STUDIA MATHEMATICA 217 (2) (2013)

Weighted estimates for the iterated commutators of multilinear maximal and fractional type operators

by

QINGYING XUE (Beijing)

Dedicated to Professor Kôzô Yabuta on his 70th birthday

Abstract. The following iterated commutators $T_{*,\Pi b}$ of the maximal operator for multilinear singular integral operators and $I_{\alpha,\Pi b}$ of the multilinear fractional integral operator are introduced and studied:

$$T_{*,\Pi b}(\vec{f})(x) = \sup_{\delta>0} \left|[b_1,[b_2,\dots[b_{m-1},[b_m,T_\delta]_m]_{m-1}\cdots]_2]_1(\vec{f})(x)\right|,$$

$$I_{\alpha,\Pi b}(\vec{f})(x) = [b_1,[b_2,\dots[b_{m-1},[b_m,I_\alpha]_m]_{m-1}\cdots]_2]_1(\vec{f})(x),$$

where T_δ are the smooth truncations of the multilinear singular integral operators and I_α is the multilinear fractional integral operator, $b_i \in \mathrm{BMO}$ for $i = 1,\dots,m$ and $\vec{f} = (f_1,\dots,f_m)$.

Weighted strong and $L(\log L)$ type end-point estimates for the above iterated commutators associated with two classes of multiple weights, $A_{\vec{p}}$ and $A_{(\vec{p},q)}$, are obtained, respectively.

1. Introduction. The multilinear Calderón–Zygmund theory is a natural generalization of the linear case. Many authors were interested in these topics ([5], [6], [4], [18], [15], [8], [19], [22], [3], [20], [25], [14], [1]). We first recall the definition of and some results on multilinear Calderón–Zygmund operators as well as the corresponding multilinear maximal operators and fractional type operators.

DEFINITION 1.1 (Multilinear Calderón–Zygmund operators). Let T be a multilinear operator initially defined on the m-fold product of the Schwartz space and taking values in the space of tempered distributions,

$$T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n).$$

As described in [6], T is an *m-linear Calderón–Zygmund operator* if for some

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 42B25.
Key words and phrases: multilinar Calderón–Zygmund operators, maximal operators, multilinear fractional type operators, commutators, multiple weights $A_{\vec{p}}$ and $A_{(\vec{p},q)}$.

DOI: 10.4064/sm217-2-1

$1 \le q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $1/q = 1/q_1 + \cdots + 1/q_m$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$,

$$|K(y_0, y_1, \dots, y_m)| \le \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}, \tag{1.1}$$

and

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y_j', \dots, y_m)| \le \frac{A|y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}} \tag{1.2}$$

for some $\varepsilon > 0$ and all $0 \le j \le m$, whenever $|y_j - y_j'| \le \frac{1}{2} \max_{0 \le k \le m} |y_j - y_k|$.

The *maximal multilinear singular integral operator* is defined by

$$T_*(\vec{f})(x) = \sup_{\delta > 0} |T_\delta(f_1, \dots, f_m)(x)|,$$

where T_δ are the smooth truncations of T given by

$$T_\delta(f_1, \dots, f_m)(x) = \int_{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\vec{y}.$$

Here, $d\vec{y} = dy_1 \cdots dy_m$.

As pointed out in [17], $T_*(\vec{f})(x)$ is pointwise well-defined when $f_j \in L^{q_j}(\mathbb{R}^n)$ with $1 \le q_j \le \infty$.

The study of multilinear singular integral operators and their maximal operators has a long history. For the maximal multilinear operator T_*, one can see for example [2], [13], [17] and [20] for more details. We list some results for T_*:

Theorem A ([17]). *Let $1 \le q_i < \infty$, $1/q = 1/q_1 + \cdots + 1/q_m$, and $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$. Let T be an m-linear Calderón–Zygmund operator. Then there exists a constant $C_{q,n} < \infty$ such that for all $\vec{f} = (f_1, \dots, f_m)$ satisfying $\|T_*(\vec{f})\|_{L^q(\omega)} < \infty$ we have*

$$\|T_*(\vec{f})\|_{L^q(\omega)} \le C_{n,q}(A + W) \prod_{i=1}^m \|f_i\|_{L^{q_i}(\omega)},$$

where W is the norm of T as a mapping $T : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$.

Theorem B ([2]). *Assume that* $1/p_1 + \cdots + 1/p_m = 1/p$ *and* $\vec{\omega} \in A_{\vec{p}}$. *Then*

(i) *If* $1 < p_1, \ldots, p_m < \infty$, *then* T_* *is bounded from* $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ *to* $L^p(\nu_{\vec{\omega}})$.
(ii) *If* $1 \le p_1, \ldots, p_m < \infty$, *then* T_* *is bounded from* $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ *to* $L^{p,\infty}(\nu_{\vec{\omega}})$.

Here, $A_{\vec{p}}$ is the multiple weights as in Definition 2.1 below, and each w_i is a nonnegative measurable function. The boundedness of T_* on Hardy spaces and weighted Hardy spaces was obtained in [13] and [21].

Now, let us recall some definitions and background for multilinear fractional type operators.

In 1992, Grafakos [12] first defined and studied the following multilinear maximal function and multilinear fractional integral:

$$M_\alpha(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} \Big| \prod_{i=1}^{m} f_i(x - \theta_i y) \Big| \, dy,$$

$$I_\alpha(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^{m} f_i(x - \theta_i y) \, dy,$$

where θ_i $(i = 1, \ldots, m)$ are fixed distinct and nonzero real numbers and $0 < \alpha < n$. If we simply take $m = 1$ and $\theta_i = 1$, then M_α and I_α are just the operators studied by Muckenhoupt and Wheeden [23]. In 1999, Kenig and Stein [18] considered another more general type of multilinear fractional integral defined by

$$I_{\alpha,A}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|(y_1, \ldots, y_m)|^{mn-\alpha}} \prod_{i=1}^{m} f_i(\ell_i(y_1, \ldots, y_m, x)) \, dy_i,$$

where ℓ_i is a linear combination of y_js and x depending on the matrix A. They showed that $I_{\alpha,A}$ was of strong type $(L^{p_1} \times \cdots \times L^{p_m}, L^q)$ and weak type $(L^{p_1} \times \cdots \times L^{p_m}, L^{q,\infty})$. When $\ell_i(y_1, \ldots, y_m, x) = x - y_i$, we denote this multilinear fractional type operator by I_α.

The question of the existence of a multiple weight theory was posed in [16], and since then it has been an open problem to give the right class of multiple weights for m-linear Calderón–Zygmund operators and multilinear fractional integral operators so that weighted estimates still hold in the multilinear setting. This was established in [19], [22], [3] and the multiple weight classes $A_{\vec{p}}$ and $A_{(\vec{p},q)}$ were introduced (see the definitions in Section 2 below).

In [19] and [3], the following commutators of T and I_α in the jth entry were defined. Weighted strong and weighted end-point $L(\log L)$ type estimates associated with $A_{\vec{p}}$ and $A_{(\vec{p},q)}$ weights were given, respectively.

Definition 1.2 (Commutators in the jth entry [19], [3]). Given a collection of locally integrable functions $\mathbf{b} = \vec{b} = (b_1, \dots, b_m)$, we define the commutators of the m-linear Calderón–Zygmund operator T and fractional integral I_α to be

$$[\vec{b}, T](\vec{f}) = T_{\vec{b}}(f_1, \dots, f_m) = \sum_{j=1}^{m} T_{\vec{b}}^{j}(\vec{f}) = T_{\Sigma\mathbf{b}}(\vec{f}),$$

$$I_{\vec{b},\alpha}(\vec{f})(x) = \sum_{i=1}^{m} I_{\vec{b},\alpha}^{j}(\vec{f})(x) = I_{\Sigma\mathbf{b},\alpha}(\vec{f})(x),$$

where each term is the commutator of b_j and T in the jth entry of T, that is,

$$T_{\vec{b}}^{j}(\vec{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m).$$

Also

$$I_{\vec{b},\alpha}^{j}(\vec{f})(x) = b_j(x) I_\alpha(f_1, \dots, f_j, \dots, f_m)(x) - I_\alpha(f_1, \dots, b_j f_j, \dots, f_m)(x).$$

Recently, the following iterated commutators of multilinear Calderón–Zygmund operators and pointwise multiplication with BMO functions were defined and studied in products of Lebesgue spaces, including strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights [25]:

$$T_{\Pi\mathbf{b}}(\vec{f})(x) = [b_1, [b_2, \dots [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2]_1(\vec{f})(x). \tag{1.3}$$

To clarify the notation, if T is associated with a Calderón–Zygmund kernel K in the usual way, then

$$T_{\Pi\mathbf{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) K(x, y_1, \dots, y_m) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y}.$$

Here, we use the notation $\Pi\mathbf{b}$ since the commutator part in the above integrand is a product of $b_j(x) - b_j(y_j)$.

Therefore, an open interesting question arises: can we establish weighted strong and end-point estimates of iterated commutators for the multilinear operator T_* and I_α? We have found no results in the literature for the commutators of the multilinear operator T_* ($m \geq 2$), even for the commutators of T_* in the jth entry.

In this article, we give a positive answer to the above question. We study iterated commutators of maximal multilinear singular integral operators and multilinear fractional integral operators defined by

$$\begin{aligned} T_{*,\Pi\mathbf{b}}(\vec{f})(x) &= \sup_{\delta>0} \big|[b_1, [b_2, \dots [b_{m-1}, [b_m, T_\delta]_m]_{m-1} \cdots]_2]_1(\vec{f})(x)\big| \\ &= \sup_{\delta>0} \Big| \int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) K(x, y_1, \dots, y_m) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y} \Big| \end{aligned} \tag{1.4}$$

and

$$I_{\alpha,\Pi\mathbf{b}}(\vec{f})(x) = [b_1,[b_2,\dots[b_{m-1},[b_m,I_\alpha]_m]_{m-1}\cdots]_2]_1(\vec{f})(x) \tag{1.5}$$
$$= \int_{(\mathbb{R}^n)^m} \frac{1}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} \prod_{j=1}^m (b_j(x)-b_j(y_j)) \prod_{i=1}^m f_i(y_i)\,d\vec{y}.$$

Remark 1.1. Note that, when $m=1$ in (1.3) and (1.5), $T_{\Pi\mathbf{b}}(\vec{f})$ and $I_{\alpha,\Pi\mathbf{b}}(\vec{f})$ coincide with the classical linear commutators $[b,T]f = bT(f) - T(bf)$ and $[b,I_\alpha]f = bI_\alpha(f) - I_\alpha(bf)$. A classical result by Coifman, Rochberg and Weiss [7] is that $[b,T]$ is L^p bounded for $1<p<\infty$ when $b\in \mathrm{BMO}$. But $[b,T]$ fails to be an operator of weak type $(1,1)$: a counterexample was given by C. Pérez [24] and an alternative $L(\log L)$ type result was obtained. In 1982, Chanillo proved that the commutator of the fractional integral operator $[b,I_\alpha]$ is bounded from L^p into L^q ($p>1, 1/q = 1/p - a/n$) when $b\in\mathrm{BMO}$. In 2002, Ding, Lu and Zhang [9] studied the continuity properties of fractional type operators. They showed that $[b,I_\alpha]$ can fail to be an operator of weak type $(L^1, L^{n/(n-\alpha),\infty})$, giving counterexamples and proving alternative $L(\log L)$ type estimates.

We now state our results:

Theorem 1.1 (Weighted strong bounds for $T_{*,\Pi\mathbf{b}}$). *Let $\vec{\omega}\in A_{\vec{p}}$, $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$ (see Definition 2.1), $1/p = 1/p_1+\cdots+1/p_m$ with $1<p_j<\infty$, $j=1,\dots,m$, and $\vec{b}\in\mathrm{BMO}^m$. Then there is a constant $C>0$ independent of $\vec{b}$ and $\vec{f}$ such that for all $\vec{f}$ in any product of $L^{p_j}_{\omega_j}(\mathbb{R}^n)$ spaces,*

$$\|T_{*,\Pi\mathbf{b}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}, \tag{1.6}$$

where $\vec{b} = (b_1,\dots,b_m)$.

Theorem 1.2 (Weighted end-point estimate for $T_{*,\Pi\mathbf{b}}$). *Let $\vec{\omega}\in A_{(1,\dots,1)}$, $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$ and $\vec{b}\in\mathrm{BMO}^m$. Then there exists a constant C depending on $\vec{b}$ but independent of $\vec{f}$ such that*

(1.7)
$$\nu_{\vec{\omega}}(\{x\in\mathbb{R}^n : T_{*,\Pi\mathbf{b}}(\vec{f})(x) > t^m\}) \le C\Big(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}(|f_i(y_i)|/t)\omega_i(y_i)\,dy_i\Big)^{1/m},$$

where $\Phi(t) = t(1+\log^+ t)$ and $\Phi^{(m)} = \overbrace{\Phi\circ\cdots\circ\Phi}^{m}$.

Remark 1.2. If $m=1$, then weighted strong L^p and weighted end-point $L(\log L)$ estimates for commutators of the classical linear operator T_* were studied in [27].

As for $I_{\alpha,\Pi\mathbf{b}}$, we get

THEOREM 1.3 (Weighted strong bounds for $I_{\alpha,\Pi\mathbf{b}}$). *Let $0 < \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \alpha/n$. For $r > 1$ with $0 < r\alpha < mn$, if $\vec{\omega}^r \in A_{(\vec{p}/r,q/r)}$, $\widetilde{\nu}_{\vec{\omega}}^q \in A_\infty$ and $\vec{b} \in \mathrm{BMO}^m$, where $\widetilde{\nu_{\vec{\omega}}} = \prod_{i=1}^m \omega_i$, there is a constant $C > 0$ independent of $\vec{b}$ and $\vec{f}$ such that*

$$\|I_{\alpha,\Pi\mathbf{b}}(\vec{f})\|_{L^q(\widetilde{\nu}_{\vec{\omega}}^q)} \le C \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \tag{1.8}$$

THEOREM 1.4 (Weighted end-point estimate for $I_{\alpha,\Pi\mathbf{b}}$). *Let $0 < \alpha < mn$, $\vec{\omega} \in A_{((1,\dots,1),n/(mn-\alpha))}$, $\widetilde{\nu}_{\vec{\omega}} = \prod_{i=1}^m \omega_i$ and $\vec{b} \in \mathrm{BMO}^m$. Then there exists a constant C depending on $\vec{b}$ but independent of $\vec{f}$ such that*

$$\begin{aligned} \widetilde{\nu}_{\vec{\omega}}^{\frac{n}{mn-\alpha}}(\{x \in \mathbb{R}^n : I_{\alpha,\Pi\mathbf{b}}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}}\}) &\le C\bigg\{\bigg[1 + \frac{\alpha}{mn}\log^+\bigg(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}(|f_i(y_i)|/t)\,dy_i\bigg)\bigg]^m \\ &\quad\times \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}(|f_j(y_j)|/t)\omega_j(y_j)\,dy_j\bigg\}^{\frac{n}{mn-\alpha}}. \end{aligned} \tag{1.9}$$

Moreover, if $0 < \alpha_j < n$ for all j and $\alpha = \sum_{j=1}^m$, we obtain

$$\begin{aligned} \widetilde{\nu}_{\vec{\omega}}^{\frac{n}{mn-\alpha}}(\{x \in \mathbb{R}^n : I_{\alpha,\Pi\mathbf{b}}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}}\}) &\le C\bigg\{\prod_{j=1}^m \bigg[1 + \frac{\alpha_j}{n}\log^+\bigg(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}(|f_i(y_i)|/t)\,dy_i\bigg)\bigg] \\ &\quad\times \int_{\mathbb{R}^n} \Phi^{(m)}(|f_j(y_j)|/t)\omega_j(y_j)\,dy_j\bigg\}^{\frac{n}{mn-\alpha}}, \end{aligned} \tag{1.10}$$

where $\Phi(t)$ and $\Phi^{(m)}$ are as in Theorem 1.2.

As a corollary of Theorems 1.3 and 1.4, similar results can be obtained for commutators of the multilinear fractional maximal operator. Let us first give its definition. Suppose each f_i $(i = 1, \dots, m)$ is locally integrable on $\mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, we define the multilinear fractional maximal operator and its commutator by

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_Q |Q|^{\alpha/n} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)|\,dy_i$$

and

$$\mathcal{M}_{\alpha,\Pi\mathbf{b}}(\vec{f})(x)=\sup_{Q}|Q|^{\alpha/n}\prod_{i=1}^{m}\frac{1}{|Q|}\int_{Q}|b_i(x)-b_i(y_i)|\,|f_i(y_i)|\,dy_i,$$

where the supremum is taken over all cubes Q containing x in $\mathbb{R}^n$ with sides parallel to the axes.

COROLLARY 1.1. *Let α, b_i, $\vec{\omega}$, p_i, q be as in Theorems* 1.3 *and* 1.4. *Then the conclusions of Theorems* 1.3 *and* 1.4 *still hold for $\mathcal{M}_{\alpha,\Pi\mathbf{b}}$.*

The article is organized as follows. In Section 2, we prepare some definitions and lemmas. Some propositions will be listed and proved in Section 3, including the main Proposition 3.1. Then, we give the proof of Theorems 1.1–1.3. Section 4 will be devoted to the study of end-point $L(\log L)$ type estimates for iterated commutators of multilinear fractional type operators.

2. Definitions and some lemmas. Let us recall the definitions of $A_{\vec{p}}$ and $A_{(\vec{p},q)}$ weights. For m exponents $p_1,\dots,p_m$, we will often write p for the number given by $1/p=1/p_1+\cdots+1/p_m$, and $\vec{p}$ for the vector $(p_1,\dots,p_m)$.

DEFINITION 2.1 (Multiple $A_{\vec{p}}$ weights [19]). Let $1\le p_1,\dots,p_m<\infty$. Given $\vec{\omega}=(\omega_1,\dots,\omega_m)$, where each ω_i $(i=1,\dots,m)$ is a nonnegative function on $\mathbb{R}^n$, set

$$\nu_{\vec{\omega}}=\prod_{j=1}^{m}\omega_j^{p/p_j}.$$

We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ *condition* if

$$\sup_{Q}\bigg(\frac{1}{|Q|}\int_{Q}\nu_{\vec{\omega}}\bigg)^{1/p}\prod_{i=1}^{m}\bigg(\frac{1}{|Q|}\int_{Q}\omega_i^{1-p_i'}\bigg)^{1/p_i'}<\infty,\tag{2.1}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

When $p_i=1$, $\big(\frac{1}{|Q|}\int_Q\omega_i^{1-p_i'}\big)^{1/p_i'}$ is understood as $(\inf_Q\omega_i)^{-1}$.

DEFINITION 2.2 (Multiple $A_{(\vec{p},q)}$ weights [3], [22]). Let $1\le p_1,\dots,p_m<\infty$, $1/p=1/p_1+\cdots+1/p_m$, and $q>0$. Suppose that $\vec{\omega}=(\omega_1,\dots,\omega_m)$ and each ω_i $(i=1,\dots,m)$ is a nonnegative function on $\mathbb{R}^n$. We say that $\vec{\omega}\in A_{(\vec{p},q)}$ if

$$\sup_{Q}\bigg(\frac{1}{|Q|}\int_{Q}\widetilde{\nu}_{\vec{\omega}}^{\,q}\bigg)^{1/q}\prod_{i=1}^{m}\bigg(\frac{1}{|Q|}\int_{Q}\omega_i^{-p_i'}\bigg)^{1/p_i'}<\infty,\tag{2.2}$$

where $\widetilde{\nu}_{\vec{\omega}}=\prod_{i=1}^{m}\omega_i$ and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

If $p_i=1$, $\big(\frac{1}{|Q|}\int_Q\omega_i^{-p_i'}\big)^{1/p_i'}$ is understood as $(\inf_Q\omega_i)^{-1}$.

Remark 2.1. In particular, when $m = 1$, $A_{\vec{p}}$ reduces to the classical A_p weights. Moreover, if $m = 1$ and $p_i = 1$, then this class of weights coincides with the classical A_1 weights. Also, when $m = 1$, $A_{(\vec{p},q)}$ reduces to the classical $A_{(p,q)}$ weights, defined in 1974 by B. Muckenhoupt and R. Wheeden [23]. We will refer to (2.1) and (2.2) as the *multilinear* $A_{\vec{p}}$ *condition* and $A_{(\vec{p},q)}$ *condition*, respectively.

We need the following $L(\log L)$ type multilinear maximal fractional operators and sharp maximal functions:

Definition 2.3. For any $\vec{f} = (f_1, \dots, f_m)$ and $0 < \alpha < mn$, two multilinear fractional $L(\log L)$ type maximal operators are defined by

$$\mathcal{M}^j_{L(\log L),\alpha}(\vec{f})(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \|f_j\|_{L(\log L),Q} \prod_{i \neq j} \frac{1}{|Q|} \int_Q |f_i|$$

and

$$\mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \prod_{i=1}^{m} \|f_i\|_{L(\log L),Q},$$

respectively. If $\alpha = 0$, we simply denote $\mathcal{M}_{L(\log L),0} = \mathcal{M}_{L(\log L)}$ and $\mathcal{M}^j_{L(\log L),0} = \mathcal{M}^j_{L(\log L)}$.

Definition 2.4 (Sharp maximal functions). For $\delta > 0$, M_δ is the maximal function

$$M_\delta f(x) = (M(|f|^\delta))^{1/\delta}(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(y)|^\delta \, dy \right)^{1/\delta}$$

where Q is a cube containing x with sides parallel to the coordinate axes; in addition, $M^\sharp$ is the sharp maximal function of Fefferman and Stein [10],

$$M^\sharp f(x) = \sup_{x \in Q} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| \, dy \approx \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy$$

and

$$M^\sharp_\delta f(x) = M^\sharp(|f|^\delta)^{1/\delta}(x),$$

where f_Q is the average of f over Q.

We prepare some lemmas which will be used later. The following Hölder inequality on Orlicz spaces can be found in [26, p. 58].

Lemma 2.1 (Generalized Hölder inequality [26]). *Let* $\phi(t) = t(1+\log^+ t)$, $\psi(t) = e^t - 1$ *and suppose that*

$$\|f\|_\phi \triangleq \inf\Big\{\lambda>0:\int_{\mathbb{R}^n}\phi(|f(x)|/\lambda)\,d\mu\le 1\Big\}<\infty,$$
$$\|g\|_\psi \triangleq \inf\Big\{\lambda>0:\int_{\mathbb{R}^n}\psi(|g(x)|/\lambda)\,d\mu\le 1\Big\}<\infty$$

with respect to some measure μ. Then for any cube Q,

$$\frac{1}{|Q|}\int_Q|fg|\le 2\|f\|_{L(\log L),Q}\|g\|_{\exp L,Q}. \tag{2.3}$$

Some other inequalities are also necessary.

LEMMA 2.2 ([3]). *Let $r>1$ and $b\in \mathrm{BMO}$. Then there is a constant $C>0$ independent of b such that for any f satisfying the assumption of the generalized Hölder inequality, the following inequalities hold:*

$$\frac{1}{|Q|}\int_Q|f|\le C\|f\|_{L(\log L),Q}, \tag{2.4}$$

$$\|f\|_{L(\log L),Q}\le C\Big(\frac{1}{|Q|}\int_Q|f|^r\Big)^{1/r}, \tag{2.5}$$

$$\frac{1}{|Q|}\int_Q|(b-b_Q)f|\le C\|b\|_{\mathrm{BMO}}\|f\|_{L(\log L),Q}, \tag{2.6}$$

$$\Big(\sup_Q\frac{1}{|Q|}\int_Q|b-b_Q|^{r-1}\Big)^{1/(r-1)}\le C\|b\|_{\mathrm{BMO}}. \tag{2.7}$$

We also need Kolmogorov's inequalities:

LEMMA 2.3 (Kolmogorov's inequality [19], [11, p. 485]).

(a) *Suppose $0<p<q<\infty$. Then*

$$\|f\|_{L^p(Q,dx/|Q|)}\le C\|f\|_{L^{q,\infty}(Q,dx/|Q|)}. \tag{2.8}$$

(b) *Suppose that $0<\alpha<n$ and $p,q>0$ satisfy $1/q=1/p-\alpha/n$. Then for any measurable function f and cube Q,*

$$\Big(\int_Q|f|^p\Big)^{1/p}\le\Big(\frac{q}{q-p}\Big)^{1/p}|Q|^{\alpha/n}\|f\|_{L^{q,\infty}(Q)}. \tag{2.9}$$

To prove Theorem 1.4, we also need the following known results,

LEMMA 2.4 (Weighted estimates for $\mathcal{M}_\alpha$ and I_α [22], [3]). *Let $0<\alpha<mn$, $1\le p_1,\dots,p_m<\infty$, $1/p=1/p_1+\dots+1/p_m$, $1/q=1/p-\alpha/n$ and*

$\vec{\omega} \in A_{(\vec{p},q)}$. *Then there is a constant* $C > 0$ *independent of* $\vec{f}$ *such that*

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}, \tag{2.10}$$

$$\|I_\alpha(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \tag{2.11}$$

3. Proof of Theorems 1.1–1.3. To begin with, we prepare a proposition which plays an important role in the proof of our theorems. The basic idea is to control the iterated commutators of T_* by certain two operators.

Let $u, v \in C^\infty([0,\infty))$ be such that $|u'(t)| \le Ct^{-1}$, $|v'(t)| \le Ct^{-1}$ and

$$\chi_{[2,\infty)}(t) \le u(t) \le \chi_{[1,\infty)}(t), \qquad \chi_{[1,2]}(t) \le v(t) \le \chi_{[1/2,3]}(t).$$

We define the maximal operators

$$U^*(\vec{f})(x) = \sup_{\eta>0} \Big| \int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m) u(\sqrt{|x-y_1|+\cdots+|x-y_m|}/\eta) \prod_{i=1}^m f_i(y_i)\, d\vec{y} \Big|,$$

$$V^*(\vec{f})(x) = \sup_{\eta>0} \Big| \int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m) v(\sqrt{|x-y_1|+\cdots+|x-y_m|}/\eta) \prod_{i=1}^m f_i(y_i)\, d\vec{y} \Big|.$$

For simplicity, we denote

$$K_{u,\eta}(x,y_1,\ldots,y_m) = K(x,y_1,\ldots,y_m) u(\sqrt{|x-y_1|+\cdots+|x-y_m|}/\eta),$$
$$K_{v,\eta}(x,y_1,\ldots,y_m) = K(x,y_1,\ldots,y_m) v(\sqrt{|x-y_1|+\cdots+|x-y_m|}/\eta),$$
$$U_\eta(\vec{f}) = \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x,y_1,\ldots,y_m) \prod_{i=1}^m f_i(y_i)\, d\vec{y},$$
$$V_\eta(\vec{f}) = \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x,y_1,\ldots,y_m) \prod_{i=1}^m f_i(y_i)\, d\vec{y}.$$

It is easy to see that $T_*(\vec{f}) \le U^*(\vec{f})(x) + V^*(\vec{f})(x)$. Moreover, $T_{*,\Pi\mathbf{b}}(\vec{f}) \le U^*_{\Pi\mathbf{b}}(\vec{f})(x) + V^*_{\Pi\mathbf{b}}(\vec{f})(x)$, where

$$\begin{aligned} U^*_{\Pi\mathbf{b}}(\vec{f})(x) &= \sup_{\eta>0} \big|[b_1,[b_2,\ldots[b_{m-1},[b_m,U_\eta]_m]_{m-1}\cdots]_2]_1(\vec{f})(x)\big| \\ &= \sup_{\eta>0} \Big| \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x,y_1,\ldots,y_m) \prod_{j=1}^m (b_j(x)-b_j(y_j)) \prod_{i=1}^m f_i(y_i)\, d\vec{y} \Big| \end{aligned}$$

and

$$V^*_{\Pi\mathbf{b}}(\vec{f})(x) = \sup_{\eta>0} \big|[b_1, [b_2, \dots [b_{m-1}, [b_m, V_\eta]_m]_{m-1} \cdots]_2]_1(\vec{f})(x)\big|$$
$$= \sup_{\eta>0} \Big| \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x, y_1, \dots, y_m) \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y} \Big|.$$

Following [25], for positive integers m and j with $1 \le j \le m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements, where always $\sigma(k) < \sigma(j)$ if $k < j$. With any $\sigma \in C^m_j$, we associate the complementary sequence $\sigma' \in C^{m-j}_j$ given by $\sigma' = \{1, \dots, m\} \backslash \sigma$ with the convention $C^m_0 = \emptyset$. Given an m-tuple $\mathbf{b}$ of functions and $\sigma \in C^m_j$, we also use the notation $\mathbf{b}_\sigma$ for the j-tuple obtained from $\mathbf{b}$ and given by $(b_{\sigma(1)}, \dots, b_{\sigma(j)})$.

Similarly to the above definitions for $U^*_{\Pi\mathbf{b}}(\vec{f})(x)$ and $V^*_{\Pi\mathbf{b}}(\vec{f})(x)$, if $\sigma \in C^m_j$, and $\mathbf{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ in BMO^j, we define the following commutators:

$$U^*_{\Pi\mathbf{b}_\sigma}(\vec{f})(x)$$
$$= \sup_{\eta>0} \Big| \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x, y_1, \dots, y_m) \prod_{i=1}^{j} (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y} \Big|,$$

$$V^*_{\Pi\mathbf{b}_\sigma}(\vec{f})(x)$$
$$= \sup_{\eta>0} \Big| \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x, y_1, \dots, y_m) \prod_{i=1}^{j} (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y} \Big|,$$

$$I_{\alpha,\Pi\mathbf{b}_\sigma}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^{j} (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \prod_{i=1}^{m} f_i(y_i)}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}}\, d\vec{y}.$$

If $\sigma = \{j\}$, it is easy to see that $U^*_{\Pi\mathbf{b}_\sigma}(\vec{f}) = U^*_{b_j}(\vec{f})$, $V^*_{\Pi\mathbf{b}_\sigma}(\vec{f}) = V^*_{b_j}(\vec{f})$ and $I_{\alpha,\Pi\mathbf{b}_\sigma}(\vec{f}) = I^j_{\vec{b},\alpha}(\vec{f})$. If $\sigma = \{1, \dots, m\}$, then $U^*_{\Pi\mathbf{b}_\sigma}(\vec{f}) = U^*_{\Pi\mathbf{b}}(\vec{f})$, $V^*_{\Pi\mathbf{b}_\sigma}(\vec{f}) = V^*_{\Pi\mathbf{b}}(\vec{f})$ and $I_{\alpha,\Pi\mathbf{b}_\sigma}(\vec{f}) = I_{\alpha,\Pi\mathbf{b}}(\vec{f})$.

Proposition 3.1 (Pointwise control of $M^\sharp_\delta(U^*_{\Pi\mathbf{b}}(\vec{f})), M^\sharp_\delta(V^*_{\Pi\mathbf{b}}(\vec{f}))$, $M^\sharp_\delta(I_{\alpha,\Pi\mathbf{b}}(\vec{f}))$). *Let $0 < \delta < \varepsilon$, $0 < \delta < 1/m$ and $0 < \alpha < mn$. Then there is a constant $C > 0$ depending on δ and ε such that*

$$\text{(3.1)}\quad M^\sharp_\delta(U^*_{\Pi\mathbf{b}}(\vec{f}))(x) \le C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \big(\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(U^*(\vec{f}))(x)\big)$$
$$+ C \sum_{j=1}^{m-1} \sum_{\sigma \in C^m_j} \prod_{i=1}^{j} \|b_{\sigma(i)}\|_{\mathrm{BMO}} M_\varepsilon(U^*_{\Pi\mathbf{b}_{\sigma'}}(\vec{f}))(x),$$

$$M_\delta^\sharp(V^*_{\Pi\mathbf{b}}(\vec{f}))(x) \le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}}\big(\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(V^*(\vec{f}))(x)\big) + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i=1}^j \|b_{\sigma(i)}\|_{\mathrm{BMO}} M_\varepsilon(V^*_{\Pi\mathbf{b}_{\sigma'}}(\vec{f}))(x), \tag{3.2}$$

$$\begin{aligned} M_\delta^\sharp(I_{\alpha,\Pi\mathbf{b}}(\vec{f}))(x) &\le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}}\big(\mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) + M_\varepsilon(I_\alpha(\vec{f}))(x)\big) \\ &\quad + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i=1}^j \|b_{\sigma(i)}\|_{\mathrm{BMO}} M_\varepsilon(I_{\alpha,\Pi\mathbf{b}_{\sigma'}}(\vec{f}))(x). \end{aligned}\tag{3.3}$$

Inequality (3.3) *still holds for* $\delta = 1/m$.

Proof. We only give the proof for $U^*_{\Pi\mathbf{b}}(\vec{f})$ and $I_{\alpha,\Pi\mathbf{b}}(\vec{f})$; the proof for $V^*_{\Pi\mathbf{b}}(\vec{f})$ is almost the same as that for $U^*_{\Pi\mathbf{b}}(\vec{f})$.

For clarity of exposition, we give the proof in the case $m = 2$. The argument can be extended to the case $m > 2$ with only trivial modifications.

We first give the proof for $U^*_{\Pi\mathbf{b}}(\vec{f})$.

Fix $b_1, b_2 \in \mathrm{BMO}$ and let $\rho_1,\ \rho_2$ denote any constants. We split $U^*_{\Pi\mathbf{b}}(\vec{f})(x)$ in the following way:

$$\begin{aligned} &U^*_{\Pi\mathbf{b}}(\vec{f})(x) \\ &\quad = \sup_{\eta>0}\big|(b_1(x)-\rho_1)(b_2(x)-\rho_2)U_\eta(\vec{f})(x) - (b_1(x)-\rho_1)U_\eta(f_1,(b_2-\rho_2)f_2)(x) \\ &\qquad - (b_2(x)-\rho_2)U_\eta((b_1-\rho_1)f_1,f_2)(x) + U_\eta((b_1-\rho_1)f_1,(b_2-\rho_2)f_2)(x)\big| \\ &\quad = \sup_{\eta>0}\big|-(b_1(x)-\rho_1)(b_2(x)-\rho_2)U_\eta(\vec{f})(x) + (b_1(x)-\rho_1)U^2_{\eta,b_2-\rho_2}(f_1,f_2)(x) \\ &\qquad + (b_2(x)-\rho_2)U^1_{\eta,b_1-\rho_1}(f_1,f_2)(x) + U_\eta((b_1-\rho_1)f_1,(b_2-\rho_2)f_2)(x)\big|. \end{aligned}$$

Here, we use the notations

$$\begin{aligned} U^1_{\eta,b_1-\rho_1}(f_1,f_2)(x) &= U_\eta((b_1-\rho_1)f_1,f_2)(x), \\ U^2_{\eta,b_2-\rho_2}(f_1,f_2)(x) &= U_\eta(f_1,(b_2-\rho_2)f_2)(x). \end{aligned}$$

Similar notations will be used throughout the remainder of this paper.

Fix $x_0 \in \mathbb{R}^n$ and let Q be a cube centered at x_0. Notice that $0<\delta<1/m$. Let $c = \sup_\eta|\sum_{j=1}^3 c_j|$, with $c_j = c_j(\eta)$ to be defined later. Then we have

$$\bigg(\frac{1}{|Q|}\int_Q \big|[U^*_{\Pi\mathbf{b}}(\vec{f})(z)]^\delta - c^\delta\big|\,dz\bigg)^{1/\delta} \le C(T_1+T_2+T_3+T_4),$$

where

$$T_1 = \left(\frac{1}{|Q|} \int_Q |(b_1(z) - \rho_1)(b_2(z) - \rho_2)|^\delta [U^*(\vec{f})(z)]^\delta \, dz \right)^{1/\delta},$$

$$T_2 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |(b_1(z) - \rho_1)[U^2_{\eta, b_2-\rho_2}(f_1, f_2)(z)]|^\delta \, dz \right)^{1/\delta},$$

$$T_3 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |(b_2(z) - \rho_2)[U^1_{\eta, b_1-\rho_1}(f_1, f_2)(z)]|^\delta \, dz \right)^{1/\delta},$$

$$T_4 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(z) - \sum_{j=1}^{3} c_j \right|^\delta dz \right)^{1/\delta}.$$

Let $\rho_j = (b_j)_{3Q}$ be the average of b_j on $3Q$ for $j = 1, 2$.

For any $1 < r_1, r_2, r_3 < \infty$ with $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $r_3 < \varepsilon/\delta$, T_1 can be estimated by using Hölder's inequality and (2.7):

$$\begin{aligned} T_1 \le C &\left(\frac{1}{|Q|} \int_Q |b_1(z) - \rho_1|^{\delta r_1} \, dz \right)^{\frac{1}{\delta r_1}} \left(\frac{1}{|Q|} \int_Q |b_2(z) - \rho_2|^{\delta r_2} \, dz \right)^{\frac{1}{\delta r_2}} \\ &\times \left(\frac{1}{|Q|} \int_Q [U^*(\vec{f})(z)]^{\delta r_3} \, dz \right)^{\frac{1}{\delta r_3}} \\ &\le C \prod_{j=1}^{2} \|b_j\|_{\mathrm{BMO}} M_\varepsilon(U^*(\vec{f}))(x_0). \end{aligned}$$

Let $1 < t_1, t_2 < \infty$ with $1 = 1/t_1 + 1/t_2$ and $t_2 < \varepsilon/\delta$. Then T_2 can be estimated by using Hölder's inequality and Jensen's inequalities:

$$\begin{aligned} T_2 \le C &\left(\frac{1}{|Q|} \int_Q |b_1(z) - \rho_1|^{\delta t_1} \, dz \right)^{\frac{1}{\delta t_1}} \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U^2_{\eta, b_2-\rho_2}(f_1, f_2)(z)|^{\delta t_2} \, dz \right)^{\frac{1}{\delta t_2}} \\ &\le C\|b_1\|_{\mathrm{BMO}} M_\varepsilon(U^{*,2}_{b_2-\rho_2}(\vec{f}))(x_0) \le C\|b_1\|_{\mathrm{BMO}} M_\varepsilon(U^{*,2}_{b_2}(\vec{f}))(x_0). \end{aligned}$$

Since T_2 and T_3 are symmetric, similarly, we have

$$T_3 \le C\|b_2\|_{\mathrm{BMO}} M_\varepsilon(U^*_{b_1-\rho_1}(\vec{f}))(x_0) \le C\|b_2\|_{\mathrm{BMO}} M_\varepsilon(U^{*,1}_{b_1}(\vec{f}))(x_0).$$

Now, we are in a position to estimate T_4. Let $f_i^0 = f_i \chi_{3Q}$ and $f_i^\infty = f_i - f_i^0$. Recall that $c = \sup_\eta |\sum_{j=1}^3 c_j|$. Then we take

$$\begin{aligned} c_1 &= U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0), \\ c_2 &= U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0), \\ c_3 &= U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0). \end{aligned}$$

Using the above notations, we may split T_4 as

$$T_4 \le T_{4,1} + T_{4,2} + T_{4,3} + T_{4,4},$$

where

$$T_{4,1} = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^0)(z)|^\delta \, dz \right)^{1/\delta},$$

$$T_{4,2} = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^\infty)(z) - c_1|^\delta \, dz \right)^{1/\delta},$$

$$T_{4,3} = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1-\rho_1)f_1^\infty, (b_2-\rho_2)f_2^0)(z) - c_2|^\delta \, dz \right)^{1/\delta}$$

$$T_{4,4} = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1-\rho_1)f_1^\infty, (b_2-\rho_2)f_2^\infty)(z) - c_3|^\delta \, dz \right)^{1/\delta}.$$

First, we shall estimate $T_{4,1}$. Use Kolmogorov's inequality, Lemma 2.2(a), Theorem B with $w_i \equiv 1$ for $m = 2$ and (2.6), it now follows that

$$\begin{aligned} T_{4,1} &\le C\left(\frac{1}{|Q|} \int_Q [U^*((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^0)(z)]^{p_0\delta} \, dz \right)^{1/p_0\delta} \\ &\le C|Q|^{-2} \|U^*((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^0)\|_{L^{1/2,\infty}(Q)} \\ &\le C|Q|^{-2} \|(b_1-\rho_1)\|f_1^0\|_{L^1(Q)} \|(b_2-\rho_2)f_2^0\|_{L^1(Q)} \\ &\le C\|b_1\|_{\mathrm{BMO}} \|f_1^0\|_{L(\log L)} \|b_2\|_{\mathrm{BMO}} \|f_2^0\|_{L(\log L)} \\ &\le C \prod_{i=1}^m \|b_i\|_{\mathrm{BMO}} \mathcal{M}_{L(\log L)}(\vec{f})(x_0). \end{aligned}$$

Secondly, we estimate $T_{4,2}$. By the mean value theorem,

$$\begin{aligned} T_{4,2} &\le \frac{C}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^\infty)(z) - c_1| \, dz \\ &\le C \frac{1}{|Q|} \int_Q \int_{3Q} |(b_1-\rho_1)f_1(y_1)| \, dy_1 \int_{(3Q)^c} \frac{|x_0-z|^\varepsilon |b_2(y_2)-\rho_2| \, |f_2(y_2)| dy_2}{(|z-y_1|+|z-y_2|)^{2n+\varepsilon}} \, dz \\ &\le C \sum_{j=1}^\infty \frac{j|Q|^{\varepsilon/n}}{(3^j|Q|^{1/n})^{2n+\varepsilon}} \int_{3^{j+1}Q} |(b_1-\rho_1)f_1(y_1)| \, dy_1 \\ &\quad \times \int_{3^{j+1}Q} |b_2(y_2)-\rho_2||f_2(y_2)| \, dy_2 \\ &\le C \sum_{j=1}^\infty \frac{1}{3^{j\varepsilon}} \prod_{i=1}^2 \|b_i\|_{\mathrm{BMO}} \|f_i\|_{L(\log L), 3^{j+1}Q} \le C \prod_{i=1}^2 \|b_i\|_{\mathrm{BMO}} \mathcal{M}_{L(\log L)}(\vec{f})(x_0). \end{aligned}$$

Similarly to $T_{4,2}$, we can estimate $T_{4,3}$.

It remains to discuss the contribution of $T_{4,4}$. Note that

$$\begin{aligned}
&\big|U_\eta((b_1-\rho_1)f_1^\infty,(b_2-\rho_2)f_2^\infty)(z)-U_\eta((b_1-\rho_1)f_1^\infty,(b_2-\rho_2)f_2^\infty)(x_0)\big| \\
&\qquad\le C\int\limits_{(\mathbb{R}^n\setminus 3Q)^2}\frac{|Q|^{\varepsilon/n}|b_1-\rho_1|\,|b_2(y_2)-\rho_2|}{|(x_0-y_1,x_0-y_2)|^{2n+\varepsilon}}\prod_{i=1}^{2}|f_i^\infty(z_i)|\,d\vec{y} \\
&\qquad\le C\sum_{k=1}^{\infty}\int\limits_{(3^{k+1}Q)^2\setminus(3^kQ)^2}\frac{|Q|^{\varepsilon/n}|b_1-\rho_1|\,|b_2(y_2)-\rho_2|}{(3^k|Q|^{1/n})^{2n+\varepsilon}}\prod_{i=1}^{2}|f_i^\infty(y_i)|\,d\vec{y} \\
&\qquad\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(\vec{f}\,)(x_0).
\end{aligned}$$

Therefore,

$$T_{4,4}\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(\vec{f}\,)(x_0).$$

The proof for $U^*_{\Pi\mathbf{b}}(\vec{f})$ is complete.

Next, we will prove (3.3) for $I_{\alpha,\Pi\mathbf{b}}(\vec{f}\,)$. To begin with, we split

$$\begin{aligned}
&I_{\alpha,\Pi\mathbf{b}}(\vec{f}\,)(x) \\
&\quad=(b_1(x)-\rho_1)(b_2(x)-\rho_2)I_\alpha(\vec{f}\,)(x)-(b_1(x)-\rho_1)I_\alpha(f_1,(b_2-\rho_2)f_2)(x) \\
&\qquad-(b_2(x)-\rho_2)I_\alpha((b_1-\rho_1)f_1,f_2)(x)+I_\alpha((b_1-\rho_1)f_1,(b_2-\rho_2)f_2)(x) \\
&\quad=-(b_1(x)-\rho_1)(b_2(x)-\rho_2)I_\alpha(\vec{f}\,)(x)+(b_1(x)-\rho_1)I^2_{b_2-\rho_2,\alpha}(f_1,f_2)(x) \\
&\qquad+(b_2(x)-\rho_2)I^1_{b_1-\rho_1,\alpha}(f_1,f_2)(x)+I_\alpha((b_1-\rho_1)f_1,(b_2-\rho_2)f_2)(x).
\end{aligned}$$

Fix $x_0\in\mathbb{R}^n$ and let Q be a cube centered at x_0. Let $\tilde{c}_j$ (for $j=1,2,3$) denote any constants and set $\tilde{c}:=\tilde{c}_1+\tilde{c}_2+\tilde{c}_3$. Since $0<\delta\le 1/m$, we have

$$\left(\frac{1}{|Q|}\int\limits_Q\big||I_{\alpha,\Pi\mathbf{b}}(\vec{f}\,)(z)|^\delta-|\tilde{c}|^\delta\big|\,dz\right)^{1/\delta}\le C(S_1+S_2+S_3+S_4),$$

where

$$S_1=\left(\frac{1}{|Q|}\int\limits_Q|(b_1(z)-\rho_1)(b_2(z)-\rho_2)|^\delta|I_\alpha(\vec{f}\,)(z)|^\delta\,dz\right)^{1/\delta},$$

$$S_2=\left(\frac{1}{|Q|}\int\limits_Q|(b_1(x)-\rho_1)I^2_{b_2-\rho_2,\alpha}(f_1,f_2)(z)|^\delta\,dz\right)^{1/\delta}.$$

$$S_3=\left(\frac{1}{|Q|}\int\limits_Q|(b_2(x)-\rho_2)I^1_{b_1-\rho_1,\alpha}(f_1,f_2)(z)|^\delta\,dz\right)^{1/\delta}$$

$$S_4=\left(\frac{1}{|Q|}\int\limits_Q|I_\alpha((b_1-\rho_1)f_1,(b_2-\rho_2)f_2)(z)-\tilde{c}|^\delta\,dz\right)^{1/\delta}.$$

Let $\rho_j=(b_j)_{3Q}$ be the average of b_j on $3Q$ for $j=1,2$.

For any $1 < r_1, r_2, r_3 < \infty$ with $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $r_3 < \varepsilon/\delta$, S_1 can be estimated by using Hölder's inequality and (2.7):

$$\begin{aligned} S_1 &\le C\bigg(\frac{1}{|Q|}\int_Q |b_1(z)-\rho_1|^{\delta r_1}\,dz\bigg)^{\frac{1}{\delta r_1}}\bigg(\frac{1}{|Q|}\int_Q |b_2(z)-\rho_2|^{\delta r_2}\,dz\bigg)^{\frac{1}{\delta r_2}} \\ &\quad\times\bigg(\frac{1}{|Q|}\int_Q |I_\alpha(\vec{f})(z)|^{\delta r_3}\,dz\bigg)^{\frac{1}{\delta r_3}} \\ &\le C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}} M_\varepsilon(I_\alpha(\vec{f}))(x_0). \end{aligned}$$

As already noted, similarly to the argument for T_2, we take $1 < t_1, t_2 < \infty$ with $1 = 1/t_1 + 1/t_2$ and $t_2 < \varepsilon/\delta$. Then

$$\begin{aligned} S_2 &= \bigg(\frac{1}{|Q|}\int_Q |(b_1(x)-\rho_1) I^2_{b_2-\rho_2,\alpha}(f_1,f_2)(z)|^\delta\,dz\bigg)^{1/\delta} \\ &\le C\|b_1\|_{\mathrm{BMO}}\mathcal{M}_{t_2\delta}(I^2_{b_2-\rho_2,\alpha}(f_1,f_2))(x_0) \\ &\le C\|b_1\|_{\mathrm{BMO}}\mathcal{M}_{\varepsilon}(I^2_{b_2-\rho_2,\alpha}(f_1,f_2))(x_0). \end{aligned}$$

Similarly, we can estimate S_3.

To analyze the contribution of S_4, we denote $f_i^0 = f_i\chi_{3Q}$, $f_i^\infty = f_i - f_i^0$ and let $\tilde{c} = (I_\alpha((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^\infty)(x_0) + I_\alpha((b_1-\rho_1)f_1^\infty, (b_2-\rho_2)f_2^0)(x_0) + I_\alpha((b_1-\rho_1)f_1^\infty, (b_2-\rho_2)f_2^\infty)(x_0)) =: \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3$. Then S_4 can be written as

$$S_4 \le S_{4,1} + S_{4,2} + S_{4,3} + S_{4,4},$$

where

$$\begin{aligned} S_{4,1} &= \bigg(\frac{1}{|Q|}\int_Q |I_\alpha((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^0)(z)|^\delta\,dz\bigg)^{1/\delta}, \\ S_{4,2} &= \bigg(\frac{1}{|Q|}\int_Q |I_\alpha((b_1-\rho_1)f_1^0, (b_2-\rho_2)f_2^\infty)(z) - \tilde{c}_1|^\delta\,dz\bigg)^{1/\delta}, \\ S_{4,3} &= \bigg(\frac{1}{|Q|}\int_Q |I_\alpha((b_1-\rho_1)f_1^\infty, (b_2-\rho_2)f_2^0)(z) - \tilde{c}_2|^\delta\,dz\bigg)^{1/\delta}, \\ S_{4,4} &= \bigg(\frac{1}{|Q|}\int_Q |I_\alpha((b_1-\rho_1)f_1^\infty, (b_2-\rho_2)f_2^\infty)(z) - \tilde{c}_3|^\delta\,dz\bigg)^{1/\delta}. \end{aligned}$$

Using Hölder's inequality, Kolmogorov's inequality (2.9) with $p = 1/2$ and

$q = n/(2n-\alpha)$, and (2.11), we have

$$\begin{aligned} S_{4,1} &\le C\bigg(\frac{1}{|Q|}\int_Q \big|I_\alpha((b_1-\rho_1)f_1^0,(b_2-\rho_2)f_2^0)(z)\big|^{1/2}\,dz\bigg)^2 \\ &\le C|Q|^{\alpha/n-2}\|I_\alpha((b_1-\rho_1)f_1^0,(b_2-\rho_2)f_2^0)\|_{L^{n/(2n-\alpha),\infty}(Q)} \\ &\le C|Q|^{\alpha/n-2}\|(b_1-\rho_1)f_1^0\|_{L^1(Q)}\|(b_2-\rho_2)f_2^0\|_{L^1(Q)} \\ &\le C|3Q|^{\alpha/n}\|b_1\|_{\mathrm{BMO}}\|f_1^0\|_{L(\log L),Q}\|b_2\|_{\mathrm{BMO}}\|f_2^0\|_{L(\log L),Q} \\ &\le C\prod_{j=1}^2\|b_j\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L),\alpha}(\vec{f})(x_0). \end{aligned}$$

By the mean value theorem again, one obtains

$$\begin{aligned} S_{4,2} &\le \frac{C}{|Q|}\int_Q |I_\alpha((b_1-\rho_1)f_1^0,(b_2-\rho_2)f_2^\infty)(z)-\tilde{c}_1|\,dz \\ &\le C\frac{1}{|Q|}\int_{3Q}|(b_1(y_1)-\rho_1)f_1(y_1)|\,dy_1 \\ &\quad\times\int_{(3Q)^c}\frac{|x_0-z|\,|b_2(y_2)-\rho_2|\,|f_2(y_2)|dy_2}{(|z_1-y_1|+|z_2-y_2|)^{2n-\alpha+1}}\,dz \\ &\le C\sum_{j=1}^\infty\frac{j}{(3^j|Q|^{1/n})^{2n-\alpha+1}}\int_{3Q}|(b_1(y_1)-\rho_1)f_1(y_1)|\,dy_1 \\ &\quad\times\int_{3^{j+1}Q}|b_2(y_2)-\rho_2|\,|f_2(y_2)|\,dy_2 \\ &\le C\prod_{j=1}^2\|b_j\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L),\alpha}(\vec{f})(x_0). \end{aligned}$$

Similarly to $S_{4,2}$, we can estimate $S_{4,3}$.

Now we estimate $S_{4,4}$. Note that

$$\begin{aligned} &\big|I_\alpha((b_1-\rho_1)f_1^\infty,(b_2-\rho_2)f_2^\infty)(z)-(I_\alpha((b_1-\rho_1)f_1^\infty,(b_2-\rho_2)f_2^\infty))(x_0)\big| \\ &\quad\le C\int_{(\mathbb{R}^n\setminus 3Q)^2}\frac{|Q|^{1/n}|b_1(y_1)-\rho_1|\,|b_2(y_2)-\rho_2|}{|(x_0-y_1,x_0-y_2)|^{2n-\alpha+1}}\prod_{i=1}^2|f_i^\infty(y_i)|\,d\vec{y} \\ &\quad\le C\sum_{k=1}^\infty\int_{(3^{k+1}Q)^2\setminus(3^kQ)^2}\frac{|Q|^{1/n}|b_1(y_1)-\rho_1|\,|b_2(y_2)-\rho_2|}{(3^k|Q|^{1/n})^{2n-\alpha+1}}\prod_{i=1}^2|f_i^\infty(y_i)|\,d\vec{y} \end{aligned}$$

$$\le C \sum_{k=1}^{\infty} \frac{k}{3^k} \|b_2\|_{\mathrm{BMO}} |3^{k+1}Q|^{\alpha/n} \prod_{j=1}^{2} \|f_j^{\infty}\|_{L(\log L),3^{k+1}Q}$$

$$\le C \prod_{j=1}^{2} \|b_j\|_{\mathrm{BMO}} \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x_0).$$

Therefore, $S_{4,i} \le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L),\alpha}(\vec{f})(x_0)$ for $i = 1, \dots, 4$.

Thus we complete the proof of Proposition 3.1.

Proposition 3.2 (Pointwise control of $M_\delta^\sharp(U^*(\vec{f})), M_\delta^\sharp(V^*(\vec{f}))$, $M_\delta^\sharp(I_\alpha(\vec{f}))$). *Let $0 < \delta < \varepsilon$, $0 < \delta < 1/m$ and $0 < \alpha < mn$. Then there is a constant $C > 0$ depending on δ and ε such that*

$$M_\delta^\sharp(U^*(\vec{f}))(x) \le C\mathcal{M}(f)(x), \tag{3.4}$$

$$M_\delta^\sharp(V^*(\vec{f}))(x) \le C\mathcal{M}(f)(x), \tag{3.5}$$

$$M_{1/m}^\sharp(I_\alpha(\vec{f}))(x) \le C\mathcal{M}_\alpha(f)(x), \tag{3.6}$$

for all bounded $\vec{f}$ with compact support.

Proof. The proof of (3.4) and (3.5) follows from similar steps in Theorem 3.2 of [19], combined with the method we used in the proof of the above proposition; we omit the details. On the other hand, (3.6) has already been obtained in [3, Proposition 5.2].

Now, we can obtain

Theorem 3.1. *Let $0 < p$ and $\omega \in A_\infty$. Suppose that $\vec{b} \in \mathrm{BMO}^m$. Then there is a constant C independent of $\vec{b}$ and a constant C_1 (possibly depending on $\vec{b}$) such that*

$$\int_{\mathbb{R}^n} |U^*_{\Pi\mathbf{b}}(\vec{f})(x)|^p \omega(x)\,dx \le C \prod_{i=1}^{m} \|b_i\|_{\mathrm{BMO}} \int_{\mathbb{R}^n} [\mathcal{M}_{L(\log L)}(f)(x)]^p \omega(x)\,dx, \tag{3.7}$$

$$\sup_{t>0} \frac{1}{\Phi^m(1/t)} \omega(\{y \in \mathbb{R}^n : |U^*_{\Pi\mathbf{b}}\vec{f}(y)| > t^m\}) \le C_1 \sup \frac{1}{\Phi^m(1/t)} \omega(\{y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}(f)(y) > t^m\}). \tag{3.8}$$

Similar results hold for $V^*_{\Pi\mathbf{b}}(\vec{f})$.

Proof. The proof of Theorem 3.1 is now standard as in the case for multilinear C-Z singular integral operators. We briefly indicate the argument in the case $m = 2$, but, as the reader will immediately notice, an iterative procedure using (3.1) and (3.2) can be followed to obtain the general case.

Using the Fefferman–Stein inequality and the pointwise estimate in Proposition 3.1 we have

$$\begin{aligned}\|U^*_{\Pi\mathbf{b}}(\vec{f})\|_{L^p(\omega)} &\le \|M_\delta(U^*_{\Pi\mathbf{b}}(\vec{f}))\|_{L^p(\omega)} \le C\|M^\sharp_\delta(U^*_{\Pi\mathbf{b}}(\vec{f}))\|_{L^p(\omega)}\\ &\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\big(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)} + \|M^\sharp_\varepsilon(U^*(\vec{f}))\|_{L^p(\omega)}\big)\\ &\quad + C\big(\|b_2\|_{\mathrm{BMO}}\|M^\sharp_\varepsilon(U^*_{b_1}(\vec{f}))\|_{L^p(\omega)} + \|b_1\|_{\mathrm{BMO}}\|M^\sharp_\varepsilon(U^*_{b_2}(\vec{f}))\|_{L^p(\omega)}\big).\end{aligned}$$

First of all, we consider the contribution of $\|M^\sharp_\varepsilon(U^*_{b_2}(\vec{f}))\|_{L^p(\omega)}$.

We define $c_\eta = U_\eta(f_1^0,(b_2-\rho_2)f_2^\infty)(x_0) + U_\eta(f_1^\infty,(b_2-\rho_2)f_2^0)(x_0) + U_\eta(f_1^\infty,(b_2-\rho_2)f_2^\infty)(x_0)$ and $c=\sup_{\eta>0}|c_\eta|$. Then

$$\begin{aligned}&|U^*_{b_2}(\vec{f})(z)-c|\\ &\quad\le \sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^2} K_{u,\eta}(z,y_1,y_2)((b_2(z)-\rho_2)-(b_2(y_2)-\rho_2))\prod_{i=1}^{2} f_i(y_i)d\vec{y}+c_\eta\Big|\\ &\quad\le C|b_2(z)-\rho_2|U^*(f_1,f_2)(z) + \sup_{\eta>0}|U_\eta(f_1,(b_2-\rho_2)f_2)(z)-c_\eta|.\end{aligned}$$

For arbitrary $0<\varepsilon'<1/2$, taking $1<t_1,t_2<\infty$ with $1=1/t_1+1/t_2$ and $t_2<\varepsilon'/\varepsilon$, we have

$$\begin{aligned}&\bigg(\frac{1}{|Q|}\int_Q |(b_2(z)-\rho_2)U^*(f_1,f_2)(z)|^\varepsilon\,dz\bigg)^{1/\varepsilon}\\ &\qquad\le \bigg(\frac{1}{|Q|}\int_Q |b_2(z)-\rho_2|^{t_1\varepsilon}\,dz\bigg)^{\frac{1}{t_1\varepsilon}}\bigg(\frac{1}{|Q|}\int_Q |U^*(f_1,f_2)(z)|^{t_2\varepsilon}\,dz\bigg)^{\frac{1}{t_2\varepsilon}}\\ &\qquad\le C\|b_2\|_{\mathrm{BMO}}\mathcal{M}_{\varepsilon'}(U^*(f_1,f_2))(x_0).\end{aligned}$$

Similar to the proof of Proposition 3.1, $U_\eta(f_1,(b_2-\rho_2)f_2)$ can be written as

$$\begin{aligned}U_\eta(f_1,(b_2-\rho_2)f_2) &= U_\eta(f_1^0,(b_2-\rho_2)f_2^0) + U_\eta(f_1^0,(b_2-\rho_2)f_2^\infty)\\ &\quad + U_\eta(f_1^\infty,(b_2-\rho_2)f_2^0) + U_\eta(f_1^\infty,(b_2-\rho_2)f_2^\infty).\end{aligned}$$

Taking $1<p_0<1/(2\varepsilon)$ and using Hölder's inequality again, we have

$$\begin{aligned}&\bigg(\frac{1}{|Q|}\int_Q \sup_{\eta>0}|U_\eta(f_1,(b_2-\rho_2)f_2)(z)-c_\eta|^\varepsilon\,dz\bigg)^{1/\varepsilon}\\ &\qquad\le \bigg(\frac{1}{|Q|}\int_Q \sup_{\eta>0}|U_\eta(f_1,(b_2-\rho_2)f_2)(z)-c_\eta|^{p_0\varepsilon}dz\bigg)^{\frac{1}{p_0\varepsilon}}\\ &\qquad\le G_1+G_2+G_3+G_4,\end{aligned}$$

where

$$G_1=\left(\frac{1}{|Q|}\int_Q \sup_{\eta>0}|U_\eta(f_1^0,(b_2-\rho_2)f_2^0)(z)|^{p_0\varepsilon}dz\right)^{\frac{1}{p_0\varepsilon}},$$

$$G_2=\left(\frac{1}{|Q|}\int_Q \sup_{\eta>0}\left|U_\eta(f_1^0,(b_2-\rho_2)f_2^\infty)(z)-U_\eta(f_1^0,(b_2-\rho_2)f_2^\infty)(x_0)\right|^{p_0\varepsilon}dz\right)^{\frac{1}{p_0\varepsilon}},$$

$$G_3=\left(\frac{1}{|Q|}\int_Q \sup_{\eta>0}\left|U_\eta(f_1^\infty,(b_2-\rho_2)f_2^0)(z)-U_\eta(f_1^\infty,(b_2-\rho_2)f_2^0)(x_0)\right|^{p_0\varepsilon}dz\right)^{\frac{1}{p_0\varepsilon}},$$

$$G_4=\left(\frac{1}{|Q|}\int_Q \sup_{\eta>0}\left|U_\eta(f_1^\infty,(b_2-\rho_2)f_2^\infty)(z)-U_\eta(f_1^\infty,(b_2-\rho_2)f_2^\infty)(x_0)\right|^{p_0\varepsilon}dz\right)^{\frac{1}{p_0\varepsilon}}.$$

A similar procedure to that for T_4 in Proposition 3.1 yields

$$G_1 \le C\|b_2\|_{\mathrm{BMO}}\mathcal{M}^2_{L(\log L)}(\vec{f})(x_0).$$

By the mean value theorem, we deduce

$$G_2 \le C\|b_2\|_{\mathrm{BMO}}\mathcal{M}^2_{L(\log L)}(\vec{f})(x_0).$$

Similarly to G_2, we can estimate G_3. Moreover

$$G_4 \le C\|b_2\|_{\mathrm{BMO}}\mathcal{M}^2_{L(\log L)}(\vec{f})(x_0).$$

This completes the analysis of $\|M_\varepsilon^\sharp(U^*_{b_2}(\vec{f}))\|_{L^p(\omega)}$.

Therefore, by Proposition 3.2, we have

$$\begin{aligned}\|M_\varepsilon^\sharp[U^*_{b_2}(\vec{f})]\|_{L^p(\omega)} &\le C\|b_2\|_{\mathrm{BMO}}(\|\mathcal{M}(\vec{f})\|_{L^p(\omega)}+\|\mathcal{M}^2_{L(\log L)}(\vec{f})\|_{L^p(\omega)})\\ &\le C\|b_2\|_{\mathrm{BMO}}\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)}.\end{aligned}$$

$\|M_\varepsilon^\sharp(U^*_{b_1}(\vec{f}))\|_{L^p(\omega)}$ can be treated in the same way as $M_\varepsilon^\sharp(U^*_{b_2}(\vec{f}))$. The desired inequality (3.7) now follows.

Since the main steps and ideas are almost the same as in [25], we omit the proof of (3.8).

Proof of Theorems 1.1–1.2. Theorem 1.1 follows from

$$T_{*,\Pi\mathbf{b}}(\vec{f}) \le U^*_{\Pi\mathbf{b}}(\vec{f})(x)+V^*_{\Pi\mathbf{b}}(\vec{f})(x),$$

Theorem 3.1 and the weighted strong boundedness of $\mathcal{M}_{L(\log L)}$ proved in [19]. Theorem 1.2 follows by repeating the same steps as in [19], [25] and the method used in [27]. We omit the details.

Proof of Theorem 1.3. Theorem 1.3 follows by using Proposition 3.1 and the estimate for $I^j_{\vec{b},\alpha}$ $(j=1,2)$ in Theorem 2.7 of [3].

4. Weighted end-point estimates for $I_{\alpha,\Pi\mathbf{b}}(\vec{f})$. Firstly, we will consider the end-point estimate for a multilinear fractional $L(\log L)$ type maximal operator.

Proposition 4.1 (Weighted end-point estimate for $\mathcal{M}_{L(\log L),\alpha}$). *Let $\Phi(t) = t(1+\log^+ t)$ and $\vec{\omega} \in A_{((1,\dots,1),n/(mn-\alpha))}$. If $0<\alpha<mn$, then there is a constant $C>0$ such that*

$$\nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}}(\{x\in\mathbb{R}^n : \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}}\}) \le C\bigg\{\bigg[1+\frac{\alpha}{mn}\log^+\bigg(\prod_{i=1}^m \int_{\mathbb{R}^n}\Phi^{(m)}(|f_i(y_i)|/t)\,dy_i\bigg)\bigg]^m \times \prod_{j=1}^m \int_{\mathbb{R}^n}\Phi^{(m)}(|f_j(y_j)|/t)\omega_j(y_j)\,dy_j\bigg\}^{\frac{n}{mn-\alpha}}. \tag{4.1}$$

If $0<\alpha_j<n$ for each $1\le j\le m$, and $\sum_{j=1}^m \alpha_j = \alpha$, then there is a constant $C>0$ such that

$$\nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}}(\{x\in\mathbb{R}^n : \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}}\}) \le C\bigg\{\prod_{j=1}^m \bigg[1+\frac{\alpha_j}{n}\log^+\bigg(\prod_{i=1}^m \int_{\mathbb{R}^n}\Phi^{(m)}(|f_i(y_i)|/t)\,dy_i\bigg)\bigg] \times \int_{\mathbb{R}^n}\Phi^{(m)}(|f_j(y_j)|/t)\omega_j(y_j)\,dy_j\bigg\}^{\frac{n}{mn-\alpha}}. \tag{4.2}$$

Proof. By homogeneity, we may assume $t=1$. We first prove (4.2). Introduce the notations

$$E_1 = \{x\in\mathbb{R}^n : \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) > 1\} \quad\text{and}\quad E_{1,k} = E_1\cap B(0,k),$$

where $B(0,k) = \{x\in\mathbb{R}^n : |x|\le k\}$. By the monotone convergence theorem, it suffices to estimate $|E_{1,k}|$.

For any $x\in E_{1,k}$, there is a cube Q_x such that

$$1 < |Q_x|^{\alpha/n}\prod_{j=1}^m \|f_j\|_{L(\log L),Q_x}. \tag{4.3}$$

Hence, $\{Q_x\}_{x\in E_{1,k}}$ is a family of cubes covering $E_{1,k}$. Using a covering argument, we obtain a finite family $\{Q_{x_l}\}$ of disjoint cubes whose dilations cover $E_{1,k}$ such that

$$|E_{1,k}| \le C\sum_l |Q_{x_l}| \quad\text{and}\quad 1 < |Q_{x_l}|^{\alpha/n}\prod_{j=1}^m \|f_j\|_{L(\log L),Q_{x_l}}. \tag{4.4}$$

We follow the main steps first as in [25] and denote by C_h^m the family of all subsets $\sigma = \{\sigma(1),\dots,\sigma(h)\}$ of $\{1,\dots,m\}$ with $1\le h\le m$ different elements. Given $\sigma \in C_h^m$ and a cube Q_{x_l}, if $|Q_{x_l}|^{\alpha_{\sigma(j)}} \|f_{\sigma(j)}\|_{L(\log L),Q_{x_l}} > 1$ for $j = 1, \ldots, h$, we say that $j \in B_\sigma$; then $|Q_{x_l}|^{\alpha_{\sigma(j)}} \|f_{\sigma(j)}\|_{L(\log L),Q_{x_l}} \leq 1$ for $j = h+1, \ldots, m$. Denote

$$A_k = \prod_{j=1}^{k} |Q_{x_l}|^{\alpha_{\sigma(j)}/n} \|f_{\sigma(j)}\|_{L(\log L),Q_{x_l}}$$

and $A_0 = 1$. Then it is easy to check that if $\sigma \in C_h^m$ and $j \in B_\sigma$, the for any $1 \leq k \leq m$, we have $A_k > 1$ and

$$1 < \prod_{j=1}^{k} |Q_{x_l}|^{\alpha_{\sigma(j)}/n} \|f_{\sigma(j)}\|_{L(\log L),Q_{x_l}} = \big\| |Q_{x_l}|^{\alpha_{\sigma(k)}/n} f_{\sigma(k)} A_{k-1} \big\|_{\Phi,Q_{x_l}},$$

or, equivalently,

(4.5)
$$1 < \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi\Big(|Q_{x_l}|^{\alpha_{\sigma(k)}/n} f_{\sigma(k)} \Big(\prod_{j=1}^{k-1} |Q_{x_l}|^{\alpha_{\sigma(j)}/n} \|f_{\sigma(j)}\|_{L(\log L),Q_{x_l}} \Big)\Big).$$

By the equivalence

$$\|f\|_{\Phi,Q} \simeq \inf_{\mu>0} \Big\{ \mu + \frac{\mu}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi(|f|/\mu) \Big\},$$

if $1 \leq j \leq m-h-1$ we obtain

$$\Phi^j(A_{m-j}) = \Phi^j\big(\big\| |Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1} \big\|_{\Phi,Q_{x_l}} \big).$$

Since $\big\| |Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1} \big\|_{\Phi,Q_{x_l}} > 1$, using the fact that Φ is submultiplicative (i.e. $\Phi(st) \leq \Phi(s)\Phi(t)$ for $s, t > 0$) and Jensen's inequality one obtains

$$\begin{aligned} \Phi^j(A_{m-j}) &= \Phi^j\big(\big\| |Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1} \big\|_{\Phi,Q_{x_l}} \big) \\ &\leq C\Phi^j \Big(1 + \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi(|Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1}) \Big) \\ &\leq C \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^{j+1}(|Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)}) \Phi^{j+1}(A_{m-j-1}). \end{aligned}$$

By iterating the inequalities above and the fact that $\big\| |Q_{x_l}|^{\alpha_{\sigma(j)}/n} f_{\sigma(j)} \big\|_{\Phi,Q_{x_l}} > 1$ for $j \in B_\sigma$, $\Phi^{j+1} \leq \Phi^m$ and $\Phi^{m-h+1} \leq \Phi^m$ for $1 \leq h \leq m$ and $0 \leq j \leq m-h-1$, we obtain

$$(4.6)\qquad \begin{aligned} 1 &< \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi(|Q_{x_l}|^{\alpha_{\sigma(m)}/n} f_{\sigma(m)})\frac{1}{|Q_{x_l}|} \\ &\quad\times \int_{Q_{x_l}} \Phi^2(|Q_{x_l}|^{\alpha_{\sigma(m-1)}/n} f_{\sigma(m-1)})\Phi^2(A_{m-2}) \\ &\le \Big(\prod_{j=0}^{m-h-1} \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi^{j+1}(|Q_{x_l}|^{\alpha_{\sigma(m-1)}/n} f_{\sigma(m-1)})\Big) \\ &\quad\times \prod_{j=1}^{h} \Phi^{m-h}\big(\big\||Q_{x_l}|^{\alpha_{\sigma(j)}/n} f_{\sigma(j)}\big\|_{\Phi,Q_{x_l}}\big) \\ &\le \Big(\prod_{j=0}^{m-h-1} \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi^{j+1}(|Q_{x_l}|^{\alpha_{\sigma(m-1)}/n} f_{\sigma(m-1)})\Big) \\ &\quad\times \prod_{j=1}^{h} \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi^{m-h+1}(|Q_{x_l}|^{\alpha_{\sigma(j)}/n} f_{\sigma(j)}) \\ &\le C\prod_{j=1}^{m} \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi^{m}(|Q_{x_l}|^{\alpha_j/n} f_j). \end{aligned}$$

This implies that

$$(4.7)\qquad \begin{aligned} 1 &< C\prod_{j=1}^{m} \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi^{m}(|Q_{x_l}|^{\alpha_j/n} f_j) \\ &\le C\prod_{j=1}^{m} \frac{1}{|Q_{x_l}|}\int_{Q_{x_l}} \Phi^{m}(|Q_{x_l}|^{\alpha_j/n})\Phi^m(f_j) \\ &\le C\prod_{j=1}^{m} \frac{1}{|Q_{x_l}|}|Q_{x_l}|^{\alpha_j/n}(1+\log^+|Q_{x_l}|^{\alpha_j/n})\int_{Q_{x_l}} \Phi^{m}(f_j). \end{aligned}$$

Since $\alpha_j < n$, there exists a constant $C_0 > 1$ and $\eta_1, \dots, \eta_m$ small enough such that

$$0 < \eta_j < 1 - \alpha_j/n, \qquad 1 + \log^+ t^{\alpha_j/n} \le t^{\eta_j} \qquad \text{if } t > C_0.$$

Denote $\eta = \sum_{j=1}^m \eta_j$. By (4.7), if $|Q_{x_l}| > C_0$, we have

$$(4.8)\qquad |Q_{x_l}|^{m-\alpha/n-\eta} \le C\prod_{j=1}^{m}\int_{Q_{x_l}} \Phi^m(f_j).$$

Therefore,

$$(m-\alpha/n-\eta)\log^+(|Q_{x_l}|^{\alpha_j/n}) \le C\frac{\alpha_j}{n}\log^+\Big(\prod_{j=1}^{m}\int_{Q_{x_l}} \Phi^m(f_j)\Big).$$

By (4.7) again, we have

$$(4.9)\qquad |Q_{x_l}|^{m-\alpha/n} \le C\prod_{j=1}^{m}\Big\{1+\frac{\alpha_j}{n}\log^+\Big(\prod_{j=1}^{m}\int_{Q_{x_l}}\Phi^m(f_j)\Big)\Big\}\int_{Q_{x_l}}\Phi^m(f_j).$$

On the other hand, if $|Q_{x_l}|\le C_0$, then it is easy to see that $1+\log^+|Q_{x_l}|^{\alpha_j/n} \le C$. Thus

$$(4.10)\qquad |Q_{x_l}|^{m-\alpha/n} \le C\prod_{j=1}^{m}\int_{Q_{x_l}}\Phi^m(f_j).$$

(4.9) and (4.10) yield

$$(4.11)\qquad |Q_{x_l}|^{m-\alpha/n} \le C\prod_{j=1}^{m}\Big\{1+\frac{\alpha_j}{n}\log^+\Big(\prod_{j=1}^{m}\int_{Q_{x_l}}\Phi^m(f_j)\Big)\Big\}\int_{Q_{x_l}}\Phi^m(f_j).$$

Finally, by (4.4) and the definition of $A_{((1,\dots,1),n/(mn-\alpha))}$, we have

$$\begin{aligned}
\Big(\int_{E_{1,k}}\nu_{\vec\omega}^{\frac{n}{mn-\alpha}}\Big)^{\frac{mn-\alpha}{n}}
&\le \Big(\sum_{h=1}^{m}\sum_{\sigma\in C_h^m}\sum_{l\in B_\sigma}\int_{Q_{x_l}}\nu_{\vec\omega}^{\frac{n}{mn-\alpha}}\Big)^{\frac{mn-\alpha}{n}}\\
&\le C\sum_{h=1}^{m}\sum_{\sigma\in C_h^m}\sum_{l\in B_\sigma}\Big(\int_{Q_{x_l}}\nu_{\vec\omega}^{\frac{n}{mn-\alpha}}\Big)^{\frac{mn-\alpha}{n}}\\
&\le C\sum_{h=1}^{m}\sum_{\sigma\in C_h^m}\sum_{l\in B_\sigma}|Q_{x_l}|^{m-\alpha/n}\prod_{j=1}^{m}\inf\omega_j\\
&\le C\sum_{h=1}^{m}\sum_{\sigma\in C_h^m}\sum_{l\in B_\sigma}\prod_{j=1}^{m}\Big\{1+\frac{\alpha_j}{n}\log^+\Big(\prod_{j=1}^{m}\int_{Q_{x_l}}\Phi^m(f_j)\Big)\Big\}\int_{Q_{x_l}}\Phi^m(f_j)\omega_j\\
&\le C\sum_{h=1}^{m}\sum_{\sigma\in C_h^m}\sum_{l\in B_\sigma}\prod_{j=1}^{m}\Big\{1+\frac{\alpha_j}{n}\log^+\Big(\prod_{j=1}^{m}\int_{\mathbb{R}^n}\Phi^m(f_j)\Big)\Big\}\int_{Q_{x_l}}\Phi^m(f_j)\omega_j\\
&\le C\prod_{j=1}^{m}\Big\{1+\frac{\alpha_j}{n}\log^+\Big(\prod_{j=1}^{m}\int_{\mathbb{R}^n}\Phi^m(f_j)\Big)\Big\}\int_{\mathbb{R}^n}\Phi^m(f_j)\omega_j.
\end{aligned}$$

The proof of inequality (4.2) is finished.

Inequality (4.1) follows by taking $\alpha_j=\alpha/m<n$ in the above proof.

Proof of Theorem 1.4 and Corollary 1.1. We obtain Theorem 1.4 by following the main steps as in [3]. Proposition 3.1 is used in the last step.

To prove Corollary 1.1, as in the linear case [9] we define

$$\overline{I_{\alpha,\Pi\mathbf{b}}}(\vec{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{\prod_{j=1}^m |b_j(x) - b_j(y_j)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} \prod_{i=1}^m |f_i(y_i)|\, d\vec{y}, \tag{4.12}$$

and a careful check of the proof of Theorems 1.3–1.4 shows that these theorems still hold for $\overline{I_{\alpha,\Pi\mathbf{b}}}$. Noting that $\mathcal{M}_{\Pi\mathbf{b},\alpha}(\vec{f})(x) \le \overline{I_{\alpha,\Pi\mathbf{b}}}(|f_1|,\ldots,|f_m|)(x)$ implies Corollary 1.1.

Acknowledgements. The author wants to express his sincere thanks to the unknown referee for his or her valuable remarks and pointing out a problem in the previous version of this manuscript.

The author was partly supported by NSFC (Key program Grant No. 10931001), CPDRFSFP (Grant No. 200902070), Beijing Natural Science Foundation (Grant 1102023), Program for Changjiang Scholars and Innovative Research Teams in Universities.

References

[1] A. Bernardis, O. Gorosito and G. Pradolini, *Weighted inequalities for multilinear potential operators and its commutators*, Potential Anal. 35 (2011), 253–274.

[2] X. Chen, *Weighted estimates for the maximal operator of a multilinear singular integral*, Bull. Polish Acad. Sci. Math. 58 (2010), 129–135.

[3] X. Chen and Q. Xue, *Weighted estimates for a class of multilinear fractional type operators*, J. Math. Anal. Appl. 362 (2010), 355–373.

[4] M. Christ and J.-L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. 159 (1987), 51–80.

[5] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. 212 (1975), 315–331.

[6] R. R. Coifman et Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) 28 (1978), no. 3, 177–202.

[7] R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) 103 (1976), 611–635.

[8] C. Demeter, T. Tao and C. Thiele, *Maximal multilinear operators*, Trans. Amer. Math. Soc. 360 (2008), 4989–5042.

[9] Y. Ding, S. Lu and P. Zhang, *Weak estimates for commutators of fractional integral operators*, Sci. China Ser. A (English Ser.) 44 (2001), 877–888.

[10] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[11] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math. Stud. 116, North-Holland, Amsterdam, 1985.

[12] L. Grafakos, *On multilinear fractional integrals*, Studia Math. 102 (1992), 49–56.

[13] L. Grafakos and N. Kalton, *Multilinear Calderón–Zygmund operators on Hardy spaces*, Collect. Math. 52 (2001), 169–179.

[14] L. Grafakos, L. Liu, C. Pérez and R. H. Torres, *The multilinear strong maximal function*, J. Geom. Anal. 21 (2011), 118–149.

[15] L. Grafakos and R. H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. 165 (2002), 124–164.
[16] L. Grafakos and R. H. Torres, *On multilinear singular integrals of Calderón–Zygmund type*, in: Proc. 6th Int. Conf. on Harmonic Analysis and Partial Differential Equations (El Escorial), Publ. Mat. 2002, Vol. Extra, 57–91.
[17] L. Grafakos and R. H. Torres, *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J. 51 (2002), 1261–1276.
[18] C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. 6 (1999), 1–15.
[19] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory*, Adv. Math. 220 (2009), 1222–1264.
[20] W. Li, Q. Xue and K. Yabuta, *Multilinear Calderón–Zygmund operators on weighted Hardy spaces*, Studia Math. 199 (2010), 1–16.
[21] W. Li, Q. Xue and K. Yabuta, *Maximal operator for multilinear Calderón–Zygmund singular integral operators on weighted Hardy spaces*, J. Math. Anal. Appl. 373 (2011), 384–392.
[22] K. Moen, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math. 60 (2009), 213–238.
[23] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. 192 (1974), 261–274.
[24] C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. 128 (1995), 163–185.
[25] C. Pérez, G. Pradolini, R. H. Torres, and R. Trujillo-González, *End-point estimates for iterated commutators of multilinear singular integrals*, arXiv:1004.4976.
[26] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monogr. Textbooks Pure Appl. Math. 146, Dekker, New York, 1991.
[27] P. Zhang, *Weighted estimates for maximal multilinear commutators*, Math. Nachr. 279 (2006), 445–462.

Qingying Xue
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing, 100875, P.R. China
E-mail: qyxue@bnu.edu.cn

Received September 22, 2010
Revised version November 19, 2011 (6993)