

## Khinchin inequality and Banach–Saks type properties in rearrangement-invariant spaces

by

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**Abstract.** We study the class of all rearrangement-invariant (= r.i.) function spaces  $E$  on  $[0, 1]$  such that there exists  $0 < q < 1$  for which  $\|\sum_{k=1}^n \xi_k\|_E \leq Cn^q$ , where  $\{\xi_k\}_{k \geq 1} \subset E$  is an arbitrary sequence of independent identically distributed symmetric random variables on  $[0, 1]$  and  $C > 0$  does not depend on  $n$ . We completely characterize all Lorentz spaces having this property and complement classical results of Rodin and Semenov for Orlicz spaces  $\exp(L_p)$ ,  $p \geq 1$ . We further apply our results to the study of Banach–Saks index sets in r.i. spaces.

**1. Introduction.** A classical result of Rodin and Semenov (see [18] or [16, Theorem 2.b.4]) says that the sequence of Rademacher functions  $\{r_k\}_{k \geq 1}$  on  $[0, 1]$  in a rearrangement-invariant (= r.i.) space  $E$  is equivalent to the unit vector basis of  $l_2$  if and only if  $E$  contains (the separable part of) the Orlicz space  $L_{N_2}(0, 1)$  (customarily denoted as  $\exp(L_2)$ ) where  $N_2(t) = e^{t^2} - 1$ . Here,  $\{r_k\}_{k \geq 1}$  may be thought of as a sequence of independent identically distributed centered Bernoulli variables on  $[0, 1]$ . A quick analysis of the proof (see e.g. [16, p. 134]) shows that the embedding  $\exp(L_2) \subseteq E$  is established there under a weaker assumption that  $\{r_k\}_{k \geq 1}$  is a 2-Banach–Saks sequence in  $E$ , that is,  $\|\sum_{k=1}^n r_k\|_E \leq Cn^{1/2}$ , where  $C > 0$  does not depend on  $n \geq 1$ . The main object of study in the present article is the class of all r.i. spaces  $E$  such that there exists  $0 < q < 1$  for which

$$(1) \quad \left\| \sum_{k=1}^n \xi_k \right\|_E \leq Cn^q,$$

where  $\{\xi_k\}_{k \geq 1} \subset E$  is an arbitrary sequence of independent identically distributed symmetric random variables on  $[0, 1]$  and  $C > 0$  does not depend on  $n$ . We completely characterize all Lorentz spaces from this class in Corol-

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lary 13 below. In Theorem 23 we obtain sharp estimates of type (1) for the Orlicz spaces  $\exp(L_p) = L_{N_p}(0, 1)$ ,  $1 \leq p < \infty$ , where  $N_p(t) = e^{t^p} - 1$ , complementing results of [18] (see also exposition in [11]). Our results also have a number of interesting implications to the study of Banach–Saks type properties in r.i. spaces.

Recall that a bounded sequence  $\{x_n\} \subset E$  is called a *p-BS-sequence* if for all subsequences  $\{y_k\} \subset \{x_n\}$  we have

$$\sup_{m \in \mathbb{N}} m^{-1/p} \left\| \sum_{k=1}^m y_k \right\|_E < \infty.$$

We say that  $E$  has the *p-BS-property* and we write  $E \in \text{BS}(p)$  if each weakly null sequence contains a *p-BS-sequence*. The set

$$\Gamma(E) = \{p : p \geq 1, E \in \text{BS}(p)\}$$

is said to be the *index set* of  $E$ , and is of the form  $[1, \gamma]$  or  $[1, \gamma)$  for some  $\gamma \geq 1$ .

If, in the preceding definition, we replace all weakly null sequences by weakly null sequences of independent random variables (respectively, by weakly null sequences of pairwise disjoint elements, or by weakly null sequences of independent identically distributed random variables), we obtain the set  $\Gamma_i(E)$  (respectively,  $\Gamma_d(E)$ ,  $\Gamma_{\text{iid}}(E)$ ). The general problem of describing and comparing the sets  $\Gamma(E)$ ,  $\Gamma_i(E)$ ,  $\Gamma_{\text{iid}}(E)$  and  $\Gamma_d(E)$  in various classes of r.i. spaces was addressed in [20, 12, 22, 1, 21, 2]. In particular, it is known [1] that  $1 \in \Gamma(E) \subseteq \Gamma_i(E) \subseteq \Gamma_{\text{iid}}(E) \subseteq [1, 2]$  and  $\Gamma_i(E) \subseteq \Gamma_d(E)$  for any r.i. space  $E$ . Moreover, the sets  $\Gamma(E)$  and  $\Gamma_i(E)$  coincide in many cases but not always. For example,  $\Gamma(L_p) = \Gamma_i(L_p) = \Gamma_{\text{iid}}(L_p)$ ,  $1 < p < \infty$  (see e.g. [21, Corollary 4.4 and Theorem 4.5] and also Theorem 18 below), whereas for the Lorentz space  $L_{2,1}$  generated by the function  $\psi(t) = t^{1/2}$ , we have  $\Gamma(L_{2,1}) = [1, 2)$  and  $\Gamma_i(L_{2,1}) = [1, 2]$  ([21, Theorem 5.9] and [1, Proposition 4.12]). It turns out that these two situations are typical [22, Theorem 9]: under the assumption that  $\Gamma(E) \neq \{1\}$ , we have either  $\Gamma_i(E) \setminus \Gamma(E) = \emptyset$  or else  $\Gamma_i(E) \setminus \Gamma(E) = \{2\}$ .

The present paper may also be considered as a contribution to the study of the class of all r.i. spaces  $E$  such that  $\Gamma_{\text{iid}}(E) = \Gamma_i(E)$ . We prove a general theorem (see Theorem 18 below) that  $\Gamma_{\text{iid}}(E) = \Gamma_i(E)$  if and only if  $\Gamma_{\text{iid}}(E) \subseteq \Gamma_d(E)$ . It is easy to see that every Lorentz space  $\Lambda(\psi)$  satisfies the latter condition and, using the main result described above, we give a complete characterization of all Lorentz spaces  $E = \Lambda(\psi)$  such that  $\Gamma_{\text{iid}}(E) \neq \{1\}$  (see Theorem 21 and Corollary 22).

It also pertinent to note here that if one views the Rademacher system as a special example of sequences of independent mean zero random variables, then a significant generalization of Khinchin inequality is due to

W. B. Johnson and G. Schechtman [13]. They introduced the r.i. space  $Z_E^2$  on  $[0, \infty)$  linked with a given r.i. space  $E$  on  $[0, 1]$  and showed that any sequence  $\{f_k\}_{k=1}^\infty$  of independent mean zero random variables in  $E$  is equivalent to the sequence of its disjoint translates  $\{\tilde{f}_k(\cdot) := f_k(\cdot - k + 1)\}_{k=1}^\infty$  in  $Z_E^2$ , provided that  $E$  contains an  $L_p$ -space for some  $p < \infty$ . This study was taken further in [6, 1, 4, 5], where the connection between this (generalized) Khinchin inequality and the so-called Kruglov property was established (we explain the latter property in the next section). We show the connection between the class of all r.i. spaces with the Kruglov property and the estimates (1) in Theorem 3. Recently, examples of r.i. spaces  $E$  such that  $\Gamma(E) = \{1\}$  but  $\Gamma_1(E) \neq \{1\}$  have been produced in [2] under the assumption that  $E$  has the Kruglov property. Our approach in this paper complements that of [2]; in particular, we present examples of Lorentz and Marcinkiewicz spaces  $E$  such that  $\Gamma_1(E) = \Gamma_{\text{iid}}(E) \neq \{1\}$  and which do not have the Kruglov property.

Finally, we show that the equality  $\Gamma_{\text{iid}}(E) = \Gamma_1(E)$  fails to hold when  $E$  is the classical space  $L_{pq}$ ,  $1 < q < p < 2$ .

## 2. Definitions and preliminaries

**2.1. Rearrangement-invariant spaces.** A Banach space  $(E, \|\cdot\|_E)$  of real-valued Lebesgue measurable functions (with identification  $m$ -a.e.) on the interval  $[0, 1]$  will be called *rearrangement-invariant* (briefly, r.i.) if

- (i)  $E$  is an ideal lattice, that is, if  $y \in E$ , and if  $x$  is any measurable function on  $[0, 1]$  with  $0 \leq |x| \leq |y|$ , then  $x \in E$  and  $\|x\|_E \leq \|y\|_E$ ;
- (ii)  $E$  is rearrangement-invariant in the sense that if  $y \in E$ , and if  $x$  is any measurable function on  $[0, 1]$  with  $x^* = y^*$ , then  $x \in E$  and  $\|x\|_E = \|y\|_E$ .

Here,  $m$  denotes Lebesgue measure and  $x^*$  denotes the non-increasing, right-continuous rearrangement of  $x$  given by

$$x^*(t) = \inf\{s \geq 0 : m(\{u \in [0, 1] : |x(u)| > s\}) \leq t\}, \quad t > 0.$$

For basic properties of r.i. spaces, we refer to the monographs [14, 16]. We note that for any r.i. space  $E$  we have  $L_\infty[0, 1] \subseteq E \subseteq L_1[0, 1]$ . We will also work with an r.i. space  $E(\Omega, \mathcal{P})$  of measurable functions on a probability space  $(\Omega, \mathcal{P})$  given by

$$E(\Omega, \mathcal{P}) := \{f \in L_1(\Omega, \mathcal{P}) : f^* \in E\}, \quad \|f\|_{E(\Omega, \mathcal{P})} := \|f^*\|_E.$$

Here, the decreasing rearrangement  $f^*$  is calculated with respect to the measure  $\mathcal{P}$  on  $\Omega$ .

Recall that for  $0 < \tau < \infty$ , the *dilation operator*  $\sigma_\tau$  is defined by setting

$$\sigma_\tau x(t) = \begin{cases} x(t/\tau), & 0 \leq t \leq \min(1, \tau), \\ 0, & \min(1, \tau) < t \leq 1. \end{cases}$$

The dilation operators  $\sigma_\tau$  are bounded in every r.i. space  $E$ . Denoting the space of all bounded linear operators on a Banach space  $E$  by  $\mathcal{L}(E)$ , we set

$$\alpha_E := \lim_{\tau \rightarrow 0} \frac{\ln \|\sigma_\tau\|_{\mathcal{L}(E)}}{\ln \tau}, \quad \beta_E := \lim_{\tau \rightarrow \infty} \frac{\ln \|\sigma_\tau\|_{\mathcal{L}(E)}}{\ln \tau}.$$

The numbers  $\alpha_E$  and  $\beta_E$  belong to the closed interval  $[0, 1]$  and are called the *Boyd indices* of  $E$ .

The *Köthe dual*  $E^\times$  of an r.i. space  $E$  on  $[0, 1]$  consists of all measurable functions  $y$  for which

$$\|y\|_{E^\times} := \sup \left\{ \int_0^1 |x(t)y(t)| dt : x \in E, \|x\|_E \leq 1 \right\} < \infty.$$

If  $E^*$  denotes the Banach dual of  $E$ , then  $E^\times \subset E^*$ , and  $E^\times = E^*$  if and only if  $E$  is separable. An r.i. space  $E$  is said to have the *Fatou property* if whenever  $\{f_n\}_{n=1}^\infty \subseteq E$  and  $f$  measurable on  $[0, 1]$  satisfy  $f_n \rightarrow f$  a.e. on  $[0, 1]$  and  $\sup_n \|f_n\|_E < \infty$ , it follows that  $f \in E$  and  $\|f\|_E \leq \liminf_{n \rightarrow \infty} \|f_n\|_E$ . It is well-known that an r.i. space  $E$  has the Fatou property if and only if the natural embedding of  $E$  into its Köthe bidual  $E^{\times \times}$  is a surjective isometry.

Let us recall some classical examples of r.i. spaces on  $[0, 1]$ . Denote by  $\Psi$  the set of all increasing continuous concave functions on  $[0, 1]$  with  $\varphi(0) = 0$ . Each function  $\varphi \in \Psi$  generates the Lorentz space  $\Lambda(\varphi)$  (see e.g. [14]) endowed with the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) d\varphi(t)$$

and the Marcinkiewicz space  $M(\varphi)$  endowed with the norm

$$\|x\|_{M(\varphi)} = \sup_{0 < \tau \leq 1} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t) dt.$$

The space  $M(\varphi)$  is not separable, but the space

$$\left\{ x \in M(\varphi) : \lim_{\tau \rightarrow 0} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t) dt = 0 \right\}$$

endowed with the norm  $\|\cdot\|_{M(\varphi)}$  is a separable r.i. space, denoted further as  $(M(\varphi)_0)$ , which coincides with the closure of  $L_\infty$  in  $(M(\varphi), \|\cdot\|_{M(\varphi)})$ .

It is well known (see e.g. [14, Section II.1]) that

$$\beta_{M(\varphi)} = 1 \Leftrightarrow \alpha_{\Lambda(\varphi)} = 0 \Leftrightarrow \forall t \in (0, 1) \exists (s_n)_{n \geq 1} \subseteq (0, 1) : \lim_{n \rightarrow \infty} \frac{\varphi(ts_n)}{\varphi(s_n)} = 1,$$

$$\alpha_{M(\varphi)} = 0 \Leftrightarrow \beta_{\Lambda(\varphi)} = 1 \Leftrightarrow \forall \tau \geq 1 \exists (s_n)_{n \geq 1} \subseteq (0, 1) : \lim_{n \rightarrow \infty} \frac{\varphi(s_n \tau)}{\varphi(s_n)} = \tau.$$

If  $M(t)$  is a convex increasing function on  $[0, \infty)$  such that  $M(0) = 0$ , then the Orlicz space  $L_M$  on  $[0, 1]$  (see e.g. [14, 16]) is an r.i. space of all  $x \in L_1[0, 1]$  such that

$$\|x\|_{L_M} := \inf \left\{ \lambda : \lambda > 0, \int_0^1 M(|x(t)|/\lambda) dt \leq 1 \right\} < \infty.$$

The function  $N_p(u) = e^{u^p} - 1$  is convex for  $p \geq 1$  and is equivalent to a convex function for  $0 < p < 1$  (see e.g. [6, 3]). The space  $L_{N_p}$ ,  $0 < p < \infty$ , is customarily denoted  $\exp(L_p)$ .

**2.2. The Kruglov property in r.i. spaces.** Let  $f$  be a random variable on  $[0, 1]$ . We denote by  $\pi(f)$  the random variable  $\sum_{i=1}^N f_i$ , where  $f_i$ 's are independent copies of  $f$ , and  $N$  is a Poisson random variable with parameter 1 independent of the sequence  $\{f_i\}$ .

DEFINITION. An r.i. space  $E$  is said to have the *Kruglov property* if and only if  $f \in E \Leftrightarrow \pi(f) \in E$ .

This property has been studied by M. Sh. Braverman [6], who uses some earlier probabilistic constructions of V. M. Kruglov [15] and in [3, 4, 5] via an operator approach. It was proved in [5] that an r.i. space  $E$  has the Kruglov property if and only if for every sequence of independent mean zero functions  $f_n \in E$  the following inequality holds:

$$(2) \quad \left\| \sum_{k=1}^n f_k \right\|_E \leq \text{const} \cdot \left\| \sum_{k=1}^n \bar{f}_k \right\|_{Z_E^2}.$$

Here,  $Z_E^2$  is an r.i. space on  $(0, \infty)$  equipped with the norm

$$\|x\| = \|x^* \chi_{[0,1]}\|_E + \|x^* \chi_{[1,\infty)}\|_{L_2},$$

and  $\{\bar{f}_k\}_{k=1}^n \subseteq Z_X^2$  is a sequence of disjoint translates of  $\{f_k\}_{k=1}^n \subseteq X$ , that is,  $\bar{f}_k(\cdot) = f_k(\cdot - k + 1)$ . Note that inequality (2) has been proved earlier in [13] (see inequality (3) there) under the more restrictive assumption that  $E \supseteq L_p$  for some  $p < \infty$ . Clearly, the latter assumption holds if  $\alpha_E > 0$ .

**3. Operators  $A_n$ ,  $n \geq 0$ .** Let  $\Omega$  be the interval  $[0, 1]$  equipped with the Lebesgue measure. Let  $E$  be an arbitrary rearrangement-invariant space on  $\Omega$ .

For every  $n \geq 1$ , we consider the operator

$$A_n : E(\Omega) \rightarrow E(\underbrace{\Omega \times \dots \times \Omega}_{2n \text{ times}})$$

given by

$$A_n f = (f \otimes r) \otimes (1 \otimes 1) \otimes \dots \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (f \otimes r) \otimes \dots \otimes (1 \otimes 1) + \dots + (1 \otimes 1) \otimes \dots \otimes (1 \otimes 1) \otimes (f \otimes r),$$

where  $r$  is a centered Bernoulli random variable. For brevity, we will also use the notation

$$A_n f = (f \otimes r)_1 + (f \otimes r)_2 + \dots + (f \otimes r)_n.$$

We set  $A_0 = 0$ .

The following theorem is the main result of the present section.

**THEOREM 1.** *The following alternative is valid in an arbitrary r.i. space  $E$ : either*

- (i)  $\|A_n\|_{\mathcal{L}(E)} = n$  for every natural  $n$ , or
- (ii) there exists a constant  $1/2 \leq q < 1$  such that  $\|A_n\|_{\mathcal{L}(E)} \leq \text{const} \cdot n^q$  for all  $n \in \mathbb{N}$ .

*Proof.* Since for all  $n, m \geq 0$ , we have

$$(3) \quad \|A_{n+m}\|_{\mathcal{L}(E)} \leq \|A_n\|_{\mathcal{L}(E)} + \|A_m\|_{\mathcal{L}(E)},$$

and since  $\|f \otimes r\|_E = \|f\|_E$ , we infer that  $\|A_n\|_{\mathcal{L}(E)} \leq n$ .

Observing that  $A_{mn}(f)$  and  $A_m(A_n(f))$  are identically distributed, we have

$$\|A_{mn}(f)\|_E = \|A_m(A_n(f))\|_E, \quad f \in E(\Omega).$$

Here, we identify the element  $A_n f \in E(\Omega \times \dots \times \Omega)$  with an element from  $E(\Omega)$  via a measure preserving transformation  $\Omega \times \overset{2n}{\dots} \times \Omega \rightarrow \Omega$ . Hence,

$$(4) \quad \|A_{mn}\|_{\mathcal{L}(E)} \leq \|A_m\|_{\mathcal{L}(E)} \cdot \|A_n\|_{\mathcal{L}(E)}.$$

Thus, we have the following alternative: either

- (i)  $\|A_n\|_{\mathcal{L}(E)} = n$  for every natural  $n$ , or
- (ii) there exists  $n_0 \geq 2$  such that  $\|A_{n_0}\|_{\mathcal{L}(E)} < n_0$ .

To finish the proof of Theorem 1, we only need to consider the second case. Suppose there exists a constant  $1/2 \leq q < 1$  such that  $\|A_{n_0}\|_{\mathcal{L}(E)} \leq n_0^q$ . By (4) we have

$$\|A_{n_0^m}\|_{\mathcal{L}(E)} \leq \|A_{n_0}\|_{\mathcal{L}(E)}^m \leq n_0^{qm}, \quad \forall m \in \mathbb{N}.$$

Every  $n$  can be written as  $\sum_{i=0}^k a_i n_0^i$ , where  $0 \leq a_i \leq n_0 - 1$  and  $a_k \neq 0$ . So, using (3) and (4), we have

$$\begin{aligned} \|A_n\|_{\mathcal{L}(E)} &\leq \sum_{i=0}^k \|A_{a_i n_0^i}\|_{\mathcal{L}(E)} \leq \sum_{i=0}^k \|A_{a_i}\|_{\mathcal{L}(E)} n_0^{qi} \\ &\leq \left( \sum_{i=0}^k n_0^{qi} \right) \max_{1 \leq s \leq n_0} \{ \|A_s\|_{\mathcal{L}(E)} \} \leq \frac{n_0^q \cdot n_0^{qk}}{n_0^q - 1} \max_{1 \leq s \leq n_0} \{ \|A_s\|_{\mathcal{L}(E)} \}. \end{aligned}$$

Now, using the fact that  $q \geq 1/2$  and  $n_0 \geq 2$ , we have  $n_0^q - 1 \geq \sqrt{2} - 1$ . So,

$$\frac{1}{n_0^q - 1} \leq \sqrt{2} + 1.$$

Since  $n_0^k \leq n$ , we have

$$\|A_n\|_{\mathcal{L}(E)} \leq (\sqrt{2} + 1)n_0^q \max_{1 \leq s \leq n_0} \{\|A_s\|_{\mathcal{L}(E)}\} \cdot n_0^{qk} \leq \text{const} \cdot n^q.$$

This proves the theorem. ■

REMARK 2. We record here an important connection between the estimates given in Theorem 1(ii) above and the set  $\Gamma_{\text{id}}(E)$ , where the r.i. space  $E$  is separable. For  $1/2 \leq q \leq 1$  the following conditions are equivalent:

- (i)  $\|A_n\|_{\mathcal{L}(E)} \leq \text{const} \cdot n^q, n \geq 1.$
- (ii)  $1/q \in \Gamma_{\text{id}}(E).$

Indeed, the implication (i) $\Rightarrow$ (ii) is obvious. Now, let the probability space  $(\Omega, \mathcal{P})$  be the infinite direct product of the measure spaces  $([0, 1], m)$ . Fix  $f \in E$  and consider the sequence  $\{(f \otimes r)_n\}_{n \geq 1} \subset E(\Omega, \mathcal{P})$ . It follows from [22, Lemma 3.4] that this sequence is weakly null in  $E(\Omega, \mathcal{P})$ . Since the spaces  $E$  and  $E(\Omega, \mathcal{P})$  are isometric, we obtain the implication (ii) $\Rightarrow$ (i) via an application of the uniform boundedness principle.

We complete this section with an estimate of  $\|A_n\|_{\mathcal{L}(E)}, n \geq 1$ , in general r.i. spaces with the Kruglov property.

THEOREM 3. *Let  $E$  be a separable r.i. space. If  $\beta_E < 1$  and if  $E$  satisfies the Kruglov property, then  $\|A_n\|_{\mathcal{L}(E)} \leq \text{const} \cdot n^q$  for all sufficiently large  $n \geq 1$  and any  $\beta_E < q < 1$ .*

*Proof.* It is proved in [2, Proposition 2.2] (see also [17, Theorem 1]) that for every r.i. space  $E$  and an arbitrary sequence of independent random variables  $\{f_k\}_{k=1}^n (n \geq 1)$  from  $E$ , the right hand side of (2) can be estimated as

$$(5) \quad \left\| \sum_{k=1}^n \bar{f}_k \right\|_{Z_E^2} \leq 6 \left\| \left( \sum_{k=1}^n f_k^2 \right)^{1/2} \right\|_E.$$

Now, assume in addition that the sequence  $\{f_k\}_{k=1}^n (n \geq 1)$  consists of independent identically distributed random variables,  $\|f_1\|_E = 1$ . Since  $\beta_E < 1$ , there exist  $N$  and  $\beta_E < q < 1$  such that for every  $k \geq N, \|\sigma_k\|_{\mathcal{L}(E)} \leq k^q$ . Fix  $\varepsilon > 0$  such that  $1/2 + \varepsilon < q$ . By [22, Theorem 9], in every separable r.i. space  $E$ , the right hand side of (5) can be estimated as

$$(6) \quad \left\| \left( \sum_{k=1}^n f_k^2 \right)^{1/2} \right\|_E \leq \frac{4}{\varepsilon} \max_{1 \leq k \leq n} \left( \frac{n}{k} \right)^{1/2+\varepsilon} \|\sigma_k\|_{\mathcal{L}(E)} =: A, \quad n \geq 1.$$

The right hand side of (6) can be estimated as

$$(7) \quad \begin{aligned} A &\leq \frac{4}{\varepsilon} n^{1/2+\varepsilon} \max \left\{ \max_{N \leq k \leq n} k^{-1/2-\varepsilon} k^q, \max_{1 \leq k \leq N} k^{-1/2-\varepsilon} \|\sigma_k\|_{\mathcal{L}(E)} \right\} \\ &= \frac{4}{\varepsilon} n^{1/2+\varepsilon} \max \{ n^{q-1/2-\varepsilon}, \text{const} \} \leq \text{const} \cdot n^q. \end{aligned}$$

Recalling the definition of the operator  $A_n$  and combining it with (2), (5), (6), (7) yields the assertion. ■

REMARK 4.

- (i) The assumption  $\beta_E < 1$  in Theorem 3 is necessary (see [1, Theorem 4.2]). For example, the space  $E = L_1$  has the Kruglov property and  $\beta_E = 1$ . However,  $\|A_n\|_{\mathcal{L}(L_1)} = n$ .
- (ii) On the other hand, the condition that  $E$  has the Kruglov property is not optimal. In the following section, we will show that there are Lorentz spaces without the Kruglov property which still satisfy the condition of Theorem 1(ii).

**4. Operators  $A_n$ ,  $n \geq 1$ , in Lorentz spaces.** We need the following technical facts. The first lemma is elementary and its proof is omitted.

LEMMA 5. *Let  $\psi$  be a concave function on  $[0, 1]$ . If there are points  $0 \leq x_1 \leq \dots \leq x_n \leq 1$  such that*

$$\frac{1}{n}(\psi(x_1) + \dots + \psi(x_n)) = \psi\left(\frac{1}{n}(x_1 + \dots + x_n)\right),$$

*then  $\psi$  is linear on  $[x_1, x_n]$ .*

LEMMA 6. *Let  $x_1, \dots, x_n$  be independent random variables. Then*

$$\mathbb{E}(|x_1 + \dots + x_n|) \leq \mathbb{E}(|x_1|) + \dots + \mathbb{E}(|x_n|).$$

*Moreover, equality holds if and only if all  $x_i$ 's are simultaneously non-negative (or non-positive).*

*Proof.* We have

$$\mathbb{E}(|x_1|) + \dots + \mathbb{E}(|x_n|) - \mathbb{E}(|x_1 + \dots + x_n|) = \mathbb{E}(|x_1| + \dots + |x_n| - |x_1 + \dots + x_n|) \geq 0.$$

From the independence of  $x_i$ 's,  $i = 1, \dots, n$ , it follows that  $\text{sign}(x_i)$ ,  $i = 1, \dots, n$ , are independent random variables. If there exists a function  $x_i$  which is neither non-negative nor non-positive, then, for every other function  $x_j$ , we have

$$\begin{aligned} m(x_i x_j < 0) &= m(\text{sign}(x_i) > 0, \text{sign}(x_j) < 0) + m(\text{sign}(x_i) < 0, \text{sign}(x_j) > 0) \\ &= m(\text{sign}(x_i) > 0)m(\text{sign}(x_j) < 0) + m(\text{sign}(x_i) < 0)m(\text{sign}(x_j) > 0) > 0. \end{aligned}$$

Hence, there exists a set  $A$  of positive measure such that  $x_i x_j < 0$  almost everywhere on  $A$ . This guarantees that  $|x_1| + \dots + |x_n| > |x_1 + \dots + x_n|$  almost everywhere on  $A$ . This is sufficient for the strict inequality to hold. ■



We need to consider the following properties of the function  $\psi$ :

$$(8) \quad a_\psi := \limsup_{u \rightarrow 0} \frac{\psi(ku)}{\psi(u)} < k,$$

$$(9) \quad c_\psi := \limsup_{u \rightarrow 0} \frac{\psi(u^l)}{\psi(u)} < 1,$$

$$(10) \quad \limsup_{u \rightarrow 0} \frac{1}{\psi(u)} \sum_{s=1}^n \psi\left(2^{1-s} \binom{n}{s} u^s\right) < n.$$

PROPOSITION 7. *Suppose there exist  $k \geq 2$  such that (8) holds and  $l \geq 2$  such that (9) holds. Then (10) holds for all sufficiently large  $n$ .*

*Proof.* Consider the sum  $\sum_{s=1}^n \psi\left(\binom{n}{s} 2^{1-s} u^s\right)$ . For any sufficiently large  $n$ , we write

$$\sum_{s=1}^n = \sum_{s=1}^{1+[n/k]} + \sum_{s=2+[n/k]}^n.$$

Consequently, the upper limit in (10) can be estimated as

$$(11) \quad \begin{aligned} \limsup_{u \rightarrow 0} \frac{1}{\psi(u)} \sum_{s=1}^n \psi\left(\binom{n}{s} 2^{1-s} u^s\right) \\ \leq \limsup_{u \rightarrow 0} \frac{1}{\psi(u)} \sum_{s=1}^{1+[n/k]} \psi\left(\binom{n}{s} 2^{1-s} u^s\right) \\ + \limsup_{u \rightarrow 0} \frac{1}{\psi(u)} \sum_{s=2+[n/k]}^n \psi\left(\binom{n}{s} 2^{1-s} u^s\right). \end{aligned}$$

Consider the first upper limit in (11). Since  $\psi$  is concave, we have

$$\begin{aligned} \sum_{s=1}^{1+[n/k]} \psi\left(\binom{n}{s} 2^{1-s} u^s\right) &\leq (1 + [n/k]) \psi\left(\frac{1}{1 + [n/k]} \sum_{s=1}^{1+[n/k]} \binom{n}{s} 2^{1-s} u^s\right) \\ &= (1 + [n/k]) \psi\left(\frac{1}{1 + [n/k]} (nu + o(u))\right) \leq (1 + [n/k]) \psi(ku(1 + o(1))). \end{aligned}$$

Hence, the first upper limit in (11) is bounded from above by

$$(1 + [n/k])a_\psi = n \cdot \frac{a_\psi}{k} + o(n).$$

Consider the second upper limit in (11). It is clear that for all  $n/k \leq s \leq n$ ,

$$\binom{n}{s} 2^{1-s} \leq 2^n \quad \text{and} \quad \binom{n}{s} 2^{1-s} u^s \leq 2^n u^{n/k} = (2^k u)^{n/k}.$$

Thus, the second upper limit in (11) can be estimated as

$$\limsup_{u \rightarrow 0} \frac{1}{\psi(u)} \sum_{s=2+\lceil n/k \rceil}^n \psi \left( \binom{n}{s} 2^{1-s} u^s \right) \leq n(1 - 1/k) \limsup_{u \rightarrow 0} \frac{\psi((2^k u)^{n/k})}{\psi(u)}.$$

Substituting  $w = 2^k u$  on the right hand side, we have the bound

$$n(1 - 1/k) \limsup_{w \rightarrow 0} \frac{\psi(w^{n/k})}{\psi(2^{-k} w)}.$$

By the concavity of  $\psi$ , we have  $\psi(2^{-k} w) \geq 2^{-k} \psi(w)$ . Therefore, the second upper limit in (11) is bounded from above by

$$n(1 - 1/k) 2^k \limsup_{w \rightarrow 0} \frac{\psi(w^{n/k})}{\psi(w)}.$$

Now, we observe that

$$(12) \quad \limsup_{w \rightarrow 0} \frac{\psi(w^m)}{\psi(w)} \leq c_\psi^{\log(m)/\log(l)-1}.$$

Indeed, let  $l^r \leq m \leq l^{r+1}$ ,

$$\frac{\psi(w^m)}{\psi(w)} \leq \frac{\psi(w^{l^r})}{\psi(w)} = \frac{\psi(w^{l^r})}{\psi(w^{l^{r-1}})} \cdots \frac{\psi(w^l)}{\psi(w)}$$

and

$$\limsup_{w \rightarrow 0} \frac{\psi(w^m)}{\psi(w)} \leq c_\psi^r \leq c_\psi^{\log(m)/\log(l)-1}.$$

If  $n$  tends to infinity, then, thanks to the assumption  $c_\psi < 1$ , we have

$$n(1 - 1/k) 2^k \limsup_{w \rightarrow 0} \frac{\psi(w^{n/k})}{\psi(w)} = o(n).$$

Therefore, the upper limit in (10) (see also (11)) is bounded from above by

$$\frac{a_\psi}{k} n + o(n).$$

Thus, the upper limit in (10) is strictly less than  $n$  for every sufficiently large  $n$ . ■

Let the function  $g_n$  be defined by

$$(13) \quad g_n(u) := \frac{\|A_n \chi_{[0,u]}\|_{\Lambda(\psi)}}{n \|\chi_{[0,u]}\|_{\Lambda(\psi)}} \\ = \frac{1}{n\psi(u)} \sum_{s=1}^n \psi(m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq s)).$$

It is obvious that  $0 \leq g_n \leq 1$ .

REMARK 8. The second equality in (13) is a corollary of [14, II.5.4].

PROPOSITION 9. For sufficiently large  $n$ , we have  $g_n(u) < 1$  for all  $u \in (0, 1]$ .

*Proof.* Since  $\psi$  is concave, we have

$$(14) \quad \sum_{s=1}^n \psi(m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq s)) \\ \leq n\psi\left(\frac{1}{n} \sum_{s=1}^n m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq s)\right).$$

Note that if a random variable  $\xi_n$  takes the values  $0, 1, \dots, n$  then

$$(15) \quad \sum_{s=1}^n m(\xi_n \geq s) = \mathbb{E}(\xi_n).$$

By (15), the right hand side of (14) equals  $n\psi\left(\frac{1}{n}\mathbb{E}(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n|)\right)$ . By Lemma 6, we have

$$(16) \quad \frac{1}{n} \mathbb{E}(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n|) < \mathbb{E}(|\chi_{[0,u]} \otimes r|) = u.$$

Applying  $\psi$ , we obtain

$$(17) \quad n\psi\left(\frac{1}{n} \mathbb{E}(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n|)\right) \leq n\psi(\mathbb{E}(|\chi_{[0,u]} \otimes r|)).$$

The right hand side of (17) is equal to  $n\psi(u)$ .

Let us assume that  $g_n(u) = 1$  for some  $u > 0$  and some  $n > 1$ . It then follows that both inequalities (14) and (17) are actually equalities.

The equality

$$\sum_{s=1}^n \psi(m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq s)) \\ = n\psi\left(\frac{1}{n} \sum_{s=1}^n m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq s)\right)$$

implies, by Lemma 5, that  $\psi$  is linear on the interval  $[a_1, b_1]$  with

$$a_1 = m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq n), \\ b_1 = m(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n| \geq 1).$$

Since the inequality in (17) is actually an equality, we derive from (16) and (17) that  $\psi$  must be a constant on the interval  $[a_2, b_2]$  with  $a_2 = (1/n)\mathbb{E}(|(\chi_{[0,u]} \otimes r)_1 + \cdots + (\chi_{[0,u]} \otimes r)_n|)$  and  $b_2 = \mathbb{E}(|\chi_{[0,u]} \otimes r|)$ . Since  $\psi$  is increasing and concave, it must be constant on  $[a_2, 1]$ .

Since, by (15),

$$\frac{1}{n} \mathbb{E}(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n|) > m(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n| \geq n)$$

and

$$\frac{1}{n} \mathbb{E}(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n|) < m(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n| \geq 1),$$

we have  $a_1 < a_2 < b_2$ . So, the intersection of the intervals  $[a_1, b_1]$  and  $[a_2, 1]$  contains an interval  $[a_3, b_3]$  with  $a_3 < b_3$ .

Since  $\psi$  is a linear function on  $[a_1, b_1]$  and is constant on  $[a_2, 1]$  it must be constant on  $[a_1, 1]$ , that is, on the interval

$$[m(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n| \geq n), 1] = [2^{1-n}u^n, 1].$$

Thus,  $\psi$  is constant on the interval  $[2^{1-n}, 1] \subset [2^{1-n}u^n, 1]$ , which is not the case for sufficiently large  $n$ . So,  $g_n(u) < 1$  for all sufficiently large  $n$ . ■

LEMMA 10. *For the function  $g_n$  defined in (13), we have*

$$\limsup_{u \rightarrow 0} g_n(u) = \limsup_{u \rightarrow 0} \frac{1}{n\psi(u)} \sum_{s=1}^n \psi\left(2^{1-s} \binom{n}{s} u^s\right).$$

*Proof.* For every  $s \geq 1$ , using a formula for conditional probabilities, we have

$$\begin{aligned} m(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n| \geq s) \\ = \sum_{k=1}^n \binom{n}{s} u^k (1-u)^{n-k} m(|r_1 + \dots + r_k| \geq s). \end{aligned}$$

Actually, the summation above is taken from  $k = s$  up to  $n$ , since we have  $m(|r_1 + \dots + r_k| \geq s) = 0$  for every  $k < s$ .

If now  $u \rightarrow 0$ , then, for every  $s \geq 1$  and  $k > s$ , we have  $\binom{n}{k} u^k (1-u)^{n-k} = o(u^s)$ . Therefore,

$$(18) \quad m(|(\chi_{[0,u]} \otimes r)_1 + \dots + (\chi_{[0,u]} \otimes r)_n| \geq s) = 2^{1-s} \binom{n}{s} u^s (1 + o(1)).$$

Since  $\psi$  is concave, we have

$$(19) \quad \psi\left(\frac{1}{m} u\right) \leq \frac{1}{m} \psi(u), \quad 0 < m \leq 1.$$

This implies

$$(20) \quad \lim_{u \rightarrow 0} \frac{\psi(u(1 + o(1)))}{\psi(u)} = 1.$$

After applying (18) and (20) to the definition of  $g_n$  in (13), we obtain the assertion of the lemma. ■

The following theorem is the main result in this section.

THEOREM 11. Let  $\psi \in \Psi$ . The following conditions are equivalent:

- (i)  $\|A_n\|_{\mathcal{L}(\Lambda(\psi))} < n$  for all sufficiently large  $n$ .
- (ii) Estimates (8) and (9) hold for some  $k \geq 2$  and  $l \geq 2$ .

REMARK 12. Note that condition (i) above is equivalent to the assumption that  $\|A_{n_0}\|_{\mathcal{L}(\Lambda(\psi))} < n_0$  for some  $n_0 > 1$  (see Theorem 1).

*Proof of Theorem 11.* We are interested in whether there exist  $n \in \mathbb{N}$  and  $c < n$  such that

$$(21) \quad \|A_n f\|_{\Lambda(\psi)} \leq c \|f\|_{\Lambda(\psi)}, \quad f \in \Lambda(\psi).$$

We will use the following known description of extreme points of the unit ball in  $\Lambda(\psi)$ . A function  $f$  is in  $\text{extr}(B_{\Lambda(\psi)}(0, 1))$  if and only if

$$|f| = \frac{\chi_A}{\|\chi_A\|_{\Lambda(\psi)}}$$

for some measurable set  $A \subset [0, 1]$ . Here  $\chi_A$  is the indicator function of the set  $A$ . This means that  $f$  is of the form

$$f = \frac{\chi_{A_1} - \chi_{A_2}}{\psi(m(A_1 \cup A_2))}$$

with  $A_1$  and  $A_2$  having empty intersection. It is sufficient to verify (21) only for functions  $f$  as above (see [14, Lemma II.5.2]).

Clearly,  $f \otimes r$  and  $|f| \otimes r$  are identically distributed random variables. Therefore,  $A_n(f)$  and  $A_n(|f|)$  are also identically distributed. Furthermore,  $\|A_m(f)\| = \|A_m(|f|)\|$  and  $\|f\| = \||f|\|$ . Thus, we need to check (21) for indicator functions only. It is sufficient to take  $A$  of the form  $[0, u]$ ,  $0 < u \leq 1$ .

Using the notation  $g_n(\cdot)$  introduced in (13), we see that (21) is equivalent to

$$(22) \quad \sup_u g_n(u) < 1.$$

Now, we are ready to finish the proof of the theorem.

(i)  $\Rightarrow$  (ii). Fix  $n$  such that  $\|A_n\|_{\mathcal{L}(\Lambda(\psi))} < n$ . It follows from the argument above that (22) holds. Now, we immediately infer from Lemma 10 and the definition of  $g_n(\cdot)$  that

$$\limsup_{u \rightarrow 0} \frac{1}{n\psi(u)} \sum_{s=1}^n \psi\left(\binom{n}{s} 2^{1-s} u^s\right) < 1,$$

which is equivalent to (10). Thus,

$$\begin{aligned} \limsup_{u \rightarrow 0} \frac{\psi(nu)}{n\psi(u)} &= \limsup_{u \rightarrow 0} \frac{\psi(2^{1-1} \binom{n}{1} u^1)}{n\psi(u)} \\ &\leq \limsup_{u \rightarrow 0} \frac{1}{n\psi(u)} \sum_{s=1}^n \psi\left(2^{1-s} \binom{n}{s} u^s\right) < 1. \end{aligned}$$

Suppose that (9) fails. We have

$$\limsup_{u \rightarrow 0} \frac{\psi(u^l)}{\psi(u)} = 1$$

for every  $l \geq 1$ . Since  $\binom{n}{s} 2^{1-s} u^s \geq u^{n+1}$  for every  $s = 1, \dots, n$  and every sufficiently small  $u$ , we have

$$\limsup_{u \rightarrow 0} \frac{1}{n\psi(u)} \sum_{s=1}^n \psi\left(\binom{n}{s} 2^{1-s} u^s\right) \geq \limsup_{u \rightarrow 0} \frac{n\psi(u^{n+1})}{n\psi(u)} = 1.$$

This contradicts (10) and completes this part of the proof.

(ii)  $\Rightarrow$  (i). Fix  $k \geq 2$  (respectively,  $l \geq 2$ ) such that (8) (respectively, (9)) holds. Then, for sufficiently large  $n$ , (10) also holds. By Lemma 10, we have

$$(23) \quad \limsup_{u \rightarrow 0} g_n(u) < 1$$

for all sufficiently large  $n$ . By Proposition 9, we have  $g_n(u) < 1$  for all sufficiently large  $n$  and for all  $u \in (0, 1]$ . Therefore, by (23), the inequality (22) holds for sufficiently large  $n$ . Then (see the argument at the beginning of the proof),  $\|A_n\|_{\mathcal{L}(\Lambda(\psi))} < n$  for sufficiently large  $n$ . ■

Combining Theorems 1 and 11, we have

**COROLLARY 13.** *For every function  $\psi$ , one of the following two mutually excluding alternatives holds:*

- (1) *There exist  $q \in [1/2, 1)$  and  $C > 0$  such that the operator  $A_n : \Lambda(\psi) \rightarrow \Lambda(\psi)$  satisfies*

$$\|A_n\|_{\mathcal{L}(E)} \leq Cn^q, \quad n \geq 1.$$

- (2) *Either for every  $k \in \mathbb{N}$ ,*

$$(24) \quad \limsup_{u \rightarrow 0} \frac{\psi(ku)}{\psi(u)} = k,$$

*or for every  $l \in \mathbb{N}$ ,*

$$(25) \quad \limsup_{u \rightarrow 0} \frac{\psi(u^l)}{\psi(u)} = 1.$$

**REMARK 14.**

- (i) Condition (24) is equivalent to the assumption  $\beta_{\Lambda(\psi)} = 1$ .  
(ii) Condition (25) implies (but is not equivalent to) the condition  $\alpha_{\Lambda(\psi)} = 0$ . In the last section of this paper, we will exhibit a  $\psi \in \Psi$  failing (25) such that the Lorentz space  $\Lambda(\psi)$  does not have the Kruglov property.

**5. Operators  $A_n$ ,  $n \geq 1$ , in Orlicz spaces  $\exp(L_p)$ .** The space  $\exp(L_p)$  has the Kruglov property if and only if  $p \leq 1$  (see [6, 3]). Furthermore,  $\exp(L_p)$  is 2-convex for all  $0 < p < \infty$  (see e.g. [16, 1.d]). Now, we immediately infer from [2] that  $\Gamma_{\text{iid}}(\exp(L_p)_0) = \Gamma_i(\exp(L_p)_0) = [1, 2]$  for all  $0 < p \leq 1$  (here,  $\exp(L_p)_0$  is the separable part of  $\exp(L_p)$ ). Using Remark 2, we have  $\|A_n\|_{\mathcal{L}(\exp(L_p)_0)} \leq \text{const} \cdot n^{1/2}$  for all  $n \geq 1$  and  $0 < p \leq 1$ . It easily follows that in fact,  $\|A_n\|_{\mathcal{L}(\exp(L_p))} \leq \text{const} \cdot n^{1/2}$  for all  $n \geq 1$  and  $0 < p \leq 1$ . In this section, we prove the estimate  $\|A_n\|_{\mathcal{L}(\exp(L_p))} \leq \text{const} \cdot n^{1/2}$  (respectively,  $\|A_n\|_{\mathcal{L}(\exp(L_p))} \leq \text{const} \cdot n^{1-1/p}$ ) for all  $1 < p \leq 2$  (respectively,  $2 \leq p < \infty$ .) To this end, it is convenient to view  $\exp(L_p)$  as a Marcinkiewicz space  $M(\psi_p)$  with  $\psi_p(t) = t \log^{1/p}(e/t)$  (see [3, Lemma 4.3]). The following simple lemma is crucial.

LEMMA 15. *There exists  $\Psi \ni \psi \sim \psi_2$  such that the random variable  $\psi' \otimes r$  is Gaussian.*

*Proof.* Setting

$$F(t) := \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-z^2} dz, \quad t \geq 0,$$

and denoting its inverse by  $G$ , we clearly see that  $G \otimes r$  is Gaussian. From the obvious inequality

$$c_1 e^{-2t^2} \leq F(t) \leq c_2 e^{-t^2},$$

substituting  $t = G(z)$ , we obtain

$$c_1 e^{-2G^2(z)} \leq z \leq c_2 e^{-G^2(z)}$$

or, equivalently,

$$\frac{1}{\sqrt{2}} \log^{1/2} \left( \frac{c_1}{z} \right) \leq G(z) \leq \log^{1/2} \left( \frac{c_2}{z} \right).$$

This means

$$\psi(t) = \int_0^t G(z) dz \sim \int_0^t \log^{1/2} \left( \frac{e}{z} \right) dz \sim t \log^{1/2} \left( \frac{e}{t} \right) = \psi_2(t). \quad \blacksquare$$

THEOREM 16.

- (i) *For every  $1 \leq p \leq 2$ , we have  $\|A_n\|_{\mathcal{L}(\exp(L_p))} \leq \text{const} \cdot \sqrt{n}$ .*
- (ii) *For every  $2 \leq p \leq \infty$ , we have  $\|A_n\|_{\mathcal{L}(\exp(L_p))} \leq \text{const} \cdot n^{1-1/p}$ .*

*Proof.* (i) By Lemma 15,  $\exp(L_2) = M(\psi)$  for some  $\psi \in \Psi$  with  $\psi' \otimes r$  Gaussian. Recall the following description of the extreme points of the unit ball in Marcinkiewicz spaces (see [19]): a function  $f$  is an extreme point of the unit ball in  $M(\psi)$  if and only if  $f^* = \psi'$ . Since  $\|A_n x\|_{M(\psi)} = \|A_n \psi'\|_{M(\psi)}$

for any  $x \in M_\psi$  with  $x^* = \psi'$ , we infer that  $\|A_n \psi'\|_{M(\psi)} = \|A_n\|_{\mathcal{L}(M(\psi))}$ ,  $n \geq 1$ . Since  $\psi' \otimes r$  is Gaussian, the function

$$\frac{(\psi' \otimes r)_1 + \cdots + (\psi' \otimes r)_n}{\sqrt{n}}$$

is also Gaussian, in particular, its rearrangement coincides with  $\psi'$ . This means  $\|A_n\|_{\mathcal{L}(M_\psi)} = \sqrt{n}$ . The result now follows by interpolation between  $\exp(L_1)$  and  $\exp(L_2)$ , since for every  $0 < p_1 \leq p_2 \leq \infty$  we have

$$[\exp(L_{p_1}), \exp(L_{p_2})]_{\theta, \infty} = \exp(L_p)$$

with  $1/p = (1 - \theta)/p_1 + \theta/p_2$  (see, for example, [8]).

(ii) Noting that  $\|A_n\|_{\mathcal{L}(L_\infty)} = n$ ,  $n \geq 1$ , we deduce the assertion from (i) by applying the real method of interpolation to the couple  $(\exp(L_2), L_\infty)$  as above. ■

**6. Applications to Banach–Saks index sets.** The first main result of this section, characterizing a subclass of the class of all r.i. spaces  $E$  such that  $\Gamma_{\text{iid}}(E) = \Gamma_1(E)$ , is given in Theorem 18 below. We first need a modification of the subsequence splitting result from [21, Theorem 3.2]. We present necessary details of the proof for the convenience of the reader.

**THEOREM 17.** *Let  $\{x_n\}_{n \geq 1}$  be a weakly null sequence of independent functions in a separable r.i. space  $E$  with the Fatou property. Then there exists a subsequence  $\{y_n\}_{n \geq 1} \subset \{x_n\}_{n \geq 1}$  which can be split as  $y_n = u_n + v_n + w_n$ ,  $n \geq 1$ . Here  $\{u_n\}_{n \geq 1}$  is a weakly null sequence of independent identically distributed functions, the sequence  $\{v_n\}_{n \geq 1}$  is also weakly null and consists of the elements with pairwise disjoint support and  $\|w_n\|_E \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* Let the probability space  $(\Omega, \mathcal{P})$  be the infinite direct product of measure spaces  $([0, 1], m)$ . Without loss of generality, we assume that  $E = E(\Omega)$  and that each function  $x_n$  depends only on the  $n$ th coordinate. That is,

$$x_n = 1 \otimes \cdots \otimes 1 \otimes h_n \otimes 1 \otimes \cdots, \quad h_n \in E(0, 1), \quad n \geq 1.$$

Consider the sequence  $\{g_n\}_{n \geq 1} = \{h_n^*\}_{n \geq 1} \subset E(0, 1)$ . Since

$$\|x_n\|_E = \|g_n\|_E \geq \|g_n \chi_{[0, s]}\|_E \geq g_n(s) \|\chi_{[0, s]}\|_E, \quad s \in [0, 1],$$

and the sequence  $\{x_n\}$  is bounded, it follows from Helly’s selection theorem that there exists a subsequence  $\{g_n^1\} \subset \{g_n\}$  which converges almost everywhere on  $[1/2, 1]$ . Repeating the argument, we get a subsequence  $\{g_n^2\} \subset \{g_n^1\}$  which converges almost everywhere on  $[1/3, 1]$ , etc. Thus, there exists a function  $h \in L_1(0, 1)$  to which the diagonal sequence  $\{g_n^n\}_{n \geq 1} = \{(h_n^n)^*\}_{n \geq 1}$  converges almost everywhere. The Fatou property of  $E$  guarantees that  $h \in E(0, 1)$  and  $\|h\|_E \leq 1$ . There is an operator  $P_n : L_1(0, 1) \rightarrow L_1(0, 1)$  of



the form  $(P_n x)(t) = \alpha(t)x(\gamma(t))$  (here  $|\alpha(t)| = 1$  and  $\gamma$  is a measure preserving transformation of the interval  $(0, 1)$  into itself), such that  $P_n g_n^n = h_n^n$ ,  $n \geq 1$  (see e.g. [14]). Now, put

$$\begin{aligned} y_n &:= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes h_n^n \otimes 1 \cdots, & n \geq 1, \\ u_n &:= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes (P_n h) \otimes 1 \cdots, & n \geq 1. \end{aligned}$$

It is clear that the functions  $u_n$  are independent. The proof is finished by repeating the remaining argument from [21, Theorem 3.2]. ■

**THEOREM 18.** *For an arbitrary separable r.i. space  $E$  with the Fatou property, we have*

$$\Gamma_{\text{iid}}(E) = \Gamma_i(E) \Leftrightarrow \Gamma_{\text{iid}}(E) \subseteq \Gamma_d(E).$$

*Proof.* If  $\Gamma_{\text{iid}}(E) = \Gamma_i(E)$ , then the embedding  $\Gamma_{\text{iid}}(E) \subseteq \Gamma_d(E)$  follows immediately from [1, Lemma 4.1(ii)]. Suppose now that  $\Gamma_{\text{iid}}(E) \subseteq \Gamma_d(E)$  and let  $\{f_k\}_{k \geq 1} \subset E$  be a normalized weakly null sequence of independent random variables on  $[0, 1]$ . Passing to a subsequence and applying the preceding theorem, we may assume that  $f_n = u_n + v_n + w_n$ ,  $n \geq 1$ , where  $\{u_n\}_{n \geq 1}$  is a weakly null sequence of independent identically distributed functions, the sequence  $\{v_n\}_{n \geq 1}$  is also weakly null and consists of elements with pairwise disjoint support, and  $\|w_n\|_E \rightarrow 0$  as  $n \rightarrow \infty$ . By the last convergence, we may assume that  $\|w_k\|_E \leq 2^{-k}$ , and so for every subsequence  $\{z_n\} \subset \{w_n\}$ , we have

$$\left\| \sum_{k=1}^n z_k \right\|_E \leq 1.$$

If, in addition,  $1/q \in \Gamma_{\text{iid}}(E)$ , then our assumptions also guarantee that there are constants  $C_2, C_3 > 0$  for which

$$\left\| \sum_{k=1}^n u_k \right\|_E \leq C_2 n^q, \quad \left\| \sum_{k=1}^n v_k \right\|_E \leq C_3 n^q. \quad \blacksquare$$

We will illustrate the result above in the settings of:  $(\alpha)$  r.i. spaces satisfying an upper 2-estimate;  $(\beta)$  Lorentz spaces  $\Lambda(\varphi)$  and Marcinkiewicz spaces  $M(\varphi)_0$ ,  $\varphi \in \Psi$ ; and  $(\gamma)$  classical  $L_{p,q}$ -spaces.

$(\alpha)$  Recall that a Banach lattice  $X$  is said to satisfy an *upper 2-estimate* if there exists a constant  $C > 0$  such that for every finite sequence  $(x_i)_{i=1}^n$  of pairwise disjoint elements in  $X$ ,

$$\left\| \sum_{j=1}^n x_j \right\|_X \leq C \left( \sum_{j=1}^n \|x_j\|_X^2 \right)^{1/2}.$$

**COROLLARY 19.** *If  $E$  is a separable r.i. space with the Fatou property and satisfying an upper 2-estimate, then  $\Gamma_{\text{iid}}(E) = \Gamma_i(E)$ .*

*Proof.* The assumption that the space  $E$  satisfies an upper 2-estimate implies immediately that  $2 \in \Gamma_d(E)$  and hence  $[1, 2] \subseteq \Gamma_d(E)$ . Since  $\Gamma_{\text{iid}}(E) \subseteq [1, 2]$  (see [1, Lemma 4.1(i)]), the result now follows from Theorem 18. ■

( $\beta$ ) Although Lorentz spaces do not satisfy an upper 2-estimate, we have

$$\Gamma_d(\Lambda(\psi)) = [1, \infty)$$

(see e.g. the proof of [1, Corollary 4.8]) and similarly,  $\Gamma_d(M(\psi)_0) = [1, \infty)$  (see e.g. [1, p. 897]) for any  $\psi \in \Psi$ . Although the Marcinkiewicz spaces  $(M(\psi)_0)$  do not have the Fatou property, applying a modification of Theorem 17 similar to [1, Lemma 3.6], we obtain the following corollary from Theorem 18.

**COROLLARY 20.** *For every  $\psi \in \Psi$ , we have  $\Gamma_i(\Lambda(\psi)) = \Gamma_{\text{iid}}(\Lambda(\psi))$  and  $\Gamma_i(M(\psi)_0) = \Gamma_{\text{iid}}(M(\psi)_0)$ .*

( $\gamma$ ) We will now show that the equality  $\Gamma_i(E) = \Gamma_{\text{iid}}(E)$  fails to hold in an important subclass of r.i. spaces which plays a significant role in interpolation theory [14, 16]. Recall the definition of the Lorentz spaces  $L_{p,q}$ ,  $1 < p, q < \infty$ :  $x \in L_{p,q}$  if and only if the quasi-norm

$$\|x\|_{p,q} = \frac{q}{p} \left( \int_0^1 (x^*(t)t^{1/p})^q \frac{dt}{t} \right)^{1/q}$$

is finite. The expression  $\|\cdot\|_{p,q}$  is a norm if  $1 \leq q \leq p$  and is equivalent to a (Banach) norm if  $q > p$ .

We will now show that  $\Gamma_i(L_{p,q}) \neq \Gamma_{\text{iid}}(L_{p,q})$  provided  $1 < q < p < 2$ . To this end, we first observe that every normalized sequence  $\{v_n\}_{n \geq 1} \subset L_{p,q}$  of functions with disjoint supports contains a subsequence spanning the space  $l_q$  (see [9, Lemma 2.1]). In particular,  $\Gamma_d(L_{p,q}) \subset \Gamma(l_q) = [1, q]$  and so, by [1, Lemma 4.1(ii)], we have  $\Gamma_i(L_{p,q}) \subseteq [1, q]$ . Next, it is proved in [10, Corollary 3.7] (see also [7, Corollary 5.2]) that if  $p < 2$  then for every sequence of identically distributed independent random variables we have

$$\left\| \sum_{k=1}^n x_k \right\|_{L_{p,q}} = o(n^{1/p}),$$

which implies, in particular, that  $[1, p] \subseteq \Gamma_{\text{iid}}(L_{p,q})$ . This shows that  $(q, p] \subseteq \Gamma_{\text{iid}}(L_{p,q}) \setminus \Gamma_i(L_{p,q})$  as soon as  $1 < q < p < 2$ . We observe that a combination of Theorem 18 above with the results given in [10] and [21, Section 5] provides a tool for a complete characterization of the sets  $\Gamma_i(L_{p,q})$ ,  $\Gamma_{\text{iid}}(L_{p,q})$  and  $\Gamma_d(L_{p,q})$  for all pairs  $(p, q)$ . However, such a characterization is beyond the scope of the present paper.

Our second main result in this section completely characterizes the subclass of all Lorentz spaces  $\Lambda(\psi)$ ,  $\psi \in \Psi$ , whose Banach–Saks index set  $\Gamma_i(\Lambda(\psi))$  is non-trivial.

**THEOREM 21.**  $\Gamma_{\text{iid}}(\Lambda(\psi)) \neq \{1\}$  if and only if the function  $\psi$  satisfies conditions (8) and (9) for some  $k, l \geq 2$ .

*Proof.* Let  $\{f_k\}_{k \geq 1} \subset \Lambda(\psi)$  be a normalized weakly null sequence of independent identically distributed random variables on  $[0, 1]$ . Note that we automatically have  $\int_0^1 f_k dm = 0, k \geq 1$ .

Using the standard symmetrization trick, we consider another sequence  $\{f'_k\}_{k \geq 1}$  of independent random variables (which is also independent with respect to the sequence  $\{f_k\}_{k \geq 1}$ ) such that  $f'_k$  is equidistributed with  $f_k$ , and define  $h_k := f_k - f'_k, k \geq 1$ . Clearly,  $\{h_k\}_{k \geq 1}$  is a sequence of independent symmetric identically distributed random variables. Note that, by [6, Proposition 11, p. 6], we have

$$\left\| \sum_{k=1}^n f_k \right\|_{\Lambda(\psi)} \leq \text{const} \cdot \left\| \sum_{k=1}^n h_k \right\|_{\Lambda(\psi)}, \quad n \geq 1.$$

Now, if  $\psi$  satisfies conditions (8) and (9), then it follows from Corollary 13 that  $\|\sum_{k=1}^n h_k\|_{\Lambda(\psi)} \leq \text{const} \cdot n^q$  for some  $q \in (0, 1)$  and hence  $1/q \in \Gamma_{\text{iid}}(\Lambda(\psi))$ .

Conversely, let  $1/q \in \Gamma_{\text{iid}}(\Lambda(\psi))$  for some  $q \in (0, 1)$ . Fix  $f \in \Lambda(\psi)$  and consider the sequence  $\{(f \otimes r)_n\}_{n \geq 1} \subset \Lambda(\psi)(\Omega, \mathcal{P})$ , where the probability space  $(\Omega, \mathcal{P})$  is the infinite direct product of measure spaces  $([0, 1], m)$ . Since the Lorentz spaces  $\Lambda(\psi)(\Omega, \mathcal{P})$  and  $\Lambda(\psi)(0, 1)$  are isometric, and since the sequence  $\{(f \otimes r)_n\}_{n \geq 1}$  is weakly null in  $\Lambda(\psi)(\Omega, \mathcal{P})$  (see e.g. [22, Lemma 3.4]), we have

$$\sup_{n \geq 1} \frac{1}{n^q} \|(f \otimes r)_1 + (f \otimes r)_2 + \dots + (f \otimes r)_n\|_{\Lambda(\psi)} \leq C(f).$$

Setting  $B_n := (1/n^q)A_n, n \geq 1$ , we have  $\|B_n f\|_{\Lambda(\psi)} \leq C(f)$  for every  $n \geq 1$ . By the uniform boundedness principle, we have  $\|B_n\|_{\mathcal{L}(\Lambda(\psi))} \leq C < \infty$  for all  $n \geq 1$ , or equivalently,  $\|A_n\|_{\mathcal{L}(\Lambda(\psi))} \leq Cn^q, n \geq 1$ . Corollary 9 now implies that the function  $\psi$  satisfies conditions (8) and (9). ■

The following corollary follows immediately from the above combined with Corollary 20.

**COROLLARY 22.**  $\Gamma_i(\Lambda(\psi)) \neq \{1\}$  if and only if the function  $\psi \in \Psi$  satisfies conditions (8) and (9) for some  $k, l \geq 2$ .

We complete this section with the description of  $\Gamma_i(\exp(L_p)_0), 1 \leq p < \infty$ .

**THEOREM 23.** For every  $1 \leq p \leq 2$ , we have  $\Gamma_{\text{iid}}(\exp(L_p)_0) = \Gamma_i(\exp(L_p)_0) = [1, 2]$ . For every  $2 \leq p < \infty$ , we have  $\Gamma_{\text{iid}}(\exp(L_p)_0) = \Gamma_i(\exp(L_p)_0) = [1, p/(p - 1)]$ .

*Proof.* The first assertion follows from Remark 2, Theorem 16 and Corollary 20. The same argument shows that  $\Gamma_i(\exp(L_p)_0) \supseteq [1, p/(p-1)]$  for every  $2 \leq p < \infty$ . The equality  $\Gamma_i(\exp(L_p)_0) = [1, p/(p-1)]$  follows from the fact that the estimate

$$\|A_n \chi_{[0,1]}\|_{\exp(L_p)_0} \leq \text{const} \cdot n^{1-1/p}, \quad n \geq 1,$$

is the best possible (see [18, Theorem 8] or [11, Theorem 15]). ■

**7. Concluding remarks and examples.** The preceding theorem shows that the set  $\Gamma_i(\exp(L_p)_0)$  is non-trivial for all  $1 \leq p < \infty$ , whereas  $\exp(L_p)$  has the Kruglov property if and only if  $0 < p \leq 1$ . This result extends and complements [2], where examples of r.i. spaces  $E$  with the Kruglov property such that  $\Gamma(E) = \{1\}$  and  $\Gamma_i(E) \neq \{1\}$  are built. We now present an example of a Lorentz space  $\Lambda(\psi)$  such that  $\Gamma_i(\Lambda(\psi)) \neq \{1\}$  and which does not have the Kruglov property.

EXAMPLE 24. Let  $\psi \in \Psi$  be given by the condition

$$\psi(t) := \frac{1}{\log^{1/2}(1/t)}, \quad t \in [0, e^{-3/2}],$$

and be linear on  $[e^{-3/2}, 1]$ . The space  $\Lambda(\psi)$  does not have the Kruglov property, although  $\Gamma_i(\Lambda(\psi)) \neq \{1\}$ .

*Proof.* Since for every  $k, l > 1$  we have

$$\begin{aligned} \lim_{u \rightarrow 0} \frac{\psi(ku)}{\psi(u)} &= \lim_{u \rightarrow 0} \left( \frac{\log(u)}{\log(ku)} \right)^{1/2} = 1 < k, \\ \lim_{u \rightarrow 0} \frac{\psi(u^l)}{\psi(u)} &= \lim_{u \rightarrow 0} \left( \frac{\log(u)}{\log(u^l)} \right)^{1/2} = \frac{1}{l^{1/2}} < 1 \end{aligned}$$

we see that  $\Gamma_i(\Lambda(\psi)) \neq \{1\}$  by Corollary 22.

By [1, Theorem 5.1] a Lorentz space  $\Lambda(\phi)$ ,  $\phi \in \Psi$ , has the Kruglov property if and only if

$$\sup_{t>0} \frac{1}{\phi(t)} \sum_{n=1}^{\infty} \phi\left(\frac{t^n}{n!}\right) < \infty.$$

In our case, for every fixed  $t \leq e^{-3/2}$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} \psi\left(\frac{t^n}{n!}\right) &= \sum_{n=1}^{\infty} \frac{1}{(\log(n!) + n \log(1/t))^{1/2}} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n \log(n)(1 + o(1)))^{1/2}} = \infty. \quad \blacksquare \end{aligned}$$

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