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Derivations on noncommutative Banach algebras

by

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Abstract. We discuss range inclusion results for derivations on noncommutative Banach algebras from the point of view of ring theory.

1. Results. Throughout, A always denotes a Banach algebra over the complex field $\mathbb{C}$. We denote by $\mathrm{rad}(A)$ the Jacobson radical of A and by $r(x)$ the spectral radius of $x \in A$. Also, let $Q(A)$ be the set of all quasinilpotent elements of A and let $\text{q-Inv}(A)$ be the set of all quasi-regular elements in A. A linear mapping $T\colon A \to A$ is called *spectrally bounded* if there exists $M \geq 0$ such that $r(T(x)) \leq Mr(x)$ for all $x \in A$. In addition, if $M = 0$ (i.e., $T(A) \subseteq Q(A)$), then T is called *spectrally infinitesimal*. It is clear that $\mathrm{rad}(A) \subseteq Q(A) \subseteq \text{q-Inv}(A)$. Therefore, we have the following implications:

$$T(A) \subseteq \mathrm{rad}(A) \;\Rightarrow\; T(A) \subseteq Q(A) \;\Rightarrow\; T(A) \subseteq \text{q-Inv}(A).$$

By a *derivation* of A we mean a $\mathbb{C}$-linear map $d\colon A \to A$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. In 1955 Singer and Wermer [13] proved that $d(A) \subseteq \mathrm{rad}(A)$ for every bounded derivation d of a commutative Banach algebra A. They also conjectured that the boundedness hypothesis is superfluous. The conjecture was proved by Thomas [14]. In the last few years, various generalizations of Thomas' theorem have been proved. We mention some related results.

Let d be a derivation of A. Mathieu and Murphy [9] proved that if d is bounded such that $d(A) \subseteq Q(A)$, then $d(A) \subseteq \mathrm{rad}(A)$. A generalization due to Brešar and Mathieu [4] states that $d(A) \subseteq \mathrm{rad}(A)$ for d a spectrally bounded derivation on A. Brešar and Vukman [5] proved that if d is bounded and if $[d(x), x] \in \mathrm{rad}(A)$ for all $x \in A$, then $d(A) \subseteq \mathrm{rad}(A)$. In [3] Brešar proved that if d is bounded and if $[d(x), x] \in Q(A)$ for all $x \in A$, then $d(A) \subseteq \mathrm{rad}(A)$. As pointed out by Brešar [3, p. 58], whether the inclusion

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$d(A) \subseteq \mathrm{rad}(A)$ holds in general is a very difficult problem. By [12, Theorem 1.8], it is equivalent to the so-called noncommutative Singer–Wermer conjecture: Every derivation on a Banach algebra leaves each primitive ideal invariant. The goal of this note is to discuss some related problems about range inclusions on Banach algebras. Precisely, we will prove the following theorems:

THEOREM 1.1. *Let A be a Banach algebra with a derivation d. If $d(A) \subseteq$ q-$\mathrm{Inv}(A)$, then $d(A) \subseteq \mathrm{rad}(A)$.*

Theorem 1.1 also generalizes Mathieu and Murphy's theorem [9] by removing the assumption of boundedness of the derivation. On the other hand, Mathieu and Murphy [9, Theorem 3.1] proved that if d is bounded and $d(x)^2 \in \mathrm{rad}(A)$ for all $x \in A$, then $d(A) \subseteq \mathrm{rad}(A)$. Since every element in the Jacobson radical of a Banach algebra is quasinilpotent, Theorem 1.1 also gives a generalization of Mathieu and Murphy's theorem: *If $d(x)^{n(x)} \in \mathrm{rad}(A)$ for all $x \in A$, where $n(x) \geq 1$ depends on x, then $d(A) \subseteq \mathrm{rad}(A)$.* Indeed, let $x \in A$. Then $d(x)^{n(x)} \in \mathrm{rad}(A) \subseteq Q(A)$ and so $d(x) \in Q(A)$. By Theorem 1.1, $d(A) \subseteq \mathrm{rad}(A)$. Paper [10] contains further information on derivations mapping into the radical and, in particular, an argument showing that an arbitrary derivation mapping into the quasinilpotent elements has to map into the radical. We let $\mathcal{I}_A$ be the ideal of A generated by $[A, A]$. Related to the noncommutative Singer–Wermer conjecture we will show the following:

THEOREM 1.2. *Let A be a Banach algebra with a derivation d. Suppose that $[d(x), x] \in Q(A)$ for all $x \in A$. Then $[d(A), A] \subseteq \mathrm{rad}(A)$ and $d(\mathcal{I}_A) \subseteq \mathrm{rad}(A)$.*

See also [2], which contains information on derivations d with spectral restrictions on the commutators $[d(x), x]$. Finally, we will prove a closely related result in general rings:

THEOREM 1.3. *Let R be a semisimple ring with a derivation d. If $[d(x), x]^{n(x)} = 0$ for all $x \in R$, where $n(x) \geq 1$ depends on x, then $d(R) \subseteq \mathcal{Z}(R)$, where $\mathcal{Z}(R)$ is the center of R.*

2. Proofs. The key point to our proofs is the simple observation: If a is a quasinilpotent element in a unital Banach algebra A, then $1 + a$ is invertible in A [1, Theorem 9 (p. 12)]. Thus we begin with the following result in general rings. Although derivations on Banach algebras are always assumed to be $\mathbb{C}$-linear here, we do not require a derivation on a k-algebra to be necessarily k-linear.

PROPOSITION 2.1. *Let R be an algebra over a field k, $|k| > 2$, and $1 \in R$, and let d be a derivation of R. Suppose that $1 + d(x)$ is invertible for all*

$x \in R$. Then, for each primitive ideal P of R, $d(P) \subseteq P$ and either $d(R) \subseteq P$ or R/P is a division ring.

Proof. Let P be a primitive ideal of R and consider the quotient ring R/P. Then $1 + v$ is invertible in R/P for all $v \in P + d(P)/P$. Note that $P + d(P)/P$ is an ideal of R/P. Thus $P + d(P)/P \subseteq \mathrm{rad}(R/P)$. Since $\mathrm{rad}(R/P) = 0$, we have $P+d(P)/P = 0$ and so $d(P) \subseteq P$. Now d canonically induces a derivation d_P on R/P such that $1+d_P(\overline{x})$ is invertible in R/P for all $x \in R$, where $\overline{x} = x+P$. Replacing R and d by R/P and d_P, respectively, we may assume that R is a primitive ring. The aim is to prove that either $d = 0$ or R is a division ring.

Suppose that R is not a division ring. Let V be a faithful irreducible left R-module with $D = \mathrm{End}({}_RV)$. Then $\dim_D V \geq 2$. Let $v \in V$ and let $a \in R$. Suppose that $av = 0$ but $d(a)v \neq 0$. Choose an element $b \in R$ such that $bd(a)v = v$. Then $d(ba)v = (bd(a) + d(b)a)v = bd(a)v = v$, implying $(1 - d(ba))v = 0$. Since $1 - d(ba)$ is invertible, $v = 0$ follows. This is a contradiction. Thus, for $a \in A$ and $v \in V$, we have $d(a)v = 0$ if $av = 0$.

Let $a \in R$ and $v \in V$. We claim that av and $d(a)v$ are linearly dependent over D. If not, then by the Density Theorem there exists $b \in R$ such that $b(av) = 0$ and $bd(a)v = -av$. By the above we see that $d(ba)v = 0 = d(b)av$. Thus $bd(a)v = 0$ and so $av = 0$, a contradiction.

Let $a \in R$ be such that $\dim_D aV \geq 2$. Then there exists $\beta \in \mathcal{Z}(D)$, the center of D, such that $d(a) = \beta a$. Indeed, choose $v, w \in V$ such that av and aw are D-independent. Then, by the above, there exist $\beta, \mu, \nu \in D$ such that $d(a)v = (av)\beta$, $d(a)w = (aw)\mu$ and $d(a)(v + w) = (a(v + w))\nu$. Thus $(av)(\nu - \beta) + (aw)(\nu - \mu) = 0$. It follows from the D-independence of av and aw that $\nu = \mu = \beta$. Let $z \in V$. If $az \neq 0$, then either av and az are D-independent or aw and az are D-independent. In either case, $d(a)z = (az)\beta$. This proves that there exists $\beta \in D$ such that $d(a)v = (av)\beta$ for all $v \in V$. Let $\eta \in D$ and $v \in V$. Then $d(a)(v\eta) = (a(v\eta))\beta = (av)\eta\beta$ and, on the other hand, $d(a)(v\eta) = (d(a)v)\eta = (av)\beta\eta$. This amounts to saying that $\beta\eta = \eta\beta$ for all $\eta \in D$, that is, $\beta \in \mathcal{Z}(D)$. This proves our claim.

Suppose next that $a \in R \setminus \mathcal{Z}(D)$ and $\dim_D aV = 1$. Then there exist D-independent $v_1, v_2 \in V$ such that $av_1 = v_2$. Write $av_2 = v_2\mu$, where $\mu \in D$. Since $|k| > 2$, we choose $\beta \in k$ with $\beta \neq 0, -\mu$. Then $(\beta + a)v_1 = \beta v_1 + v_2$ and $(\beta + a)v_2 = v_2(\beta + \mu) \neq 0$. Thus $\beta + a$ has rank ≥ 2 and so $d(\beta+a) \in \mathcal{Z}(D)(\beta+a)$ by the above argument. Thus $d(a) \in \mathcal{Z}(D)a+\mathcal{Z}(D)$.

So we always have $[d(y), y] = 0$ for all $y \in R$. By Posner's theorem [11], either $d = 0$ or R is commutative. The latter case cannot occur since $\dim_D V \geq 2$. So $d = 0$ follows. This finishes the proof. ∎

Proof of Theorem 1.1. Clearly, we may assume that A is unital. Let $x \in A$. By assumption, $d(x) \in \text{q-Inv}(A)$ and so $1 + d(x)$ is invertible in A.

Let P be a primitive ideal of A with $d(A) \not\subseteq P$. By Proposition 2.1, we have $d(P) \subseteq P$ and $A/P \cong \mathbb{C}$. Thus d canonically induces a derivation d_P: $A/P \to A/P$, implying that $d_P(A/P) = 0$, that is, $d(A) \subseteq P$, a contradiction. Therefore, we have proved that $d(A) \subseteq P$ for all primitive ideals P of A. So $d(A) \subseteq \mathrm{rad}(A)$, as asserted. ∎

In order to prove Theorems 1.2 and 1.3 we need one more lemma.

LEMMA 2.2. *Let R be a primitive ring with* 1 *and let $a, u \in R$ with u a unit. Suppose that $u + ax - xa$ is invertible for all $x \in I$, a nonzero ideal of R. Then either a is central in R or R is a division ring.*

Proof. Let ${}_RV$ be a faithful irreducible left R-module and let $D = \mathrm{End}({}_RV)$. By the Jacobson density theorem, R acts densely on V. Suppose that a is not central. We claim that R is a division ring, that is, $\dim_D V = 1$. Suppose not. Then there exists $v \in V$ such that av and v are D-independent. Since I is a nonzero ideal of R, I also acts densely on V. Choose an element $x \in I$ such that $xv = 0$ and $x(av) = -uv$. Now we compute $(u + ax - xa)v = uv + 0 - uv = 0$, implying that $v = 0$ as $u + ax - xa$ is invertible. ∎

Let R be a ring. We denote by $\mathrm{M}_n(R)$ the $n \times n$ matrix ring with entries in R.

PROPOSITION 2.3. *Let R be an algebra over a field k, $|k| > 2$ and $1 \in R$. Suppose that d is a derivation of R such that $1 + [d(x), x]$ is invertible for all $x \in R$. Then, for each primitive ideal P of R, either $d(R) \subseteq P$ or $R/P \cong \mathrm{M}_n(D)$ where $n \leq 2$ and where D is a division ring, depending on P. For the latter case, $d(P) \subseteq P$ if $n = 2$.*

Proof. Let P be a primitive ideal of R. Suppose that R/P is not a division ring. We claim that $d(P) \subseteq P$. Let $x \in R$ and $p, q \in P$. Then, by assumption, $1 + [d(x + p), x + p]$ is invertible. Note that

$$\begin{aligned} 1 + [d(x+p), x+p] &\equiv 1 + [d(x), x] + [d(p), x] \\ &\equiv 1 + [d(x), x] + [d(p) + q, x] \bmod P. \end{aligned}$$

In R/P we let $u = (1 + [d(x), x]) + P$ and $\overline{x} = x + P$. Then $u + \overline{x}z - z\overline{x}$ is invertible for all $z \in P + d(P)/P$, an ideal of R/P. By Lemma 2.2, if $d(P) \not\subseteq P$, then $\overline{x}$ is central for all $x \in R$ or R/P is a division ring. In either case, R/P is a division ring, a contradiction. Thus $d(P) \subseteq P$ and hence d canonically induces a derivation on R/P.

Passing from R to R/P we assume that R is a primitive ring, not a division ring. Let V be a faithful irreducible right R-module with $D = \mathrm{End}({}_RV)$. Thus $\dim_D V \geq 2$. The aim is then to prove that either $d = 0$ or $\dim_D V = 2$. Suppose that $\dim_D V \geq 3$. Let $x \in R$ and $v \in V$ be such that $xv = 0$. We claim that $d(x)v = 0$.

CASE 1. Assume $\dim_D xV \geq 2$. Suppose on the contrary that $d(x)v \neq 0$. Since $\dim_D xV \geq 2$, we have $xV \nsubseteq (d(x)v)D$. Choose $w \in V$ such that $xw \notin (d(x)v)D$. Since R acts densely on V_D, we can choose $y \in R$ satisfying $y(d(x)v) = w$ and $y(xw) = v$. We compute $(1+[d(yx), yx])v = v - yx(yd(x) + d(y)x)v = v - y(xw) = 0$. Since $1 + [d(yx), yx]$ is invertible, we have $v = 0$, a contradiction.

CASE 2. Assume $\dim_D xV = 1$. Note that $\dim_D \ker(x) \geq 2$ as $\dim_D V \geq 3$. Choose $u \in \ker(x)$ such that u, v are D-independent. Clearly, there exist D-independent $w, w' \in V$ such that $xw = w'$. Note that u, v and w are D-independent. Applying the dense action of R on V_D, we can choose $y \in R$ such that $yu = w'$, $yv = 0$ and $yw = w$. Thus $(x + y)u = w'$, $(x + y)v = 0$ and $(x + y)w = w + w'$. Then $\dim_D(x + y)V \geq 2$ and $\dim_D yV \geq 2$. Since $(x + y)v = 0 = yv$, by Case 1 we see that $d(x + y)v = 0 = d(y)v$ and so $d(x)v = 0$.

This proves our claim. Let $x \in R$ and $v \in V$. We claim that $d(x)v$ and xv are D-independent. Suppose not. Choose $y \in R$ such that $y(xv) = 0$ and $y(d(x)v) = v$. Applying the above claim, we see that $d(y)xv = 0 = d(yx)v$ and hence $y(d(x)v) = 0$, a contradiction. Applying the same argument given in Proposition 2.1 shows that $d(x) \in \mathcal{Z}(D)x + \mathcal{Z}(D)$. In particular, $[d(x), x] = 0$ for all $x \in R$. By Posner's theorem [11], either $d = 0$ or R is commutative. This proves the proposition. ∎

LEMMA 2.4. *Let $R = \mathrm{M}_2(D)$, where D is a division ring, and let d be a derivation of R. If $[d(x), x]^2 = 0$ for all $x \in R$, then $d = 0$.*

Proof. By assumption, $[d(x), x]^2 = 0$ for all $x \in R$. If d is outer, by Kharchenko's theorem [6] we see that $[y, x]^2 = 0$ for all $x, y \in R$, implying that R is commutative. This is absurd. So d must be inner. Write $d(x) = [a, x]$, where $a \in R$. Let $e = e^2 \in R$. Expanding $(1-e)[[a, exe], exe]^2(1-e) = 0$ yields $(1 - e)a(ex)^4ea(1 - e) = 0$ for all $x \in R$. Applying [7, Theorem 2] or [8, Theorem], we see that $(1 - e)a(ex)ea(1 - e) = 0$ for all $x \in R$. By the primeness of R, we have

$$(*) \qquad \text{either} \quad (1 - e)ae = 0 \quad \text{or} \quad ea(1 - e) = 0.$$

Write $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in R$. We take $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $(*)$. We see that $\beta = 0$ or $\gamma = 0$. Without loss of generality, we may assume that $\beta = 0$. Let $x = \left(\begin{smallmatrix} \eta & 0 \\ 0 & \lambda \end{smallmatrix}\right) \in R$. Expanding $[[a, x], x]^2 = 0$ and regarding its $(1, 1)$ and $(2, 2)$ entries we obtain $[[\alpha, \eta], \eta] = 0$ and $[[\delta, \lambda], \lambda] = 0$ for all $\eta, \lambda \in D$. By Posner's theorem [11], $\alpha, \delta \in \mathcal{Z}(D)$. Since a and $a - \alpha$ define the same inner derivation of R, by replacing a with $a - \alpha$ we may assume $\alpha = 0$, that is, $a = \left(\begin{smallmatrix} 0 & 0 \\ \gamma & \delta \end{smallmatrix}\right)$. The aim is to prove $a = 0$, that is, $\gamma = 0 = \delta$.

Case 1. Assume that $\delta = 0$. Let u be a unit in R. Then $[[uau^{-1}, x], x]^2 = 0$ for all $x \in R$. By $(*)$, given an idempotent $e \in R$ we see that

$$(**) \qquad \text{either} \quad e(uau^{-1})(1-e) = 0 \quad \text{or} \quad (1-e)(uau^{-1})e = 0.$$

In particular, we choose $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $uau^{-1} = \left(\begin{smallmatrix} \gamma & -\gamma \\ \gamma & -\gamma \end{smallmatrix}\right)$. Choose $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $(**)$. Then $\gamma = 0$ follows and so $a = 0$, as asserted.

Case 2. Assume that $\delta \neq 0$. Since $\delta \in \mathcal{Z}(D)$, replacing a by $\delta^{-1}a$ we may assume that $\delta = 1$, that is, $a = \left(\begin{smallmatrix} 0 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$. We will derive a contradiction to exclude this case. Note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} a \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma+1 \\ \gamma & -\gamma+1 \end{pmatrix}.$$

By $(**)$, either $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, we compute

$$\left[\left[a, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right], \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right]^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 0,$$

implying $\operatorname{char} D = 5$. On the other hand,

$$\left[\left[a, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\right], \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\right]^2 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \neq 0,$$

a contradiction. Suppose that $\gamma = 1$. Since

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we reduce to the case that $\gamma = 0$. This finishes the proof. ∎

Proof of Theorem 1.2. Without loss of generality we may assume that A is a unital Banach algebra. Let $x \in A$. Since $[d(x), x] \in Q(A)$, $1 + [d(x), x]$ is invertible. Suppose that $d(A) \nsubseteq P$. In view of Proposition 2.3, $A/P \cong \mathrm{M}_n(\mathbb{C})$, where $n \leq 2$, depending on P. If $d(P) \subseteq P$, d induces canonically a derivation d_P on A/P and $[d_P(\overline{x}), \overline{x}] \in Q(A/P)$ for all $\overline{x} \in A/P$. However, in $\mathrm{M}_n(\mathbb{C})$, an element is quasinilpotent if and only if it is nilpotent. Thus, $[d_P(\overline{x}), \overline{x}]^2 = 0$ for all $\overline{x} \in A/P$. By Lemma 2.4, $d_P = 0$ and so $d(A) \subseteq P$, a contradiction. Thus $d(P) \nsubseteq P$ and so $A/P \cong \mathbb{C}$ by Proposition 2.3 again. In this case, $[A, A] \subseteq P$ follows. Thus, in either case, $d(A)[A, A] \subseteq P$ and $[d(A), A] \subseteq P$ for each primitive ideal P of A. Hence, $d(A)[A, A] \subseteq \operatorname{rad}(A)$ and $[d(A), A] \subseteq \operatorname{rad}(A)$. Note that $\mathcal{I}_A = [A, A] + A[A, A]$ since $A[A, A]A \subseteq A[[A, A], A] + A^2[A, A] \subseteq A[A, A]$. Thus $d(\mathcal{I}_A) \subseteq \operatorname{rad}(A)$. ∎

Proof of Theorem 1.3. We consider the ring $S = R \times \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers, with the usual multiplicative rule. Then $\operatorname{rad}(S) = \operatorname{rad}(R)$ and d is uniquely extended to a derivation of S. By assumption, it is clear that $[d(x), x]^{n(x)} = 0$ for all $x \in S$.

Thus we may assume, without loss of generality, that $1 \in R$. By assumption, $1 + [d(x), x]$ is invertible for all $x \in R$. Let P be a primitive ideal of R. Suppose that $d(R) \nsubseteq P$. In view of Proposition 2.3, $R/P \cong \mathrm{M}_n(D)$, where $n \leq 2$ and D is a division ring, depending on P. Moreover, R/P is a division ring if $d(P) \nsubseteq P$.

Case 1. Assume that $R/P \cong D$. Then $[d(x), x] \in P$ for all $x \in R$. If $d(P) \subseteq P$, then $[d_P(\overline{x}), \overline{x}] = 0$ for all $\overline{x} \in R/P$. By Posner's theorem [11], $d_P = 0$ or R/P is commutative. Since $d(R) \nsubseteq P$, $[R, R] \subseteq P$ follows. Suppose next that $d(P) \nsubseteq P$, then $d(P) + P = R$. Let $x, y \in R$. Write $y = d(p) + q$, where $p, q \in P$. Then

$$\begin{aligned}[d(x+p), x+p] &\equiv [d(x+p), x] = [d(x), x] + [d(p), x]\\ &\equiv [d(p), x] \equiv [d(p) + q, x] \equiv [y, x] \bmod P.\end{aligned}$$

That is, $[R, R] \subseteq P$. In either case, $[d(R), R] \subseteq P$.

Case 2. Assume that $R/P \cong \mathrm{M}_2(D)$. By Proposition 2.3, we have $d(P) \subseteq P$. Replacing R with R/P we may assume $P = 0$. Now we have $[d(x), x]^2 = 0$ for all $x \in R$. In view of Lemma 2.4, $d = 0$ follows.

By the two cases above, $[d(R), R] \subseteq P$ for all primitive ideals P of R. But R is semisimple, $[d(R), R] = 0$ and so $d(R)$ is contained in the center of R. ∎

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