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The space of maximal Fourier multipliers as a dual space

by

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Abstract. Figà-Talamanca characterized the space of Fourier multipliers as the dual space of a certain Banach space. In this paper, we characterize the space of maximal Fourier multipliers as a dual space.

1. Introduction. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. The space $M_p(\mathbb{R}^n)$ of Fourier multipliers consists of all $m \in L^\infty(\mathbb{R}^n)$ such that T_m is bounded on $L^p(\mathbb{R}^n)$, where T_m is defined by $T_m f = \mathcal{F}^{-1}[m\widehat{f}]$ for $f \in \mathcal{S}(\mathbb{R}^n)$. We define the norm on $M_p(\mathbb{R}^n)$ by $\|m\|_{M_p} = \sup \|T_m f\|_{L^p}$, where the supremum is taken over all $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f\|_{L^p} = 1$. Let $C_0(\mathbb{R}^n)$ be the space of all continuous functions such that $\lim_{|x|\to\infty} f(x) = 0$. For $1 < p < \infty$, p' is the conjugate exponent of p (that is, $1/p + 1/p' = 1$). Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of all integers and positive integers, respectively. The space $A_p(\mathbb{R}^n)$ consists of all $f \in C_0(\mathbb{R}^n)$ which can be written as $f = \sum_{i\in\mathbb{N}} f_i * g_i$ in $L^\infty(\mathbb{R}^n)$, where $\{f_i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ and $\sum_{i\in\mathbb{N}} \|f_i\|_{L^p}\|g_i\|_{L^{p'}} < \infty$. Then the norm $\|f\|_{A_p}$ is the infimum of the last sums over all representations of f.

In [6], Figà-Talamanca proved that $M_p(\mathbb{R}^n) = A_p(\mathbb{R}^n)^*$, where $A_p(\mathbb{R}^n)^*$ is the dual space of $A_p(\mathbb{R}^n)$ (see also Larsen [10]). Berkson, Paluszyński and Weiss applied Figà-Talamanca's result to wavelet theory [2] (for other applications, see Asmar, Berkson and Gillespie [1] and Figà-Talamanca and Gaudry [7]).

Maximal functions generated by Fourier multipliers were studied by, for example, Christ, Grafakos, Honzík and Seeger [3], Dappa and Trebels [4] and Kenig and Tomas [9]. For $m \in L^\infty(\mathbb{R}^n)$, the dyadic maximal Fourier

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multiplier operator M_m is defined by

$$M_m f(x) = \sup_{j\in\mathbb{Z}} |T_{m(2^j\cdot)} f(x)| = \sup_{j\in\mathbb{Z}} |\mathcal{F}^{-1}[m(2^j\cdot)\widehat{f}](x)|$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ ([3], [4]). We denote by $\max M_p(\mathbb{R}^n)$ the space of all $m \in L^\infty(\mathbb{R}^n)$ such that M_m is bounded on $L^p(\mathbb{R}^n)$. We define the norm on $\max M_p(\mathbb{R}^n)$ by

$$\|m\|_{\max M_p} = \sup\{\|M_m f\|_{L^p} : f \in \mathcal{S}(\mathbb{R}^n),\ \|f\|_{L^p} = 1\}.$$

Then $\max M_p(\mathbb{R}^n)$ is a Banach space (Proposition 3.1). The purpose of this paper is to characterize $\max M_p(\mathbb{R}^n)$ as the dual space of a certain normed space. The space $\widetilde{A}_p(\mathbb{R}^n)$ consists of all $f \in C_0(\mathbb{R}^n)$ which can be written as

$$f = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) \quad \text{in } L^\infty(\mathbb{R}^n),$$

where $\{f_i\}_{i\in\mathbb{N}}, \{g_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ and $\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \|f_i\|_{L^p} \|g_{i,j}\|_{L^{p'}} < \infty$. Note that, if the last condition is satisfied, then $\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) \in C_0(\mathbb{R}^n)$ and $\sum_{i\in\mathbb{N}} \|f_i\|_{L^p} \|\{g_{i,j}\}_{j\in\mathbb{Z}}\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} < \infty$, where the norm $\|\{g_j\}_{j\in\mathbb{Z}}\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))}$ is, by definition, $\{\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} |g_j(x)|)^{p'}\, dx\}^{1/p'}$. We define the norm on $\widetilde{A}_p(\mathbb{R}^n)$ by

$$\|f\|_{\widetilde{A}_p} = \inf\Big\{ \sum_{i\in\mathbb{N}} \|f_i\|_{L^p} \|\{g_{i,j}\}_{j\in\mathbb{Z}}\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} : f = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) \Big\}.$$

Then $\widetilde{A}_p(\mathbb{R}^n)$ is a normed space (Proposition 3.2). Also, $A_p(\mathbb{R}^n)$ is continuously embedded in $\widetilde{A}_p(\mathbb{R}^n)$. For $m \in \max M_p(\mathbb{R}^n)$, we define a linear functional φ_m on $\widetilde{A}_p(\mathbb{R}^n)$ by

$$\varphi_m(f) = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i * g_{i,j}(0) \tag{1}$$

for $f = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) \in \widetilde{A}_p(\mathbb{R}^n)$. We note that the right hand side of (1) is independent of the representation of f (Lemma 3.6). Our main result is the following.

THEOREM 1. *Let $1 < p < \infty$. If $m \in \max M_p(\mathbb{R}^n)$, then $\varphi_m \in \widetilde{A}_p(\mathbb{R}^n)^*$ and $\|\varphi_m\|_{(\widetilde{A}_p)^*} = \|m\|_{\max M_p}$. Conversely, if $\varphi \in \widetilde{A}_p(\mathbb{R}^n)^*$, then there exists $m \in \max M_p(\mathbb{R}^n)$ such that $\varphi = \varphi_m$. In this sense, $\max M_p(\mathbb{R}^n) = \widetilde{A}_p(\mathbb{R}^n)^*$.*

2. Preliminaries. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx, \qquad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

We also define the Fourier transform $\mathcal{F}u$ and the inverse Fourier transform $\mathcal{F}^{-1}u$ of $u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \mathcal{F}u, \psi \rangle = \langle u, \mathcal{F}\psi \rangle, \quad \langle \mathcal{F}^{-1}u, \psi \rangle = \langle u, \mathcal{F}^{-1}\psi \rangle \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^n).$$

Note that, if u is an appropriate function, then $\langle u, \psi \rangle = \int_{\mathbb{R}^n} u(x)\psi(x)\,dx$. For $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, the convolution $u * \psi$ is defined by $u * \psi(x) = \langle u, \tau_x \check{\psi} \rangle$, where $\tau_x \check{\psi}(y) = \check{\psi}(y - x)$ and $\check{\psi}(y) = \psi(-y)$. As usual, for a function ψ on $\mathbb{R}^n$ and $t > 0$, we write $\psi_t(x) = t^{-n}\psi(x/t)$.

The Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-y)|\,dy$$

for all locally integrable functions f on $\mathbb{R}^n$, where $B(0,r)$ is the ball of radius r centered at the origin and $|B(0,r)|$ denotes the Lebesgue measure of $B(0,r)$. The following lemma appears as [5, Proposition 2.7].

LEMMA 2.1. *Let ψ be a function on $\mathbb{R}^n$ which is dominated by a non-negative, radial, decreasing* (*as a function on* $(0,\infty)$) *and integrable function. Then there exists a constant $C > 0$ such that*

$$\sup_{t>0} |(\psi_t * f)(x)| \le CMf(x)$$

for all locally integrable functions f.

3. Proofs. Throughout the rest of the paper, we always assume $1 < p < \infty$.

PROPOSITION 3.1. $\max M_p(\mathbb{R}^n)$ *is a Banach space.*

Proof. We first check that $\|\cdot\|_{\max M_p}$ is a norm. Since $\|\cdot\|_{M_p} \le \|\cdot\|_{\max M_p}$ and $\|\cdot\|_{L^\infty} \le \|\cdot\|_{M_p}$ ([8, p. 217]), it follows that if $\|m\|_{\max M_p} = 0$ then $m = 0$. Let $m, m_1, m_2 \in \max M_p(\mathbb{R}^n)$ and $\alpha \in \mathbb{C}$. Then $M_{\alpha m} f = |\alpha| M_m f$ and $M_{m_1+m_2} f \le M_{m_1} f + M_{m_2} f$ give $\|\alpha m\|_{\max M_p} = |\alpha|\,\|m\|_{\max M_p}$ and $\|m_1 + m_2\|_{\max M_p} \le \|m_1\|_{\max M_p} + \|m_2\|_{\max M_p}$.

We next check that $\max M_p(\mathbb{R}^n)$ is complete. Let $\{m_k\} \subset \max M_p(\mathbb{R}^n)$ be a Cauchy sequence. Since $M_p(\mathbb{R}^n)$ is complete, and $\|\cdot\|_{M_p} \le \|\cdot\|_{\max M_p}$, we see that there exists $m \in M_p(\mathbb{R}^n)$ such that $m_k \to m$ in $M_p(\mathbb{R}^n)$ as $k \to \infty$. From $\|\cdot\|_{L^\infty} \le \|\cdot\|_{M_p}$ it follows that $m_k \to m$ in $L^\infty(\mathbb{R}^n)$ as $k \to \infty$. Hence, $m_k \to m$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$. Since $m_k(2^j\cdot) \to m(2^j\cdot)$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$ for all $j \in \mathbb{Z}$, we see that $T_{m_k(2^j\cdot)} f(x) \to T_{m(2^j\cdot)} f(x)$ as $k \to \infty$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. This gives

$$\begin{aligned} |T_{m_k(2^j\cdot)} f(x) - T_{m(2^j\cdot)} f(x)| &= \lim_{k'\to\infty} |T_{m_k(2^j\cdot)} f(x) - T_{m_{k'}(2^j\cdot)} f(x)| \\ &= \liminf_{k'\to\infty} |T_{m_k(2^j\cdot)} f(x) - T_{m_{k'}(2^j\cdot)} f(x)| \le \liminf_{k'\to\infty} M_{m_k - m_{k'}} f(x), \end{aligned}$$

so $M_{m_k-m}f \le \liminf_{k'\to\infty} M_{m_k-m_{k'}}f$. On the other hand, since $\{m_k\}$ is a Cauchy sequence, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|m_k - m_{k'}\|_{\max M_p} = \sup \|M_{m_k-m_{k'}}f\|_{L^p} < \varepsilon$$

for all $k, k' \ge N$, where the supremum is taken over all $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f\|_{L^p} = 1$. Therefore, by Fatou's lemma, we get

$$\|M_{m_k-m}f\|_{L^p} \le \|\liminf_{k'\to\infty} M_{m_k-m_{k'}}f\|_{L^p} \le \liminf_{k'\to\infty} \|M_{m_k-m_{k'}}f\|_{L^p} \le \varepsilon$$

for all $k \ge N$ and $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f\|_{L^p} = 1$. The proof is complete. ∎

Proposition 3.2. *$\widetilde{A}_p(\mathbb{R}^n)$ is a normed space.*

Proof. We only prove that, if $f \in \widetilde{A}_p(\mathbb{R}^n)$ and $\|f\|_{\widetilde{A}_p} = 0$, then $f = 0$. We note that $\mathcal{S}(\mathbb{R}^n) \subset \max M_p(\mathbb{R}^n)$. Indeed, from Lemma 2.1, for $\psi \in \mathcal{S}(\mathbb{R}^n)$ we have $M_\psi f(x) \le CMf(x)$, where M is the Hardy–Littlewood maximal operator (see Section 2). Since M is bounded on $L^p(\mathbb{R}^n)$ ([5, Theorem 2.5]), we see that M_ψ is bounded on $L^p(\mathbb{R}^n)$. Let $f \in \widetilde{A}_p(\mathbb{R}^n)$ and $\|f\|_{\widetilde{A}_p} = 0$. For $\varepsilon > 0$, we can find $\{f_{\varepsilon,i}\}, \{g_{\varepsilon,i,j}\} \subset \mathcal{S}(\mathbb{R}^n)$ such that $f = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_{\varepsilon,i} * g_{\varepsilon,i,j}(2^j\cdot)$ in $L^\infty(\mathbb{R}^n)$, $\sum_{i\in\mathbb{N}} \|f_{\varepsilon,i}\|_{L^p}\|\{g_{\varepsilon,i,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} < \varepsilon$ and $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_{\varepsilon,i}\|_{L^p}\|g_{\varepsilon,i,j}\|_{L^{p'}} < \infty$. Since $f \in C_0(\mathbb{R}^n)$, it is enough to prove that $\langle f, \psi\rangle = 0$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\begin{aligned}
\langle f, \psi\rangle &= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} f_{\varepsilon,i} * g_{\varepsilon,i,j}(2^j x)\psi(x)\,dx \\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f^\vee_{\varepsilon,i}(y-x) g_{\varepsilon,i,j}(y)\,dy\Big)\psi_{2^j}(x)\,dx \\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \psi_{2^j} * f^\vee_{\varepsilon,i}(y) g_{\varepsilon,i,j}(y)\,dy = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} T_{\widehat{\psi}(2^j\cdot)} f^\vee_{\varepsilon,i}(y)\, g_{\varepsilon,i,j}(y)\,dy,
\end{aligned}$$

we see that

$$\begin{aligned}
|\langle f, \psi\rangle| &\le \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} M_{\widehat{\psi}} f^\vee_{\varepsilon,i}(y) \sum_{j\in\mathbb{Z}} |g_{\varepsilon,i,j}(y)|\,dy \\
&\le \sum_{i\in\mathbb{N}} \|M_{\widehat{\psi}} f^\vee_{\varepsilon,i}\|_{L^p} \|\{g_{\varepsilon,i,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} \\
&\le \|\widehat{\psi}\|_{\max M_p} \sum_{i\in\mathbb{N}} \|f_{\varepsilon,i}\|_{L^p} \|\{g_{\varepsilon,i,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} < \|\widehat{\psi}\|_{\max M_p}\,\varepsilon.
\end{aligned}$$

Hence, the arbitrariness of ε gives $\langle f, \psi\rangle = 0$. The proof is complete. ∎

The following lemma appears as [8, (1.2)].

Lemma 3.3. *If $m \in M_p(\mathbb{R}^n)$, then $\|\psi * m\|_{M_p} \le \|\psi\|_{L^1}\|m\|_{M_p}$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$.*

LEMMA 3.4. *If* $m \in M_p(\mathbb{R}^n)$, *then* $\|\psi m\|_{M_p} \le \|\mathcal{F}^{-1}\psi\|_{L^1}\|m\|_{M_p}$ *for all* $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Use the fact that $T_{\psi m}f = [\mathcal{F}^{-1}\psi] * T_m f$. ■

LEMMA 3.5. *Let* $m \in M_p(\mathbb{R}^n)$. *If* $\{f_i\}_{i\in\mathbb{N}}, \{g_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ *satisfy* $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i\|_{L^p}\|g_{i,j}\|_{L^{p'}} < \infty$ *and* $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) = 0$ *in* $L^\infty(\mathbb{R}^n)$, *then*

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)}f_i * g_{i,j}(0) = 0.$$

Proof. Let $m \in M_p(\mathbb{R}^n)$ and ψ be a $C^\infty(\mathbb{R}^n)$-function such that $\psi(\xi) = 1$ if $|\xi| \le 1$, $\psi(\xi) = 0$ if $|\xi| \ge 2$. Also, let $\widetilde{\psi}$ be a radial $C^\infty(\mathbb{R}^n)$-function such that $\widetilde{\psi}(\xi) = 0$ if $|\xi| \ge 1$ and $\int_{\mathbb{R}^n} \widetilde{\psi}(\xi)\, d\xi = 1$. Then we set $\varrho_{(\varepsilon)} = \psi(\varepsilon\cdot)[\widetilde{\psi}_\varepsilon * m]$ for $\varepsilon > 0$, where $\widetilde{\psi}_\varepsilon = \varepsilon^{-n}\widetilde{\psi}(\cdot/\varepsilon)$. Since $\widetilde{\psi}_\varepsilon * [\psi(\varepsilon\cdot)\, f] \to f$ in $\mathcal{S}(\mathbb{R}^n)$ as $\varepsilon \to 0$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, we see that

$$\begin{aligned} (2)\quad T_{\varrho_{(\varepsilon)}(2^j\cdot)}f * g(0) &= \langle [\mathcal{F}^{-1}\varrho_{(\varepsilon)}]_{2^j}, \check{f} * \check{g}\rangle = \langle \psi(\varepsilon\cdot)[\widetilde{\psi}_\varepsilon * m], \mathcal{F}^{-1}[\check{f} * \check{g}(2^j\cdot)]\rangle \\ &\to \langle m, \mathcal{F}^{-1}[\check{f} * \check{g}(2^j\cdot)]\rangle = T_{m(2^j\cdot)}f * g(0) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. Since $\|m(t\cdot)\|_{M_p} = \|m\|_{M_p}$ for all $t > 0$, by Lemmas 3.3 and 3.4, we also have

$$\begin{aligned} \|\varrho_{(\varepsilon)}(2^j\cdot)\|_{M_p} = \|\varrho_{(\varepsilon)}\|_{M_p} &\le \|\mathcal{F}^{-1}[\psi(\varepsilon\,\cdot)]\|_{L^1}\|\widetilde{\psi}_\varepsilon * m\|_{M_p} \\ &\le \|[\mathcal{F}^{-1}\psi]_\varepsilon\|_{L^1}\|\widetilde{\psi}_\varepsilon\|_{L^1}\|m\|_{M_p} = \|\mathcal{F}^{-1}\psi\|_{L^1}\|\widetilde{\psi}\|_{L^1}\|m\|_{M_p}. \end{aligned}$$

This gives

$$(3)\qquad |T_{\varrho_{(\varepsilon)}(2^j\cdot)}f * g(0)| \le \|\mathcal{F}^{-1}\psi\|_{L^1}\|\widetilde{\psi}\|_{L^1}\|m\|_{M_p}\|f\|_{L^p}\|g\|_{L^{p'}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. Let $\{f_i\}_{i\in\mathbb{N}}, \{g_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ satisfy $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i\|_{L^p}\|g_{i,j}\|_{L^{p'}} < \infty$ and $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) = 0$ in $L^\infty(\mathbb{R}^n)$. By (2) and (3), we get

$$T_{\varrho_{(\varepsilon)}(2^j\cdot)}f_i * g_{i,j}(0) \to T_{m(2^j\cdot)}f_i * g_{i,j}(0) \quad \text{as } \varepsilon \to 0$$

and

$$|T_{\varrho_{(\varepsilon)}(2^j\cdot)}f_i * g_{i,j}(0)| \le \|\mathcal{F}^{-1}\psi\|_{L^1}\|\widetilde{\psi}\|_{L^1}\|m\|_{M_p}\|f_i\|_{L^p}\|g_{i,j}\|_{L^{p'}}$$

for each $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Hence, by the Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{\varrho_{(\varepsilon)}(2^j\cdot)}f_i * g_{i,j}(0) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)}f_i * g_{i,j}(0).$$

Since $\mathcal{F}^{-1}\varrho_{(\varepsilon)} \in L^1(\mathbb{R}^n)$ and $\sum_{i\le N}\sum_{|j|\le N} f_i * g_{i,j}(2^j\cdot) \to 0$ in $L^\infty(\mathbb{R}^n)$ as

$N \to \infty$, we see that

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{\varrho_{(\varepsilon)}(2^j\cdot)} f_i * g_{i,j}(0) = \lim_{N\to\infty} \sum_{i\le N}\sum_{|j|\le N} \int_{\mathbb{R}^n} [\mathcal{F}^{-1}\varrho_{(\varepsilon)}](-x) f_i * g_{i,j}(2^j x)\,dx$$
$$= \lim_{N\to\infty} \int_{\mathbb{R}^n} [\mathcal{F}^{-1}\varrho_{(\varepsilon)}](-x)\Big(\sum_{i\le N}\sum_{|j|\le N} f_i * g_{i,j}(2^j x)\Big)dx = 0.$$

This completes the proof. ∎

LEMMA 3.6. *Let $m \in M_p(\mathbb{R}^n)$. Then we can define a linear functional φ_m on $\widetilde{A}_p(\mathbb{R}^n)$ by* (1).

Proof. To define φ_m, we need to show that, if $\{f_i^{(1)}\}, \{f_i^{(2)}\}, \{g_{i,j}^{(1)}\}, \{g_{i,j}^{(2)}\} \subset \mathcal{S}(\mathbb{R}^n)$ satisfy $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i^{(1)}\|_{L^p}\|g_{i,j}^{(1)}\|_{L^{p'}}, \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i^{(2)}\|_{L^p}\|g_{i,j}^{(2)}\|_{L^{p'}} < \infty$ and $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i^{(1)} * g_{i,j}^{(1)}(2^j\cdot) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i^{(2)} * g_{i,j}^{(2)}(2^j\cdot)$ in $L^\infty(\mathbb{R}^n)$, then

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i^{(1)} * g_{i,j}^{(1)}(0) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i^{(2)} * g_{i,j}^{(2)}(0).$$

To do this, we define $\{f_i^{(3)}\}_i, \{\{g_{i,j}^{(3)}\}_j\}_i \subset \mathcal{S}(\mathbb{R}^n)$ by $\{f_i^{(3)}\}_i = \{f_1^{(1)}, f_1^{(2)}, f_2^{(1)}, f_2^{(2)}, \ldots\}$, and $\{\{g_{i,j}^{(3)}\}_j\}_i = \{\{g_{1,j}^{(1)}\}_j, \{-g_{1,j}^{(2)}\}_j, \{g_{2,j}^{(1)}\}_j, \{-g_{2,j}^{(2)}\}_j, \ldots\}$. Then we have

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i^{(3)}\|_{L^p}\|g_{i,j}^{(3)}\|_{L^{p'}}$$
$$= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i^{(1)}\|_{L^p}\|g_{i,j}^{(1)}\|_{L^{p'}} + \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \|f_i^{(2)}\|_{L^p}\|g_{i,j}^{(2)}\|_{L^{p'}} < \infty$$

and

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i^{(3)} * g_{i,j}^{(3)}(2^j\cdot) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i^{(1)} * g_{i,j}^{(1)}(2^j\cdot) - \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i^{(2)} * g_{i,j}^{(2)}(2^j\cdot) = 0.$$

Hence, by Lemma 3.5, we get

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i^{(1)} * g_{i,j}^{(1)}(0) - \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i^{(2)} * g_{i,j}^{(2)}(0)$$
$$= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i^{(3)} * g_{i,j}^{(3)}(0) = 0.$$

Thus, the values $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_i * g_{i,j}(0)$ are independent of the representations of f. In the same way, we can prove the linearity of φ_m. ∎

We are now ready to prove Theorem 1 given in the introduction.

Proof of Theorem 1. We first prove that, if $m \in \max M_p(\mathbb{R}^n)$, then $\varphi_m \in \widetilde{A}_p(\mathbb{R}^n)^*$ and $\|m\|_{\max M_p} = \|\varphi_m\|_{(\widetilde{A}_p)^*}$. Let $m \in \max M_p(\mathbb{R}^n)$. By

Lemma 3.6, we see that φ_m is a linear functional on $\widetilde{A}_p(\mathbb{R}^n)$. Let $f = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot) \in \widetilde{A}_p(\mathbb{R}^n)$. Since

$$\begin{aligned}|\varphi_m(f)| &\le \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |T_{m(2^j\cdot)} f_i(x) g_{i,j}(-x)|\,dx \\ &\le \sum_{i\in\mathbb{N}} \|M_m f_i\|_{L^p} \|\{g_{i,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} \\ &\le \|m\|_{\max M_p} \sum_{i\in\mathbb{N}} \|f_i\|_{L^p} \|\{g_{i,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))},\end{aligned}$$

taking the infimum over all the representations of f, we have $|\varphi_m(f)| \le \|m\|_{\max M_p}\|f\|_{\widetilde{A}_p}$, so $\varphi_m \in \widetilde{A}_p(\mathbb{R}^n)^*$ and $\|\varphi_m\|_{(\widetilde{A}_p)^*} \le \|m\|_{\max M_p}$. To prove $\|\varphi_m\|_{(\widetilde{A}_p)^*} \ge \|m\|_{\max M_p}$, we use the duality $L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))^* = L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))$ ([12, Proposition, 2.11.1]), that is,

$$\begin{aligned}\|m\|_{\max M_p} &= \sup \|\{T_{m(2^j\cdot)} f\}_{j\in\mathbb{Z}}\|_{L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))} \\ &= \sup \Big| \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f(x) g_j(x)\,dx \Big|,\end{aligned}$$

where the supremum is taken over all $f \in \mathcal{S}(\mathbb{R}^n)$ and finitely supported sequences $\{g_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ such that $\|f\|_{L^p} = \|\{g_j\}_{j\in\mathbb{Z}}\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} = 1$. For $\varepsilon > 0$, we can find $f_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$ and a finitely supported sequence $\{g_{\varepsilon,j}\} \subset \mathcal{S}(\mathbb{R}^n)$ such that $\|f_\varepsilon\|_{L^p} = \|\{g_{\varepsilon,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))} = 1$ and

$$\|m\|_{\max M_p} - \varepsilon < \Big| \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_\varepsilon(x) g_{\varepsilon,j}(x)\,dx \Big|.$$

Since $\{g_{\varepsilon,j}\} \subset \mathcal{S}(\mathbb{R}^n)$ is a finitely supported sequence, we have $\sum_{j\in\mathbb{Z}} f_\varepsilon * g^\vee_{\varepsilon,j}(2^j\cdot) \in \widetilde{A}_p(\mathbb{R}^n)$ and $\|\sum_{j\in\mathbb{Z}} f_\varepsilon * g^\vee_{\varepsilon,j}(2^j\cdot)\|_{\widetilde{A}_p} \le \|f_\varepsilon\|_{L^p}\|\{g^\vee_{\varepsilon,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))}$. Hence, we get

$$\begin{aligned}\|m\|_{\max M_p} &< \Big| \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_\varepsilon(x) g_{\varepsilon,j}(x)\,dx \Big| + \varepsilon \\ &= \Big| \sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)} f_\varepsilon * g^\vee_{\varepsilon,j}(0) \Big| = \Big| \varphi_m\Big(\sum_{j\in\mathbb{Z}} f_\varepsilon * g^\vee_{\varepsilon,j}(2^j\cdot) \Big) \Big| + \varepsilon \\ &\le \|\varphi_m\|_{(\widetilde{A}_p)^*} \Big\| \sum_{j\in\mathbb{Z}} f_\varepsilon * g^\vee_{\varepsilon,j}(2^j\cdot) \Big\|_{\widetilde{A}_p} + \varepsilon \le \|\varphi_m\|_{(\widetilde{A}_p)^*} + \varepsilon.\end{aligned}$$

Hence, the arbitrariness of ε gives $\|\varphi_m\|_{(\widetilde{A}_p)^*} \ge \|m\|_{\max M_p}$.

We next prove that, if $\varphi \in \widetilde{A}_p(\mathbb{R}^n)^*$, then there exists $m \in \max M_p(\mathbb{R}^n)$ such that $\varphi = \varphi_m$. We note that, if $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, then $f * g(2^j\cdot) \in \widetilde{A}_p(\mathbb{R}^n)$ and $\|f * g(2^j\cdot)\|_{\widetilde{A}_p} \le \|f\|_{L^p}\|g\|_{L^{p'}}$. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, we can define a linear functional $L_f^{(j)}$ on the dense subspace $\mathcal{S}(\mathbb{R}^n)$ of $L^{p'}(\mathbb{R}^n)$ by $L_f^{(j)}(g) = \varphi(f * g(2^j\cdot))$ for $g \in \mathcal{S}(\mathbb{R}^n)$. Since

$$|L_f^{(j)}(g)| = |\varphi(f * g(2^j\cdot))| \le \|\varphi\|_{(\widetilde{A}_p)^*}\|f * g(2^j\cdot)\|_{\widetilde{A}_p} \le \|\varphi\|_{(\widetilde{A}_p)^*}\|f\|_{L^p}\|g\|_{L^{p'}}$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$, it follows that $L_f^{(j)} \in L^{p'}(\mathbb{R}^n)^*$ and its norm satisfies $\|L_f^{(j)}\|_{(L^{p'})^*} \le \|\varphi\|_{(\widetilde{A}_p)^*}\ \|f\|_{L^p}$. Since $L^{p'}(\mathbb{R}^n)^* = L^p(\mathbb{R}^n)$, we can find $h_f^{(j)} \in L^p(\mathbb{R}^n)$ such that $\|h_f^{(j)}\|_{L^p} = \|L_f^{(j)}\|_{(L^{p'})^*}$ and

$$L_f^{(j)}(g) = \int_{\mathbb{R}^n} h_f^{(j)}(x)g(x)\,dx \quad \text{for all } g \in \mathcal{S}(\mathbb{R}^n).$$

We define a linear operator T_j from $\mathcal{S}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ by $T_jf = (h_f^{(j)})^\vee$. Then we have

$$\|T_jf\|_{L^p} = \|(h_f^{(j)})^\vee\|_{L^p} = \|L_f^{(j)}\|_{(L^{p'})^*} \le \|\varphi\|_{(\widetilde{A}_p)^*}\|f\|_{L^p}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. That is, T_j is bounded on $L^p(\mathbb{R}^n)$. Since $\tau_xf * g(2^j\cdot) = f * \tau_xg(2^j\cdot)$, the equations

$$\varphi(\tau_xf * g(2^j\cdot)) = L_{\tau_xf}^{(j)}(g) = \int_{\mathbb{R}^n} T_j[\tau_xf](y)g(-y)\,dy$$

and

$$\varphi(f * \tau_xg(2^j\cdot)) = L_f^{(j)}(\tau_xg) = \int_{\mathbb{R}^n} T_jf(y)[\tau_xg](-y)\,dy$$

give $T_j\tau_x = \tau_xT_j$. Since T_j is bounded on $L^p(\mathbb{R}^n)$ and commutes with translations, by [11, Chapter 1, Theorem 3.16], we can find $m_j \in L^\infty(\mathbb{R}^n)$ such that $T_{m_j} = T_j$. We next show that $m_j = m_0(2^j\cdot)$ for all $j \in \mathbb{Z}$. Since $f * g(2^j\cdot) = [f_{2^{-j}} * g(2^j\cdot)](2^0\cdot)$, the equations

$$\varphi(f * g(2^j\cdot)) = L_f^{(j)}(g) = \int_{\mathbb{R}^n} T_{m_j}f(x)g(-x)\,dx$$

and

$$\begin{aligned}\varphi([f_{2^{-j}} * g(2^j\cdot)](2^0\cdot)) &= L_{f_{2^{-j}}}^{(0)}(g(2^j\cdot)) \\ &= \int_{\mathbb{R}^n} T_{m_0}f_{2^{-j}}(x)g(-2^jx)\,dx = \int_{\mathbb{R}^n} T_{m_0(2^j\cdot)}f(x)g(-x)\,dx\end{aligned}$$

give $m_j = m_0(2^j\cdot)$. We write $m = m_0$. Then we have

$$\varphi(f * g(2^j\cdot)) = \int_{\mathbb{R}^n} T_{m(2^j\cdot)}f(x)\,g(-x)\,dx = T_{m(2^j\cdot)}f * g(0) \tag{4}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. To show $m \in \max M_p(\mathbb{R}^n)$, we define a space S by $S = \{\{g_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n) : \{g_j\}_{j\in\mathbb{Z}}$ is a finitely supported sequence$\}$. We note that, if $f \in \mathcal{S}(\mathbb{R}^n)$ and $\{g_j\}_{j\in\mathbb{Z}} \in S$, then $\sum_{j\in\mathbb{Z}} f * g_j(2^j\cdot) \in \widetilde{A}_p(\mathbb{R}^n)$

and $\|\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot))\|_{\widetilde{A}_p} \le \|f\|_{L^p}\|\{g_j\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))}$. For $f\in\mathcal{S}(\mathbb{R}^n)$, we can define a linear functional L_f on the dense subspace S of $L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))$ by $L_f(\{g_j\}_j)=\varphi(\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot))$ for $\{g_j\}_{j\in\mathbb{Z}}\in S$. From the boundedness of φ, it follows that

$$\begin{aligned}|L_f(\{g_j\}_j)| &\le \|\varphi\|_{(\widetilde{A}_p)^*}\Big\|\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot))\Big\|_{\widetilde{A}_p}\\ &\le \|\varphi\|_{(\widetilde{A}_p)^*}\|f\|_{L^p}\|\{g_j\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))}\end{aligned}$$

for all $\{g_j\}_{j\in\mathbb{Z}}\in S$, so that $L_f\in L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))^*$ and $\|L_f\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))^*}\le \|\varphi\|_{(\widetilde{A}_p)^*}\|f\|_{L^p}$. By the duality $L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))^*=L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))$, we can find $\{h_j\}_{j\in\mathbb{Z}}\in L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))$ such that $\|\{h_j\}_j\|_{L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))}=\|L_f\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))^*}$ and

$$L_f(\{g_j\}_j)=\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}} h_j(x)g_j(x)\,dx \quad \text{for all } \{g_j\}_j\in S.$$

Now, (4) and

$$\varphi(f*g(2^{j_0}\cdot))=L_f(\{\delta_{j,j_0}g\}_j)=\int_{\mathbb{R}^n} h_{j_0}(x)g(x)\,dx$$

give $T_{m(2^{j_0}\cdot)}f=h_{j_0}^\vee$ for all $j_0\in\mathbb{Z}$, where $\delta_{j,j_0}=1$ if $j=j_0$ and $\delta_{j,j_0}=0$ if $j\ne j_0$. So, we get

$$\begin{aligned}\|M_m f\|_{L^p} &= \|\{T_{m(2^j\cdot)}f\}_j\|_{L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))}=\|\{h_j^\vee\}_j\|_{L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}))}\\ &= \|L_f\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))^*}\le \|\varphi\|_{(\widetilde{A}_p)^*}\|f\|_{L^p}.\end{aligned}$$

That is, $m\in \max M_p(\mathbb{R}^n)$. Finally, we prove $\varphi_m=\varphi$. Let $f\in\mathcal{S}(\mathbb{R}^n)$ and $\{g_j\}_j\subset\mathcal{S}(\mathbb{R}^n)$ satisfy $\|f\|_{L^p}\sum_{j\in\mathbb{Z}}\|g_j\|_{L^{p'}}<\infty$. We note that $\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot)\in\widetilde{A}_p(\mathbb{R}^n)$. Since $\sum_{|j|\le N} f*g_j(2^j\cdot)\to\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot)$ in $\widetilde{A}_p(\mathbb{R}^n)$ as $N\to\infty$, using the continuity and linearity of φ and (4), we have

$$\begin{aligned}(5)\quad \varphi\Big(\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot)\Big) &= \lim_{N\to\infty}\sum_{|j|\le N}\varphi(f*g_j(2^j\cdot))\\ &= \lim_{N\to\infty}\sum_{|j|\le N} T_{m(2^j\cdot)}f*g_j(0)=\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)}f*g_j(0).\end{aligned}$$

Let $f=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot)\in\widetilde{A}_p(\mathbb{R}^n)$, where $\{f_i\},\{g_{i,j}\}_{i,j}\subset\mathcal{S}(\mathbb{R}^n)$ satisfy $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}}\|f_i\|_{L^p}\|g_{i,j}\|_{L^{p'}}<\infty$. Since

$$\sum_{i\in\mathbb{N}}\|f_i\|_{L^p}\|\{g_{i,j}\}_j\|_{L^{p'}(\mathbb{R}^n,\ell^1(\mathbb{Z}))}\le\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}}\|f_i\|_{L^p}\|g_{i,j}\|_{L^{p'}}<\infty,$$

we see that $\sum_{i\le N}\sum_{j\in\mathbb{Z}} f*g_j(2^j\cdot)\to f$ in $\widetilde{A}_p(\mathbb{R}^n)$ as $N\to\infty$. Hence, from the continuity and linearity of φ and (5), we get

$$\begin{aligned}\varphi(f) &= \lim_{N\to\infty}\sum_{i\le N}\varphi\Big(\sum_{j\in\mathbb{Z}} f_i * g_{i,j}(2^j\cdot)\Big)\\ &= \lim_{N\to\infty}\sum_{i\le N}\sum_{j\in\mathbb{Z}} T_{m(2^j\cdot)}f_i * g_{i,j}(0) = \varphi_m(f).\end{aligned}$$

The proof is complete.

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References

[1] N. Asmar, E. Berkson and T. A. Gillespie, *Summability methods for transferring Fourier multipliers and transference of maximal inequalities*, in: Analysis and Partial Differential Equations: A Collection of Papers Dedicated to Mischa Cotlar, ed. by C. Sadosky, Lecture Notes in Pure and Appl. Math. 122, Dekker, New York, 1990, 1–34.

[2] E. Berkson, M. Paluszyński and G. Weiss, *Wavelet decompositions of Fourier multipliers*, Proc. Amer. Math. Soc. 125 (1997), 2395–2399.

[3] M. Christ, L. Grafakos, P. Honzík and A. Seeger, *Maximal functions associated with Fourier multipliers of Mikhlin–Hörmander type*, Math. Z. 249 (2005), 223-240.

[4] H. Dappa and W. Trebels, *On maximal functions generated by Fourier multipliers*, Ark. Mat. 23 (1985), 241–259.

[5] J. Duoandikoetxea, *Fourier Analysis*, Grad. Stud. Math. 29, Amer. Math. Soc., Providence, RI, 2001.

[6] A. Figà-Talamanca, *Translation invariant operators in L^p*, Duke Math. J. 32 (1965), 495-501.

[7] A. Figà-Talamanca and G. I. Gaudry, *Multipliers of L^p which vanish at infinity*, J. Funct. Anal. 7 (1971), 475–486.

[8] M. Jodeit, *Restrictions and extensions of Fourier multipliers*, Studia Math. 34 (1970), 215–226.

[9] C. Kenig and P. Tomas, *Maximal operators defined by Fourier multipliers*, ibid. 68 (1980), 79–83.

[10] R. Larsen, *An Introduction to the Theory of Multipliers*, Springer, Berlin, 1971.

[11] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

[12] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

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