Linearization and compactness

by

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Abstract. This paper is devoted to several questions concerning linearizations of function spaces. We first consider the relation between linearizations of a given space when it is viewed as a function space over different domains. Then we study the problem of characterizing when a Banach function space admits a Banach linearization in a natural way. Finally, we consider the relevance of compactness properties in linearizations, more precisely, the relation between different compactness properties of a mapping, and compactness of its associated linear operator.

Introduction. Let $\mathcal{F}(U)$ be a linear space of continuous complex-valued functions on a topological space $U$. By a linearization of $\mathcal{F}(U)$ over $U$ we understand a pair $(Z,e)$, where $Z$ is a locally convex vector space and $e : U \to Z$ is a continuous map satisfying

(i) For every continuous linear functional $L \in Z'$ we have $L \circ e \in \mathcal{F}(U)$.

(ii) For each $f \in \mathcal{F}(U)$ there exists a unique continuous linear functional $L_f \in Z'$ such that $f = L_f \circ e$, that is, the following diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{f} & C \\
e & \downarrow & \\
Z & \xrightarrow{L_f} & \\
\end{array}
\]

In this way, $\mathcal{F}(U)$ is identified algebraically with the dual space of $Z$. Linearization can be a useful tool for the study of function spaces, since it enables the application of linear functional analysis to problems concerning non-linear functions.

Tensor products are a typical example of such an object, but many other linearizations have been constructed for various kinds of function spaces. For example, when $\mathcal{F}(U)$ is the space of Lipschitz functions on a metric space $U$, 2000 Mathematics Subject Classification: 46E10, 46E50, 47B07.

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the classical Arens–Eells construction provides a linearization of $\mathcal{F}(U)$ (see [23, Theorem 2.2.4]). For the space of continuous homogeneous polynomials on a Banach space, a linearization has been given by Ryan [21]. In the holomorphic setting, linearizations have been constructed by Mazet [15] and by Mujica and Nachbin [18] for spaces of holomorphic functions on finite- or infinite-dimensional domains. The case of bounded holomorphic functions was considered by Mujica [16], and the space of holomorphic functions of bounded type was studied by Galindo, García and Maestre [11] and by Mujica [17] (see also [3]). Some related work can be seen in [2]. Linearizations of several spaces of holomorphic functions have also been considered in [4], [5], and [12], where various properties of the corresponding linearizations are studied in connection with properties of the underlying spaces.

In [6] Carando and Zalduendo develop a general linearization procedure by constructing a canonical linearization $(\mathcal{F}_*(U), e)$ for any linear space of continuous functions $\mathcal{F}(U)$, which encompasses the above mentioned examples. This linearization also produces a factorization for vector-valued mappings in the following way. If $F$ is a locally convex space we denote by $\omega \mathcal{F}(U, F)$ the space of all continuous mappings $f : U \to F$ such that $\varphi \circ f \in \mathcal{F}(U)$ for every continuous linear functional $\varphi \in F'$. It is proved in [6, Theorem 3] that for each $f \in \omega \mathcal{F}(U, F)$ there exists a continuous linear operator $L_f : \mathcal{F}(U) \to F$ such that the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{f} & F \\
\downarrow{e} & & \downarrow{L_f} \\
\mathcal{F}_*(U) & & \\
\end{array}
$$

In this way, we can identify algebraically $\omega \mathcal{F}(U, F)$ with the space of continuous linear operators $\mathcal{L}(\mathcal{F}_*(U), F)$.

One of the purposes of this paper is to explore the relationship between compactness and linearizations, in particular, the relationship between different compactness properties of a mapping $f : U \to F$ and compactness (or weak compactness) of the corresponding operator $L_f : \mathcal{F}_*(U) \to F$. This problem has been studied by Pełczyński [20], Ryan [22], Aron and Schottenhofer [1], and Mujica [16] for spaces of polynomials and holomorphic mappings in infinite dimensions. We provide here a general approach which extends and unifies some previous results.

For convenience, we begin by recalling the linearization procedure in [6]. We fix a topological space $U$ and a linear space $\mathcal{F}(U)$ of complex-valued continuous functions defined over $U$.

Consider first the vector space $\mathbb{C}^U$ of finitely supported families of $U$-indexed complex numbers. A typical element will be denoted by $s = \sum_{x \in U} a_x e_x$, with $e_x(y) = \delta_{xy}$. Note that the sum is finite. For any given
$f \in \mathcal{F}(U)$, define the seminorm

$$p_f(s) = \left| \sum_{x \in U} a_x f(x) \right|,$$

and

$$\mathcal{N} = \{ s \in \mathbb{C}^U : p_f(s) = 0 \text{ for all } f \in \mathcal{F}(U) \}.$$

Now define

$$X = \mathbb{C}^U / \mathcal{N}.$$ 

Continue to denote the class of $e_x$ by $e_x$, and the class of $s$ by $s = \sum_{x \in U} a_x e_x$. Note that now this way of writing the class of $s$ need not be unique. However, the seminorms on $X$, coming from $|\sum_{x \in U} a_x f(x)|$ (which we continue to denote $p_f$) are well-defined and provide a Hausdorff locally convex space structure for $X$. Also, define

$$e : U \to X, \quad e(x) = e_x.$$ 

It is clear that a function $f \in \mathcal{F}(U)$ factors through $e$ in the following way:

$$\begin{array}{ccc}
U & \xrightarrow{f} & \mathbb{C} \\
\downarrow{e} & & \downarrow{L_f} \\
X & & \\
\end{array}$$

where $L_f$ is the continuous linear form defined by $L_f(s) = \sum_x a_x f(x)$ if $s = \sum_x a_x e_x$. The pairing $\langle X, \mathcal{F}(U) \rangle$ given by $\langle s, f \rangle = L_f(s)$ is a dual pairing, so algebraically we have $X' = \mathcal{F}(U)$ (see [14]).

Consider now the topology $\tau$ on $X$ defined by means of the seminorms $p_f$, where $f$ ranges over all functions of $\mathcal{F}(U)$. There are of course many topologies on $X$ which are compatible with $\tau$ (i.e., topologies with the same continuous linear functionals). For any one of them the dual space $X'$ is identified algebraically with $\mathcal{F}(U)$.

Define on $X$ the strongest locally convex topology compatible with $\tau$ for which the map $e : U \to X$ is continuous. We call it the $\alpha$-topology on $X$. Now $\mathcal{F}_*(U)$ is defined to be the completion of $(X, \alpha)$. This is the construction in [6]. Note that, as shown in [6], all the linearizations mentioned above are particular cases of this abstract linearization. We will use a few results from [6]. First, we have a characterization of the $\alpha$-topology:

**Fact 0.1** ([6, Proposition 3]). The topology $\alpha$ on $\mathcal{F}_*(U)$ is the topology of uniform convergence on the equicontinuous pointwise compact disks of $\mathcal{F}(U)$.

The following uniqueness result will be also useful:

**Fact 0.2** ([6, Corollary 2]). If $(Y, \widehat{e})$ is a linearization of $\mathcal{F}(U)$ and $Y$ is a Fréchet space, then there exists a topological isomorphism $T : \mathcal{F}_*(U) \to Y$. 

such that the following diagram commutes.

\[ \begin{array}{ccc}
F_*(U) & \xrightarrow{T} & Y \\
\downarrow^{e} & & \downarrow^{\hat{e}} \\
U & \xleftarrow{\check{e}} & \end{array} \]

Now suppose that \( F(U) \) is endowed with a locally convex topology. A linearization \((Z,e)\) of \( F(U) \) is said to be strong if \( F(U) \) is topologically isomorphic to the strong dual \((Z',\beta)\). The next result gives a sufficient condition for \( F_*(U) \) to be strong.

Recall from [6] that a subset \( B \subset X \) is called \( F \)-bounded if it is bounded in the \( \alpha \)-topology (or any compatible topology), while a subset \( A \subset U \) is called \( F \)-bounding if every \( f \in F(U) \) is bounded on \( A \). We say that \( U \) has the \( BBF \)-property if for each \( F \)-bounded subset \( B \) of \( X \), there is an \( F \)-bounding subset \( A \) of \( U \) and an \( r > 0 \) such that

\[ B \subset r \cdot \text{coe}(e(A)), \]

where \( \text{coe}(e(A)) \) denotes the closed absolutely convex hull of \( e(A) \). Then we have

**Fact 0.3** ([6, Theorem 1]). \( \text{Suppose that } F(U) \text{ is barreled, and its topology is that of uniform convergence on } F \)-bounding subsets of } U. Then

(i) \( F(U) \) is isomorphic to the strong dual \((F_*(U)',\beta)\),

(ii) \( U \) has the \( BBF \)-property.

The contents of the paper are as follows. Since many spaces can be modeled as function spaces in several different ways, we begin in Section 1 by studying the linearizations obtained when considering the same function space over several different domains. In Section 2 we consider the following problem: when \( F(U) \) is a Banach space of continuous functions on \( U \), a linearization is in general only a locally convex space. We are interested in giving conditions under which \( F(U) \) has a strong Banach linearization. Of course \( F(U) \) must be a dual Banach space, but we will see that this is not enough; a Banach space may have a Banach predual without having a linearizing Banach predual. We obtain a characterization in terms of compactness of the unit ball of \( F(U) \) for pointwise and compact-open topologies. This is akin to the Dixmier–Ng Theorem [19], but the result presented here is independent. In Section 3 we focus on the relationship between compactness properties of a mapping and compactness of its linearization. Apart from the usual definition of compactness, we also consider what we call boundedly compact mappings.

**1. Function spaces over different domains.** Many spaces can be viewed as spaces of continuous functions over several different domains, which can give rise to different linearizations.
EXAMPLE 1. Consider the Banach space \( L^1[0, 1] \). Any Banach space \( E \) may be viewed as a space of continuous functions on \((B_{E'}, \omega^*)\), by identifying each \( x \in E \) with \( \hat{x} \) where \( \hat{x}(\gamma) = \gamma(x) \) for every \( \gamma \in B_{E'} \). Thus in particular

\[
L^1[0, 1] = \mathcal{F}(B_{L^\infty}).
\]

If \( E \) is any Banach space with a Schauder basis \((v_k)_{k \in \mathbb{N}}\), we denote by \((v_k')_{k \in \mathbb{N}}\) the corresponding coordinate functionals, i.e. \( x = \sum_{k=1}^{\infty} v_k' (x) v_k \) for each \( x \in E \). We may then consider \( E \) as a space of (continuous) functions over \( \mathbb{N} \) by identifying each element \( x \in E \) with the sequence of its coordinates \((v_1'(x), v_2'(x), \ldots)\). In this way we see that

\[
L^1[0, 1] = \mathcal{F}(\mathbb{N}).
\]

Finally, if \( A, B \in \mathcal{B} \) (the Borel \( \sigma \)-algebra of \([0,1]\)), put \( d(A, B) = m(A \triangle B) \), and \( A \sim B \) if \( d(A, B) = 0 \). Then \( d \) is a metric on \( \mathcal{B}/\sim \), and each \( f \in L^1[0, 1] \) may be identified with the following map on \( \mathcal{B}/\sim \), which is continuous:

\[
A \mapsto \int_A f \, dm.
\]

In this way we find that

\[
L^1[0, 1] = \mathcal{F}(\mathcal{B}/\sim).
\]

The construction of the linearization \( \mathcal{F}_*(U) \) corresponding to \( \mathcal{F}(U) \) heavily depends on the domain \( U \), and even on the topology of \( U \), as the following simple example shows.

EXAMPLE 2. Let \( E \) be a Banach space, and consider the two topological spaces \( U = (E, w) \) and \( V = (E, \| \|) \). The dual \( E' \) may then be viewed as

\[
E' = \mathcal{F}(U) : \text{the space of weakly continuous linear forms on } E, \text{ or}
\]

\[
E' = \mathcal{G}(V) : \text{the space of norm continuous linear forms on } E.
\]

On linearizing, one obtains \( \mathcal{F}_*(U) = U = (E, w) \) and \( \mathcal{G}_*(V) = V = (E, \| \|) \).

We must, therefore, consider linearizations of function spaces over different domains, and ask ourselves when such linearizations coincide. We have the following result.

**Proposition 1.1.** Let \( U \) and \( V \) be topological spaces, \( \mathcal{F}(U) \) and \( \mathcal{G}(V) \) linear spaces of functions which are continuous over \( U \) and \( V \) respectively, and \( \varphi : V \to U \) continuous and such that \( f \circ \varphi \in \mathcal{G}(V) \) for each \( f \in \mathcal{F}(U) \). Suppose that the transpose \( \varphi^t : \mathcal{F}(U) \to \mathcal{G}(V) \) given by \( \varphi^t(f) = f \circ \varphi \) is an algebraic isomorphism, and \( \mathcal{F}_*(U) \) is a Fréchet space. Then there exists a topological isomorphism \( T_\varphi : \mathcal{G}_*(V) \to \mathcal{F}_*(U) \) such that following diagram
commutes:

$$
\begin{array}{c}
V \xrightarrow{\varphi} U \\
e_V \downarrow \quad \downarrow e_U \\
G_*(V) \xrightarrow{T_\varphi} F_*(U)
\end{array}
$$

Proof. In what follows, we index with $U$ and $V$ objects appearing in the constructions of $F(U)$ and $G(V)$ respectively. Define $\tilde{\varphi} : \mathbb{C}^V \to \mathbb{C}^U$ by $\tilde{\varphi}(\sum_{y \in V} a_y e_y) = \sum_{y \in V} a_y e_{\varphi(y)}$. It is easily seen that $\tilde{\varphi}$ passes to the quotient, producing a map

$$
\varphi : X_V \to X_U
$$

for which $L_f \circ \varphi = L_{f \circ \varphi}$. Also, this map is continuous for the corresponding $\alpha$-topologies in $X_V$ and $X_U$. To see this, it will be enough to check that given any equicontinuous $U$-pointwise compact disk $A \subset F(U)$, the set $\varphi^t(A) = \{f \circ \varphi : f \in A\}$ is an equicontinuous $V$-pointwise compact disk in $G(V)$. It is clearly a disk, and pointwise compactness follows from the continuity (in the pointwise topologies) of $\varphi^t : F(U) \to G(V)$. To see the equicontinuity, fix $y \in V$ and $\varepsilon > 0$. There is a neighborhood $W$ of $\varphi(y)$ such that for all $x \in W$ and $f \in A$, $|f(\varphi(y)) - f(x)| < \varepsilon$. Thus for all $z$ in the neighborhood $\varphi^{-1}(W)$ of $y$, and for all $f \in A$,

$$
|f(\varphi(y)) - f(\varphi(z))| < \varepsilon.
$$

Thus a map $\varphi : G_*(V) \to F_*(U)$ is induced which is linear and continuous, and such that the transpose $\varphi^t$ “coincides” with $\varphi^t$ in the sense that

$$
\varphi^t(L_f) = L_f \circ \varphi = L_{f \circ \varphi} = L_{\varphi^t(f)}.
$$

Now, since $\varphi^t : F(U) \to G(V)$ is an algebraic isomorphism, so is $\varphi^t : F_*(U)' \to G_*(V)'$. Then considering for each $f \in F(U)$ the commutativity of the diagram

$$
\begin{array}{c}
V \xrightarrow{f \circ \varphi} \mathbb{C} \\
e_U \circ \varphi \downarrow \quad \downarrow L_f \\
F_*(U)
\end{array}
$$

we see that $(F_*(U), e_U \circ \varphi)$ is a linearization of $G(V)$. Since $F_*(U)$ is a Fréchet space, Fact 0.2 gives the desired isomorphism $T_\varphi : G_*(V) \to F_*(U)$. $lacksquare$

A situation where Proposition 1.1 applies is the following. Let $U$ be an open connected subset of a locally convex space and $F(U)$ a linear space of holomorphic functions on $U$ such that $F_*(U)$ is a Fréchet space. If now $V \subset U$ is any non-empty open subset, let us consider the inclusion map $\iota : V \hookrightarrow U$ and the space

$$
G(V) = \{f|_V : f \in F(U)\}.
$$
Since $U$ is connected, the restriction map $\mathcal{F}(U) \to \mathcal{G}(V)$ is an algebraic isomorphism and thus $\mathcal{G}_*(V)$ is topologically isomorphic to $\mathcal{F}_*(U)$ through the associated $T_i$ which makes the corresponding diagram commutative.

2. Banach linearization. Suppose that $(\mathcal{F}(U), \| \cdot \|)$ is a Banach space of continuous functions on a topological space $U$. In this section we consider the problem of when $\mathcal{F}(U)$ admits a strong Banach linearization $(Z, e)$. Recall that this means that $Z$ is isomorphic to a Banach space and the Banach dual $Z'$ is isomorphic to $\mathcal{F}(U)$. In particular, we study when $\mathcal{F}_*(U)$ is a strong Banach linearization. Note that the topology of $\mathcal{F}(U)$ plays no role whatsoever in the construction of $\mathcal{F}_*(U)$, so in order to obtain our results relating the topologies of $\mathcal{F}_*(U)$ and $\mathcal{F}(U)$, we must necessarily impose some topological conditions on $\mathcal{F}(U)$.

For the case in which $U$ is a $k$-space, we obtain the following sufficient condition. Here, we denote by $\tau_{co}$ the compact-open topology and by $\tau_p$ the pointwise topology on $\mathcal{F}(U)$.

**Theorem 2.1.** Let $U$ be a $k$-space and $(\mathcal{F}(U), \| \cdot \|)$ a Banach space of continuous functions on $U$. If the ball of $\mathcal{F}(U)$ is $\tau_{co}$-compact, then $\mathcal{F}_*(U)$ is a strong Banach linearization.

**Proof.** Let $B$ be the unit ball of $\mathcal{F}(U)$, which is $\tau_{co}$-compact. For each $s \in X$, we may consider $s : \mathcal{F}(U) \to \mathbb{C}$ given by $s(f) = L_f(s)$. In this way, $s$ can be seen as a $\sigma(\mathcal{F}(U), X)$-continuous linear functional, so by compactness $|s|$ attains its maximum on $B$. Define $\|s\|_B = \max_B |s|.$ This defines a norm on $X$. Note that for each $x \in U$, if $f \in \mathcal{F}(U)$ and $f \neq 0$, we have $|e_x(f)| = \|f\| \cdot \left| e_x \left( \frac{f}{\|f\|} \right) \right| \leq \|f\| \cdot \|e_x\|_B.$ This shows that $e_x \in \mathcal{F}(U)'$. We claim that $e : U \to (\mathcal{F}(U)', \| \cdot \|)$ is continuous. Since $U$ is a $k$-space, it is enough to show that for each compact subset $K \subset U$, $e|_K : K \to (\mathcal{F}(U)', \| \cdot \|)$ is continuous. By the Ascoli–Arzelà Theorem, $B|_K$ is equicontinuous on $K$. Hence, for every $\varepsilon > 0$ and every $x \in K$, there exists a neighborhood $W^x$ such that if $y \in K \cap W^x$, then $\sup\{|f(x) - f(y)| : f \in B\} < \varepsilon;$ thus, $\|e_x - e_y\|_{\mathcal{F}(U)'} = \sup_{f \in B} |e_x(f) - e_y(f)| < \varepsilon.$ This establishes the claim.
As a consequence, $B$ is in fact equicontinuous as a set of functions on $U$. Indeed, for $\varepsilon > 0$ and every $x \in U$, there exists a neighborhood $V^x$ such that if $y \in V^x$, then
\[
\sup_{f \in B} |f(x) - f(y)| = \sup_{f \in B} |e_x(f) - e_y(f)| = \|e_x - e_y\|_\mathcal{F}(U)' < \varepsilon.
\]

In this way we see that, since $B$ is an equicontinuous $\tau_p$-compact disk, the norm $\|\cdot\|_B$ is continuous for the $\alpha$-topology. If we denote by $Y$ the completion of $(X,\|\cdot\|_B)$ we find that $e : U \to (X,\alpha) \to Y$ is continuous. Furthermore, each $f \in \mathcal{F}(U)$ linearizes through $Y$; indeed,
\[
|L_f(s)| = |s(f)| = \|f\| \left| s\left(\frac{f}{\|f\|}\right)\right| \leq \|f\| \cdot \|s\|_B,
\]
so $L_f$ is $\|\cdot\|_B$-continuous. Now by Fact 0.2 we conclude that $\mathcal{F}_*(U)$ is isomorphic to the Banach space $Y$. Next, we are going to see that $\mathcal{F}_*(U)$ is strong. Note that $\mathcal{F}(U)$ can be algebraically identified with the dual space $Y'$, and we can consider on $\mathcal{F}(U)$ the norm $\|\cdot\|_d$ which is dual to the norm of $Y$. It is clear that for every $x \in U$ and every $f \in \mathcal{F}(U)$,
\[
|f(x)| \leq \|e_x\|_B \cdot \|f\|_d.
\]
As a consequence, the topology induced by $\|\cdot\|_d$ on $\mathcal{F}(U)$ is finer than the $\tau_p$-topology. On the other hand, as we have seen above, for every $x \in U$ and every $f \in \mathcal{F}(U)$,
\[
|f(x)| \leq \|e_x\|_B \cdot \|f\|,
\]
so the topology induced by $\|\cdot\|$ on $\mathcal{F}(U)$ is also finer than the $\tau_p$-topology. Now, by applying the Closed Graph Theorem we find that, in fact, $\|\cdot\|_d$ is equivalent to $\|\cdot\|$ on $\mathcal{F}(U)$.

In the next theorem, by an equivalent ball in $\mathcal{F}(U)$ we mean the unit ball of a norm in $\mathcal{F}(U)$ which is equivalent to the original norm. We have to consider equivalent balls since, in general, a dual Banach space may have equivalent norms which are not dual norms (see, e.g., [10]).

**Theorem 2.2.** Let $(\mathcal{F}(U),\|\cdot\|)$ be a Banach space of continuous functions on $U$. The following conditions are equivalent.

(i) $\mathcal{F}(U)$ admits a strong Banach linearization.

(ii) $\mathcal{F}_*(U)$ is a strong Banach linearization.

(iii) $\mathcal{F}(U)$ admits an equicontinuous and $\tau_p$-compact equivalent ball.

(iv) $\mathcal{F}(U)$ admits an equivalent ball which is $\tau_p$-compact, and the evaluation map $\delta : U \to (\mathcal{F}(U)',\|\cdot\|)$ is continuous.

Furthermore, all of these conditions imply

(v) $\mathcal{F}(U)$ admits a $\tau_{co}$-compact equivalent ball.
Proof. (i)⇔(ii) follows from Fact 0.2.

(ii)⇒(iii): Consider $\mathcal{F}(U)$ as the dual space of the Banach space $\mathcal{F}_*(U)$, endowed with the dual norm. The $w^*$-topology on $\mathcal{F}(U)$ is finer than $\sigma(\mathcal{F}(U), X)$, that is, the identity mapping

$$(\mathcal{F}(U), w^*) \to (\mathcal{F}(U), \sigma(\mathcal{F}(U), X))$$

is continuous. Note that the topology $\sigma(\mathcal{F}(U), X)$ coincides with $\tau_p$ on $\mathcal{F}(U)$. Since the unit ball $B$ of $\mathcal{F}(U)$ is $w^*$-compact, it is $\tau_p$-compact. On the other hand, $B$ is equicontinuous as a set of functions on $\mathcal{F}_*(U)$; since $e : U \to \mathcal{F}_*(U)$ is continuous, $B$ is an equicontinuous subset of $\mathcal{F}(U)$, considered as a set of functions on $U$.

(iii)⇒(ii): We follow the lines of Theorem 2.1. Let $B$ be an equivalent ball in $\mathcal{F}(U)$ which is equicontinuous and $\tau_p$-compact, and thus $\sigma(\mathcal{F}(U), X)$-compact. Each $s \in X$ can be viewed as a $\sigma(\mathcal{F}(U), X)$-continuous linear functional $s : \mathcal{F}(U) \to \mathbb{C}$, given by $s(f) = L_f(s)$. So $|s|$ attains its maximum on $B$. Define

$$\|s\|_B = \max_B |s|.$$

This is a norm on $X$ and we denote the corresponding completion of $X$ by $Y$. First, we check that $\mathcal{F}_*(U)$ is isomorphic to the Banach space $Y$. Note that the norm $\|\cdot\|_B$ is continuous for the $\alpha$-topology, since $B$ is an equicontinuous $\tau_p$-compact disk. Therefore, the map $e : U \to Y$ is continuous. Also, each $f \in \mathcal{F}(U)$ linearizes through $Y$: if $f \in \mathcal{F}(U)$, and $f \neq 0$, then for some $c > 0$ we have $f/c\|f\| \in B$, and

$$|L_f(s)| = |s(f)| = c \cdot \|f\| \cdot \left|s\left(\frac{f}{c\|f\|}\right)\right| \leq c \cdot \|f\| \cdot \|s\|_B,$$

so $L_f$ is $\|\cdot\|_B$-continuous. Now by Fact 0.2, $\mathcal{F}_*(U)$ is isomorphic to $Y$. Finally, it can be proved as in Theorem 2.1 that $\mathcal{F}_*(U)$ is, in fact, a strong linearization.

(iii)⇔(iv): Let $B$ be a $\tau_p$-compact equivalent ball in $\mathcal{F}(U)$. Each evaluation functional $\delta_x : \mathcal{F}(U) \to \mathbb{C}$ is bounded on $B$ and therefore norm-continuous, where $\delta_x(f) = f(x)$ for each $x \in U$ and $f \in \mathcal{F}(U)$. It is clear that the evaluation map

$$\delta : U \to (\mathcal{F}(U)', \|\cdot\|)$$

is continuous if and only if $B$ is an equicontinuous set of functions on $U$.

(iii)⇒(v): This is clear, since on an equicontinuous set, the topologies $\tau_p$ and $\tau_{co}$ coincide. \textbf{■}

Two direct consequences are the following.

COROLLARY 2.3. If $U$ is a $k$-space, (i) through (v) in Theorem 2.2 above are all equivalent.
Corollary 2.4. Let $K$ be an infinite compact set, and $\mathcal{F}(K)$ an infinite-dimensional closed subspace of $(C(K), \| \cdot \|_\infty)$. Then $\mathcal{F}(K)$ does not admit a Banach linearization.

Proof. Since $K$ is a compact set, $\tau_{co} = \| \cdot \|_\infty$ on $C(K)$. Now if $\mathcal{F}(K)$ admits a Banach linearization $(Z, e)$, then $\mathcal{F}(K)$ can be algebraically identified with the dual space $Z'$, and we can consider on $\mathcal{F}(K)$ the norm $\| \cdot \|$ which is dual to the norm of $Z$. By Theorem 2.2, since $(Z, e)$ is a strong Banach linearization of $(\mathcal{F}(K), \| \cdot \|)$, this space admits an equivalent ball $B$ which is compact for $\tau_{co} = \| \cdot \|_\infty$. In particular, $B$ is $\| \cdot \|_\infty$-bounded and by the Open Mapping Theorem, $\| \cdot \|$ is equivalent to $\| \cdot \|_\infty$. The compactness of $B$ implies that $\mathcal{F}(K)$ is finite-dimensional. ■

Note, however, that when $K$ is hyperstonean, $C(K)$ is a dual Banach space by the Dixmier–Grothendieck Theorem [13]. Thus, admitting a strong Banach linearization is strictly stronger than admitting a Banach predual. A further example in this line is the following. We can consider any Banach space $E = \mathcal{F}(\mathcal{B}_{E'})$ as a space of continuous functions on the dual unit ball $\mathcal{B}_{E'}$ with the $w^*$-topology, through the canonical inclusion map $E \hookrightarrow C(\mathcal{B}_{E'})$. Now, $E = \mathcal{F}(\mathcal{B}_{E'})$ admits a Banach predual whenever $E$ is a dual space. Nevertheless, by Corollary 2.4, $\mathcal{F}(\mathcal{B}_{E'})$ admits a Banach linearization only when $E$ is finite-dimensional.

As mentioned before, many Banach spaces $E$ may be viewed as a space of functions $\mathcal{F}(U)$ in different ways. In particular, if $E$ is a Banach space with Schauder basis $(v_k)_{k \in \mathbb{N}}$ and we denote by $(v'_k)_{k \in \mathbb{N}}$ the coordinate functionals, we may consider $E = \mathcal{F}(\mathbb{N})$ as a space of (continuous) functions over $\mathbb{N}$ by identifying each element $x \in E$ with the sequence of its coordinates $(v'_1(x), v'_2(x), \ldots)$. Note that, in this case, the evaluation map $\delta : \mathbb{N} \to (\mathcal{F}(\mathbb{N})', \| \cdot \|)$ is well-defined and continuous.

We ask when $\mathcal{F}_*(\mathbb{N})$ is a strong Banach linearization. This is not always the case as, for example, when $\mathcal{F}(\mathbb{N}) = c_0$. Indeed, if $B$ is a ball equivalent to the unit ball of $c_0$, then $B$ contains a sequence of the form $x_n = (a, (a'), a, 0, \ldots)$ (where $a \neq 0$, and there are $n$ $a$’s, followed by $0$’s). This sequence converges pointwise to $x = (a, a, a, \ldots)$, which is not an element of $c_0$, so $B$ is not pointwise compact. More generally, we have the following corollary. Here, (iii)$\Rightarrow$(i) is essentially Alaoglu’s Theorem (Theorem 6.10 in [10]).

Corollary 2.5. Let $E$ be a Banach space with a Schauder basis $(v_k)_{k \in \mathbb{N}}$, and consider $E = \mathcal{F}(\mathbb{N})$ as before. The following conditions are equivalent.

(i) $\mathcal{F}_*(\mathbb{N})$ is a strong Banach linearization.
(ii) Some equivalent ball of $E$ is $\tau_p$-compact.
(iii) $(v_k)_{k \in \mathbb{N}}$ is boundedly complete.
Proof. (i)⇔(ii) follows from Theorem 2.2.

To prove (ii)⇔(iii), suppose (ii) holds and let ∥·∥ be an equivalent norm on E whose unit ball B is τ_p-compact. Let (z_n) be a sequence of complex numbers such that
\[ \sup_n \left\| \sum_{k=1}^n z_k v_k \right\| < \infty. \]
We may—by normalizing—suppose this supremum to be 1. The sequence \( (x_n) \) given by
\[ x_n = \sum_{k=1}^n z_k v_k \]
is in B and for each k, \( v'_k(x_n) \to z_k \) as \( n \to \infty \) (indeed, the sequence is eventually \( z_k \)). By compactness, there is a subnet \( x_{n_i} \) that converges pointwise to some \( z \in B \). For each k, \( v'_k(x_{n_i}) \to v'_k(z) \), so that \( v'_k(z) = z_k \), and the series \( \sum_{k=1}^\infty z_k v_k \) converges, for this is just the series of \( z_k \).

Suppose now that (iii) holds, and renorm E so that the basis \( (v_k)_{k \in \mathbb{N}} \) is monotone. Let B be the unit ball of this norm ∥·∥. Since B is τ_p-bounded, we only have to prove its pointwise closedness. Let \( (x_i) \subset B \) be a net such that for each k, \( v'_k(x_i) \to z_k \), and consider any \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). Then
\[ \left\| \sum_{k=1}^n z_k v_k \right\| \leq \left\| \sum_{k=1}^n (z_k - v'_k(x_i)) v_k \right\| + \left\| \sum_{k=1}^n v'_k(x_i) v_k \right\| \leq \varepsilon + 1, \]
where \( i \) is large enough. Since \( (v_k) \) is boundedly complete, \( \sum_{k=1}^\infty z_k v_k \) converges, say to \( z \in E \). Now
\[ \|z\| = \lim_{n \to \infty} \left\| \sum_{k=1}^n z_k v_k \right\| \leq \varepsilon + 1 \]
as before, and therefore \( z \in B \). □

In the remainder of this section we will consider a Banach space of continuous functions \( (F(U), \|\cdot\|) \) satisfying the following two conditions:

(a) the norm ∥·∥ is finer than the pointwise topology τ_p on \( F(U) \),
(b) the evaluation map \( \delta : U \to (F(U)', \|\cdot\|) \) is continuous.

Note that, in this case, the unit ball B of \( (F(U), \|\cdot\|) \) is an equicontinuous τ_p-bounded set. Our purpose is to construct a Banach space containing \( F(U) \) which admits a strong Banach linearization and which is minimal in some sense. By analogy with [7], we define \( F_{pb}(U) \) as the space of functions on U which are approximable pointwise by bounded nets in \( F(U) \). Each \( f \in F_{pb}(U) \) is then the pointwise limit of an equicontinuous net of functions on U and is therefore also continuous on U.

For \( f \in F_{pb}(U) \) we define the triple norm |||f||| as follows. Let
\[ B_{pb} = \{ f : U \to \mathbb{C} : f \text{ is a pointwise limit of a net } (f_i) \subset B \}. \]
Thus,
\[ \mathcal{F}_{pb}(U) = \bigcup_{r>0} rB_{pb} \]
and \( B_{pb} \) is a convex, balanced, absorbing subset of \( \mathcal{F}_{pb}(U) \). We let \( ||| \cdot ||| \) be its Minkowski functional, i.e., \( |||f||| \) is the infimum of the constants \( c > 0 \) such that there is a net in \( cB \) converging pointwise to \( f \) on \( U \). Now, the following observations will be useful.

(i) When considering \( f \in B_{pb} \) as the pointwise limit of a net \( (f_i)_{i \in I} \subset B \), the same index set \( I \) may be used for all \( f \). Indeed, if each \( f \in B_{pb}(U) \) is \( f = \lim_{i \in I} f_i \), take
\[ I = \prod_{f \in B_{pb}} I_f, \]
ordered by \( i \geq j \Leftrightarrow i_f \geq j_f \) for all \( f \in B_{pb} \).

Now if \( f = \lim_{j \in I_f} f_j \), define \( h_i = f_{i_f} \), and we have \( f = \lim_{i \in I} h_i \).

(ii) Conditions (a) and (b) above also hold for \( (\mathcal{F}_{pb}(U), ||| \cdot |||) \):

For (a), if \( f_n \to 0 \) in \( ||| \cdot ||| \), let \( x \in U \) and \( \varepsilon > 0 \). Choose \( n_\varepsilon \) so that
\[ |||f_n||| < \frac{\varepsilon}{2\|\delta_x\|} \]
for all \( n \geq n_\varepsilon \).

For each \( n \), take a net \( (f_{n,i}) \subset \frac{\varepsilon}{2\|\delta_x\|}B \) converging pointwise to \( f_n \). Then
\[ |f_n(x)| \leq |f_n(x) - f_{n,i}(x)| + |f_{n,i}(x)| = |f_n(x) - f_{n,i}(x)| + |\delta(x,f_{n,i})| \]
\[ \leq |f_n(x) - f_{n,i}(x)| + \|\delta_x\| \cdot |||f_{n,i}|||, \]
but the second term is smaller than \( \varepsilon/2 \) for \( n \geq n_\varepsilon \), and the first is also less than \( \varepsilon/2 \) for \( i \geq i(n,\varepsilon) \). Thus \( \delta : U \to (\mathcal{F}_{pb}(U), ||| \cdot |||) \) is well-defined.

To see (b), fix \( x \in U \) and \( \varepsilon > 0 \). There is a neighborhood \( V_x \) of \( x \) such that \( \|\delta_x - \delta_y\| < \varepsilon/3 \) in \( (\mathcal{F}(U), ||| \cdot |||) \) for all \( y \in V_x \). Now for any \( f \) with \( |||f||| < 1 \), let \( (f_i) \subset B \) converge pointwise to \( f \). Then for any \( y \in V_x \),
\[ |f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \]
\[ \leq |f(x) - f_i(x)| + |\delta_x - \delta_y|(f_i) + |f_i(y) - f(y)| \]
\[ \leq |f(x) - f_i(x)| + \|\delta_x - \delta_y\| + |f_i(y) - f(y)|. \]

Now, the second term is smaller than \( \varepsilon/3 \) for all \( y \in V_x \), independently of \( f \), while the first and last terms can be made smaller than \( \varepsilon/3 \) for sufficiently large \( i \).

(iii) The inclusion \( (\mathcal{F}_{pb}(U), ||| \cdot |||) \to (C(U), \tau_{co}) \) is continuous: for any compact subset \( K \) of \( U \), and \( \varepsilon > 0 \),
\[ |||f|||_K = \sup_{x \in K} |\delta_x(f)| \leq \sup_{x \in K} \|\delta_x\| \cdot |||f||| = C_K |||f|||. \]

(iv) \( B_{pb} \) is the closed unit ball of the norm \( ||| \cdot ||| \). Indeed, since this norm is the Minkowski functional of \( B_{pb} \), one has
\[ \{ f \in \mathcal{F}_{pb}(U) : |||f||| < 1 \} \subset B_{pb} \subset \{ f \in \mathcal{F}_{pb}(U) : |||f||| \leq 1 \}; \]
but $B_{pb}$ is pointwise closed, and $(\mathcal{F}_{pb}(U), ||\cdot||) \to (\mathcal{F}_{pb}(U), \tau_{pb})$ is continuous, so $B_{pb}$ is $||\cdot||$-closed, and is therefore equal to the closed $||\cdot||$-ball.

We also have

**Proposition 2.6.** $(\mathcal{F}_{pb}(U), ||\cdot||)$ is a Banach space, and the inclusion mapping $i : (\mathcal{F}(U), ||\cdot||) \hookrightarrow (\mathcal{F}_{pb}(U), ||\cdot||)$ is continuous.

**Proof.** First, we check the completeness. So let $(f_n)$ be a $||\cdot||$-Cauchy sequence in $\mathcal{F}_{pb}(U)$. By passing to a subsequence we may suppose that $||f_n - f_{n+1}|| < 1/2^n$ for all $n$. Note also that for each $x \in U$, $(f_n(x))$ is Cauchy, so the $f_n$’s converge pointwise to a function $f \in \mathcal{F}_{pb}(U)$. We must show that it converges also in $||\cdot||$.

We define inductively a sequence of nets in $\mathcal{F}(U)$ as follows: for $n = 1$ let $(f_{1,i})$ be a bounded net converging pointwise to $f_1$. Suppose we have defined nets $(f_{1,i}), \ldots, (f_{n,i})$ such that for each $k = 2, \ldots, n$ we have $(f_{k-1,i} - f_{k,i}) \subset (1/2^{k-1})B$ and $(f_{k,i})$ converges pointwise to $f_k$. Now since $||f_n - f_{n+1}|| < 1/2^n$, there is a net $(h_{n,i}) \subset (1/2^n)B$ converging pointwise to $f_n - f_{n+1}$. We define $f_{n+1,i} = f_{n,i} - h_{n,i}$. Thus $f_{n+1,i}$ converges pointwise to $f_n - f_{n+1} = f_{n+1}$, and $(f_{n,i} - f_{n+1,i}) \subset (1/2^n)B$. Now, for each fixed $i$, the sequence $(f_{n,i})$ is $||\cdot||$-Cauchy in $\mathcal{F}(U)$. Indeed, if $m > n$,

$$||f_{n,i} - f_{m,i}|| \leq ||f_{n,i} - f_{n+1,i}|| + ||f_{n+1,i} - f_{n+2,i}|| + \cdots + ||f_{m-1,i} - f_{m,i}|| \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} < \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^n-1}.$$ 

Since $\mathcal{F}(U)$ is a Banach space, $(f_{n,i})$ converges in the $||\cdot||$-norm, as $n \to \infty$, to a function $f_i \in \mathcal{F}(U)$. From this and $||f_{n,i} - f_{m,i}|| < 1/2^{n-1}$ for all $m > n$, we deduce that $||f_{n,i} - f_i|| < 1/2^{n-1}$ for all $i$. Note also that the net $(f_i)$ converges pointwise to $f$. Indeed, fix $x \in U$ and $\varepsilon > 0$. Then

$$|f(x) - f_i(x)| \leq |f(x) - f_n(x)| + |f_n(x) - f_{n,i}(x)| + |f_{n,i}(x) - f_i(x)|$$

$$= |f(x) - f_n(x)| + |f_n(x) - f_{n,i}(x)| + \delta_x(f_{n,i} - f_i)|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_{n,i}(x)| + \delta_x||f_{n,i} - f_i||$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_{n,i}(x)| + \delta_x||f_{n,i} - f_i|| \leq |f(x) - f_n(x)| + |f_n(x) - f_{n,i}(x)| + \delta_x||f_{n,i} - f_i|| \leq \frac{1}{2n-1}.$$ 

Choose $n$ so large that the first and last terms are smaller than $\varepsilon/3$; then the middle term will also be smaller than $\varepsilon/3$ for $i \geq i(n, x)$.

Thus, for each $n$, we have a net $(f_{n,i} - f_i)$ converging pointwise to $f_n - f$, and such that $(f_{n,i} - f_i) \subset (1/2^{n-1})B$, in other words, $||f_n - f|| \leq 1/2^{n-1}$. Hence $(\mathcal{F}_{pb}(U), ||\cdot||)$ is a Banach space.

Finally, it is clear that $\mathcal{F}(U) \subset \mathcal{F}_{pb}(U)$ and $||f|| \leq ||f||$ for every $f$ in $\mathcal{F}(U)$. ■
Corollary 2.7. The space \((\mathcal{F}_{pb}(U), |||\cdot|||)\) always admits a strong Banach linearization. Moreover, \((\mathcal{F}(U), ||\cdot||)\) admits a strong Banach linearization if and only if \(\mathcal{F}(U) = \mathcal{F}_{pb}(U)\).

Proof. The unit ball \(B_{pb}\) of \(\mathcal{F}_{pb}(U)\) is \(\tau_p\)-compact, since it is the \(\tau_p\)-closure of the ball \(B\) of \(\mathcal{F}(U)\). Since \(\mathcal{F}_{pb}(U)\) satisfies condition (b), a direct application of Theorem 2.2 shows that \(\mathcal{F}_{pb}(U)\) admits a strong Banach linearization.

On the other hand, if \(\mathcal{F}(U)\) admits a strong Banach linearization, it has an equivalent \(\tau_p\)-compact unit ball, so \(B_{pb} = B\) and \(\mathcal{F}(U) = \mathcal{F}_{pb}(U)\). Conversely, if this equality holds, by the Open Mapping Theorem, the norms \(\|\cdot\|\) and \(|||\cdot|||\) are equivalent and the result follows.

The minimality of \(\mathcal{F}_{pb}(U)\) has to be understood in the following sense.

Proposition 2.8. Let \(G(U)\) be a Banach space of continuous functions on \(U\) containing \(\mathcal{F}(U)\) with continuous inclusion \(\iota: \mathcal{F}(U) \hookrightarrow G(U)\). If \(G(U)\) admits a strong Banach linearization, then \(G(U)\) also contains \(\mathcal{F}_{pb}(U)\) with continuous inclusion.

Proof. By Theorem 2.2, we can consider an equivalent norm \(\|\cdot\|_G\) on \(G(U)\), whose unit ball is \(\tau_p\)-compact. By the continuity of \(\iota\), there exists \(C > 0\) such that \(\|f\|_G \leq C\|f\|\) for any \(f \in \mathcal{F}(U)\). Now, each \(f \in \mathcal{F}_{pb}(U)\) belongs to \(G(U)\), as \(f\) is the \(\tau_p\)-limit of a \(\|\cdot\|_G\)-bounded net in \(\mathcal{F}(U)\). Finally, it is easy to see that \(\|f\|_G \leq C\|f\|\) for every \(f \in \mathcal{F}_{pb}(U)\).

In this sense, for Banach spaces of continuous functions satisfying conditions (a) and (b), to admit a strong Banach linearization is equivalent to being saturated with respect to the pointwise limits of bounded nets. This is the case for the spaces of \(k\)-homogeneous polynomials on a Banach space, the space of Lipschitz functions on a metric space, and the space of bounded holomorphic functions on the unit ball of a Banach space (endowed with their natural norms). Nevertheless, if we consider, as in [7], a dual Banach space \(Z\) and the uniform algebra \(A(B)\) generated by the weak-star continuous linear functionals on the closed unit ball \(B\) of \(Z\), this is a non-saturated subalgebra of \(H^\infty(B)\). In this case, \(A_{pb}(B) = H^\infty(B)\) only under certain assumptions on \(Z\) (for example, when \(Z\) has the metric approximation property; see [7, Theorem 4.4]).

Finally, we mention the case of Banach spaces \(E = \mathcal{F}(\mathbb{N})\). Those which admit a strong Banach linearization are those saturated with respect to the pointwise limits of bounded nets (for example, \(\ell^\infty\)). As mentioned before, \(c_0\) does not admit a strong Banach linearization and it is non-saturated.

3. Compactness properties. We now focus on compactness properties of mappings and of their linearizations. In this section, \(\mathcal{F}(U)\) denotes
a locally convex space of continuous functions on $U$, which need not be a Banach space. We begin with the definition of several kinds of compactness.

**Definition.** Let $U$ be a topological space, and $E$ and $F$ locally convex spaces. We say that a mapping $f : U \to F$ is *compact* if for every $x \in U$ there is a neighborhood $U_x$ of $x$ such that $f(U_x)$ is precompact. Similarly, a linear operator $L : E \to F$ is *compact* if there is a neighborhood $V$ of 0 such that $L(V)$ is precompact.

We say that a mapping $f : U \to F$ is *$F$-boundedly compact* if for every $F$-bounding subset $A \subset U$, $f(A)$ is precompact. Similarly, a linear operator $L : E \to F$ will be called *boundedly compact* if for all bounded subsets $B \subset E$, $L(B)$ is precompact.

Clearly, compact linear operators are boundedly compact, and the two notions coincide when $E$ is a Banach space.

We are going to see that $F$-bounded compactness of $f$ is equivalent to bounded compactness of $L_f$ in quite general situations. We suppose that $F(U)$ is barreled and has the topology of uniform convergence on $F$-bounding subsets of $U$ in order to apply Fact 0.3.

For a linear operator $L : E \to F$ we denote its transpose by $L^t : F^t \to E^t$. In a similar way, for $f \in wF(U,F)$, we denote by $f^t : F^t \to \mathcal{F}(U)$ the transpose of $f$ defined by $f^t(\varphi) = \varphi \circ f$. The following result relates bounded compactness of a mapping, of its linearization, and of its transpose. Here we denote by $\beta$ and $\tau_{co}$ the strong and the compact-open topologies, respectively.

**Theorem 3.1.** Suppose $\mathcal{F}(U)$ is barreled and has the topology of uniform convergence on $\mathcal{F}$-bounding subsets of $U$. Let $F$ be a locally convex space and $f \in w\mathcal{F}(U,F)$. The following conditions are equivalent.

(i) The mapping $f : U \to F$ is $\mathcal{F}$-boundedly compact.

(ii) The linearization $L_f \in \mathcal{L}(\mathcal{F}_*(U), F)$ is boundedly compact.

If, in addition, $F$ is a complete barreled space, these conditions are also equivalent to the following:

(iii) The transpose operator $(L_f)^t : (F^t, \beta) \to (\mathcal{F}_*(U)^t, \beta)$ is boundedly compact.

(iv) The transpose map $f^t : (F^t, \beta) \to \mathcal{F}(U)$ is boundedly compact.

(v) The transpose map $f^t : (F^t, \tau_{co}) \to \mathcal{F}(U)$ is continuous.

**Proof.** (i)$\Rightarrow$(ii): Let $f$ be $\mathcal{F}$-boundedly compact and $B \subset \mathcal{F}_*(U)$ bounded. By Fact 0.3, $U$ has the $BB\mathcal{F}$-property, that is, there exist an $\mathcal{F}$-bounding subset $A \subset U$ and an $r > 0$ such that $B \subset r \text{co}e(e(A))$. Then $$L_f(B) \subset L_f(r \text{co}e(e(A))) = r \text{co}e(L_f(e(A))) = r \text{co}e(f(A)),$$ which is precompact, since $f(A)$ is. Thus $L_f$ is boundedly compact.
(ii)⇒(i): Suppose \( L_f \) is boundedly compact and \( A \subset U \) \( \mathcal{F} \)-bounding. Since for any \( L_g \in \mathcal{F}_*(U)' \), \( L_g(e(A)) = g(A) \) is bounded, \( e(A) \) is \( \mathcal{F} \)-bounded in \( \mathcal{F}_*(U) \). Thus \( f(A) = L_f(e(A)) \) is precompact, and \( f \) is boundedly compact.

(ii)⇔(iii): If \( F \) is barreled, it follows from Proposition 2.1 in [8] that \( L_f \) is boundedly compact if, and only if, \( (L_f)^t \) is boundedly compact. Note that in [8] boundedly compact operators are called bpc-operators.

(iii)⇔(iv): Note that for every \( \gamma \in F' \) we have \( \gamma \circ L_f = L_{\gamma \circ f} \). Since \( \mathcal{F}(U) \) is isomorphic to \( (\mathcal{F}_*(U)', \beta) \) by Fact 0.3, we can identify \( f^t \) with \( (L_f)^t \), and the result follows.

(i)⇒(v): Suppose \( f \) is \( \mathcal{F} \)-boundedly compact and let \( B \) be an \( \mathcal{F} \)-bounding subset of \( U \). Since \( f(B) \) is precompact, its closure \( K \) is a compact subset of the complete space \( F \) for which

\[
\sup_{x \in B} \{ |f^t(\varphi)(x)| \} = \sup_{x \in B} \{|(\varphi \circ f)(x)|\} \leq |\varphi|_K := \sup_{x \in K} \{|\varphi(x)|\} \text{ for all } \varphi \in F'.
\]

This proves the continuity of \( f^t \).

(v)⇒(i): If \( f^t \) is \( \tau_{\text{co}} \)-continuous, then for any \( \mathcal{F} \)-bounding subset \( B \) of \( U \) there exist a compact set \( K \subset F \) and \( C > 0 \) so that

\[
\sup_{x \in B} \{|(\varphi \circ f)(x)|\} \leq C|\varphi|_K \quad \text{for all } \varphi \in F'.
\]

This implies that \( f(B) \subset C \text{ coe}(K) \) and \( f \) is \( \mathcal{F} \)-boundedly compact.

The above result applies, for example, to the Fréchet space \( \mathcal{F}(U) = \mathcal{H}_b(U) \), the space of holomorphic functions of bounded type on a balanced open subset \( U \) of a Banach space; to \( \mathcal{F}(U) = \mathcal{P}(kE) \), the space of continuous \( k \)-homogeneous polynomials on a Banach space \( E \); and to \( \mathcal{F}(U) = \mathcal{H}^\infty(U) \), the space of bounded holomorphic functions on an open subset of a Banach space. (See Examples 1, 3 and 4 in [6].)

As an application, we have the following characterization of \( C_*(K) \).

**Corollary 3.2.** \( C_*(K) \) is the space of regular Borel measures on \( K \) with the topology of uniform convergence over compact subsets of \( C(K) \).

**Proof.** Consider \( \mathcal{F}(K) = C(K) \), and take \( F = C_*(K) \). It is clear that \( e : K \to C_*(K) \) is \( \mathcal{F} \)-boundedly compact. By Theorem 3.1, the linearization \( L_e = \text{id} : C_*(K) \to C_*(K) \) is boundedly compact, and hence \( C_*(K) \) is semi-Montel, and thus semireflexive. Hence, algebraically \( C_*(K) = C(K)' = \mathcal{M}(K) \), the space of regular Borel measures on \( K \). \( C_*(K) \) is then the space of regular Borel measures on \( K \) with the \( \alpha \)-topology: uniform convergence over compact subsets of \( C(K) \).

A result analogous to Theorem 3.1, but for true compactness, seems to require more of the functions in \( \mathcal{F}(U) \), and certainly does not hold for the space of all continuous functions:
Example 3. Consider $\mathcal{F}(K) = C(K)$, and $F = C_\ast(K)$ as in Corollary 3.2. It is clear that $e : K \to C_\ast(K)$ is compact. Nevertheless, the linearization $L_e = \text{id} : C_\ast(K) \to C_\ast(K)$ is compact only when $\dim C_\ast(K) < \infty$, or equivalently, when $K$ is finite.

In the context of holomorphic functions, we have the following theorem.

Theorem 3.3. Suppose that $E$ and $F$ are locally convex spaces, $U$ is a connected open subset of $E$, and $\mathcal{F}(U)$ is a linear space of holomorphic functions on $U$. For $f \in w\mathcal{F}(U, F)$, the following are equivalent.

(i) The mapping $f : U \to F$ is compact.

(ii) For every $x \in U$ there is a neighborhood $W$ of $x$ such that the linearization $L_{f|W} \in \mathcal{L}(\mathcal{F}_\ast(W), F)$ is compact, where $\mathcal{F}(W) = \{g|_W : g \in \mathcal{F}(U)\}$.

Proof. (i)$\Rightarrow$(ii): Let $W$ be a neighborhood of $x$ such that $f(W)$ is precompact. Note that since $U$ is connected, we can identify $\mathcal{F}(U) = \mathcal{F}(W)$ and therefore $(\mathcal{F}_\ast(W), e)$ can be seen as a linearization of $\mathcal{F}(U)$. Denote by $\Delta$ the closed unit disk in $\mathbb{C}$, and define

$$D = \{g \in \mathcal{F}(U) : g(W) \subset \Delta\}.$$ 

We begin by noting that $D$ is an equicontinuous $\tau_\alpha$-compact disk of $\mathcal{F}(W)$ ([9, Lemma 3.25]). Thus, the set $V = D^\circ$ is a neighborhood of zero in the $\alpha$-topology, by [6]. Note that $e(W)^{\circ} = D$, so $V = e(W)^{\circ\circ} = \text{coe}(e(W))$ by the bipolar theorem. Denote by $L_f$ the linearization of $f$ by $\mathcal{F}_\ast(W)$. Now

$$L_f(V) = L_f(\text{coe}(e(W))) = \text{coe}(L_f(e(W))) = \text{coe}(f(W))$$

is precompact (see, e.g., [14]). Hence $L_f$ is compact.

(ii)$\Rightarrow$(i): For each $x \in U$ let $W$ be as in (ii), and $L_f$ the compact linearization of $f$ by $\mathcal{F}_\ast(W)$. Suppose $V$ is a neighborhood of $0$ such that $L_f(V)$ is precompact, and take $W_x = e^{-1}(e_x + V)$. Then $W_x$ is a neighborhood of $x$, and

$$f(W_x) = (L_f \circ e)(e^{-1}(e_x + V)) \subset L_f(e_x + V) = f(x) + L_f(V)$$

is precompact. □

For the space $\mathcal{P}^k(E, F)$ of continuous $k$-homogeneous polynomials between Banach spaces and for others with Banach linearizations, such as $\mathcal{H}_\infty(U)$, a bit more can be said:

Corollary 3.4. Suppose $U$ is a connected open subset of a Banach space, $\mathcal{F}(U)$ is a linear space of holomorphic functions on $U$, and $\mathcal{F}_\ast(U)$ is a Fréchet space. Let $F$ be a locally convex space and $f \in w\mathcal{F}(U, F)$. The following are equivalent.

(i) The mapping $f : U \to F$ is compact.

(ii) The linearization $L_f \in \mathcal{L}(\mathcal{F}_\ast(U), F)$ is compact.
Proof. Simply apply Theorem 3.3 and Proposition 1.1.

It is easily seen that the above corollary is false without the Fréchet condition on $\mathcal{F}_*(U)$:

**Example 4.** Take as $U$ any locally compact space and $\mathcal{F}(U)$ an infinite-dimensional space of holomorphic functions on $U$ (for example, $\mathcal{F}(U) = \mathcal{H}(\mathbb{C})$, the entire functions on the complex plane). Choose $F$ to be $\mathcal{F}_*(U)$. Then $e : U \to \mathcal{F}_*(U)$ is compact, but its linearization is the identity $I : \mathcal{F}_*(U) \to \mathcal{F}_*(U)$, which cannot be compact.

**Remark.** Note that in the previous corollary, analyticity of the functions in $\mathcal{F}(U)$ was used only to prove that the set $D = \{ g \in \mathcal{F}(U) : g(W) \subset \Delta \}$ is equicontinuous. There are other spaces $\mathcal{F}(U)$ where this is so, and therefore the theorem is also valid for them. A case in point is the space Lip($U$) of Lipschitz functions on a metric space $U$, endowed with its natural norm.

We now combine our results with factorization theorems for linear operators to obtain the following result, which should be compared with [22]. Here, we say that a mapping $f : U \to F$ is *weakly compact* if for every $x \in U$ there is a neighborhood $U_x$ of $x$ such that $f(U_x)$ is weakly precompact. Similarly, a linear operator $L : E \to F$ is *weakly compact* if there is a neighborhood $V$ of $0$ such that $L(V)$ is weakly precompact.

**Corollary 3.5.** Suppose $U$ is a connected open subset of a Banach space, $\mathcal{F}(U)$ is a linear space of holomorphic functions on $U$, and $\mathcal{F}_*(U)$ is a Fréchet space. Let $F$ be a Fréchet space and $f \in w\mathcal{F}(U,F)$. The following are equivalent.

(i) The mapping $f : U \to F$ is weakly compact.

(ii) The linearization $L_f \in L(\mathcal{F}_*(U), F)$ is weakly compact.

(iii) The mapping $f : U \to F$ factors through a reflexive Banach space.

That is, there exist a reflexive Banach space $Z$, a mapping $g \in w\mathcal{F}(U,Z)$, and a continuous linear operator $T : Z \to F$ such that $f = T \circ g$.

**Proof.** The equivalence between (i) and (ii) holds for every locally convex space $F$, and follows from Corollary 3.4 applied to the space $(F, \text{weak})$. To prove (ii)$\Rightarrow$(iii), we apply the Davis–Figiel–Johnson–Peleczyński factorization theorem for linear operators (see e.g. [14, Theorem 17.2.9]). It yields a reflexive Banach space $Z$ and continuous linear operators $S : \mathcal{F}_*(U) \to Z$ and $T : Z \to F$ such that $L_f = T \circ S$. Then $f$ factors through $Z$ since $f = T \circ (S \circ e)$ and $S \circ e \in w\mathcal{F}(U,Z)$. Finally, (iii)$\Rightarrow$(i) is clear.

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