STUDIA MATHEMATICA 167 (2) (2005)

L^1 factorizations, moment problems and invariant subspaces

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Abstract. For an absolutely continuous contraction T on a Hilbert space $\mathcal{H}$, it is shown that the factorization of various classes of L^1 functions f by vectors x and y in $\mathcal{H}$, in the sense that $\langle T^n x, y\rangle = \widehat{f}(-n)$ for $n \geq 0$, implies the existence of invariant subspaces for T, or in some cases for rational functions of T. One of the main tools employed is the operator-valued Poisson kernel. Finally, a link is established between L^1 factorizations and the moment sequences studied in the Atzmon–Godefroy method, from which further results on invariant subspaces are derived.

1. Introduction. The starting point of the Scott Brown method is the following result, which can be found, for example, in [6] and [9, Sec. 4].

LEMMA 1.1. *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Suppose that there exist $\lambda \in \mathbb{D}$ and $n_0 \in \mathbb{N}$ for which we can find two nonzero vectors $x, y \in \mathcal{H}$ satisfying $\langle T^n x, y\rangle = \lambda^n$ for all $n \geq n_0$. Then T has a nontrivial closed invariant subspace.*

Indeed, to show this we take $\mathcal{M} = \langle (T - \lambda\,\mathrm{Id})T^{n_0}x\rangle$, where $\langle x\rangle$ denotes $\mathrm{span}\{T^n x : n \geq 0\}$, and span means the closed linear hull of a set. This is a closed T-invariant subspace, and we note that $y \perp \mathcal{M}$ and so $\mathcal{M} \neq \mathcal{H}$. On the other hand, if $\mathcal{M} = \{0\}$, then $(T - \lambda\,\mathrm{Id})T^{n_0}x = 0$, whereas $x \neq 0$, and so the cyclic subspace generated by x is finite-dimensional.

In the Banach space situation we obtain the following result, which is proved in an identical manner.

LEMMA 1.2. *Let E be a complex Banach space and suppose that there exist $\psi_0 \in E^* \setminus \{0\}$, $x_0 \in E \setminus \{0\}$, $\lambda \in \mathbb{D}$ and $n_0 \in \mathbb{N}$ such that $\langle T^n x_0, \psi_0\rangle = \lambda^n$ for all $n \geq n_0$. Then T has a nontrivial closed invariant subspace.*

2000 *Mathematics Subject Classification*: 47A15, 47A57, 30D55.

Key words and phrases: invariant subspace, operator-valued Poisson kernel, moment sequence, absolutely continuous contraction.

This work was partially supported by the European Research Training Network in Analysis and Operators, Contract No. HPRN-CT-2000-00116.

Now we specialize to the case of a complex separable Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = \infty$, and consider T an absolutely continuous contraction (see [18]). In this case we know that for all $x, y \in \mathcal{H}$ one can give a meaning to $x \overset{T}{\cdot} y$, as a function $f \in L^1(\mathbb{T}) = L^1(\mathbb{T}, dm)$, where $\mathbb{T}$ denotes the unit circle and m is normalized Lebesgue measure. The Fourier coefficients of f are given by

$$\widehat{f}(n) = \begin{cases} \langle T^{*n}x, y\rangle & \text{if } n \geq 0, \\ \langle T^{|n|}x, y\rangle & \text{if } n \leq 0. \end{cases}$$

One can also define the function $x \overset{T}{\cdot} y$ via the operator Poisson kernel (see, for example, [16])

$$\begin{aligned} K_{r,t}(T) &= (\mathrm{Id} - re^{-it}T)^{-1} + (\mathrm{Id} - re^{it}T^*)^{-1} - \mathrm{Id} \\ &= (\mathrm{Id} - re^{it}T^*)^{-1}(\mathrm{Id} - r^2T^*T)(\mathrm{Id} - re^{-it}T)^{-1} \end{aligned}$$

for $r \in (0,1)$ and $t \in [0, 2\pi)$ in the following way:

$$x \overset{T}{\cdot} y(e^{it}) = \lim_{r \to 1^-} \langle K_{r,t}(T)x, y\rangle. \tag{1}$$

Note that $x \overset{T}{\cdot} y = \overline{y \overset{T}{\cdot} x}$.

We recall that the condition $\|T\| \leq 1$ is equivalent to saying that $\sigma(\mathbb{T}) \subset \overline{\mathbb{D}}$ and $K_{r,t}(T) \geq 0$ for all $re^{it} \in \mathbb{D}$ (see [9], for example).

We may reinterpret Lemma 1.1 in the language of Hardy spaces. As usual, we define the Hardy space $H^p = H^p(\mathbb{D})$ for $1 \leq p < \infty$ as the space of all analytic functions for which the norm

$$\|f\|_p = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}$$

is finite. The space $H^\infty = H^\infty(\mathbb{D})$ comprises all the bounded analytic functions in $\mathbb{D}$, with norm $\|f\|_\infty = \sup\{|f(z)| : |z| < 1\}$.

It is well known [15, 12] that $H^p(\mathbb{D})$ may be regarded isometrically as a closed subspace of $L^p(\mathbb{T}, dm)$, by identifying the Taylor coefficients of f with the Fourier coefficients of an $L^p(\mathbb{T})$ function.

For $|\lambda| < 1$ we write P_λ for the Poisson kernel, i.e.,

$$P_\lambda(z) = \frac{1 - |\lambda|^2}{|1 - \overline{\lambda} z|^2} \qquad (|z| \leq 1).$$

Lemma 1.1 is then equivalent to the following statement: *If T is an absolutely continuous contraction for which there exist $n_0 \in \mathbb{N}$ and $x, y \in \mathcal{H}$ such that*

$$[x \overset{T}{\cdot} y]_{L^1/z^{-n_0}H^1_0} = [x \overset{T}{\cdot} y]_{L^1/z^{1-n_0}H^1} = [P_\lambda]_{L^1/z^{-n_0}H^1_0},$$

then T has a nontrivial invariant subspace. In general we shall write $[f]$ to denote $[f]_{L^1/H^1_0}$, unless specified otherwise.

QUESTION 1.1. *Let T be an absolutely continuous contraction on $\mathcal{H}$. Take $g \in L^1(\mathbb{T})$ and suppose that there exist $x, y \in \mathcal{H}$ such that*

$$[g] = [x \overset{T}{\cdot} y] \tag{2}$$

(equivalently, $\langle T^n x, y\rangle = \widehat{g}(-n)$ for all $n \geq 0$). For which functions $g \neq 0$ does the identity (2) *imply that T has a nontrivial invariant subspace?*

With a view to providing some answers to Question 1.1, we begin in Section 2 by showing that if T factorizes a function of the form h_1/h_2 with $h_1, h_2 \in H^\infty$, then T has invariant subspaces. Examples of such functions include various ℓ^1 sums of Poisson kernels. The functions h_1/h_2 cannot equal zero on sets of positive measure, and so we next consider the implications of being able to factorize functions that vanish on an arc I of the circle; we shall use a supplementary spectral condition of the form $I \cap \sigma(T) = \emptyset$, which is very natural, since one already knows from [7] that contractions with spectrum containing $\mathbb{T}$ always have invariant subspaces.

Motivated by an observation due to Foiaş and Pearcy [13], we show that in many cases the possibility of factorizing functions using $b(T)$ (where b is a finite Blaschke product) implies that T itself has invariant subspaces.

In Section 3 we regard L^1 factorizations in another light by linking them with the results of Atzmon and Godefroy [1] that guarantee the existence of real invariant subspaces when a Banach-space operator admits a moment sequence on the real line. These fit well into the context of the earlier part of this paper, provided that the given operators have a suitable algebraic structure, and we are thus able to deduce further results on the link between L^1 factorizations and the existence of invariant subspaces.

2. L^1 factorizations via the operator-valued Poisson kernel. Our first result gives a significant extension of Lemma 1.1. Let $L^1(\mathbb{T}) \cap \frac{H^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}$ denote the set of functions $f \in L^1(\mathbb{T})$ that can be written as $f = h_1/h_2$ for some $h_1, h_2 \in H^\infty(\mathbb{T})$, regarding $H^\infty(\mathbb{T})$ as the closed subspace of $L^\infty(\mathbb{T})$ consisting of boundary values of functions in $H^\infty(\mathbb{D})$.

THEOREM 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction and suppose that there exist nonzero vectors $x, y \in H$, a function $f \in L^1(\mathbb{T}) \cap \frac{H^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}$, and an $n_0 \in \mathbb{N}$ such that*

$$\langle T^n x, y\rangle = \widehat{f}(-n) \quad \text{for all } n \geq n_0.$$

Then T has a nontrivial closed invariant subspace.

Proof. We write $f = h_1/h_2$ as in the definition of $L^1(\mathbb{T}) \cap \frac{H^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}$, and consider the subspace $\mathcal{M} = \langle h_2(T) T^{n_0+1} x\rangle$. It is clearly T-invariant. Moreover, for $k \geq 1$,

$$\begin{aligned}\langle h_2(T)T^{n_0+k}x, y\rangle &= \langle h_2(z)z^{n_0+k}, x \overset{T}{\cdot} y\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} h_2(e^{i\theta})e^{i(n_0+k)\theta}(x \overset{T}{\cdot} y)(e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} h_2(e^{i\theta})e^{i(n_0+k)\theta}\,\frac{h_1(e^{i\theta})}{h_2(e^{i\theta})}\,d\theta, \\ &\qquad\qquad \text{since } \; z^{n_0}(x \overset{T}{\cdot} y - h_1/h_2) \in H^1 \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n_0+k)\theta}h_1(e^{i\theta})\,d\theta \\ &= \widehat{h}_1(-n_0-k) = 0.\end{aligned}$$

Hence $\mathcal{M} \neq \mathcal{H}$ and if $\mathcal{M} \neq \{0\}$, then T has a nontrivial closed invariant subspace.

Suppose now that $\mathcal{M} = \{0\}$. Then $h_2(T)T^{n_0+1}x = 0$. Let

$$\mathcal{N} = \operatorname{Ker}(h_2(T)T^{n_0+1}) \neq \{0\}.$$

If $\mathcal{N} \neq \mathcal{H}$, then it is a nontrivial closed invariant subspace for T.

There remains the possibility that $\mathcal{N} = \mathcal{H}$, i.e., that $h_2(T)T^{n_0+1} = 0$. Then T is a C_0 operator and in this case T has a nontrivial invariant subspace (cf. [18] and [3, Chap. II]). Indeed, it is even hyperinvariant if T is not a scalar multiple of the identity. ∎

Corollary 2.1. *Let T be an absolutely continuous contraction on $\mathcal{H}$ and suppose that there exist $x, y \in \mathcal{H}$ such that* (2) *holds for a nonzero function $g = \sum_{n\geq 1} c_n P_{\lambda_n}$, where $(\lambda_n)_{n\geq 1}$ is a Blaschke sequence* (*possibly finite*) *and $\sum_{n\geq 1} |c_n| < \infty$. Then T has a nontrivial invariant subspace.*

Proof. If f is a single Poisson kernel P_λ, then taking $b \in H^\infty$ as the Blaschke product $b(z) = (z-\lambda)/(1-\overline{\lambda}z)$, and $|z| = 1$, we have

$$f(z)b(z) = \frac{1-|\lambda|^2}{(1-\overline{\lambda}z)(1-\lambda\overline{z})}\,\frac{z-\lambda}{1-\overline{\lambda}z} = \frac{(1-|\lambda|^2)z}{(1-\overline{\lambda}z)^2} \in H^\infty.$$

The general case follows on noting that $gB \in H^\infty$, where B is a Blaschke product with zeroes $(\lambda_n)_{n\geq 1}$, as is now easily verified. ∎

It is not possible to extend the above methods to arbitrary sequences $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$, since any function in $C(\mathbb{T})$ can be written as an ℓ^1 sum of Poisson kernels (see, for example, [5, 10]), including some functions which are not in $H^\infty(\mathbb{T})/H^\infty(\mathbb{T})$, since they are zero on subintervals of positive measure but not identically zero.

It is clear that the class $L^1(\mathbb{T})\cap\frac{H^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}$ is dense in $L^1(\mathbb{T})$, since it contains all rational functions without poles on the unit circle. However, we do not know an explicit characterization of it. If $f\in L^1(\mathbb{T})\cap\frac{H^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}$ then certainly $\log|f|\in L^1(\mathbb{T})$, but this is not a sufficient condition, as the example $1+\chi_\Omega$ shows, with Ω being a closed subset of normalized measure strictly between 0 and 1. For if $1+\chi_\Omega=h_1/h_2$, then $h_1=h_2$ on a set of uniqueness, and hence almost everywhere, which is absurd.

The next proposition collects together some standard facts, which will show that the hypotheses in the theorem that follows it are very natural.

PROPOSITION 2.1. (i) *Suppose that for some* $\lambda\in\mathbb{D}$ *and vectors* $x,y\in\mathcal{H}$, *we have* $[P_\lambda]=[x\overset{T}{\cdot}y]$. *Then there is a vector* $\widetilde{x}\in\mathcal{H}$ *such that* $[P_\lambda]=[\widetilde{x}\overset{T}{\cdot}\widetilde{x}]$.

(ii) *Let* $f\in L^1(\mathbb{T})$ *and* $x\in\mathcal{H}$; *then the following assertions are equivalent*:

1. $f\geq 0$ *a.e. and* $[f]=[x\overset{T}{\cdot}x]$.
2. $f=x\overset{T}{\cdot}x$.

Proof. (i) Without loss of generality we may suppose that $y\in\langle x\rangle$. By the functional calculus we have

$$\langle f(T)x,y\rangle=\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})P_\lambda(e^{it})\,dt=f(\lambda)$$

for all $f\in H^\infty$, and so $y\perp\mathcal{V}:=\langle(T-\lambda\,\mathrm{Id})x\rangle$. Let $\widetilde{x}$ be a unit vector in the one-dimensional space $\langle x\rangle\ominus\mathcal{V}$; then one can easily check that $[P_\lambda]=[\widetilde{x}\overset{T}{\cdot}\widetilde{x}]$.

(ii) Since the operator-valued Poisson kernel is positive, we have

$$(x\overset{T}{\cdot}x)(e^{it})=\lim_{r\to 1-}\|K_{r,t}^{1/2}(T)x\|^2\geq 0,$$

and thus condition 2 implies condition 1.

Conversely, if condition 1 is satisfied, then $\langle T^n x,x\rangle=\widehat{f}(-n)$ for all $n\geq 0$, and since f is real-valued we have

$$\widehat{f}(n)=\overline{\widehat{f}(-n)}=\langle T^{*n}x,x\rangle=(x\overset{T}{\cdot}x)^\wedge(n)\quad\text{for } n\geq 0,$$

i.e., $f=x\overset{T}{\cdot}x$. ∎

Recall that if T is a contraction and $\sigma(T)\supseteq\mathbb{T}$, then T has a nontrivial invariant subspace. We now analyse some of the possibilities when $\sigma(T)\cap\mathbb{T}$ is not the whole circle. To do this, we recall the following result.

LEMMA 2.1 ([8, Lem. 5.1]). *Let* $T\in\mathcal{L}(\mathcal{H})$ *be a contraction with* $\Gamma(T):=\sigma(T)\cap\mathbb{T}\neq\mathbb{T}$. *For any* $x,y\in\mathcal{H}$ *and any closed arc* $I\subset\mathbb{T}\setminus\Gamma(T)$ *the function* $x\overset{T}{\cdot}y$ *extends analytically in a neighbourhood of* I.

THEOREM 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction. Suppose that there is an open arc $I \subseteq \mathbb{T} \setminus \sigma(T)$ and $x \in \mathcal{H}$ such that $[f] = [x \overset{T}{\cdot} x]$, where f is a nonzero real function vanishing on I. Then T has a nontrivial invariant subspace.*

Proof. Using Proposition 2.1 we have $f = x \overset{T}{\cdot} x$ and $f \geq 0$. By Lemma 2.1, since $I \subseteq \mathbb{T} \setminus \sigma(T)$, the function $x \overset{T}{\cdot} x$ extends analytically in a neighbourhood of I. For $e^{it} \in I$ we have

$$\begin{aligned} 0 = (x \overset{T}{\cdot} x)(e^{it}) &= \langle (\mathrm{Id} - T^*T)(\mathrm{Id} - e^{-it}T)^{-1}x, (\mathrm{Id} - e^{-it}T)^{-1}x \rangle \\ &= \|(\mathrm{Id} - T^*T)^{1/2}(\mathrm{Id} - e^{-it}T)^{-1}x\|^2 \\ &= \|(\mathrm{Id} - T^*T)^{1/2}(e^{it}\,\mathrm{Id} - T)^{-1}x\|^2. \end{aligned}$$

Hence

$$(\mathrm{Id} - T^*T)^{1/2}(e^{it}\,\mathrm{Id} - T)^{-1}x = 0 \quad \text{for all } t \text{ with } e^{it} \in I.$$

On taking successive derivatives with respect to t, we obtain

$$(\mathrm{Id} - T^*T)^{1/2}(e^{it}\,\mathrm{Id} - T)^{-n}x = 0 \quad \text{for all } n \geq 1, \text{ and } t \text{ with } e^{it} \in I.$$

By multiplying T by a unimodular constant, we may reduce to the case when

$$(\mathrm{Id} - T^*T)^{1/2}(\mathrm{Id} - T)^{-n}x = 0 \quad \text{for all } n \geq 1.$$

Let $\mathcal{M} = \mathrm{span}\{(\mathrm{Id} - T)^{-n}x : n \geq 1\}$. Clearly $\mathcal{M} \neq \{0\}$, since it contains the nonzero vector $(\mathrm{Id} - T)^{-1}x$. If $\mathcal{M} = \mathcal{H}$, then it follows that $\mathrm{Id} - T^*T = 0$ and thus T is an isometry, and therefore has a rich lattice of invariant subspaces. Otherwise, set $A = (\mathrm{Id} - T)^{-1}$. Then $\mathcal{M}$ is a nontrivial invariant subspace for A. Now $\sigma(A) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$, and so, by the holomorphic functional calculus, there is a sequence $(p_n)_{n \geq 1}$ of complex polynomials converging uniformly to the function $z \mapsto 1 - 1/z$ on a neighbourhood of $\sigma(A)$. Then $\|p_n(A) - T\| \to 0$ and hence $\mathcal{M}$ is a nontrivial invariant subspace for T as well. ■

A simple example of the above situation is obtained by taking $\mathcal{H} = L^2(\Omega)$, where Ω is a proper closed subset of $\mathbb{T}$. Let T be the operator of multiplication by the independent variable. Then for $x, y \in \mathcal{H}$ one has $(x \overset{T}{\cdot} y)(e^{it}) = (x\overline{y})(e^{it})$, where x and y are regarded as functions in $L^2(\mathbb{T})$ that vanish on the complement of Ω.

We can obtain a corollary of Theorem 2.2 by exploiting the following result.

THEOREM 2.3 ([11, Thm. 3.1]). *Let $T \in \mathcal{L}(\mathcal{H})$ be any absolutely continuous contraction, and let b be a finite Blaschke product. Then, for every*

$x, y \in \mathcal{H}$, we have

$$(x \overset{b(T)}{\cdot} y)(e^{it}) = \sum_{j=1}^{d} \frac{(x \overset{T}{\cdot} y)(\xi_j)}{|b'(\xi_j)|} \quad a.e.,$$

where $\xi_1, \ldots, \xi_d$ are the solutions of $b(z) = e^{it}$.

Corollary 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction. Suppose that there is an open arc $J \subseteq \mathbb{T} \setminus \sigma(b(T))$ and $x \in \mathcal{H}$ such that $[f] = [x \overset{b(T)}{\cdot} x]$, where f is a nonzero real function vanishing on J. Then T has a nontrivial invariant subspace.*

Proof. Suppose that $e^{it} \in J$. Then

$$(x \overset{b(T)}{\cdot} x)(e^{it}) = \sum_{j=1}^{d} \frac{(x \overset{T}{\cdot} x)(\xi_j)}{|b'(\xi_j)|} = 0 \quad \text{a.e.},$$

and so $x \overset{T}{\cdot} x = 0$ for almost all ξ with $b(\xi) \in J$, since $x \overset{T}{\cdot} x \geq 0$ almost everywhere. The hypotheses of Theorem 2.2 are now satisfied on taking I to be any interval contained in $b^{-1}(J)$. ∎

In particular, if $x \overset{T^2}{\cdot} x$ is a function vanishing on an arc disjoint from the spectrum of T^2, then T has nontrivial invariant subspaces. This can be viewed in the context of a result of Foiaş and Pearcy [13, Cor. 2.3], which asserts that if it is the case that every invertible contraction T with $T^2 \in \mathbb{A}_{\aleph_0}$ has a nontrivial invariant subspace, then in fact every contraction in $\mathcal{L}(\mathcal{H})$ with spectral radius 1 also has a nontrivial invariant subspace. We omit the definition of the class $\mathbb{A}_{\aleph_0}$ and refer the reader to [4] for further information; the important points are that if $T^2 \in \mathbb{A}_{\aleph_0}$, then for every $g \in L^1(\mathbb{T})$ there exist $x, y \in \mathcal{H}$ with $g = x \overset{T^2}{\cdot} y$, and moreover, that we can write $f = x \overset{T^2}{\cdot} x$ for some $x \in \mathcal{H}$ if and only if f is lower semicontinuous and strictly positive (see [10, Cor. 4.4 and Thm. 5.3]). It is possible that the methods employed here may shed further light on the question whether every contraction with spectral radius 1 has a nontrivial invariant subspace.

The hypotheses of Theorem 2.2 require that $\sigma(T) \cap \mathbb{T}$ is reasonably rich, as the following result shows.

Proposition 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction such that $\Gamma(T) := \sigma(T) \cap \mathbb{T}$ contains at most one point. Suppose that $f \in L^1(\mathbb{T})$, $[f] \neq 0$, but f vanishes on a set $\Omega \subset \mathbb{T}$ with $m(\Omega) > 0$. Then there do not exist $x, y \in \mathcal{H}$ with $[f] = [x \overset{T}{\cdot} y]$.*

Proof. Suppose to the contrary that there is a function $h \in H^1_0$ such that $f = x \overset{T}{\cdot} y + h$. By Lemma 2.1, $x \overset{T}{\cdot} y$ extends analytically to a neighbourhood of $\mathbb{T} \setminus \Gamma(T)$, which can be chosen to be connected, since $\Gamma(T)$ is either empty or a single point. Since $x \overset{T}{\cdot} y = -h$ on Ω, which is a uniqueness set for H^1, we deduce that $x \overset{T}{\cdot} y = -h$ on $\mathbb{T}$. It follows that $[f] = [0]$, which is absurd. ■

We now look at some of the consequences of factorizing a characteristic function.

PROPOSITION 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction, and let $x, y \in \mathcal{H}$ be such that $[x \overset{T}{\cdot} y]_{L^1/H^1_0} = [\chi_\Omega]_{L^1/H^1_0}$ for some subset $\Omega \subset \mathbb{T}$ of normalized measure strictly between 0 and 1. Then by considering $z \overset{T}{\cdot} y$ for different $z \in \mathcal{H}$ but with the same y one can factorize all functions in $H^\infty_{|\Omega}$; this is a subset of $L^\infty(\Omega)$ that is not dense in $L^\infty(\Omega)$ in the L^∞ norm but is dense in $L^1(\Omega)$ in the L^1 norm.*

Proof. We have, for $n \geq 0$ and $k \in H^\infty$,

$$(k(T)x \overset{T}{\cdot} y)^\wedge(-n) = \langle T^n k(T)x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} k(e^{i\theta})(x \overset{T}{\cdot} y)(e^{i\theta})\, d\theta$$
$$= (k.(x \overset{T}{\cdot} y))^\wedge(-n),$$

and hence $[k(T)x \overset{T}{\cdot} y]_{L^1/H^1_0} = [k.(x \overset{T}{\cdot} y)]_{L^1/H^1_0}$. The fact that $H^1_{|\Omega}$ is dense in $L^1(\Omega)$ but $H^\infty_{|\Omega}$ is not dense in $L^\infty(\Omega)$ may be found in [2]. ■

Another consequence of factorizing χ_Ω in certain special cases is the following.

PROPOSITION 2.4. *Let $n \geq 2$ be an integer and let Ω be a subarc of the circle such that $m(\Omega) = 1/n$. Suppose that there exist $x, y \in \mathcal{H}$ such that $[\chi_\Omega]_{L^1/H^1_0} = [x \overset{T}{\cdot} y]_{L^1/H^1_0}$. Then T^n has a nontrivial invariant subspace.*

Proof. We know from Theorem 2.3 that, writing $\xi_j = e^{(it+2j\pi)/n}$ for $j = 1, \ldots, n$, we have

$$(x \overset{T^n}{\cdot} y)(e^{it}) = \frac{1}{n}\sum_{j=1}^{n}(x \overset{T}{\cdot} y)(\xi_j) = \frac{1}{n}\sum_{j=1}^{n}(\chi_\Omega(\xi_j) + \xi_j h(\xi_j)),$$

where $h \in H^1(\mathbb{T})$. If $h(z) = \sum_{r=0}^\infty a_r z^r$, then

$$\sum_{j=1}^{n} \xi_j h(\xi_j) = \sum_{r=0}^{\infty} a_r \sum_{j=1}^{n} \xi_j^{r+1} = n\sum_{k=1}^{\infty} a_{nk-1} e^{ikt},$$

which, regarded as a function of e^{it}, lies in H^1_0. Here we have used the fact that

$$\sum_{j=1}^{n} \xi_j^{r+1} = \begin{cases} 0 & \text{if } n \nmid (r+1), \\ ne^{ikt} & \text{if } r+1 = nk \text{ with } k \in \mathbb{N}. \end{cases}$$

We conclude that

$$(x \overset{T^n}{\cdot} y)(e^{it}) - \frac{1}{n}\sum_{j=1}^{n} \chi_\Omega(\xi_j) \in H_0^1,$$

and so $[x \overset{T^n}{\cdot} y]_{L^1/H_0^1} = [P_0]_{L^1/H_0^1}$. The result now follows from Corollary 2.1. ■

3. L^1 factorizations and moment problems. The following result is due to Atzmon and Godefroy [1, Cor. 4].

THEOREM 3.1. *Let X be a real Banach space and let $A \in \mathcal{L}(X)$ be an operator which admits a real moment sequence, in the sense that there exists a positive Borel measure μ on $\mathbb{R}$ and elements $x \in X\setminus\{0\}$ and $x^* \in X^*\setminus\{0\}$ such that for each $n \geq 0$ we have*

$$\langle x_0^*, A^n x_0\rangle = \int_{\mathbb{R}} u^n \, d\mu(u).$$

Then A has a nontrivial invariant subspace.

We may use the above result to deduce further results linking L^1 factorizations with the existence of (complex) invariant subspaces.

THEOREM 3.2. *Let $\mathcal{H}$ be a complex Hilbert space that decomposes as the topological direct sum $\mathcal{K} \oplus i\mathcal{K}$ of real closed subspaces. Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction such that $T\mathcal{K} \subseteq \mathcal{K}$ and $-1 \notin \sigma(T)$, and suppose that there exist nonzero vectors $x, y \in \mathcal{K}$ and a nonnegative function $f \in L^1(\mathbb{T})$ vanishing on an open arc containing -1 such that $[f] = [x \overset{T}{\cdot} y]$. Then the operator $(\mathrm{Id} - T)^2(\mathrm{Id} + T)^{-2}$ has a nontrivial closed invariant subspace.*

Proof. Let $B_r := (\mathrm{Id} - rT)(\mathrm{Id} + rT)^{-1}$ for $0 \leq r \leq 1$ with $B = B_1$; then $\|B_r - B\| \to 0$ as $r \to 1^-$. By the H^∞ functional calculus, we have

$$\langle B_r^n x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1 - re^{it}}{1 + re^{it}}\right)^n f(e^{it})\, dt \qquad \text{for } n = 0, 1, \ldots$$

whenever $0 \leq r < 1$, and hence, by the hypotheses on f, we deduce that

$$\langle B^n x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1 - e^{it}}{1 + e^{it}}\right)^n f(e^{it})\, dt.$$

Note that $(1 - e^{it})/(1 + e^{it}) = -i\tan(t/2)$, and so, setting $u = \tan(t/2)$, we obtain

$$\langle B^n x, y\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-iu)^n g(u)\, \frac{2\, du}{1 + u^2},$$

where $g(u) = f(e^{it})$. Hence

$$\langle (-B^2)^k x, y\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} u^{2k} g(u)\,\frac{2\,du}{1+u^2} \quad \text{for } k = 0, 1, \ldots .$$

Splitting the range of integration into two pieces, namely $(-\infty, 0]$ and $[0, \infty)$, and letting $v = u^2$ in each integral, we obtain

$$\langle (-B^2)^k x, y\rangle = \frac{1}{2\pi}\int_0^{\infty} v^k (g(\sqrt{v}) + g(-\sqrt{v}))\,\frac{dv}{(1+v)\sqrt{v}},$$

which is easily seen to correspond to the moment sequence of a positive L^1 function. Hence, by Theorem 3.1, the operator $A := -B^2$ (and hence also the operator B^2) has a nontrivial real invariant subspace $\mathcal{E} \subseteq \mathcal{K}$. It now follows that $\mathcal{E} \oplus i\mathcal{E}$ is a nontrivial complex invariant subspace for B^2 (this technique of passing from real to complex invariant subspaces appears to originate in [14]). ■

Such ideas can be used to prove more general results as follows. We work with rational functions ϕ of the form

$$\phi(z) = \frac{\sum_{n=0}^{N} a_n (1-z)^{2n}(1+z)^{2N-2n}}{\sum_{n=0}^{N} b_n (1-z)^{2n}(1+z)^{2N-2n}}, \tag{3}$$

with real coefficients (a_n) and (b_n). Note that ϕ maps $\mathbb{T}$ to $\mathbb{R} \cup \{\infty\}$, since $(1-z)/(1+z) \in i\mathbb{R} \cup \{\infty\}$ when $z \in \mathbb{T}$. Simple examples of such functions include $\phi(z) = z + z^{-1}$, and $z(1+z)^{-2}$.

Theorem 3.3. *Let $\mathcal{H}$ be a complex Hilbert space that decomposes as the topological direct sum $\mathcal{K} \oplus i\mathcal{K}$ of real closed subspaces. Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction such that $T\mathcal{K} \subseteq \mathcal{K}$. Suppose that there exist nonzero vectors $x, y \in \mathcal{K}$ and a nonnegative function $f \in L^1(\mathbb{T})$, supported on a closed subset F of $\mathbb{T}$, such that $[f] = [x \overset{T}{\cdot} y]$. Let ϕ be a function given by* (3) *that has all its poles in the unbounded component of $\mathbb{C} \setminus (\sigma(T) \cup F)$. Then the operator $\phi(T)$ has a nontrivial closed invariant subspace.*

Proof. Write $B = \phi(T)$. Using Mergelyan's theorem [17, Thm. 20.5] and the holomorphic functional calculus, we see that $z \mapsto \phi(z)$ is a uniform limit of polynomials on a neighbourhood of $\sigma(T) \cup F$ and we have

$$\langle B^n x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} (\phi(e^{it}))^n f(e^{it})\,dt \quad \text{for } n = 0, 1, \ldots .$$

We set $u = \phi(e^{it})$ and change the integration variable, splitting the circle into finitely many arcs $(I_j)_{j=1}^{M}$ on each of which $\phi(e^{it})$ varies monotonically with t. We obtain

$$\langle B^n x, y\rangle = \frac{1}{2\pi}\sum_{j=1}^{M}\int_{\phi(I_j)} u^n g_j(u)\,du,$$

where the intervals $\phi(I_j)$ are taken in the positive direction and

$$g_j(u) = \frac{f(e^{it})}{\left|\frac{d}{dt}\phi(e^{it})\right|} \geq 0$$

on the arc I_j. Hence we obtain the moment sequence of a positive L^1 function. As in Theorem 3.2, the operator B has a nontrivial real invariant subspace $\mathcal{E} \subseteq \mathcal{K}$ and $\mathcal{E} \oplus i\mathcal{E}$ is a nontrivial complex invariant subspace. ∎

Remark 3.1. The method is well illustrated by the example $\phi(z) = z + z^{-1}$. Write $B = T + T^{-1}$ and set $u = e^{it} + e^{-it} = 2\cos t$. The arcs $I_1 = [0, \pi]$ and $I_2 = [\pi, 2\pi]$ both map under ϕ to $[-2, 2]$. We obtain

$$\langle B^n x, y\rangle = \frac{1}{2\pi}\int_{-2}^{2} u^n h_1(u)\,\frac{du}{\sqrt{4-u^2}} + \frac{1}{2\pi}\int_{-2}^{2} u^n h_2(u)\,\frac{du}{\sqrt{4-u^2}},$$

where $h_j(u) = f(e^{it})$ on the arc I_j for $j = 1, 2$.

Note that it is not necessary for f to vanish at ± 1, because the integrals converge absolutely for $f \in L^1(\mathbb{T})$.

Remark 3.2. Under further spectral assumptions the hypotheses of Theorem 3.3 would imply that T also has an invariant subspace, since T will be the limit of polynomials in B provided that the function ϕ^{-1} is analytic in a neighbourhood of $\sigma(B)$ containing all the bounded components of $\mathbb{C} \setminus \sigma(B)$.

Remark 3.3. Finally, it is possible to prove Banach space versions of the above results, although, since the main interest of this paper has been with Hilbert space operators, we shall not give full details. One requires a complex Banach space $\mathcal{X}$ that decomposes as the topological direct sum $\mathcal{Y} \oplus i\mathcal{Y}$ of real closed subspaces. Note that this situation arises naturally if $\mathcal{X}$ has an unconditional basis $(x_m)_{m\geq 1}$ and one defines $\mathcal{Y}$ to be the closed linear span of those linear combinations of the (x_m) that use only real coefficients. It is then easy to see that the natural real-linear projection from $\mathcal{X}$ onto $\mathcal{Y}$ is bounded. Then $T \in \mathcal{L}(\mathcal{X})$ should be a contraction such that $T\mathcal{X} \subseteq \mathcal{X}$. In this situation we say that T factorizes a suitable nonnegative function $f \in L^1(\mathbb{T})$ if for some vector $y_0 \in \mathcal{Y}$ and functional $y_0^* \in \mathcal{X}^*$ such that y_0^* takes real values on $\mathcal{Y}$, one has $\widehat{f}(-n) = \langle y_0^*, T^n y_0\rangle$ for $n \geq 0$.

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Received April 26, 2004
Revised version October 27, 2004 (5409)