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How far is $C(\omega)$ from the other $C(K)$ spaces?

by

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Abstract. Let us denote by $C(\alpha)$ the classical Banach space $C(K)$ when K is the interval of ordinals $[1,\alpha]$ endowed with the order topology. In the present paper, we give an answer to a 1960 Bessaga and Pełczyński question by providing tight bounds for the Banach–Mazur distance between $C(\omega)$ and any other $C(K)$ space which is isomorphic to it. More precisely, we obtain lower bounds $L(n,k)$ and upper bounds $U(n,k)$ on $d(C(\omega),C(\omega^n k))$ such that $U(n,k)-L(n,k)<2$ for all $1\le n,k<\omega$.

1. Introduction. We follow the standard notation and terminology of Banach spaces theory that can be found in [6]. Let K be a compact Hausdorff space. We denote by $C(K)$ the Banach space of all continuous scalar valued functions defined on K, endowed with the supremum norm. The variation of a measure μ will be denoted by $|\mu|$. The symbol $\mathbb{K}$ may denote the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. If α is an ordinal number, then $[1,\alpha]$ denotes the interval $\{\gamma : 1\le\gamma\le\alpha\}$ endowed with the order topology. The space $C([1,\alpha])$ will be denoted by $C(\alpha)$. As usual, ω denotes the first infinite ordinal and ω_1 the first uncountable ordinal. For isomorphic Banach spaces X and Y (written $X\sim Y$), let $d(X,Y)$ denote the Banach–Mazur distance between them, defined to be $\inf\{\|T\|\,\|T^{-1}\|\}$ where the infimum is taken over all isomorphisms T from X onto Y.

In this work we are mainly interested in studying the Banach–Mazur distances between $C(\omega)$ and other $C(K)$ spaces which are isomorphic to it. First of all notice that in this case by the well-known Mazurkiewicz and Sierpiński theorem [7] and the classical isomorphic classification of $C(\alpha)$ spaces, $\omega\le\alpha<\omega_1$, due to Bessaga and Pełczyński [1], it follows that $C(K)$ is isomorphic to some $C(\omega^n k)$ space with $1\le n,k<\omega$.

The motivation for this research comes from [1]. There, the authors stated that if $\omega\le\alpha\le\beta<\omega_1$, then $C(\alpha)$ is isomorphic to $C(\beta)$ if and only if there exists $1\le n<\omega$ such that $\alpha^n\le\beta<\alpha^{n+1}$. Moreover, in this case, they

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proved that

$$n \leq d(C(\alpha), C(\beta)) \leq 4^{n+3}.$$

It was also indicated in [1, p. 59] that it would be interesting to obtain an estimate of the form

$$G(n) \leq d(C(\alpha), C(\beta)) \leq H(n),$$

where

$$\sup(H(n)/G(n)) < \infty. \tag{1.1}$$

The purpose of the present paper is to get such an estimate in the case where $\alpha = \omega$. Thus, we focus on lower bounds $L(n,k)$ and upper bounds $U(n,k)$ on the distances between $C(\omega)$ and $C(\omega^n k)$, $1 \leq n, k < \omega$. In this case, we establish something more than (1.1). Namely, our estimate satisfies

$$U(n,k) - L(n,k) < 2, \qquad \forall 1 \leq n, k < \omega. \tag{1.2}$$

Indeed, we find the following bounds:

$$L(n,k) = \begin{cases} 1 & \text{if } n = 1,\ k = 1, \\ 3 & \text{if } n = 1,\ k > 1, \\ 2n-1 & \text{if } n > 1,\ k = 1, \\ 2n+1 & \text{if } n > 1,\ k > 1, \end{cases}$$

and

$$U(n,k) = \begin{cases} 1 & \text{if } n = 1,\ k = 1, \\ 3 & \text{if } n = 1,\ k = 2, \\ 2 + \sqrt{5} & \text{if } n = 1,\ k > 2, \\ n + \sqrt{(n-1)(n+3)} & \text{if } n > 1,\ k = 1, \\ n + 1 + \sqrt{n(n+4)} & \text{if } n > 1,\ k > 1. \end{cases}$$

Therefore it is easy to check that (1.2) holds. We stress that in [5] it has already been proved that $d(C(\omega), C(\omega 2)) = 3$ and $d(C(\omega), C(\omega k)) \geq 3$ for every $k \geq 3$.

Of course, the above tight bounds lead naturally to the following conjecture on the exact values of the distances between the Banach spaces we are considering.

Conjecture 1.1. *Let $n \geq 2$ and $k \geq 2$ be integers. Then*

(a) *$d(C(\omega), C(\omega(k+1))$ is equal to 3, 4 or $2 + \sqrt{5}$.*
(b) *$d(C(\omega), C(\omega^n))$ is equal to $2n-1$, $2n$ or $n + \sqrt{(n-1)(n+3)}$.*
(c) *$d(C(\omega), C(\omega^n k))$ is equal to $2n+1$, $2n+2$ or $n+1+\sqrt{n(n+4)}$.*

The paper is organized as follows. In Section 2, inspired by [2] and [3], we prove the following result which was our guide in the search for tight bounds satisfying (1.2). Recall that for a positive integer n, the nth derived of K, $K^{(n)}$, is defined by induction: $K^{(0)} = K$, $K^{(1)}$ is the set of non-isolated points of K, and $K^{(n+1)} = (K^{(n)})^{(1)}$. The cardinality of a set Γ will be denoted by $|\Gamma|$.

THEOREM 1.2. *Let F be an infinite compact Hausdorff space with $F^{(2)} = \emptyset$. Then for every compact Hausdorff space K and $1 \leq n < \omega$, we have*

$$C(K) \sim C(F) \text{ and } |K^{(n)}| > |F^{(1)}| \Rightarrow d(C(K), C(F)) \geq 2n + 1.$$

Thus, since $[1, \omega^n k]^{(n)} = \{\omega^n, \omega^n 2, \ldots, \omega^n k\}$, $1 \leq n, k < \omega$, as an immediate consequence of Theorem 1.2 we obtain the following lower bounds on distances between $C(\omega)$ and the $C(\omega^n k)$ spaces, $1 < n, k < \omega$.

COROLLARY 1.3. *Suppose that $1 < n, k < \omega$. Then*

(a) $d(C(\omega), C(\omega k)) \geq 3$.
(b) $d(C(\omega), C(\omega^n)) \geq 2n - 1$.
(c) $d(C(\omega), C(\omega^n k)) \geq 2n + 1$.

In Section 3, we turn our attention to upper bounds on the distances between $C(\omega)$ and $C(\omega^n k)$, $1 \leq n, k < \omega$. In view of Corollary 1.3, our task is to search for isomorphisms T from $C(\omega)$ onto $C(\omega^n)$ (resp. $C(\omega^n k)$) such that the product $\|T\| \, \|T^{-1}\|$ is not too far from $2n - 1$ (resp. $2n + 1$). In Theorem 3.1 we present some special such isomorphisms. Finally, in Section 4, as an immediate consequence, we prove the following result.

THEOREM 1.4. *Suppose that $1 < n, k < \omega$. Then*

(a) $d(C(\omega), C(\omega k)) \leq 2 + \sqrt{5}$.
(b) $d(C(\omega), C(\omega^n)) \leq n + \sqrt{(n-1)(n+3)}$.
(c) $d(C(\omega), C(\omega^n k)) \leq n + 1 + \sqrt{n(n+4)}$.

2. A lower bound on $d(C(K), C(F))$ where $F^{(2)} = \emptyset$. The main aim of this section is to prove Theorem 1.2. Before, we need to state two auxiliary results. For a subset A of a topological space K we denote by $\mathring{A}$ the set of interior points of A. Recall that an isomorphism T of $C(K)$ into $C(F)$ is said to be *norm-increasing* if $\|f\| \leq \|Tf\|$ for every $f \in C(K)$.

PROPOSITION 2.1. *Let F be an infinite compact Hausdorff space with $F^{(2)} = \emptyset$, K a compact Hausdorff space and T a norm-increasing isomorphism from $C(K)$ into $C(F)$. Let $1 < n < \omega$, $x_0 \in K^{(n)}$, K_0 a compact neighborhood of x_0, $0 < \epsilon < 1$ and $h_0 \in C(K)$ such that $0 \leq h_0 \leq 1$, $h_0(x) = 1$ for each $x \in K_0$ and $|Th_0(y)| < \epsilon$ for every $y \in F^{(1)}$. Then there are points $x_1, \ldots, x_{n-1} \in K$, compact subsets $K_1, \ldots, K_{n-1}$ of K and functions $h_1, \ldots, h_{n-1}$ in $C(K)$ satisfying*

(a) $x_i \in \mathring{K}_i \cap K^{(n-i)}$ *for* $0 \leq i \leq n - 1$.
(b) $K_i \subset \mathring{K}_{i-1}$ *for* $1 \leq i \leq n - 1$.
(c) $0 \leq h_i \leq 1$, $h_i(x) = 1$ *if* $x \in K_i$, *and* $h_i(x) = 0$ *if* $x \notin \mathring{K}_{i-1}$, *for* $1 \leq i \leq n - 1$.

(d) *The sets* $G_i = \{y \in F : |Th_i(y)| \geq \epsilon\}$, $0 \leq i \leq n-1$, *are non-empty and pairwise disjoint sets of isolated points.*

Proof. Since T is norm-increasing and $0 < \epsilon < 1$, the set G_0 is clearly non-empty. Moreover, it is finite: otherwise we would have $G_0 \cap F^{(1)} \neq \emptyset$, contrary to our hypothesis. Next, given $0 \leq r < n-1$, suppose that we have obtained points $x_0, x_1, \ldots, x_r$, compact sets $K_0, K_1, \ldots, K_r$, and functions $h_0, h_1, \ldots, h_r$ in $C(K)$ satisfying (a)–(d) above.

Since K is a compact Hausdorff space and $K^{(n)} \neq \emptyset$, it is possible to find points $b_1, b_2, \ldots$ in $(\mathring{K}_r \setminus \{x_r\}) \cap K^{(n-r-1)}$, pairwise disjoint open sets $U_1, U_2, \ldots$ and compact subsets $M_1, M_2, \ldots$ such that

$$b_i \in \mathring{M}_i \subset M_i \subset U_i \subset \mathring{K}_r, \quad i \in \mathbb{N}.$$

By the Urysohn Lemma [4, Theorem 1.5.11, p. 41], we can find functions $g_1, g_2, \ldots \in C(K)$ such that, for every $i \in \mathbb{N}$, $0 \leq g_i \leq 1$, $g_i(x) = 1$ if $x \in M_i$, and $g_i(x) = 0$ if $x \notin U_i$. Since $U_i \cap U_j = \emptyset$ if $i \neq j$, we have $g_i \cdot g_j = 0$ if $i \neq j$. Recalling that T is norm-increasing and $0 < \epsilon < 1$, the sets $\{y \in F : |Tg_i(y)| \geq \epsilon\}$ are non-empty for every $i \in \mathbb{N}$.

Next, define $G = G_0 \cup G_1 \cup \cdots \cup G_r$. We claim that there exists $s \in \mathbb{N}$ such that

$$\{y \in F : |Tg_s(y)| \geq \epsilon\} \cap (G \cup F^{(1)}) = \emptyset. \tag{2.1}$$

Indeed, otherwise, assuming that $G \cup F^{(1)} = \{y_1, \ldots, y_t\}$ and denoting

$$\Gamma_i = \{j \in \mathbb{N} : |Tg_j(y_i)| \geq \epsilon\}, \quad 1 \leq i \leq t,$$

we would obtain

$$\mathbb{N} \subseteq \Gamma_1 \cup \cdots \cup \Gamma_t,$$

and so Γ_p must be infinite for some $1 \leq p \leq t$. Let $p_1, p_2, \ldots$ be distinct integers in Γ_p.

Pick $m \in \mathbb{N}$ satisfying $\epsilon m > \|T\|$. For each $1 \leq i \leq m$ let r_i be a scalar such that

$$r_i \cdot Tg_{p_i}(y_p) = |Tg_{p_i}(y_p)|.$$

Since $g_i \cdot g_j = 0$ if $i \neq j$, the function $g = \sum_{i=1}^m r_i \cdot g_{p_i} \in C(K)$ is such that $\|g\| \leq 1$. However,

$$\begin{aligned} \|T\| \geq \|Tg\| &\geq \Big| T\Big(\sum_{i=1}^m r_i \cdot g_{p_i}\Big)(y_p) \Big| \\ &= \Big| \sum_{i=1}^m r_i \cdot Tg_{p_i}(y_p) \Big| = \sum_{i=1}^m |Tg_{p_i}(y_p)| > \|T\|; \end{aligned}$$

this contradiction establishes our claim.

Finally, let $s \in \mathbb{N}$ be chosen to satisfy (2.1). We set $x_{r+1} = b_s$, $K_{r+1} = M_s$, $h_{r+1} = g_s$ and $G_{r+1} = \{y \in F : |Tg_s(y)| \geq \epsilon\}$. It is easy to check that (a)–(d) hold for $r+1$, so the proposition is proved. ∎

Proposition 2.2. *Let F be an infinite compact Hausdorff space with $F^{(2)} = \emptyset$, K a compact Hausdorff space and T an isomorphism from $C(K)$ into $C(F)$. Suppose that $|K^{(n)}| > |F^{(1)}|$ for some $1 \le n < \omega$. Then for every $\epsilon > 0$ there exists $x_0 \in K^{(n)}$, a compact neighborhood K_0 of x_0 and a function $h \in C(K)$ such that $0 \le h \le 1$, $h(x) = 1$ for every $x \in K_0$ and $|Th(y)| < \epsilon$ for every $y \in F^{(1)}$.*

Proof. Towards a contradiction suppose $\epsilon > 0$ is such that $|Th(y)| \ge \epsilon$ for some $y \in F^{(1)}$ whenever $h \in C(K)$ is such that $0 \le h \le 1$ and $h(x) = 1$ for every x in a closed set K_0 satisfying $\mathring{K}_0 \cap K^{(n)} \neq \emptyset$.

Assume that $|F^{(1)}| = m$ and pick distinct points $x_1, \ldots, x_{m+1}$ in $K^{(n)}$ with respective pairwise disjoint compact neighborhoods $A_1, \ldots, A_{m+1}$. By applying the Urysohn Lemma, we find functions $h_i \in C(K)$, $1 \le i \le m+1$, such that $0 \le h_i \le 1$, $h_i(x) = 1$ for every $x \in A_i$ and moreover $h_i \cdot h_j = 0$ if $i \neq j$.

Let l_∞^{m+1} be the space $\mathbb{K}^{m+1}$ provided with the maximum norm. For each $a = (a_1, \ldots, a_{m+1}) \in l_\infty^{m+1}$ consider the function

$$\gamma_a = \sum_{i=1}^{m+1} a_i \cdot h_i \in C(K).$$

Notice that $\|\gamma_a\| = \|a\|$. We can identify, in the usual manner, the space $C(F^{(1)})$ with l_∞^m . Now define $S : l_\infty^{m+1} \to l_\infty^m$ by

$$S(a) = T\gamma_a|_{F^{(1)}}, \qquad a \in l_\infty^{m+1}.$$

Clearly S is a linear operator. From our assumption, for every $a \in l_\infty^{m+1}$ there is a $y \in F^{(1)}$ such that

$$\|S(a)\| = |T\gamma_a(y)| \ge \epsilon\|a\|.$$

Hence, S is an isomorphism of l_∞^{m+1} into l_∞^m, which is impossible. ∎

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We will assume the existence of an isomorphism T of $C(K)$ onto $C(F)$ such that $\|T\|\,\|T^{-1}\| < 2n+1$ and obtain a contradiction.

Without loss of generality we may assume that $\|T^{-1}\| = 1$ so that T is norm-increasing. Otherwise we may simply replace T by the isomorphism $\|T^{-1}\|T$.

Pick $0 < \epsilon < 1$ and $\eta > 0$ such that

$$\|T\| < (2n+1)\frac{1-\epsilon}{1+\epsilon} \quad \text{and} \quad \eta < \min\left\{\epsilon, \frac{(2n+1)(1-\epsilon) - \|T\|}{2}\right\}.$$

By Proposition 2.2, there is $x_0 \in K^{(n)}$, a compact neighborhood K_0 of x_0 and a function $h_0 \in C(K)$ such that $0 \le h_0 \le 1$, $h_0(x) = 1$ for every $x \in K_0$, and $|Th_0(y)| < \epsilon$ for every $y \in F^{(1)}$. Related to x_0, K_0, h_0 and $\epsilon > 0$, consider points $x_1, \dots, x_{n-1} \in K$, compact subsets $K_1, \dots, K_{n-1} \subset K$, functions $h_1, \dots, h_{n-1} \in C(K)$ and subsets $G_0, G_1, \dots, G_{n-1} \subset F$ satisfying the statements (a)–(d) of Proposition 2.1. For each $0 \le i \le n-1$, define

$$g_i = \chi_{G_i} \cdot Th_i,$$

where χ_{G_i} stands for the characteristic function of G_i. Notice that $g_i \in C(F)$ for each $0 \le i \le n-1$.

Let G be the finite set $\bigcup_{i=0}^{n-1} G_i$. For each $y \in G$ let δ_y be the unit point mass at y. By the Riesz Representation Theorem [8, Theorem 18.4.1, p. 312] we identify δ_y with a linear functional in $C(F)^*$. Then it is clear that

$$H = \bigcup_{y \in G} \{x \in K : |T^*(\delta_y)|(\{x\}) > \eta\}$$

is a finite set. Hence, there is $z \in \mathring{K}_{n-1} \setminus H$ such that $|T^*(\delta_y)|(\{z\}) < \eta$ for each $y \in G$. By regularity, we can find an open neighborhood of z, $U \subset K_{n-1}$, such that $|T^*(\delta_y)|(U) < \eta$ for every $y \in G$.

Thanks to the Urysohn Lemma, we can take $h_n \in C(K)$ such that $0 \le h_n \le 1$, $h_n(z) = 1$ and $h_n(x) = 0$ if $x \notin U$. Let $\alpha \in C(F)$ be defined by

$$\alpha(y) = g_0(y) + 2\sum_{i=1}^{n-1} g_i(y) + 2Th_n(y), \qquad y \in F.$$

Claim 1. $\|\alpha\| = \max\{2\|Th_n\|, |\alpha(y)| : y \in G\}$.

In order to establish this, notice that for every $y \in G$,

$$|Th_n(y)| = \Big|\int Th_n \, d\delta_y\Big| = \Big|\int h_n \, dT^*(\delta_y)\Big| \le |T^*(\delta_y)|(U) < \eta < 1. \tag{2.2}$$

On the other hand, if $y \mapsto |Th_n(y)|$ attains its maximum at $y_0 \in F$, since T is norm-increasing we have

$$|Th_n(y_0)| = \|Th_n\| \ge 1, \tag{2.3}$$

and hence $y_0 \in F \setminus G$. Since $\alpha(y) = 2Th_n(y)$ for $y \in F \setminus G$, our claim is established.

Claim 2. $\|\alpha\| \ge (2n+1) - (2n-1)\epsilon$.

Since $\|T^{-1}\|$=1, we have

$$\begin{aligned} \|\alpha\| &= \Big\|g_0 + 2\sum_{i=1}^{n-1} g_i + 2Th_n\Big\| \ge \Big\|T^{-1}g_0 + 2\sum_{i=1}^{n-1} T^{-1}g_i + 2h_n\Big\| \\ &\ge \Big|\Big(h_0(z) + 2\sum_{i=1}^{n} h_i(z)\Big) - (h_0(z) - T^{-1}g_0(z)) - 2\sum_{i=1}^{n-1}(h_i(z) - T^{-1}g_i(z))\Big| \\ &\ge \Big|h_0(z) + 2\sum_{i=1}^{n} h_i(z)\Big| - |h_0(z) - T^{-1}g_0(z)| - 2\sum_{i=1}^{n-1} |h_i(z) - T^{-1}g_i(z)|, \end{aligned} \tag{2.4}$$

and since $\|f\| \le \|Tf\|$, $f \in C(K)$, we have, for each $0 \le i \le n-1$,

$$\begin{aligned}(2.5)\qquad |h_i(z) - T^{-1}g_i(z)| &\le \|h_i - T^{-1}g_i\| \le \|Th_i - g_i\| \\ &= \|(1-\chi_{G_i})\cdot Th_i\| \le \epsilon.\end{aligned}$$

Putting (2.4) and (2.5) together and recalling the definition of h_i we see that Claim 2 is true.

In view of Claims 1 and 2 there are two possibilities:

(i) $2\|Th_n\| \ge (2n+1) - (2n-1)\epsilon$,
(ii) $|\alpha(y)| \ge (2n+1) - (2n-1)\epsilon$ for some $y \in G$.

We will show that both lead to a contradiction.

Suppose first that (i) holds. Set $A = T^{-1}g_0 - 2h_n$. Since $0 \le h_n \le h_0 \le 1$ and $\|f\| \le \|Tf\|$ for all $f \in C(K)$, for every $x \in K$ we have

$$\begin{aligned}|T^{-1}g_0(x) - 2h_n(x)| &\le |h_0(x) - 2h_n(x)| + |T^{-1}g_0(x) - h_0(x)| \\ &\le 1 + \|T^{-1}g_0 - h_0\| \le 1 + \|g_0 - Th_0\| \le 1+\epsilon.\end{aligned}$$

So $\|A\| \le 1+\epsilon$.

Recalling (2.2) and (2.3), we can fix $y_0 \in F \setminus G$ such that $\|Th_n\| = |Th_n(y_0)|$. It follows that

$$|T(A)(y_0)| = 2|Th_n(y_0)| = 2\|Th_n\| \ge (2n+1) - (2n-1)\epsilon > (2n+1)(1-\epsilon).$$

Consequently,

$$\|T\| \ge \left\| T\left(\frac{1}{1+\epsilon}A\right)\right\| > (2n+1)\frac{1-\epsilon}{1+\epsilon},$$

contradicting the choice of ϵ.

Now, assume that (ii) holds. We distinguish two cases.

Case 1: $\|\alpha\| = |\alpha(y_0)|$ for some $y_0 \in G_0$. Since $G_0, G_1, \ldots, G_{n-1}$ are pairwise disjoint we have

$$|\alpha(y_0)| = |g_0(y_0) + 2Th_n(y_0)| \ge (2n+1) - (2n-1)\epsilon.$$

By the choice of η we deduce

$$\begin{aligned}|g_0(y_0)| &\ge (2n+1) - (2n-1)\epsilon - 2|Th_n(y_0)| \\ &> (2n+1) - (2n-1)\epsilon - 2\eta > \|T\|.\end{aligned}$$

Therefore,

$$\|T\| \ge \|Th_0\| \ge |Th_0(y_0)| = |g_0(y_0)| > \|T\|,$$

a contradiction.

Case 2: $\|\alpha\| = |\alpha(y_0)|$ for some $y_0 \in G_i$, $i > 0$. Since $G_0, G_1, \ldots, G_{n-1}$ are pairwise disjoint we have

$$|\alpha(y_0)| = |2g_i(y_0) + 2Th_n(y_0)| \ge (2n+1) - (2n-1)\epsilon.$$

By recalling (2.2) and since $\eta < \epsilon$, we infer

$$2|g_i(y_0)| \geq (2n+1) - (2n-1)\epsilon - 2|Th_n(y_0)| > (2n+1)(1-\epsilon). \tag{2.6}$$

Next, set

$$B_i = T^{-1}g_0 - 2h_i.$$

Recalling that $0 \leq h_i \leq h_0 \leq 1$ and $\|f\| \leq \|Tf\|$ for all $f \in C(K)$, for every $x \in K$ we have

$$\begin{aligned} |T^{-1}g_0(x) - 2h_i(x)| &\leq |h_0(x) - 2h_i(x)| + |T^{-1}g_0(x) - h_0(x)| \\ &\leq 1 + \|T^{-1}g_0 - h_0\| \leq 1 + \|g_0 - Th_0\| \leq 1 + \epsilon. \end{aligned}$$

It follows that $\|B_i\| \leq 1 + \epsilon$. Moreover, from (2.6), we conclude that

$$|TB_i(y_0)| = 2|Th_i(y_0)| = 2|g_i(y_0)| > (2n+1)(1-\epsilon).$$

Thus,

$$\|T\| \geq \left\| T\left(\frac{1}{1+\epsilon}B_i\right)\right\| > (2n+1)\frac{1-\epsilon}{1+\epsilon},$$

a contradiction.

This completes the proof of Theorem 1.2. ∎

3. Special isomorphisms between $C(\omega)$ and $C(\omega^n k)$. The purpose of this section is to prove Theorem 3.1. It establishes the existence of some special isomorphisms between $C(\omega)$ and $C(\omega^n k)$, where $1 \leq n, k < \omega$, and it will be the key ingredient in proving Theorem 1.4 in the next section.

THEOREM 3.1. *Let $A > 1$ be a real number and $1 \leq k, n < \omega$ ordinal numbers. There is an isomorphism T of $C(\omega^n k)$ onto $C(\omega)$ such that*

$$\|T\|\,\|T^{-1}\| = \begin{cases} \max\left\{\dfrac{2nA}{A-1} + 1, 2A-1\right\} & \text{if } k > 1, \\ \max\left\{\dfrac{2(n-1)A}{A-1} + 1, 2A-1\right\} & \text{if } k = 1 \text{ and } n > 1. \end{cases}$$

We start by proving two preliminary results on sequences of ordinal numbers (Propositions 3.4 and 3.5).

In order to simplify the notation of certain sequences of ordinal numbers we will introduce some new terminology. First, we recall that each ordinal number $1 \leq \xi < \omega^\omega$ can be written in a unique way in *Cantor normal form* (see [8, p. 153])

$$\xi = \omega^{n_k} m_k + \cdots + \omega^{n_1} m_1 \tag{3.1}$$

where $0 \leq n_1 < \cdots < n_k < \omega$, $1 \leq m_1 < \omega, \ldots, 1 \leq m_k < \omega$ and $1 \leq k < \omega$.

Definition 3.2. For each ordinal number $1 \leq \xi < \omega^\omega$, written in Cantor normal form, as in (3.1), we set $\xi^{[0]} = \xi$ and by induction

$$\xi^{[r]} = \begin{cases} \omega^{n_k} m_k + \cdots + \omega^{n_2} m_2 + \omega^{n_1+1} & \text{if } r = 1, \\ (\xi^{[r-1]})^{[1]} & \text{if } 1 \leq r < \omega. \end{cases}$$

Remark 3.3. By using the Cantor normal form, it is easy to see that each ordinal number $1 \leq \xi < \omega^{n+1}$ admits a unique representation in the form

$$\xi = \omega^n i_0 + \omega^{n-1} i_1 + \cdots + \omega^{n-(j-1)} i_{j-1} + \omega^{n-j} i_j \tag{3.2}$$

where $0 \leq j \leq n$, $1 \leq i_j < \omega$ and $0 \leq i_r < \omega$ if $0 \leq r \leq j-1$.

This alternative representation is more convenient for the function $\xi \mapsto \xi^{[1]}$ of Definition 3.2. For an ordinal $1 \leq \xi < \omega^{n+1}$ written as in (3.2), we have

$$\begin{aligned} \xi^{[1]} &= \omega^n i_0 + \omega^{n-1} i_1 + \cdots + \omega^{n-(j-1)}(i_{j-1} + 1), \\ \xi^{[2]} &= \omega^n i_0 + \omega^{n-1} i_1 + \cdots + \omega^{n-(j-2)}(i_{j-2} + 1), \\ &\vdots \\ \xi^{[j-1]} &= \omega^n i_0 + \omega^{n-1}(i_1 + 1), \\ \xi^{[j]} &= \omega^n (i_0 + 1). \end{aligned}$$

Proposition 3.4. *Let A and B be real numbers and $1 \leq n < \omega$. For each $f \in C(\omega^n)$ consider the sequence $(a_\xi)_{1 \leq \xi \leq \omega^n}$ defined by*

$$a_\xi = \begin{cases} A & \text{if } \xi = \omega^n, \\ B(f(\xi) - f(\xi^{[1]})) & \text{if } 1 \leq \xi < \omega^n. \end{cases}$$

Then for each $\epsilon > 0$ there are only a finite number of ordinals $1 \leq \xi \leq \omega^n$ such that $|a_\xi| \geq \epsilon$.

Proof. We will argue by finite induction on n. Clearly, the proposition is true for $n = 1$. Assume that it is true for $n-1$ with $n \geq 2$. Fix $f \in C(\omega^n)$ and consider the sequence $(a_\xi)_{1 \leq \xi \leq \omega^n}$ defined as in the statement.

Given $\epsilon > 0$, by the continuity of f there is $1 \leq m < \omega$ such that

$$\xi \in \,]\omega^{n-1} m, \omega^n] \;\Rightarrow\; |f(\xi) - f(\omega^n)| < \frac{\epsilon}{2(|B|+1)}.$$

If $\xi \in \,]\omega^{n-1} m, \omega^n[$, then $\xi^{[1]} \in \,]\omega^{n-1} m, \omega^n]$. Thus

$$\begin{aligned} |a_\xi| &= |B|\,|f(\xi) - f(\xi^{[1]})| \leq |B|\big(|f(\xi) - f(\omega^n)| + |f(\xi^{[1]}) - f(\omega^n)|\big) \\ &< \frac{|B|\epsilon}{2(|B|+1)} + \frac{|B|\epsilon}{2(|B|+1)} < \epsilon. \end{aligned}$$

For each $1 \leq r \leq m$ define $g_r \in C(\omega^{n-1})$ by

$$g_r(\xi) = f(\omega^{n-1}(r-1) + \xi), \qquad 1 \leq \xi \leq \omega^{n-1},$$

and consider the sequence $(a^r_\xi)_{1\le\xi\le\omega^{n-1}}$ given by

$$a^r_\xi = \begin{cases} A & \text{if } \xi=\omega^{n-1},\\ B(g_r(\xi)-g_r(\xi^{[1]})) & \text{if } 1\le\xi<\omega^{n-1}.\end{cases}$$

According to the induction hypothesis, there are only a finite number of ordinals $1\le\xi\le\omega^{n-1}$ such that $|a^r_\xi|\ge\epsilon$. Moreover, by construction,

$$a^r_\xi = a_{\omega^{n-1}(r-1)+\xi}, \qquad 1\le\xi<\omega^{n-1}.$$

So, we deduce that for each $1\le r\le m$ there are only a finite number of ordinals ξ in the interval $[\omega^{n-1}(r-1)+1,\omega^{n-1}r]$ satisfying $|a_\xi|\ge\epsilon$. Since $[1,\omega^n]$ is the union of the intervals $[1,\omega^{n-1}],\ldots,[\omega^{n-1}(m-1)+1,\omega^{n-1}m]$ and $[\omega^{n-1}m+1,\omega^n]$, we are done. ∎

Proposition 3.5. *Let A,B,C,D,E be real numbers and $1\le n,k<\omega$. For each $f\in C(\omega^n k)$ consider the sequence $(a_\xi)_{1\le\xi\le\omega^n k}$ given by*

$$\begin{cases} A & \text{if } \xi=\omega^n k,\\ B(f(\xi)-f(\xi^{[1]})) & \text{if } \xi=\omega^n(k-1)+\omega^{n-1}i,\ i\ge1,\\ C(f(\xi)-f(\xi^{[1]})) & \text{if } \xi\in\,]\omega^n(k-1)+\omega^{n-1}(i-1),\omega^n(k-1)+\omega^{n-1}i[,\ i\ge1,\\ D(f(\omega^n r)-f(\omega^n k)) & \text{if } \xi=\omega^n r,\ 1\le r\le k-1,\\ E(f(\xi)-f(\xi^{[1]})) & \text{if } \xi\in\,]\omega^n(r-1),\omega^n r[,\ 1\le r\le k-1.\end{cases}$$

Then for each $\epsilon>0$ there are only a finite number of ordinals $1\le\xi\le\omega^n k$ such that $|a_\xi|\ge\epsilon$.

Proof. Fix $f\in C(\omega^n k)$ and consider the sequence $(a_\xi)_{1\le\xi\le\omega^n k}$ defined as in the statement. Let $g\in C(\omega^n)$ be defined by

$$g(\xi)=f(\omega^n(k-1)+\xi), \qquad 1\le\xi\le\omega^n,$$

and let $(b_\xi)_{1\le\xi\le\omega^n}$ be given by

$$\begin{cases} A & \text{if } \xi=\omega^n,\\ B(g(\xi)-g(\xi^{[1]})) & \text{if } \xi=\omega^{n-1}i,\ i\ge1,\\ C(g(\xi)-g(\xi^{[1]})) & \text{if } \xi\in\,]\omega^{n-1}(i-1),\omega^{n-1}i[,\ i\ge1.\end{cases}$$

According to Proposition 3.4 there are only a finite number of ordinals $1\le\xi\le\omega^n$ such that $|b_\xi|\ge\epsilon$. Since

$$b_\xi=a_{\omega^n(k-1)+\xi}, \qquad 1\le\xi\le\omega^n,$$

we deduce that there are only a finite number of ordinals ξ in the interval $[\omega^n(k-1)+1,\omega^n k]$ satisfying $|a_\xi|\ge\epsilon$.

If $k>1$, for each $1\le r\le k-1$ define $h_r\in C(\omega^n)$ as follows:

$$h_r(\xi)=f(\omega^n(r-1)+\xi), \qquad 1\le\xi\le\omega^n.$$

Next, consider the sequence $(c^r_\xi)_{1\le\xi\le\omega^n}$ given by

$$c^r_\xi=\begin{cases} D(f(\omega^n r)-f(\omega^n k)) & \text{if } \xi=\omega^n,\\ E(h_r(\xi)-h_r(\xi^{[1]})) & \text{if } 1\le\xi<\omega^n.\end{cases}$$

Once more, by Proposition 3.4, for each $1 \le r \le k-1$, there are only a finite number of ordinals $1 \le \xi \le \omega^n$ such that $|c^r_\xi| \ge \epsilon$. Since

$$c^r_\xi = a_{\omega^n(r-1)+\xi}, \qquad 1 \le \xi \le \omega^n,$$

we conclude that, for each $1 \le r \le k-1$, there are only a finite number of ordinals ξ in the interval $[\omega^n(r-1)+1, \omega^n r]$ satisfying $|a_\xi| \ge \epsilon$. Moreover, since $[1, \omega^n k]$ is the union of the intervals $[1, \omega^n], \ldots, [\omega^n(k-2)+1, \omega^n(k-1)]$ and $[\omega^n(k-1)+1, \omega^n k]$, we are done. ∎

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Given $1 \le k, n < \omega$, let $\Gamma_{n,k}$, $1 \le k, n < \omega$, be the interval $]0, \omega^n k[$ endowed with the discrete topology. We denote by $K_{n,k} = \Gamma_{n,k} \dot\cup \{\omega^n k\}$ the Aleksandrov compactification of $\Gamma_{n,k}$. In order to simplify the proof, we will replace the space $C(\omega)$ by $C(K_{n,k})$. These spaces are isometrically isomorphic.

Let $A > 1$. For each $f \in C(\omega^n k)$ consider the function $T(f) : K_{n,k} \to \mathbb{K}$ defined by

$$T(f)(\xi) = \begin{cases} f(\omega^n k) & \text{if } \xi = \omega^n k, \\ Af(\xi) - (A-1)f(\xi^{[1]}) & \text{if } \xi = \omega^n(k-1) + \omega^{n-1} i,\ i \ge 1, \\ \dfrac{(n-1)A}{A-1}(f(\xi) - f(\xi^{[1]})) + f(\omega^n k) & \text{if } \xi \in]\omega^n(k-1)+\omega^{n-1}(i-1), \omega^n(k-1)+\omega^{n-1} i[, \\ & i \ge 1, \\ Af(\omega^n r) - (A-1)f(\omega^n k) & \text{if } \xi = \omega^n r,\ 1 \le r \le k-1, \\ \dfrac{nA}{A-1}(f(\xi) - f(\xi^{[1]})) + f(\omega^n k) & \text{if } \xi \in]\omega^n(r-1), \omega^n r[,\ 1 \le r \le k-1. \end{cases}$$

We have to demonstrate that $T(f) \in C(K_{n,k})$ for every $f \in C(\omega^n k)$. Indeed, given $f \in C(\omega^n k)$ consider the function

$$G = T(f) - f(\omega^n k).$$

More explicitly,

$$G(\xi) = \begin{cases} 0 & \text{if } \xi = \omega^n k, \\ A(f(\xi) - f(\xi^{[1]})) & \text{if } \xi = \omega^n(k-1) + \omega^{n-1} i,\ i \ge 1, \\ \dfrac{(n-1)A}{A-1}(f(\xi) - f(\xi^{[1]})) & \text{if } \xi \in]\omega^n(k-1)+\omega^{n-1}(i-1), \omega^n(k-1)+\omega^{n-1} i[,\ i \ge 1, \\ A(f(\omega^n r) - f(\omega^n k)) & \text{if } \xi = \omega^n r,\ 1 \le r \le k-1, \\ \dfrac{nA}{A-1}(f(\xi) - f(\xi^{[1]})) & \text{if } \xi \in]\omega^n(r-1), \omega^n r[,\ 1 \le r \le k-1. \end{cases}$$

According to Proposition 3.5, for each $\epsilon > 0$ there are only a finite number of ordinals ξ in the interval $[1, \omega^n k]$ satisfying $|G(\xi)| \ge \epsilon$. It follows that $T(f)$ is continuous at $\omega^n k$. Hence $T(f) \in C(K_{n,k})$.

Now it is easy to check that T defines a bounded linear operator from $C(\omega^n k)$ to $C(K_{n,k})$. Moreover, if $k>1$, then

$$\|T\| = \max\left\{\frac{2nA}{A-1}+1, 2A-1\right\}, \tag{3.3}$$

while if $k=1$ and $n>1$, then

$$\|T\| = \max\left\{\frac{2(n-1)A}{A-1}+1, 2A-1\right\}. \tag{3.4}$$

Next, we recall Remark 3.3 and use the fact that every ordinal number $1 \le \xi < \omega^{n+1}$ admits a unique representation in the form

$$\xi = \omega^n i_0 + \omega^{n-1} i_1 + \cdots + \omega^{n-(j-1)} i_{j-1} + \omega^{n-j} i_j \tag{3.5}$$

where $0 \le j \le n$, $1 \le i_j < \omega$ and $0 \le i_r < \omega$ if $0 \le r \le j-1$.

For each $g \in C(K_{n,k})$ we consider the function $S(g) : [1, \omega^n k] \to \mathbb{K}$ defined by

$$S(g)(\xi) = \begin{cases} g(\omega^n k) & \text{if } \xi = \omega^n k, \\ \dfrac{1}{A} g(\xi) + \dfrac{A-1}{A} g(\omega^n k) & \text{if } \xi = \omega^n (k-1) + \omega^{n-1} i,\ i \ge 1, \\ \dfrac{1}{A} g(\omega^n r) + \dfrac{A-1}{A} g(\omega^n k) & \text{if } \xi = \omega^n r,\ 1 \le r \le k-1. \end{cases}$$

If $\xi \in \,]\omega^n(k-1) + \omega^{n-1}(i-1), \omega^n(k-1) + \omega^{n-1} i[$, $1 \le i < \omega$, and ξ is written as in (3.5), then

$$S(g)(\xi) = \frac{A-1}{A(n-1)} \sum_{s=0}^{j-2} (g(\xi^{[s]}) - g(\omega^n k)) + \frac{1}{A} g(\xi^{[j-1]}) + \frac{A-1}{A} g(\omega^n k),$$

and if $\xi \in \,]\omega^n(r-1), \omega^n r[, 1 \le r \le k-1$, and ξ is written as in (3.5), then

$$S(g)(\xi) = \frac{A-1}{An} \sum_{s=0}^{j-1} (g(\xi^{[s]}) - g(\omega^n k)) + \frac{1}{A} g(\xi^{[j]}) + \frac{A-1}{A} g(\omega^n k).$$

We will check that $S(g)$ is continuous on $[1, \omega^n k]$ for every $g \in C(K_{n,k})$. Fix $g \in C(K_{n,k})$ and ξ_0 a non-isolated point of the interval $[1, \omega^n k]$. Given $\epsilon > 0$ define

$$\Lambda_\epsilon = \{1 \le \xi \le \omega^n k : |g(\xi) - g(\omega^n k)| \ge \epsilon/n\}.$$

We distinguish two cases.

Case 1: $\xi_0 = \omega^n k$. Since Λ_ϵ is a finite set, there is $1 \le m < \omega$ such that

$$]\omega^n(k-1) + \omega^{n-1} m, \omega^n k[\cap \Lambda_\epsilon = \emptyset.$$

It follows from the definition of $S(g)$ that if $\xi \in \,]\omega^n(k-1) + \omega^{n-1} m, \omega^n k[$, then

$$|S(g)(\xi) - S(g)(\xi_0)| \le |g(\xi_1) - g(\omega^n k)| + \cdots + |g(\xi_s) - g(\omega^n k)|,$$

where $1 \le s \le n$ and $\xi = \xi_1 < \cdots < \xi_s < \omega^n k$. Then

$$|S(g)(\xi) - S(g)(\xi_0)| < \epsilon.$$

CASE 2: $1 \le \xi_0 < \omega^n k$. We write $\xi_0 = \omega^n i_0 + \omega^{n-1} i_1 + \cdots + \omega^{n-j} i_j$, $0 \le j < n$, $0 \le i_0 \le k-1$, $1 \le i_j < \omega$ and $0 \le i_r < \omega$ if $1 \le r \le j-1$.

Since Λ_ϵ is a finite set, there is $1 \le m < \omega$ such that

$$]\omega^n i_0 + \cdots + \omega^{n-j}(i_j - 1) + \omega^{n-(j+1)} m, \omega^n i_0 + \cdots + \omega^{n-j} i_j[\cap \Lambda_\epsilon = \emptyset.$$

On the other hand, if

$$\xi \in]\omega^n i_0 + \cdots + \omega^{n-j}(i_j - 1) + \omega^{n-(j+1)} m, \omega^n i_0 + \cdots + \omega^{n-j} i_j[,$$

then there is $1 \le s \le n-j$ such that $\xi^{[s]} = \xi_0$. By the definition of $S(g)$, we have

$$|S(g)(\xi) - S(g)(\xi_0)| \le |g(\xi_1) - g(\omega^n k)| + \cdots + |g(\xi_s) - g(\omega^n k)|,$$

where $\xi = \xi_1 < \cdots < \xi_s < \xi_0$. Hence,

$$|S(g)(\xi) - S(g)(\xi_0)| < \epsilon,$$

so that $S(g)$ is continuous at ξ_0. Therefore, S defines a function from $C(K_{n,k})$ to $C(\omega^n k)$.

Next, we will check that $S \circ T$ and $T \circ S$ are, respectively, the identity operators in $C(\omega^n k)$ and $C(K_{n,k})$. Indeed, let $f \in C(\omega^n k)$ and $\xi \in [1, \omega^n k]$.

If $\xi = \omega^n k$, then

$$(S \circ T)(f)(\omega^n k) = T(f)(\omega^n k) = f(\omega^n k).$$

If $\xi = \omega^n (k-1) + \omega^{n-1} i$, $1 \le i < \omega$, then

$$\begin{aligned}(S \circ T)(f)(\xi) &= \frac{1}{A} T(f)(\xi) + \frac{A-1}{A} T(f)(\omega^n k) \\ &= \frac{1}{A}(A f(\xi) - (A-1) f(\omega^n k)) + \frac{A-1}{A} f(\omega^n k) = f(\xi).\end{aligned}$$

If $\xi = \omega^n r$, $1 \le r \le k-1$, then

$$\begin{aligned}(S \circ T)(f)(\xi) &= \frac{1}{A} T(f)(\omega^n r) + \frac{A-1}{A} T(f)(\omega^n k) \\ &= \frac{1}{A}(A f(\omega^n r) - (A-1) f(\omega^n k)) + \frac{A-1}{A} f(\omega^n k) = f(\xi).\end{aligned}$$

If $\xi \in]\omega^n (k-1) + \omega^{n-1}(i-1), \omega^n (k-1) + \omega^{n-1} i[, 1 \le i < \omega$, and ξ is written as in (3.5), then $(S \circ T)(f)(\xi)$ is equal to

$$\frac{A-1}{A(n-1)}\sum_{s=0}^{j-2}\big(T(f)(\xi^{[s]})-T(f)(\omega^n k)\big)+\frac{1}{A}T(f)(\xi^{[j-1]})+\frac{A-1}{A}T(f)(\omega^n k)$$
$$=\frac{A-1}{A(n-1)}\sum_{s=0}^{j-2}\left(\frac{(n-1)A}{A-1}(f(\xi^{[s]})-f(\xi^{[s+1]}))\right)+f(\xi^{[j-1]})$$
$$=(f(\xi)-f(\xi^{[j-1]}))+f(\xi^{[j-1]})=f(\xi).$$

If $\xi\in\,]\omega^n(r-1),\omega^n r[, 1\le r\le k-1$, and ξ is written as in (3.5), then $(S\circ T)(f)(\xi)$ is equal to

$$\frac{A-1}{An}\sum_{s=0}^{j-1}\big(T(f)(\xi^{[s]})-T(f)(\omega^n k)\big)+\frac{1}{A}T(f)(\xi^{[j]})+\frac{A-1}{A}T(f)(\omega^n k)$$
$$=\frac{A-1}{An}\sum_{s=0}^{j-1}\left(\frac{nA}{A-1}(f(\xi^{[s]})-f(\xi^{[s+1]}))\right)+f(\xi^{[j]})$$
$$=(f(\xi)-f(\xi^{[j]}))+f(\xi^{[j]})=f(\xi).$$

We conclude that $(S\circ T)(f)=f$ for all $f\in C(\omega^n k)$.

Now, let $g\in C_0(K_{n,k})$ and $\xi\in K_{n,k}$.

If $\xi=\omega^n k$, then

$$(T\circ S)(g)(\omega^n k)=S(g)(\omega^n k)=g(\omega^n k).$$

If $\xi=\omega^n(k-1)+\omega^{n-1}i$, $1\le i<\omega$, then

$$(T\circ S)(g)(\xi)=AS(g)(\xi)-(A-1)S(g)(\omega^n k)$$
$$=A\left(\frac{1}{A}g(\xi)+\frac{A-1}{A}g(\omega^n k)\right)-(A-1)g(\omega^n k)=g(\xi).$$

If $\xi=\omega^n r$, $1\le r\le k-1$, then

$$(T\circ S)(g)(\xi)=AS(g)(\omega^n r)-(A-1)S(g)(\omega^n k)$$
$$=A\left(\frac{1}{A}g(\omega^n r)+\frac{A-1}{A}g(\omega^n k)\right)-(A-1)g(\omega^n k)=g(\xi).$$

If $\xi\in\,]\omega^n(k-1)+\omega^{n-1}(i-1),\omega^n(k-1)+\omega^{n-1}i[,\ 1\le i<\omega$, and ξ is written as in (3.5), then

$$(T\circ S)(g)(\xi)=\frac{(n-1)A}{A-1}\big(S(g)(\xi)-S(g)(\xi^{[1]})\big)+S(g)(\omega^n k)$$
$$=\frac{(n-1)A}{A-1}\left(\frac{A-1}{A(n-1)}(g(\xi)-g(\omega^n k))\right)+g(\omega^n k)=g(\xi).$$

If $\xi \in \,]\omega^n(r-1), \omega^n r[$, $1 \le r \le k-1$, and ξ is written as in (3.5), then

$$\begin{aligned}(T\circ S)(g)(\xi) &= \frac{nA}{A-1}\big(S(g)(\xi) - S(g)(\xi^{[1]})\big) + S(g)(\omega^n k)\\ &= \frac{nA}{A-1}\left(\frac{A-1}{An}(g(\xi) - g(\omega^n k))\right) + g(\omega^n k) = g(\xi).\end{aligned}$$

We infer that $(T\circ S)(g) = g$ for every $g \in C(K_{n,k})$ and $(S\circ T)(f) = f$ for each $f \in C(\omega^n k)$. Therefore, S is the inverse operator of T; moreover, S is linear and satisfies

$$\|S\| = 1. \tag{3.6}$$

Combining the relations (3.3), (3.4) and (3.6), we are done. ■

4. An upper bound on $d(C(\omega), C(\omega^n k))$ where $1 \le n, k < \omega$. In this last section we prove Theorem 1.4.

Proof of Theorem 1.4. First we will prove (a) and (c). Notice that according to Theorem 3.1, if $k > 1$ and $n \ge 1$ then

$$d(C(\omega), C(\omega^n k)) \le \inf_{A>1} \max\left\{\frac{2nA}{A-1} + 1, 2A-1\right\}. \tag{4.1}$$

Now observe that for $A > 1$ the function $f(A) = \frac{2nA}{A-1} + 1$ is strictly decreasing while $g(A) = 2A-1$ is strictly increasing. So, the infimum in (4.1) is attained when $f(A) = g(A)$ and this happens when

$$A = \frac{n+2+\sqrt{n(n+4)}}{2}.$$

Hence

$$d(C(\omega), C(\omega^n k)) \le n+1+\sqrt{n(n+4)}.$$

In particular, when $n = 1$ and $k > 1$ it follows that

$$d(C(\omega), C(\omega k)) \le 2+\sqrt{5},$$

and thus (a) and (c) hold.

Finally, we prove (b). Pick $k = 1$ and $n > 1$. Then, once more by Theorem 3.1, we see that

$$d(C(\omega), C(\omega^n)) \le \inf_{A>1} \max\left\{\frac{2(n-1)A}{A-1} + 1, 2A-1\right\}. \tag{4.2}$$

Next notice that for $A > 1$ the function $f(A) = \frac{2(n-1)A}{A-1} + 1$ is strictly decreasing while $g(A) = 2A-1$ is strictly increasing. Thus, the infimum in (4.2) is attained when $f(A) = g(A)$, that is, when

$$A = \frac{n+1+\sqrt{(n-1)(n+3)}}{2}.$$

Consequently,

$$d(C(\omega), C(\omega^n)) \leq n + \sqrt{(n-1)(n+3)},$$

therefore (b) holds, and the proof is complete. ∎

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