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A lower bound on the radius of analyticity of a power series in a real Banach space

by

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Abstract. Let F be a power series centered at the origin in a real Banach space with radius of uniform convergence ϱ. We show that F is analytic in the open ball B of radius $\varrho/\sqrt{e}$, and furthermore, the Taylor series of F about any point $a \in B$ converges uniformly within every closed ball centered at a contained in B.

1. Introduction. Let X, Y be Banach spaces over a field $K = \mathbb{R}$ or $\mathbb{C}$. We will denote the norms on X and Y both by $|\cdot|$ since there should be no confusion as to which Banach space elements are in. We quickly review the notion of a power series (see [3] for further background).

A *power series* centered at $a \in X$ is a formal sum

$$\sum_{m=0}^{\infty} P_m(x-a) \tag{1}$$

where for each m, $P_m : X \to Y$ is a continuous homogeneous polynomial of degree m defined as follows: If $L_m : X^m \to Y$ is a symmetric m-linear map, we write $L_m(x_1^{k_1}, \ldots, x_i^{k_i})$ as shorthand for $L_m(x_1, \ldots, x_1, \ldots, x_i, \ldots, x_i)$, where each x_j appears k_j times for $1 \le j \le i$, $k_1 + \cdots + k_i = m$. A map $P_m : X \to K$ is a *homogeneous polynomial of degree* m if there exists a symmetric m-linear map L_m such that $P_m(x) = L_m(x^m)$.

Define the *radius of uniform convergence* of the power series (1) to be

$$\varrho := \sup\{r : (1) \text{ converges uniformly on } |x-a| \le r\}.$$

There is the following standard formula for the radius of uniform convergence of (1):

$$\varrho = 1/\limsup_{m\to\infty} \|P_m\|^{1/m} \tag{2}$$

where $\|P_m\| = \sup_{|x|=1} |P_m(x)|$ is the polynomial norm of P_m. If $\varrho > 0$, then for every $0 < r < \varrho$, (1) is a uniformly and absolutely convergent series for

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every $x \in B_r(a)$, the open ball of radius r centered at a. Hence in this case, (1) is more than just a formal sum: it defines a function on $B_\varrho(a)$ taking values in Y.

The *Taylor series* of an infinitely differentiable function F defined in a neighborhood of a is the power series defined by

$$T_aF(x) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m F(a)((x-a)^m). \tag{3}$$

Here $D^mF(a) : X^m \to Y$ is the symmetric m-linear map given by taking the Fréchet derivative of F m times. We say that F is *analytic at* a if $T_aF(x)$ has a positive radius of uniform convergence and equals $F(x)$ within the domain of uniform convergence. If $U \subset X$ is open, we say F is *analytic in* U if it is analytic at every $a \in U$. If furthermore $T_aF(x)$ converges uniformly in every closed ball centered at a contained in U, for each $a \in U$, we say F is *fully analytic in* U ([1]).

If a power series centered at a has positive radius of uniform convergence, then this series defines an infinitely differentiable function whose Taylor series at a is equal to the original power series as one would expect from the classical one-dimensional setting (see Lemma 2).

Suppose we are given a power series $F(x)$ centered at zero that has radius of uniform convergence $\varrho > 0$. If we pick any $a \in B_\varrho(0)$ different from 0, it follows that since the polynomials appearing in the series $T_aF(x)$ have a complicated dependence on the polynomials appearing in the power series $F(x)$, then just using the definition (2), it is a priori unclear what the relationship is between the radius of uniform convergence for $T_aF(x)$ and that of $F(x)$. However, if $K = \mathbb{C}$, we get the expected result from the classical theory of one complex variable, namely that $F(x)$ is fully analytic in $B_\varrho(0)$ so that the radius of uniform convergence of $T_a(x)$ is at least $\varrho - |a|$. In the one-dimensional setting $X, Y = \mathbb{C}$, this follows from the Cauchy integral formula. For general complex Banach spaces, one gets the same result by applying the Cauchy integral formula inside every complex plane in the Banach space, namely by applying the estimate to $g(z) = F(zx + (1-z)y)$ for every $x, y \in X$ (see [3, Chapter 13]).

However, the situation is unclear when $K = \mathbb{R}$ and we are in a real Banach space. As we no longer have the Cauchy integral formula, it is unclear what happens to the radius of uniform convergence when we reexpand $F(x)$ about a new point a, i.e. when we form the power series $T_aF(x)$.

([1]) A function may be analytic in all of X without being fully analytic, since its Taylor series about any point might only have a finite radius of uniform convergence. See [3, Section 15.5] for an example.

Definition 1. Let $F(x)$ be a power series centered at a with radius of uniform convergence $\varrho > 0$. The *radius of analyticity* $\varrho_A = \varrho_A(F)$ of $F(x)$ at a is the largest $r > 0$ such that $F(x)$ is fully analytic in $B_r(a)$.

As mentioned above, in the complex setting we have $\varrho_A = \varrho$. In the real case, it is known that $\varrho_A \geq \varrho/e$ ([3, Theorem, p. 156]). Our main result is an improvement of this theorem:

Theorem 1. *Let $F(x)$ be a power series in a Banach space X, which we may take to be centered at the origin. Let $\varrho > 0$ denote its radius of uniform convergence and $\varrho_A = \varrho_A(F)$. Then*

(i) $\varrho_A \geq \varrho/\sqrt{e}$;
(ii) *for every n, the nth Fréchet derivative $D^nF : X \to \mathcal{L}_n(X, Y)$ of $F(x)$, viewed as a map from X to the Banach space $\mathcal{L}_n(X, Y)$ of continuous n-linear maps from X into Y, has a Taylor series centered at the origin with radius of uniform convergence at least $\varrho/\sqrt{e}$. Moreover, the radius of analyticity of this power series is also at least $\varrho/\sqrt{e}$.*

When X is a (real or complex) Hilbert space, one has $\varrho_A = \varrho$ (see the discussion after the proof of Theorem 1). However, the author does not know if $\varrho_A = \varrho$ holds for a general real Banach space.

The proof of the theorem involves understanding certain norms of multilinear maps which allow us to control the polynomial terms occurring in a Taylor series.

2. Multilinear map estimates. Given an m-homogeneous polynomial P_m, there is a unique symmetric m-linear map L_m such that $P_m = L_m(x^m)$. This is because one can recover L_m from P_m via the polarization identity:

$$L_m(x_1, \ldots, x_m) = \frac{1}{2^m m!} \sum_{e_1,\ldots,e_m \in \{\pm 1\}} e_1 \cdots e_m P_m(e_1 x_1 + \cdots + e_m x_m). \tag{4}$$

Given a symmetric m-linear map, let $\widetilde{L}$ denote the polynomial associated to L, namely $\widetilde{L}(x) = L(x^m)$. We define the following norms:

$$\|L\| = \sup_{|x_1|,\ldots,|x_m|=1} |L(x_1, \ldots, x_m)|, \tag{5}$$

$$\|L\|_{(2)} = \sup_{0\leq k\leq m} \sup_{|x_1|,|x_2|=1} |L(x_1^k, x_2^{m-k})|, \tag{6}$$

$$\|\widetilde{L}\| = \sup_{|x|=1} |\widetilde{L}(x)|. \tag{7}$$

Clearly, $\|\widetilde{L}\| \leq \|L\|_{(2)} \leq \|L\|$. An easy consequence of the polarization identity (4) is that

$$\|L\| \leq \frac{m^m}{m!}\,\|\widetilde{L}\|. \tag{8}$$

This bound is sharp, i.e., there are examples where equality is achieved [3, 4.14]. However, this bound can be improved for certain Banach spaces X. If X is a (real or complex) Hilbert space, we in fact have $\|L\| = \|\widetilde{L}\|$ (see [1, Proposition 1.1]). Several other authors have investigated when (8) can be improved (see [6], [8]).

The norm we need to control when we expand a power series at a new point is $\|L\|_{(2)}$. To see this, consider the power series (1) centered at $a = 0$, which we may rewrite as

$$F(x) = \sum_{m=0}^{\infty} L_m(x^m). \tag{9}$$

Observe that by the binomial formula, given any $y \in X$,

$$L_m(x^m) = L_m((y + x - y)^m) = \sum_{k=0}^{m} \binom{m}{k} L_m(y^{m-k}, (x-y)^k). \tag{10}$$

Then if in

$$\sum_{m=0}^{\infty} L_m(x^m) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \binom{m}{k} L_m(y^{m-k}, (x-y)^k) \tag{11}$$

the double series on the right converges absolutely, we can interchange summations and obtain

$$F(x) = \sum_{m=0}^{\infty} L_m(x^m) = \sum_{k=0}^{\infty}\sum_{m=k}^{\infty} \binom{m}{k} L_m(y^{m-k}, (x-y)^k). \tag{12}$$

Thus, if we can perform this change of summation for all $x \in B_r(y)$, for some $r > 0$, then we will have expressed $F(x)$ as a power series centered at y whose k-homogeneous polynomial coefficients are given by

$$A_k(z) := \sum_{m=k}^{\infty} \binom{m}{k} L_m(y^{m-k}, z^k). \tag{13}$$

Observe that the absolute convergence of the double sum (11) for $x \in B_r(y)$ implies the absolute convergence of the A_k in $B_r(0)$ and hence on all of X by homogeneity.

Absolute convergence of (11) holds if

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m} \binom{m}{k} \|L_m\|_{(2)} |y|^{m-k} |x-y|^k = \sum_{m=0}^{\infty} \|L_m\|_{(2)} (|y| + |x-y|)^m < \infty. \tag{14}$$

This holds when

$$(15)\qquad |y| + |x-y| < \frac{1}{\limsup \|L_m\|_{(2)}^{1/m}}.$$

Choose a subsequence m_j such that

$$\lim_{j\to\infty} \|L_{m_j}\|_{(2)}^{1/m_j} = \limsup \|L_m\|_{(2)}^{1/m}.$$

Let $\varrho = 1/\limsup \|\widetilde{L}_m\|^{1/m} > 0$ be the radius of uniform convergence of (9).

Suppose $\varrho < \infty$. Then (15) is satisfied if

$$(16)\qquad |y| + |x-y| < \varrho \frac{\limsup_{j\to\infty} \|\widetilde{L}_{m_j}\|^{1/m_j}}{\lim_{j\to\infty} \|L_{m_j}\|_{(2)}^{1/m_j}} = \varrho \limsup_{j\to\infty} \left(\frac{\|\widetilde{L}_{m_j}\|}{\|L_{m_j}\|_{(2)}} \right)^{1/m_j} =: \bar{\varrho}.$$

Thus, for $|y| < \bar{\varrho}$ and $|x-y| < \bar{\varrho} - |y|$, the series (11) converges absolutely.

Altogether then, we have shown that for any fixed $|y| < \bar{\varrho}$, we have

$$(17)\qquad F(x) = \sum_{k=0}^{\infty} A_k(x-y)$$

for $|x-y| < \bar{\varrho} - |y|$.

LEMMA 2 ([3, Corollary 1, p. 165]). *For any power series as in* (17) *centered at* y *with positive radius of uniform convergence*, $A_k = (1/k!)D^k F(y)$ *as* k*-homogeneous polynomials.*

LEMMA 3. *Let* $L : X^m \to Y$ *be an* m*-linear map. Define the norm* $\|L\|_{(n)}$ *as in* (6) *but with up to* n *distinct arguments possible. Then* $(\|L\|_{(n)}/\|\widetilde{L}\|)^{1/m} \le C\sqrt{e}$ *where* $C = C(m,n)$ *is independent of* L, X, *and* Y *and tends to* 1 *as* $m \to \infty$ *for any fixed* n.

Given Lemmas 2 and 3, our main theorem easily follows:

Proof of Theorem 1. Suppose $\varrho < \infty$. Lemma 3 implies

$$(18)\qquad \limsup_{m\to\infty} (\|\widetilde{L}_m\|/\|L_m\|_{(2)})^{1/m} \ge 1/\sqrt{e},$$

hence $\bar{\varrho} \ge \varrho/\sqrt{e}$ by (16). Thus the preceding analysis in (12)–(16) shows that for $|y| < \bar{\varrho}$, the Taylor series (17) is absolutely convergent in $\{x : |x-y| < \varrho/\sqrt{e} - |y|\}$. From this, we get uniform convergence of (17) in $\{x : |x-y| \le r\}$ for every $r < \varrho/\sqrt{e} - |y|$, since one can bound the tail of

(17) as follows:

$$\Big|\sum_{k=N}^{\infty} A_k(x-y)\Big| \le \sum_{k=N}^{\infty}\sum_{m=k}^{\infty}\binom{m}{k}|L_m(y^{m-k},(x-y)^k)| \le \sum_{m=N}^{\infty}\|L_m\|_{(2)}(|y|+|x-y|)^m \tag{19}$$

where (19) tends to zero uniformly in x as $N \to 0$ so long as $|y|+|x-y| \le |y|+r$ is bounded away from $\bar{\varrho} \ge \varrho/\sqrt{e}$. This follows because we have shown that (19) viewed as power series in a single real variable has radius of uniform convergence at least $\bar{\varrho}$. Finally, Lemma 2 implies that the power series (17) is the Taylor series of $F(x)$ centered at y. So for $\varrho < \infty$, this proves (i) since we have shown $\varrho_A \ge \varrho/\sqrt{e}$.

For the remaining case $\varrho = \infty$, i.e. $\lim_{m\to\infty}\|\widetilde{L}_m\|^{1/m} = 0$, (18) also implies that $\lim_{m\to\infty}\|L_m\|_{(2)}^{1/m} = 0$. From (15), we can apply the previous analysis for every $\bar{\varrho} > 0$, whence $\varrho_A = \varrho = \infty$. Altogether, this proves (i).

By [3, Corollary 1, p. 165], given any power series $F(x) = \sum_{m=0}^{\infty} L_m(x^m)$ with radius of uniform convergence $\varrho > 0$, for every n, $D^nF(x)$ has a Taylor series centered at the origin given by

$$T_0D^nF(x) = n!\sum_{m=0}^{\infty}\binom{m+n}{n}L_{m+n}(x^m). \tag{20}$$

In [3], (8) is used to show that the radius of uniform convergence of this power series is at least ϱ/e . However, in (20) the linear maps L_{m+n} are only evaluated on at most $n+1$ distinct arguments ($D^nF(x)$ takes values in n-linear maps), so by Lemma 3, we can improve the lower bound for the radius of uniform of convergence of (20) to $\varrho/\sqrt{e}$ since

$$\limsup_{m\to\infty}\Big\|\binom{m+n}{n}L_{m+n}\Big\|_{(n+1)}^{1/m} \le (\limsup_{m\to\infty} C(m,n+1)\sqrt{e})(\limsup_{m\to\infty}\|\widetilde{L}_{m+n}\|^{1/m}) \le \frac{\sqrt{e}}{\varrho}.$$

By Lemma 2 and the formula (13) for the A_k, it follows that $T_0D^nF(y) = D^nF(y)$ for all y such that $|y| < \varrho/\sqrt{e}$.

To complete the proof of (ii), we also need to show that the radius of analyticity of (20) is at least $\varrho/\sqrt{e}$. For this, we need to control $\|L_{m+n}\|_{(n+2)}$, since this is precisely the $\|\cdot\|_{(2)}$-norm of L_{m+n} viewed as a map from $X \to \mathcal{L}_n(X,Y)$. We apply Lemma 3 once again. ∎

As mentioned earlier, when X is a Hilbert space, $\|\widetilde{L}\| = \|L\|$ and so $\|L\|_{(n)} = \|\widetilde{L}\|$ for all n. Thus, $\varrho_A = \varrho$.

Proof of Lemma 3. We can assume $L \not\equiv 0$. We first prove the lemma in the case $n = 2$. Let $x, y \in X$ be unit vectors and let $p + q = m$. By the polarization identity,

$$L(x^p, y^q) = \frac{1}{2^m m!} \sum_{e_i, f_j = \pm 1} [e_1 \cdots e_p f_1 \cdots f_q] \times [L(((e_1 + \cdots + e_p)x + (f_1 + \cdots + f_q)y)^m)], \tag{21}$$

$$|L(x^p, y^q)| \le \frac{1}{m!} \sum_{e_i, f_j = \pm 1} \frac{1}{2^m} (|e_1 + \cdots + e_p| + |f_1 + \cdots + f_q|)^m \|\widetilde{L}\|. \tag{22}$$

The last sum is an expectation with respect to the uniform probability measure on the space of outcomes for the independent Rademacher random variables $e_1, \ldots, e_p, f_1, \ldots, f_q$, which each take on the values ± 1 with probability $1/2$. We have the following well-known large deviation estimate (Hoeffding's inequality) for such random variables:

$$P(|e_1 + \cdots + e_p| \ge x) \le 2 \exp(-x^2/2p).$$

Let $A = e_1 + \cdots + e_p$ and $B = f_1 + \cdots + f_q$. Thus, $\lambda_A(x) := P(|A| \ge x)$ is bounded from above by Hoeffding's inequality. Using the binomial theorem and the independence of A and B, we can rewrite (22) as

$$\frac{|L(x^p, y^q)|}{\|\widetilde{L}\|} \le \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \mathbb{E}(|A|^k) \mathbb{E}(|B|^{m-k}). \tag{23}$$

Next,

$$\mathbb{E}(|A|^k) = k \int_0^\infty x^{k-1} \lambda_A(x)\, dx \le 2k \int_0^\infty x^{k-1} \exp(-x^2/2p)\, dx = k(2p)^{k/2} \Gamma(k/2), \tag{24}$$

where the first equality is a general equality relating the kth moment of a random variable with an appropriately weighted integral of its "distribution function" $\lambda_A(x)$ (see e.g. [5, Section 6.4]).

Below, we write C_k to denote some positive constants depending only on k such that $\limsup_{k\to\infty} C_k^{1/k} \le 1$. The precise value of C_k is unimportant and may change from line to line.

By Stirling's formula,

$$\Gamma(k) \le C_k \left(\frac{k}{e}\right)^k.$$

Thus,

$$\mathbb{E}(|A|^k) \le C_k \left(\frac{pk}{e}\right)^{k/2}.$$

Since the same estimate holds for $\mathbb{E}(|B|^{m-k})$, we have

$$
(25) \quad \begin{aligned}
\frac{|L(x^p,y^q)|}{\|\widetilde{L}\|} &\le \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}C_k\Big(\frac{pk}{e}\Big)^{k/2}C_{m-k}\Big(\frac{q(m-k)}{e}\Big)^{(m-k)/2}\\
&= e^{m/2}\sum_{k=0}^{m}\bigg\{\Big[C_k\cdot\frac{1}{k!}\Big(\frac{k}{e}\Big)^{k}\Big]\Big[C_{m-k}\cdot\frac{1}{(m-k)!}\Big(\frac{m-k}{e}\Big)^{m-k}\Big]\\
&\qquad\qquad\times\Big(\frac{p}{k}\Big)^{k/2}\Big(\frac{q}{m-k}\Big)^{(m-k)/2}\bigg\}\\
&\le e^{m/2}\Big(\sup_{0\le k\le m}C_kC_{m-k}\Big)\sum_{k=0}^{m}\Big(\frac{p}{k}\Big)^{k/2}\Big(\frac{q}{m-k}\Big)^{(m-k)/2}
\end{aligned}
$$

where in the last line, we used Stirling's formula once again. If we view the (last) summand appearing in (25) as a function of k, then setting its derivative equal to zero, one finds that the maximum value of the summand is achieved at $k = pm/(p+q) = p$ since $p+q=m$. At this value of k, the summand is one. Thus, since

$$
\limsup_{m\to\infty}\Big(\sup_{0\le k\le m}C_kC_{m-k}\Big)^{1/m} = \limsup_{m\to\infty}\Big(\sup_{0\le k\le m}C_k^{1/m}C_{m-k}^{1/m}\Big)\le 1,
$$

as $m\ge k, m-k$, (25) implies

$$
(|L(x^p,y^q)|/\|\widetilde{L}\|)^{1/m}\le C(m)\sqrt{e},
$$

where $C(m)$ is some absolute constant and $\limsup_{m\to\infty}C(m)\le 1$.

The proof is similar for the case of general n. We now have n independent random variables obeying the estimate (24). Instead of a binomial sum, we have a multinomial sum, which by the same calculation is also bounded by $e^{m/2}$ times a subexponential factor as $m\to\infty$ for fixed n. Explicitly, if we have unit vectors $x_1,\dots,x_n\in X$ and positive integers $p_1,\dots,p_n$ with $p_1+\dots+p_n=m$, then proceeding similarly to the above we get

$$
\frac{|L(x_1^{p_1},\dots,x_n^{p_n})|}{\|\widetilde{L}\|}\le e^{m/2}C_m\sum_{\substack{m_1+\dots+m_n=m\\ m_1,\dots,m_n\ge 0}}\Big(\frac{p_1}{m_1}\Big)^{m_1/2}\cdots\Big(\frac{p_n}{m_n}\Big)^{m_n/2}.
$$

One can show that the maximum of the summand in the above is achieved at $m_i=p_i$ for all i (for example, using the method of Lagrange multipliers), in which case, the summand is equal to one. This implies that the entire sum is bounded by $\binom{m+n-1}{n-1}$. This is subexponential in m for fixed n. ■

REMARK 1. One can try to complexify the Banach space X and extend the power series $F(x)$ to the complexification $\widetilde{X}$ of X. One could then hope to apply Cauchy estimates to the function $F(x)$ viewed as a holomorphic function on an open subset of $\widetilde{X}$. However, the problem of complexification of a Banach space, namely, of choosing a suitable complexification norm, is a subtle issue (e.g., see [7]). In particular, [7] has shown that the unique complex extension $\widetilde{P} : \widetilde{X} \to \widetilde{Y}$ of a real polynomial $P : X \to Y$ obeys (under a suitable choice of norms on $\widetilde{X}$ and $\widetilde{Y}$)

$$\|\widetilde{P}\|_* \leq 2^{n-1}\|P\|$$

where n is the degree of P and $\|\cdot\|_*$ is the polynomial norm taken with respect to $\widetilde{X}, \widetilde{Y}$. This bound is sharp, i.e., there are examples of polynomials defined on a fixed Banach space that achieve equality in the above. Thus, if one replaces P_m with $\widetilde{P}_m$ in the definition of F, the radius of uniform convergence on $\widetilde{X}$ for the complex extension may a priori decrease to $\varrho/2$, which is worse than $\varrho/\sqrt{e}$.

Since the estimates made in Lemma 3 are far from sharp, the ratio $\varrho_A/\varrho \geq 1/\sqrt{e}$ may not be optimal. Ideally, one would like to have $\varrho_A = \varrho$ as in the complex case.

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References

[1] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings in topological vector spaces*, Studia Math. 39 (1971), 59–76.

[2] —, —, *Analytic functions in topological vector spaces*, ibid., 77–112.

[3] S. B. Chae, *Holomorphy and Calculus in Normed Spaces*, Monogr. Textbooks Pure Appl. Math. 92, Dekker, 1985.

[4] R. M. Dudley and R. Norvaiša, *Concrete Functional Calculus*, book manuscript, to appear.

[5] G. Folland, *Real Analysis*, Wiley, 1999.

[6] L. A. Harris, *Bounds on the derivatives of holomorphic functions of vectors*, in: Analyse fonctionnelle et applications (Rio de Janeiro, 1972), Actualités Sci. Indust. 1367, Hermann, Paris, 1975, 145–163.

[7] G. A. Muñoz, Y. Sarantopoulos and A. Tonge, *Complexifications of real Banach spaces, polynomials and multilinear maps*, Studia Math. 134 (1999), 1–33.

[8] I. Sarantopoulos, *Estimates for polynomial norms on $L^p(\mu)$ spaces*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 263–271.

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