamma^{(j)}(t))$$

for p in the set of continuous seminorms on E and $k \in \mathbb{N}_0$ with $k \le r$. We abbreviate $C([0,1], E) := C^0([0,1], E)$. Three folklore lemmas concerning these function spaces will be used (the proofs can be found in Appendix A):

LEMMA 2.2. *Let E and F be locally convex spaces, $\alpha\colon E \to F$ be a continuous linear map, and $r \in \mathbb{N}_0 \cup \{\infty\}$. Then also the map*

$$\alpha_* := C^r([0,1], \alpha)\colon C^r([0,1], E) \to C^r([0,1], F), \qquad \gamma \mapsto \alpha \circ \gamma,$$

is continuous and linear. If α is a topological embedding (*i.e., a homeomorphism onto its image*)*, then also α_* is a topological embedding.*

LEMMA 2.3. *If E is a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$, then the topology on the space $C^r([0,1], E)$ is initial with respect to the mappings $(\pi_p)_* : C^r([0,1], E) \to C^r([0,1], E_p)$, $\gamma \mapsto \pi_p \circ \gamma$, for $p \in P(E)$.*

LEMMA 2.4. *If $r \in \mathbb{N}_0 \cup \{\infty\}$ and E is a locally convex space which is complete* (*resp., Mackey-complete*)*, then also $C^r([0,1], E)$ is complete* (*resp., Mackey-complete*)*.*

3. Picard iteration of paths in a topological algebra

SETTING 3.1. Let A be a locally convex topological algebra over $\mathbb{C}$, i.e., a unital, associative, complex algebra, equipped with a Hausdorff locally convex vector topology making the map $A\times A\to A$, $(x,y)\mapsto xy$, continuous. We assume that condition $(*)$ from Theorem 1.1 is satisfied (7).

If E is a locally convex space, then a function $\gamma\colon[0,1]\to E$ is a *Lipschitz curve* if $\bigl\{\frac{\gamma(t)-\gamma(s)}{t-s} : s\neq t\in[0,1]\bigr\}$ is bounded in E (cf. [13, p. 9]). For our current purposes, we endow the space $\mathrm{Lip}([0,1],E)$ of all such curves with the topology $\mathcal{O}_{C^0}$ induced by $C^0([0,1],E)$.

Lemma 3.2 (Picard Iteration). *Let A be as in* 3.1. *If A is sequentially complete and $\gamma\in C([0,1],A)$, we can define a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $C^1([0,1],A)$ via*

$$\eta_0(t):=1,\qquad \eta_n(t):=1+\int_0^t\eta_{n-1}(t_n)\gamma(t_n)\,dt_n\quad \textit{for } t\in[0,1] \textit{ and } n\in\mathbb{N}.$$

Then:

(a) *The limit $\eta:=\eta_\gamma:=\lim_{n\to\infty}\eta_n$ exists in $C^1([0,1],A)$.*

(b) $\eta_n(t)=1+\sum_{k=1}^n\int_0^t\int_0^{t_k}\cdots\int_0^{t_2}\gamma(t_1)\cdots\gamma(t_k)\,dt_1\cdots dt_k$ *for all $n\in\mathbb{N}_0$ and $t\in[0,1]$, and thus*

$$\eta(t)=1+\sum_{n=1}^{\infty}\int_0^t\int_0^{t_n}\cdots\int_0^{t_2}\gamma(t_1)\cdots\gamma(t_n)\,dt_1\cdots dt_n. \tag{3.1}$$

(c) $\eta'(t)=\eta(t)\gamma(t)$ *and* $\eta(0)=1$.

(d) *The map $\Phi\colon C([0,1],A)\to C^1([0,1],A)$, $\gamma\mapsto\eta_\gamma$, is $\mathbb{C}$-analytic.*

If A is not sequentially complete, but Mackey-complete, then the $(\eta_n)_{n\in\mathbb{N}_0}$ can be defined and (a)–(c) *hold for each $\gamma\in\mathrm{Lip}([0,1],A)$. Moreover,*

(d)′ $\Phi\colon(\mathrm{Lip}([0,1],A),\mathcal{O}_{C^0})\to C^1([0,1],A)$, $\gamma\mapsto\eta_\gamma$, *is $\mathbb{C}$-analytic.*

Proof. If A is sequentially complete, set $X:=C([0,1],A)$; otherwise, set $X:=\mathrm{Lip}([0,1],A)$. Let $\widetilde{A}$ be a completion of A such that $A\subseteq\widetilde{A}$. Then the inclusion map $\phi\colon C^1([0,1],A)\to C^1([0,1],\widetilde{A})$ is a topological embedding (Lemma 2.2) and $C^1([0,1],\widetilde{A})$ is complete (Lemma 2.4). Hence also the closure $Y\subseteq C^1([0,1],\widetilde{A})$ of the image $\mathrm{im}(\phi)$ is complete, and thus Y is a completion of $C^1([0,1],A)$.

To prove (a), (b), (d), (d)′, let $\gamma\in X$. Then all integrals needed to define η_n exist, and each η_n is C^1, by the Fundamental Theorem of Calculus. A trivial induction shows that

$$\eta_n(t)=1+\sum_{k=1}^{n}\int_0^t\int_0^{t_k}\cdots\int_0^{t_2}\gamma(t_1)\cdots\gamma(t_k)\,dt_1\cdots dt_k \tag{3.2}$$

(7) Note that A is not assumed to be a continuous inverse algebra in this section.

(as asserted in (b)). Likewise, if $n \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_n \in X$, then the weak integrals needed to define $\tau_n(\gamma_1, \ldots, \gamma_n)\colon [0,1] \to A$,

$$t \mapsto \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} \gamma_1(t_1) \cdots \gamma_n(t_n)\, dt_1 \cdots dt_n,$$

exist and $\tau_n(\gamma_1, \ldots, \gamma_n)$ is a C^1-map. Since $\tau_n\colon X \to C^1([0,1], A)$, $(\gamma_1, \ldots, \gamma_n) \mapsto \tau_n(\gamma_1, \ldots, \gamma_n)$, is an n-linear mapping, it follows that the map $\sigma_n\colon X \to C^1([0,1], A)$, $\sigma_n(\gamma) := \tau_n(\gamma, \ldots, \gamma)$, is a homogeneous polynomial of degree n (and this conclusion also holds for $n = 0$, if we define $\sigma_0(\gamma) := 1$). If $p \in P(A)$, there is $q \in P(A)$ and $M \in [0, \infty[$ such that

$$(\forall n \in \mathbb{N}) \qquad \|\mu_n\|_{p,q} \leq M^n,$$

as a consequence of condition $(*)$. Applying p to the iterated integral defining $\sigma_n(\gamma)(t)$, we deduce that

$$p(\sigma_n(\gamma)(t)) \leq \frac{t^n}{n!} \|\mu_n\|_{p,q} \|\gamma\|^n_{C^0,q} \leq \frac{t^n M^n}{n!} \|\gamma\|^n_{C^0,q}$$

for each $t \in [0,1]$ and thus

$$\|\sigma_n(\gamma)\|_{C^0,p} \leq \frac{M^n}{n!} \|\gamma\|^n_{C^0,q}. \tag{3.3}$$

Also, $\sigma_0(\gamma)' = 0$, $\sigma_1(\gamma)'(t) = \gamma(t)$ and

$$\sigma_n(\gamma)'(t) = \int_0^t \int_0^{t_{n-1}} \ldots \int_0^{t_2} \gamma(t_1) \cdots \gamma(t_{n-1}) \gamma(t)\, dt_1 \cdots dt_{n-1} \tag{3.4}$$

if $n \geq 2$, by the Fundamental Theorem of Calculus. Thus $\sigma_n(\gamma)' = \sigma_{n-1}(\gamma) \cdot \gamma$ for all $n \in \mathbb{N}$. Using $\eta_n = \sum_{k=0}^n \sigma_k(\gamma)$, we infer that

$$(\forall n \in \mathbb{N}) \qquad \eta_n'(t) = \eta_{n-1}(t)\gamma(t), \tag{3.5}$$

which will be useful later. By (3.4), also

$$p(\sigma_n(\gamma)'(t)) \leq \frac{t^{n-1}}{(n-1)!} \|\mu_n\|_{p,q} \|\gamma\|^n_{C^0,q}$$

and thus

$$\|\sigma_n(\gamma)'\|_{C^0,p} \leq \frac{M^n}{(n-1)!} \|\gamma\|^n_{C^0,q}. \tag{3.6}$$

Combining (3.3) and (3.6), we see that

$$\|\sigma_n(\gamma)\|_{C^1,p} \leq \frac{M^n}{(n-1)!} \|\gamma\|^n_{C^0,q}. \tag{3.7}$$

Therefore $\sigma_n\colon X \to C^1([0,1], A)$ is a continuous homogeneous polynomial. Moreover, we obtain

$$\sum_{n=1}^{\infty}\|\sigma_n(\gamma)\|_{C^1,p}\le\sum_{n=1}^{\infty}\frac{M^n\|\gamma\|^n_{C^0,q}}{(n-1)!}=M\|\gamma\|_{C^0,q}e^{M\|\gamma\|_{C^0,q}}<\infty.$$

This estimate entails that the series $\sum_{n=0}^{\infty}\sigma_n(\gamma)$ converges absolutely in the completion Y of $C^1([0,1],A)$. In particular, the limit

$$\Phi(\gamma):=\sum_{n=0}^{\infty}\sigma_n(\gamma)=\lim_{n\to\infty}\eta_n$$

exists in Y, and defines a function $\Phi\colon X\to Y$. We claim that Φ is continuous. If this is true, then we can exploit that $C^1([0,1],A)$ is Mackey-complete by Lemma 2.4, and each σ_n takes its values inside $C^1([0,1],A)$. Thus all hypotheses of Lemma 2.1 are satisfied, and we deduce that $\Phi(\gamma)\in C^1([0,1],A)$ for each γ (entailing (a) and (b)), and that the map $\Phi\colon X\to C^1([0,1],A)$ is complex-analytic (establishing (d) and (d)$'$). To establish the claim, we need only show that Φ is continuous as a map to $C^1([0,1],\widetilde{A})$. Identify $p\in P(A)$ with its continuous extension to a seminorm on $\widetilde{A}$. Let $\pi_p\colon\widetilde{A}\to((\widetilde{A})_p,\|\cdot\|_p)$ be the canonical map. By Lemma 2.3, Φ will be continuous if the maps $h:=(\pi_p)_*\circ\Phi\colon X\to C^1([0,1],(\widetilde{A})_p)$ are continuous. It suffices that h is continuous on the ball $B_R:=\{\gamma\in X:\|\gamma\|_{C^0,q}<R\}$ for each $R>0$. However,

$$h(\gamma)=\sum_{n=0}^{\infty}\pi_p\circ\sigma_n(\gamma)$$

for $\gamma\in B_R$, where

$$\|\pi_p\circ\sigma_n(\gamma)\|_{C^1,\|\cdot\|_p}=\|\sigma_n(\gamma)\|_{C^1,p}\le\frac{M^n}{(n-1)!}\|\gamma\|^n_{C^0,q}\le\frac{M^n}{(n-1)!}R^n$$

for $n\in\mathbb{N}$, by (3.7). Hence

$$\sum_{n=0}^{\infty}\sup\{\pi_p\circ\sigma_n(\gamma):\gamma\in B_R\}\le p(1)+MRe^{RM}<\infty,$$

entailing that $\sum_{k=0}^{n}((\pi_p)_*\circ\sigma_n|_{B_R})\to h|_{B_R}$ uniformly. Thus $h|_{B_R}$ is continuous, being a uniform limit of continuous functions.

To prove (c), observe that because $\eta_n\to\eta$ in $C^1([0,1],A)$, we have $\eta_n'\to\eta'$ uniformly (and thus pointwise). Letting $n\to\infty$ in (3.5), we deduce that $\eta'(t)=\eta(t)\gamma(t)$. ∎

4. Proof of Theorem 1.1. We establish our theorem as a special case of a more general result (Proposition 4.4). The latter deals with certain strengthened regularity properties (as used earlier in [7] and [3]):

Definition 4.1. Let G be a Lie group modelled on a locally convex space, with Lie algebra $\mathfrak{g}$, and $k\in\mathbb{N}_0\cup\{\infty\}$.

(a) G is called *strongly C^k-regular* if every curve $\gamma \in C^k([0,1],\mathfrak{g})$ admits an evolution $\mathrm{Evol}(\gamma) \in C^1([0,1],G)$ and the mapping $\mathrm{evol}\colon C^k([0,1],\mathfrak{g}) \to G$, $\gamma \mapsto \mathrm{Evol}(\gamma)(1)$, is smooth.

(b) G is called *C^k-regular* if each $\gamma \in C^\infty([0,1],\mathfrak{g})$ has an evolution and the map $\mathrm{evol}\colon (C^\infty([0,1],\mathfrak{g}),\mathcal{O}_{C^k}) \to G$, $\gamma \mapsto \mathrm{Evol}(\gamma)(1)$, is smooth, where $\mathcal{O}_{C^k}$ denotes the topology induced by $C^k([0,1],\mathfrak{g})$ on $C^\infty([0,1],\mathfrak{g})$.

The reader is referred to [8] and [9] for a discussion of these regularity properties (and applications depending thereon). Both C^∞-regularity and strong C^∞-regularity coincide with regularity in the usual sense. If $k \le l$ and G is (strongly) C^k-regular, then G is also (strongly) C^l-regular.

Remark 4.2. If A is a continuous inverse algebra, we identify the tangent bundle $T(A^\times)$ of the open set $A^\times$ with $A^\times \times A$ in the natural way. Let $\eta\colon [0,1] \to A^\times$ be a C^1-curve and $\gamma\colon [0,1] \to A$ be continuous. Then $\eta'(t) = \eta(t)\gamma(t)$ holds in $T(A^\times)$ (using $\eta'\colon [0,1] \to T(A^\times)$, and identifying the range A of γ with $\{1\} \times A \subseteq T_1(A^\times)$) if and only if $\eta'(t) = \eta(t)\gamma(t)$ holds in A (where the product simply refers to the algebra multiplication, and $\eta'\colon [0,1] \to A$ is the derivative of the A-valued C^1-curve η).

The next lemma will help us to see that the A-valued map η associated to γ in Lemma 3.2 actually takes its values in $A^\times$ if A is a continuous inverse algebra. Hence η will be the evolution of γ, by Remark 4.2.

Lemma 4.3. *Let A be a continuous inverse algebra, $\gamma\colon [0,1] \to A$ be continuous and $\eta\colon [0,1] \to A$ as well as $\zeta\colon [0,1] \to A$ be C^1-curves. Assume that $\eta(0) = \zeta(0) = 1$ and*

$$\eta'(t) = \eta(t)\gamma(t) \quad \text{and} \quad \zeta'(t) = \zeta(t)\gamma(t) \quad \text{for all } t \in [0,1]. \tag{4.1}$$

If $\zeta([0,1]) \subseteq A^\times$, then $\eta = \zeta$.

Proof. Recall from [6, proof of Lemma 3.1] that the differential of the inversion map $\iota\colon A^\times \to A$ is given by $d\iota(a,b) = -a^{-1}ba^{-1}$ for $a \in A^\times$ and $b \in A$. As a consequence, the derivative of the C^1-curve $\iota \circ \zeta\colon [0,1] \to A^\times$, $t \mapsto \zeta(t)^{-1}$, is given by

$$(\iota \circ \zeta)'(t) = -\zeta(t)^{-1}\zeta'(t)\zeta(t)^{-1}. \tag{4.2}$$

Now consider the C^1-curve $\theta\colon [0,1] \to A$, $\theta(t) := \eta(t)\zeta(t)^{-1}$. Using the Product Rule, (4.2) and (4.1), we obtain

$$\begin{aligned}\theta'(t) &= \eta'(t)\zeta(t)^{-1} - \eta(t)\zeta(t)^{-1}\zeta'(t)\zeta(t)^{-1}\\ &= \eta(t)\gamma(t)\zeta(t)^{-1} - \eta(t)\zeta(t)^{-1}\zeta(t)\gamma(t)\zeta(t)^{-1}\\ &= \eta(t)\gamma(t)\zeta(t)^{-1} - \eta(t)\gamma(t)\zeta(t)^{-1} = 0.\end{aligned}$$

Hence $\theta(t) = \theta(0) = \eta(0)\zeta(0)^{-1} = 1$ for all $t \in [0,1]$ and thus $\eta = \zeta$. ∎

PROPOSITION 4.4. *Let A be a continuous inverse algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ which satisfies the condition $(*)$ described in Theorem* 1.1.

(a) *If A is sequentially complete, then A is strongly C^0-regular and the map* $\mathrm{Evol}\colon C^0([0,1],A) \to C^1([0,1],A^\times)$ *is $\mathbb{K}$-analytic.*
(b) *If A is Mackey-complete, then A is C^0-regular and strongly C^1-regular. Further, each $\gamma \in \mathrm{Lip}([0,1],A)$ has an evolution $\mathrm{Evol}(\gamma) \in C^1([0,1],A^\times)$, and $\mathrm{Evol}\colon (\mathrm{Lip}([0,1],A),\mathcal{O}_{C^0}) \to C^1([0,1],A^\times)$ is $\mathbb{K}$-analytic.*

Proof. If A is sequentially complete, let $X := C([0,1],A)$; otherwise, let $X := (\mathrm{Lip}([0,1],A),\mathcal{O}_{C^0})$.

We assume first that $\mathbb{K} = \mathbb{C}$. Let $\Phi\colon X \to C^1([0,1],A)$ be the mapping provided by Lemma 3.2. Note that $C^1([0,1],A^\times) \subseteq C^1([0,1],A)$ is an identity neighbourhood, $\Phi(0) = 1$ (cf. (3.1)) and Φ is $\mathbb{C}$-analytic (see (d) or (d)$'$ of Lemma 3.2) and hence continuous. Therefore, there exists an open 0-neighbourhood $\Omega \subseteq X$ such that $\Phi(\Omega) \subseteq C^1([0,1],A^\times)$. By Lemma 3.2(c), $\mathrm{Evol}(\gamma) := \Phi(\gamma)$ is an evolution for $\gamma \in \Omega$. Moreover, $\mathrm{evol}\colon \Omega \to A^\times$, $\gamma \mapsto \mathrm{Evol}(\gamma)(1) = \Phi(\gamma)(1)$, is $\mathbb{C}$-analytic, since Φ and the continuous linear point evaluation $\mathrm{ev}_1\colon C^1([0,1],A) \to A$, $\zeta \mapsto \zeta(1)$, are $\mathbb{C}$-analytic.

If A is sequentially complete, Proposition 1.3.10 in [3] now shows that $A^\times$ is strongly C^0-regular ([8]).

If A is Mackey-complete, we see as in the proof of [3, Proposition 1.3.10] that each $\gamma \in \mathrm{Lip}([0,1],A)$ has an evolution $\mathrm{Evol}(\gamma) \in C^1([0,1],A^\times)$.

In either case, we deduce with Lemmas 3.2 (c) and 4.3 that $\mathrm{Evol} = \Phi$. As a consequence, $\mathrm{Evol}\colon X \to C^1([0,1],A^\times)$ is $\mathbb{C}$-analytic and thus (a) holds. In the situation of (b), note that also $\mathrm{evol} := \mathrm{ev}_1 \circ \mathrm{Evol}\colon \mathrm{Lip}([0,1],A) \to A^\times$ is $\mathbb{C}$-analytic. The inclusion maps $(C^\infty([0,1],A),\mathcal{O}_{C^0}) \to (\mathrm{Lip}([0,1],A),\mathcal{O}_{C^0})$ and $C^1([0,1],A) \to (\mathrm{Lip}([0,1],A),\mathcal{O}_{C^0})$ being continuous linear and hence $\mathbb{C}$-analytic, it follows that also the maps $\mathrm{evol}\colon (C^\infty([0,1],A),\mathcal{O}_{C^0}) \to A^\times$ and $\mathrm{evol}\colon C^1([0,1],A) \to A^\times$ are $\mathbb{C}$-analytic and thus smooth. Hence $A^\times$ is C^0-regular and strongly C^1-regular.

If $\mathbb{K} = \mathbb{R}$, then also the complexification $A_{\mathbb{C}}$ of A is a continuous inverse algebra (see, e.g., [6, Proposition 3.4]) with the same completeness properties. In (a), we can identify $X_{\mathbb{C}}$ with $C^0([0,1],A_{\mathbb{C}})$; in the situation of (b), we can identify $X_{\mathbb{C}}$ with $\mathrm{Lip}([0,1],A_{\mathbb{C}})$. For $p \in P(A)$, let $p_{\mathbb{C}} \in P(A_{\mathbb{C}})$ be the seminorm defined via

$$p_{\mathbb{C}}(a+ib) := \inf\Big\{\sum_j |z_j|p(x_j) : a+ib = \sum_j z_j x_j,\ x_j \in A,\ z_j \in \mathbb{C}\Big\}$$

for $a,b \in A$ (which satisfies $\max\{p(a),p(b)\} \le p_{\mathbb{C}}(a+ib) \le p(a)+p(b)$). Then also $A_{\mathbb{C}}$ satisfies $(*)$, as $\|(\mu_n)_{\mathbb{C}}\|_{p_{\mathbb{C}},q_{\mathbb{C}}} = \|\mu_n\|_{p,q}$. Let $\Phi\colon X_{\mathbb{C}} \to C^1([0,1],A_{\mathbb{C}})$

([8]) Compare already [13, p. 409] and [18, Lemma 3] for similar arguments.

be the complex-analytic map provided by Lemma 3.2 (applied to $A_{\mathbb{C}}$ in place of A). By the complex case just discussed,

$$\Phi = \mathrm{Evol}_{(A_{\mathbb{C}})^\times}\colon X_{\mathbb{C}} \to C^1([0,1],(A_{\mathbb{C}})^\times).$$

If $\gamma \in X$, then $\Phi(\gamma)$ takes only values in the closed vector subspace A of $A_{\mathbb{C}} = A \oplus iA$, as is clear from (3.1). Hence $\Phi(\gamma) \in C^1([0,1],A)$ (see [10] or [1, Lemma 10.1]) and thus $\Phi(\gamma) \in C^1([0,1],A^\times)$, using the fact that $A \cap (A_{\mathbb{C}})^\times = A^\times$ for any unital algebra ([9]). We deduce that the map $\Phi|_X\colon X \to C^1([0,1],A^\times)$ is the evolution map $\mathrm{Evol}_{A^\times}$ of $A^\times$. Note that $\mathrm{Evol}_{A^\times}$ is $\mathbb{R}$-analytic, because $\Phi\colon X_{\mathbb{C}} \to C^1([0,1],A)_{\mathbb{C}}$ is a $\mathbb{C}$-analytic extension of $\mathrm{Evol}_{A^\times}$. As $\mathrm{ev}_1\colon C^1([0,1],A) \to A$, $\zeta \mapsto \zeta(1)$, is continuous linear and so $\mathbb{R}$-analytic, also $\mathrm{evol}_{A^\times} := \mathrm{ev}_1 \circ \mathrm{Evol}_{A^\times}\colon X \to A^\times$ is $\mathbb{R}$-analytic (and hence smooth). In the situation of (a), this completes the proof. In (b), compose $\mathrm{evol}_{A^\times}$ with the continuous linear inclusion map $C^1([0,1],A) \to \mathrm{Lip}([0,1],A)$ (resp., $(C^\infty([0,1],A),\mathcal{O}_{C^0}) \to \mathrm{Lip}([0,1],A)$) to see that also the evolution mapping on $C^1([0,1],A)$ (resp., on $(C^\infty([0,1],A^\times),\mathcal{O}_{C^0})$) is $\mathbb{R}$-analytic and hence C^∞. ■

Appendix A. Proofs of the lemmas from Section 2. It is useful to recall that a locally convex space E is Mackey-complete (in the sense presented in the introduction) if and only if every Mackey–Cauchy sequence in E converges, i.e., every sequence $(x_n)_{n\in\mathbb{N}}$ in E for which there exists a bounded subset $B \subseteq E$ and a double sequence $(r_{n,m})_{n,m\in\mathbb{N}}$ of real numbers $r_{n,m} \geq 0$ such that $x_n - x_m \in r_{n,m}B$ for all $n,m \in \mathbb{N}$, and $r_{n,m} \to 0$ as both $n,m \to \infty$ (cf. [13, Theorem 2.14]).

Proof of Lemma 2.1. Given $x \in U$, there exists $r \in\,]1,\infty[$ such that $rx \in U$. Thus $\sum_{n=0}^\infty r^n p_n(x)$ converges and hence $C := \{r^n p_n(x)\colon n \in \mathbb{N}_0\}$ is a bounded subset of F. Then also the absolutely convex hull B of C is bounded. For all $n,m \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{n+m} p_k(x) - \sum_{k=0}^{n} p_k(x) = \sum_{k=n+1}^{n+m} p_k(x) = r^{-n-1}\sum_{k=n+1}^{n+m} r^{n+1-k} r^k p_k(x)$$
$$\in r^{-n-1}\Big(\sum_{j=0}^{m-1}(1/r)^j\Big)B \subseteq \frac{r^{-n-1}}{1-1/r}B.$$

Hence $(\sum_{k=0}^n p_k(x))_{n\in\mathbb{N}_0}$ is a Mackey–Cauchy sequence in F and hence convergent. Thus $f(x) \in F$. By [2, Theorems 5.1 and 6.1(i)], f is $\mathbb{C}$-analytic as a map to $\widetilde{F}$. Hence, if $x \in U$, then $f(x+y) = \sum_{n=0}^\infty (1/n!)\delta_x^n(f)(y)$ for all y in some 0-neighbourhood, where $\delta_x^n f(y) := d^{(n)}f(x,y,\ldots,y)$ is the

([9]) If $x,a,b \in A$ and $x(a+ib) = (a+ib)x = 1$, then $xa + ixb = 1$ and $ax + ibx = 1$. Hence $xa = ax = 1$, i.e., $x^{-1} = a \in A$.

nth Gâteaux differential of f at x. Given $y \in E$, there is $s > 0$ such that $x + zy \in U$ for all $z \in \mathbb{C}$ such that $|z| \leq s$. For each $n \in \mathbb{N}_0$, Cauchy's Integral Formula for higher derivatives now shows that

$$\delta_x^n(f)(y) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(x + se^{it}y)}{(se^{it})^{n+1}} sie^{it}\, dt,$$

which lies in F since the integrand is a Lipschitz curve in F and F is Mackey-complete ([10]). Hence each $\delta_x^n(f)$ is a continuous homogeneous polynomial from E to F and thus f is complex-analytic as a map from E to F. ∎

Proof of Lemma 2.2. Let p be a continuous seminorm on F and $k \in \mathbb{N}_0$ be such that $k \leq r$. Then $q := p \circ \alpha$ is a continuous seminorm on E. Let $\gamma \in C^r([0,1], E)$. For each $j \in \mathbb{N}_0$ such that $j \leq k$, we have $(\alpha \circ \gamma)^{(j)} = \alpha \circ \gamma^{(j)}$ and thus $\|(\alpha \circ \gamma)^{(j)}\|_{C^0,p} = \|\alpha \circ \gamma^{(j)}\|_{C^0,p} = \|\gamma^{(j)}\|_{C^0,p\circ\alpha} = \|\gamma^{(j)}\|_{C^0,q}$, entailing that $\|\alpha \circ \gamma\|_{C^k,p} = \|\gamma\|_{C^k,q}$. Hence α_* is continuous.

If α is an embedding and Q is a continuous seminorm on $C^r([0,1], E)$, then there exists $k \in \mathbb{N}_0$ such that $k \leq r$ and a continuous seminorm q on E such that $Q \leq \|\cdot\|_{C^k,q}$. Since α is an embedding, there exists a continuous seminorm p on F such that $p(\alpha(x)) \geq q(x)$ for all $x \in E$ (because α^{-1} is continuous linear). Hence $\|(\alpha \circ \gamma)^{(j)}\|_{C^0,p} = \|\gamma^{(j)}\|_{C^0,p\circ\alpha} \geq \|\gamma^{(j)}\|_{C^0,q}$ for each $j \in \mathbb{N}_0$ such that $j \leq k$ and thus $\|\alpha \circ \gamma\|_{C^k,p} \geq \|\gamma\|_{C^k,q} \geq Q(\gamma)$, entailing that α_* is a topological embedding. ∎

Proof of Lemma 2.3. Let $p \in P(E)$ and $k \in \mathbb{N}_0$ be such that $k \leq r$. Since $p = \|\cdot\|_p \circ \pi_p$, we have

$$\|(\pi_p \circ \gamma)^{(j)}\|_{C^0,\|\cdot\|_p} = \|\pi_p \circ \gamma^{(j)}\|_{C^0,\|\cdot\|_p} = \|\gamma^{(j)}\|_{C^0,\|\cdot\|_p\circ\pi_p} = \|\gamma^{(j)}\|_{C^0,p}$$

for each $\gamma \in C^r([0,1], E)$ and $j \in \{0, 1, \ldots, k\}$, whence $\|(\pi_p)_*(\gamma)\|_{C^k,\|\cdot\|_p} = \|\gamma\|_{C^k,p}$. The assertion follows. ∎

Remark A.1. Before we turn to the proof of Lemma 2.4, it is useful to record some simple observations:

(a) It is clear from the definitions that the map

$$h\colon C^k([0,1], E) \to C([0,1], E) \times C^{k-1}([0,1], E), \qquad \gamma \mapsto (\gamma, \gamma'),$$

is linear and a homeomorphism onto its image, for each $k \in \mathbb{N}$.

(b) The image $\mathrm{im}(h)$ of h consists of all pairs (γ, η) such that $\gamma(t) = \gamma(0) + \int_0^t \eta(s)\, ds$ for each $t \in [0,1]$. Since point evaluations and the linear mappings $\eta \mapsto \int_0^t \eta(s)\, ds$ (with $p(\int_0^t \eta(s)\, ds) \leq \|\eta\|_{C^0,p}$) are continuous, it follows that $\mathrm{im}(h)$ is a closed vector subspace of $C([0,1], E) \times C^{k-1}([0,1], E)$.

([10]) The integrand is a C^∞-curve in $\widetilde{F}$ and hence a Lipschitz curve in $\widetilde{F}$, with image in F.

Proof of Lemma 2.4. Because direct products of Mackey-complete locally convex spaces are Mackey-complete, and so are closed vector subspaces, also projective limits of Mackey-complete locally convex spaces are Mackey-complete. Since $C^\infty([0,1],E) = \varprojlim C^k([0,1],E)$ (with the appropriate inclusion maps as the limit maps), we therefore only need to prove Mackey-completeness if $k := r \in \mathbb{N}_0$. Likewise in the case of completeness.

Case $k = 0$. If E is complete, then also $C([0,1],E)$ is complete, as is well known (cf. [12, Chapter 7, Theorem 10]). If E is merely Mackey-complete, let $\widetilde{E}$ be a completion of E which contains E. Then $C([0,1],\widetilde{E})$ is complete. The inclusion map $\phi\colon C([0,1],E) \to C([0,1],\widetilde{E})$ is a topological embedding, by Lemma 2.2. If $(\gamma_n)_{n\in\mathbb{N}}$ is a Mackey–Cauchy sequence in $C([0,1],E)$, then $(\phi\circ\gamma_n)_{n\in\mathbb{N}} = (\gamma_n)_{n\in\mathbb{N}}$ is a Mackey–Cauchy sequence in $C([0,1],\widetilde{E})$, hence convergent to some $\gamma \in C([0,1],\widetilde{E})$. For each $t \in [0,1]$, the point evaluation $\varepsilon_t\colon C([0,1],\widetilde{E}) \to \widetilde{E}$, $\eta \mapsto \eta(t)$, is continuous and linear. Hence $(\gamma_n(t))_{n\in\mathbb{N}}$ is a Mackey–Cauchy sequence in E and hence convergent in E. Since $\gamma_n(t) = \varepsilon_t(\gamma_n) \to \varepsilon_t(\gamma) = \gamma(t)$, we deduce that $\gamma(t) \in E$. Therefore $\gamma \in C([0,1],E)$ and it is clear that $\gamma_n \to \gamma$ in $C([0,1],E)$.

Induction step. If $C^{k-1}([0,1],E)$ is (Mackey-)complete, then so is $C^k([0,1],E)$, being isomorphic to a closed vector subspace of the (Mackey-) complete direct product $C([0,1],E)\times C^{k-1}([0,1],E)$ (see Remark A.1(b)). ∎

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References

[1] W. Bertram, H. Glöckner and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. 22 (2004), 213–282.

[2] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1971), 77–112.

[3] R. Dahmen, *Direct limit constructions in infinite-dimensional Lie theory*, doctoral dissertation, Univ. Paderborn, 2011; http://nbn-resolving.deurn:nbn:de:hbz:466:2-239.

[4] H. Glöckner, *Infinite-dimensional Lie groups without completeness restrictions*, in: Banach Center Publ. 55, Inst. Math., Polish Acad. Sci., Warszawa, 2002, 43–59.

[5] H. Glöckner, *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. 194 (2002), 347–409.

[6] H. Glöckner, *Algebras whose groups of units are Lie groups*, Studia Math. 153 (2002), 147–177.

[7] H. Glöckner, *Direct limits of infinite-dimensional Lie groups*, in: Developments and Trends in Infinite-Dimensional Lie Theory, K.-H. Neeb and A. Pianzola (eds.), Progr. Math. 288, Birkhäuser, Boston, 2011, 243–280.

[8] H. Glöckner, *Notes on regularity properties of infinite-dimensional Lie groups*, arXiv: 1208.0715.

[9] H. Glöckner, *Regularity in Milnor's sense for direct limits of infinite-dimensional Lie groups*, manuscript in preparation.

[10] H. Glöckner and K.-H. Neeb, *Infinite-Dimensional Lie Groups*, Vol. I, book in preparation.

[11] M. Hervé, *Analyticity in Infinite-Dimensional Spaces*, de Gruyter, Berlin, 1989.

[12] J. L. Kelley, *General Topology*, Springer, 1955.

[13] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*, Amer. Math. Soc., Providence, RI, 1997.

[14] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).

[15] P. W. Michor, *Manifolds of Differentiable Mappings*, Shiva Publ., Orpington, 1980.

[16] J. Milnor, *Remarks on infinite-dimensional Lie groups*, in: Relativité, groupes et topologie II (Les Houches, 1983), B. S. DeWitt and R. Stora (eds.), North-Holland, Amsterdam, 1984, 1007–1057.

[17] K.-H. Neeb, *Towards a Lie theory of locally convex groups*, Japan. J. Math. 1 (2006), 291–468.

[18] Th. Robart, *On Milnor's regularity and the path-functor for the class of infinite dimensional Lie algebras of CBH type*, Algebras Groups Geom. 21 (2004), 367–386.

[19] Ph. Turpin, *Une remarque sur les algèbres à inverse continu*, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1686–A1689.

[20] L. Waelbroeck, *Les algèbres à inverse continu*, C. R. Acad. Sci. Paris 238 (1954), 640–641.

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