Some locally mean ergodic theorems

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Abstract. The notion of local mean ergodicity is introduced. Some general locally mean ergodic theorems for linear and affine operators are presented. Locally mean ergodic theorems for affine operators whose linear parts are compact or similar to subnormal operators on a Hilbert space are given.

1. Introduction and preliminaries. Let H (respectively, X) be a Hilbert space (respectively, normed space). We shall denote by $\mathcal{B}(H)$ (respectively, $\mathcal{B}(X)$) the space of all bounded linear operators on H (respectively, on X). If $A \in \mathcal{B}(H)$ (respectively, $A \in \mathcal{B}(X)$), $\sigma(A)$ denotes the spectrum of A and $r_\sigma(A) := \lim_{n\to\infty} \|A^n\|^{1/n}$ denotes the spectral radius of A. An operator $A \in \mathcal{B}(H)$ is *subnormal* if there exists a Hilbert space K and a normal operator $\widetilde{A} \in \mathcal{B}(K)$ such that K contains H as a (closed) subspace, and $\widetilde{A} = A$ on H (cf. [C, p. 42] for a characterization). It is easy to see that an operator B which is similar to a normal (respectively, subnormal) operator need not be normal (respectively, subnormal).

Let X be a normed space and $T : X \to X$. Then T is said to have the *boundedness stability property* [ERT, p. 111], or T has (BSP) or $T \in$ BSP for short, if for each $x \in X$, the sequence $(T^n x)_{n=1}^\infty$ is bounded whenever it has a bounded subsequence. It is clear from the definition that if $S, T \in \mathcal{B}(X)$ are similar and T has (BSP), then S also has (BSP).

If X is a linear space, then $T : X \to X$ is *affine* if $A_T : X \to X$ defined by $A_T x = Tx - T0$ for all $x \in X$ is linear. In this case, A_T is called the

linear part of T. If $a = T0$, it is easily verified that for each $z \in X$,

$$(1.1) \qquad T^n z = A_T^n z + \sum_{k=1}^{n} A_T^{k-1} a \quad \text{for all } n \geq 1.$$

Thus, if X is a normed space, T is affine and $(\sum_{k=1}^{n} A_T^k a)_{n=1}^{\infty}$ is bounded, then A_T has (BSP) if and only if T has (BSP).

The following is Corollary 2 in [ERT, p. 120]:

LEMMA A. *Let H be a Hilbert space and $A \in \mathcal{B}(H)$ be similar to a subnormal operator. Then A has* (BSP).

In the literature, mean ergodic theorems for linear operators (respectively, for affine operators) usually deal with operators which are (respectively, whose linear parts are) power bounded (see, e.g., [YK] and [Z], respectively, [E]). In this paper, we introduce the notion of local mean ergodicity. Some general locally mean ergodic theorems are then given. Locally mean ergodic theorems for linear operators (respectively, affine operators whose linear parts are) similar to subnormal operators on a Hilbert space are presented. A result on affine operators whose linear parts are compact is also established.

2. Locally mean ergodic theorems. Let X be a non-empty set and $T : X \to X$. Then $x \in X$ is a *fixed point* of T if $T(x) = x$. We shall denote by $\mathrm{Fix}(T)$ the set of all fixed points of T in X.

The following can be easily verified:

LEMMA 2.1. *Let X be a vector space, $T : X \to X$ be affine and $a = T0$. Then $\mathrm{Fix}(A_T) = \ker(I - A_T)$ and $\mathrm{Fix}(T) = (I - A_T)^{-1}(\{a\}) = \xi + \mathrm{Fix}(A_T)$ for any $\xi \in \mathrm{Fix}(T)$, where I denotes the identity operator on X. Moreover, $\mathrm{Fix}(T) \neq \emptyset$ if and only if $a \in (I - A_T)X$.*

We now introduce the notion of local mean ergodicity as follows:

DEFINITION 2.2. Let X be a normed space and $A : X \to X$. Then A is said to be *locally mean ergodic*, or A is LME or $A \in$ LME for short, if for each $x \in X$, the sequence $(n^{-1} \sum_{k=1}^{n} A^k x)_{n=1}^{\infty}$ converges in X whenever the sequence $(A^n x)_{n=1}^{\infty}$ contains a bounded subsequence.

Let X be a normed space. It is clear from the definition that if $A : X \to X$ is an arbitrary mapping such that $\|A^n x\| \to \infty$ for each $x \in X$, then $A \in$ LME. Also, if $A, B \in \mathcal{B}(X)$ are similar and A is locally mean ergodic, then B is also locally mean ergodic.

The following result is Theorem 2.4 of [RTT]:

LEMMA 2.3. *Let X be a Banach space and $A \in \mathcal{B}(X)$ be compact. Let $x \in X$ be such that a subsequence of $(A^n x)_{n=1}^{\infty}$ is bounded. Then the sequence $(n^{-1} \sum_{k=1}^{n} A^k x)_{n=1}^{\infty}$ converges to a fixed point of A. In particular, $A \in$ LME.*

LEMMA 2.4. *Let* X *be a normed space,* $T : X \to X$ *be a continuous affine operator,* $x \in X$ *and* $x_n = n^{-1} \sum_{k=1}^{n} T^k x$ *for* $n = 1, 2, \ldots$ *Suppose any one of the following conditions is satisfied*:

(a) $(T^n x)_{n=1}^{\infty}$ *is bounded and a subsequence* $(x_{n_i})_{i=1}^{\infty}$ *of* $(x_n)_{n=1}^{\infty}$ *is weakly convergent to some* $\overline{x}$ *in* X,

(b) *the sequence* $(x_n)_{n=1}^{\infty}$ *is weakly convergent to some* $\overline{x}$ *in* X.

Then $\overline{x} \in \text{Fix}(T)$.

Proof. Suppose the condition (a) is satisfied. Since T is weakly continuous, $(T x_{n_i})_{i=1}^{\infty}$ converges weakly to $T\overline{x}$. Note that

$$T x_{n_i} = \frac{1}{n_i} \sum_{k=1}^{n_i} T^{k+1} x = \frac{1}{n_i} \Big[\Big(\sum_{k=1}^{n_i} T^k x \Big) + T^{n_i+1} x - Tx \Big]$$

$$= x_{n_i} + \frac{1}{n_i} (T^{n_i+1} x) - \frac{1}{n_i} Tx \to \overline{x} + 0 - 0 = \overline{x} \quad \text{weakly as } i \to \infty$$

as the sequence $(T^n x)_{n=1}^{\infty}$ is bounded. Therefore $T\overline{x} = \overline{x}$.

Now suppose the condition (b) is satisfied. Since T is weakly continuous, $(T x_n)_{n=1}^{\infty}$ converges weakly to $T\overline{x}$. Note that

$$T x_n = \frac{1}{n} \sum_{k=1}^{n} T^{k+1} x = \frac{n+1}{n} \Big(\frac{1}{n+1} \sum_{k=1}^{n+1} T^k x \Big) - \frac{1}{n} Tx$$

$$= \frac{n+1}{n} x_{n+1} - \frac{1}{n} Tx \to \overline{x} - 0 = \overline{x} \quad \text{weakly as } n \to \infty.$$

Therefore $T\overline{x} = \overline{x}$. ∎

Lemma 2.4(a) improves Theorem 1(1) of [TT] where "$(\sum_{k=1}^{n} A_T^k a)_{n=1}^{\infty}$ is bounded where $a = T0$" is additionally assumed.

As an immediate consequence of Lemma 2.4(a), we have the following result which is Theorem 5 of [ET]; for its generalization to semigroups, we refer to Theorem 5′ of [ET].

COROLLARY 2.5. *Let* X *be a reflexive Banach space and* $T : X \to X$ *be a continuous affine operator. If* $(T^n x)_{n=1}^{\infty}$ *is bounded for some* $x \in X$, *then* $\text{Fix}(T) \neq \emptyset$.

Proof. Since $(T^n x)_{n=1}^{\infty}$ is bounded, the sequence $(x_n)_{n=1}^{\infty}$ is also bounded where $x_n = n^{-1} \sum_{k=1}^{n} T^k x$ for $n = 1, 2, \ldots$ Since X is reflexive, there is a subsequence $(x_{n_i})_{i=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ that weakly converges to some $\overline{x} \in X$. By Lemma 2.4(a), $T\overline{x} = \overline{x}$. ∎

As another immediate consequence of Lemma 2.4(b), we have

COROLLARY 2.6. *Let* X *be a normed space and* $T : X \to X$ *be a continuous affine operator such that* $T \in \text{LME}$. *For each* $x \in X$, *if* $(T^n x)_{n=1}^{\infty}$ *has*

a bounded subsequence, then $(n^{-1} \sum_{k=1}^{n} T^k x)_{n=1}^{\infty}$ *converges to a fixed point of* T *in* X.

We now present a very general locally mean ergodic theorem for affine operators:

THEOREM 2.7. *Let* X *be a normed space and* $T : X \to X$ *be an affine operator* (*not necessarily continuous*) *such that* $A_T \in$ LME *and* Fix$(T) \neq \emptyset$. *Then* $T \in$ LME.

Proof. Suppose $x \in X$ is such that $(T^n x)_{n=1}^{\infty}$ contains a bounded subsequence. Since Fix$(T) \neq \emptyset$, choose any $\overline{x} \in X$ such that $T\overline{x} = \overline{x}$.

By (1.1), for each $n \geq 1$,

$$(2.2) \qquad A_T^n(x - \overline{x}) = A_T^n x - A_T^n \overline{x} = T^n x - T^n \overline{x} = T^n x - \overline{x}.$$

Thus the sequence $(A_T^n(x - \overline{x}))_{n=1}^{\infty}$ has a bounded subsequence. Since $A_T \in$ LME, $n^{-1} \sum_{k=1}^{n} A_T^k(x - \overline{x})$ converges, say to $\widehat{x} \in X$. Now (2.2) implies $n^{-1} \sum_{k=1}^{n} T^k x - \overline{x} = n^{-1} \sum_{k=1}^{n} A_T^k(x - \overline{x}) \to \widehat{x}$ so that $n^{-1} \sum_{k=1}^{n} T^k x \to \widehat{x} + \overline{x}$. Therefore $T \in$ LME. ∎

COROLLARY 2.8. *Let* X *be a normed space,* $T : X \to X$ *be a continuous affine operator such that* $A_T \in$ LME *and for some* $x \in X$, $(T^n x)_{n=1}^{\infty}$ *has a bounded subsequence. Then* Fix$(T) \neq \emptyset$ *if and only if* $(n^{-1} \sum_{k=1}^{n} T^k x)_{n=1}^{\infty}$ *is convergent.*

Proof. Suppose $(n^{-1} \sum_{k=1}^{n} T^k x)_{n=1}^{\infty}$ is convergent to $\overline{x} \in X$. Then by Lemma 2.4(b), $\overline{x} \in$ Fix(T) so that Fix$(T) \neq \emptyset$.

Conversely, suppose Fix$(T) \neq \emptyset$. Then by Theorem 2.7, $T \in$ LME. Since $(T^n x)_{n=1}^{\infty}$ has a bounded subsequence, it follows from local mean ergodicity of T that $(n^{-1} \sum_{k=1}^{n} T^k x)_{n=1}^{\infty}$ is convergent. ∎

THEOREM 2.9. *Let* X *be a normed space and* $T : X \to X$ *be an affine operator* (*not necessarily continuous*) *such that* $A_T \in$ LME *and* $I - T$ *has a closed range, where* I *denotes the identity operator on* X. *Then* $T \in$ LME.

Proof. Suppose Fix$(T) = \emptyset$. Then by Theorem 2 of [ET], for each $x \in X$, $\|T^n x\| \to \infty$ as $n \to \infty$. Thus $T \in$ LME.

On the other hand, if Fix$(T) \neq \emptyset$, then $T \in$ LME by Theorem 2.7. ∎

The following is the locally mean ergodic theorem for affine operators whose linear parts are compact:

THEOREM 2.10. *Let* X *be a Banach space and* $T : X \to X$ *be a continuous affine operator such that* A_T *is compact. Then* $T \in$ LME.

Proof. Since A_T is compact, $A_T \in$ LME by Lemma 2.3 and $I - A_T$ has a closed range (cf. [R, p. 101]). Hence $I - T$ has a closed range. Thus $T \in$ LME by Theorem 2.9. ∎

Theorem 2.10 improves Theorem 5 of [TT] where "$(\sum_{k=1}^{n} A_T^k a)_{n=1}^{\infty}$ is bounded where $a = T0$" is additionally assumed.

The following is another locally mean ergodic theorem for an affine operator:

THEOREM 2.11. *Let X be a reflexive Banach space and $T : X \to X$ be a continuous affine operator such that $A_T \in$ LME. Suppose any one of the following conditions is satisfied*:

(a) $T \in$ BSP,
(b) *the sequence $(\sum_{k=1}^{n} A_T^k a)_{n=1}^{\infty}$ is bounded where $a = T0$.*

Then $T \in$ LME.

Proof. Suppose the condition (a) is satisfied. Let $x \in X$ be such that $(T^n x)_{n=1}^{\infty}$ contains a bounded subsequence. Since $T \in$ BSP, $(T^n x)_{n=1}^{\infty}$ is itself bounded. By Corollary 2.5, $\mathrm{Fix}(T) \neq \emptyset$. By Theorem 2.7, $T \in$ LME.

Next suppose (b) is satisfied. Note that $(T^n 0)_{n=1}^{\infty} = (\sum_{k=0}^{n-1} A_T^k a)_{n=1}^{\infty}$ is bounded. By Corollary 2.5, $\mathrm{Fix}(T) \neq \emptyset$. Now Theorem 2.7 implies that $T \in$ LME. ∎

To end this section, we have the following strengthened converse of Theorem 2.11(b) (here X is not assumed to be reflexive):

THEOREM 2.12. *Let X be a normed space and $T : X \to X$ be a continuous affine operator such that $(\sum_{k=1}^{n} A_T^k a)_{n=1}^{\infty}$ is bounded in X, where $a = T0$. If T is locally mean ergodic, then A_T is locally mean ergodic and $\mathrm{Fix}(T) \neq \emptyset$.*

Proof. Let $x \in X$ be such that $(A_T^n x)_{n=1}^{\infty}$ contains a bounded subsequence $(A_T^{n_i} x)_{i=1}^{\infty}$. Then by (1.1) the subsequence $(T^{n_i} x)_{i=1}^{\infty}$ of $(T^n x)_{n=1}^{\infty}$ is bounded. Hence by Corollary 2.6, the sequence $(x_n)_{n=1}^{\infty}$ converges to some $\overline{x} \in \mathrm{Fix}(T)$, where $x_n = n^{-1} \sum_{k=1}^{n} T^k x$. By (1.1) again, we have $A_T^k \overline{x} = \overline{x} - \sum_{l=0}^{k-1} A_T^l a = \overline{x} - T^k 0$ for $k \geq 1$ so that

$$\frac{1}{n} \sum_{k=1}^{n} A_T^k \overline{x} = \overline{x} - \frac{1}{n} \sum_{k=1}^{n} T^k 0.$$

Since $(T^n 0)_{n=1}^{\infty} = (\sum_{k=0}^{n-1} A_T^k a)_{n=1}^{\infty}$ is bounded and $T \in$ LME, it follows that $(n^{-1} \sum_{k=1}^{n} T^k 0)_{n=1}^{\infty}$ converges, say to $\widehat{x} \in X$. Thus $(n^{-1} \sum_{k=1}^{n} A_T^k \overline{x})_{n=1}^{\infty}$ converges to $\overline{x} - \widehat{x}$. Therefore A_T is locally mean ergodic. ∎

3. Subnormal operators. The following result is a mean ergodic theorem for an operator which is similar to a subnormal operator on a Hilbert space:

THEOREM 3.1. *Let H be a Hilbert space and $A \in \mathcal{B}(H)$ be similar to a subnormal operator. Let $x \in H$ be such that a subsequence of $(x_n)_{n=1}^{\infty}$, where $x_n = n^{-1} \sum_{k=1}^{n} A^k x$ for all $n = 1, 2, \ldots$, is bounded. Then*

(1) *the sequence $(x_n)_{n=1}^{\infty}$ converges to a fixed point $\widehat{x}$ of A in H, and*

(2) *for each $y \in H$ with $x - \widehat{x} = (I - A)y$, where I denotes the identity operator on H, the sequence $(A^n y)_{n=1}^{\infty}$ is also bounded.*

Proof. Case 1. Suppose A is a subnormal operator.

First consider the case where H is complex. Let K be a complex Hilbert space containing H (as a subspace), and $\widetilde{A} \in B(K)$ be normal with $\widetilde{A} = A$ on H. Let $\sigma = \sigma(\widetilde{A})$ and let $\widetilde{A} = \int_{\sigma} \lambda \, dP_{\lambda}$ be the spectral representation of $\widetilde{A}$. Then for any $x \in H$ and any $k = 1, 2, \ldots$, $A^k x = \int_{\sigma} \lambda^k \, dP_{\lambda} x$, so that for each $n = 1, 2, \ldots$,

$$x_n = \frac{1}{n} \sum_{k=1}^{n} A^k x = \int_{\sigma} f_n(\lambda) \, dP_{\lambda} x, \quad \text{where} \quad f_n(\lambda) = \frac{1}{n} \sum_{k=1}^{n} \lambda^k.$$

Let $\sigma_0 = \{\lambda \in \sigma : |\lambda| \leq 1 \text{ and } \lambda \neq 1\}$, $\sigma_1 := \{\lambda \in \sigma : \lambda = 1\}$, and $\sigma_2 := \{\lambda \in \sigma : |\lambda| > 1\}$. Since for each $n = 1, 2, \ldots$ and for each $\lambda \neq 1$,

$$f_n(\lambda) = \frac{\lambda(1 - \lambda^n)}{n(1 - \lambda)} = \frac{\lambda}{n} (1 + \lambda + \ldots + \lambda^{n-1}),$$

we have $|f_n(\lambda)| \leq 1$ on σ_0; $f_n(\lambda) = 1$ on σ_1; and

$$|f_n(\lambda)| \geq \frac{|\lambda|(|\lambda|^n - 1)}{n(|\lambda| + 1)} \geq \frac{(n-1)(1/p)^2}{2(1 + \|\widetilde{A}\|)} \quad \text{if } |\lambda| \geq 1 + \frac{1}{p} \text{ for some } p = 1, 2, \ldots$$

Thus $\int_{\sigma_0 \cup \sigma_1} |f_n(\lambda)|^2 \, d(P_{\lambda} x \,|\, x)$ is uniformly bounded (w.r.t. $n \in \mathbb{N}$); since a subsequence of $(x_n)_{n=1}^{\infty}$ is bounded, and since

$$\|x_n\|^2 = \int_{\sigma_0 \cup \sigma_1} |f_n(\lambda)|^2 \, d(P_{\lambda} x \,|\, x) + \int_{\sigma_2} |f_n(\lambda)|^2 \, d(P_{\lambda} x \,|\, x),$$

we must have $(P(\sigma_2)x \,|\, x) = 0$, i.e., $P(\sigma_2)x = 0$. Thus $x_n = \int_{\sigma_1} dP_{\lambda} x + \int_{\sigma_0} f_n(\lambda) \, dP_{\lambda} x$. As $|f_n(\lambda)| \leq 1$ and $\lim_{n \to \infty} f_n(\lambda) = 0$ for each $\lambda \in \sigma_0$, by [DS, Corollary 8(iii), p. 899] we have $x_n \to \widehat{x}$ where $\widehat{x} := \int_{\sigma_1} dP_{\lambda} x$. Since $x_n \in H$, we also have $\widehat{x} \in H$. Now $A\widehat{x} = \int_{\sigma} \lambda \chi_{\sigma_1}(\lambda) \, dP_{\lambda} x = \int_{\sigma_1} dP_{\lambda} x = \widehat{x}$, so (1) is established.

To establish (2), let $B := \chi_{\sigma_2}(\widetilde{A})$. Then $Bx = \int_{\sigma_2} dP_{\lambda} x = P(\sigma_2)x = 0$ and $B\widehat{x} = \int_{\sigma} \chi_{\sigma_2}(\lambda) \chi_{\sigma_1}(\lambda) \, dP_{\lambda} x = 0$. Suppose $y \in H$ is such that $x - \widehat{x} = (I - A)y$. Then $0 = B(x - \widehat{x}) = B(I - \widetilde{A})y = (I - \widetilde{A})By$, so that $\widetilde{A}By = By$. Hence for all $n = 1, 2, \ldots$, $\widetilde{A}^n By = By$ and

$$\|\widetilde{A}^n By\| = \|By\| = \left[\int_{\sigma_2} d(P_{\lambda} y \,|\, y) \right]^{1/2} = \|P(\sigma_2)y\| \leq \|y\|,$$

also

$$\|\widetilde{A}^n(I-B)y\| = \Big[\int_{\sigma_0 \cup \sigma_1} |\lambda|^{2n} \, d(P_\lambda y \, | \, y) \Big]^{1/2} \leq \|P(\sigma_0 \cup \sigma_1)y\| \leq \|y\|.$$

Hence $\|A^n y\| \leq \|\widetilde{A}^n By\| + \|\widetilde{A}^n(I-B)y\| \leq 2\|y\|$ and (2) is established.

Consider now the case of a real Hilbert space H, and let $A \in \mathcal{B}(H)$ be subnormal. Let $x \in H$ be such that a subsequence of $(x_n)_{n=1}^\infty$ is bounded. Let H_c, respectively, A_c, be the complexification of H, respectively, A. Then A_c is subnormal (see, e.g., [ERT, p. 119], or verified directly) and $\|n^{-1} \sum_{k=1}^n A_c^k(x,0)\| = \|n^{-1} \sum_{k=1}^n A^k x\|$, $n = 1, 2, \ldots$ By the preceding result for the complex case, $(n^{-1} \sum_{k=1}^n A^k x, 0)_{n=1}^\infty = (n^{-1} \sum_{k=1}^n A_c^k(x,0))_{n=1}^\infty$ converges to a fixed point $\widehat{(x,0)}$ of A_c in H_c. Hence $(n^{-1} \sum_{k=1}^n A^k x)_{n=1}^\infty$ (is Cauchy in H, so it) converges to some $\widehat{x}$ in H, and $\widehat{(x,0)} = (\widehat{x},0)$. Hence $(A\widehat{x},0) = A_c(\widehat{x},0) = (\widehat{x},0)$, and $\widehat{x}$ is a fixed point of A. Moreover if $y \in H$ satisfies $x - \widehat{x} = (I-A)y$, then $(x,0) - \widehat{(x,0)} = (x-\widehat{x},0) = (I_c - A_c)(y,0)$, so that by the result for the complex case, $(A_c^n(y,0))_{n=1}^\infty$ is bounded. But $\|A^n y\| = \|A_c^n(y,0)\|$ for all $n = 1, 2, \ldots$, so $(A^n y)_{n=1}^\infty$ is also bounded.

Case 2. Now suppose A is similar to a subnormal operator $B \in \mathcal{B}(H)$.

Let $R \in \mathcal{B}(H)$ be invertible such that $A = RBR^{-1}$ or $B = R^{-1}AR$. Let $z = R^{-1}x$ and $z_n := n^{-1} \sum_{k=1}^n B^k z = R^{-1}x_n$ for each $n \geq 1$. Since a subsequence of $(x_n)_{n=1}^\infty$ is bounded, a subsequence of $(z_n)_{n=1}^\infty$ is bounded. By Case 1, $(z_n)_{n=1}^\infty$ converges to a fixed point $\widehat{z}$ of B in H. Hence $(x_n)_{n=1}^\infty = (Rz_n)_{n=1}^\infty$ converges to $\widehat{x} = R\widehat{z}$ in H. But $A\widehat{x} = (RBR^{-1})R\widehat{z} = RB\widehat{z} = R\widehat{z} = \widehat{x}$; i.e., $\widehat{x}$ is a fixed point of A. Moreover, for $y \in H$ with $x - \widehat{x} = (I-A)y$, we have $z - \widehat{z} = R^{-1}(x - \widehat{x}) = R^{-1}(I-A)y = (I-B)R^{-1}y$ so that by Case 1, $(B^n R^{-1}y)_{n=1}^\infty$ is bounded. But then $(A^n y)_{n=1}^\infty = (RB^n R^{-1}y)_{n=1}^\infty$ is bounded. ∎

The following locally mean ergodic theorem for linear operators which are similar to subnormal operators is an immediate consequence of Theorem 3.1 and Lemma A:

THEOREM 3.2. *Let H be a Hilbert space and $A \in \mathcal{B}(H)$ be similar to a subnormal operator. Let $x \in H$ be such that a subsequence of $(A^n x)_{n=1}^\infty$ is bounded. Then*

(1) *the sequence $(n^{-1} \sum_{k=1}^n A^k x)_{n=1}^\infty$ converges to a fixed point $\widehat{x}$ of A in H, and*

(2) *for each $y \in H$ with $x - \widehat{x} = (I-A)y$, where I denotes the identity operator in $\mathcal{B}(H)$, the sequence $(A^n y)_{n=1}^\infty$ is also bounded.*

In particular, $A \in$ LME.

COROLLARY 3.3. *Let H be a Hilbert space and $B \in \mathcal{B}(H)$ be similar to a subnormal operator in $\mathcal{B}(H)$. For $a \in H$, define $T : H \to H$ by $Tz = (I - B)z + a$ for each $z \in H$ where I is the identity operator on H. Suppose the sequence $(\sum_{k=1}^{n}(I - B)^k a)_{n=1}^{\infty}$ contains a bounded subsequence and for some $x \in H$, the sequence $(T^n x)_{n=1}^{\infty}$ is bounded. Then the sequence $(n^{-1} \sum_{k=1}^{n} T^k x)_{n=1}^{\infty}$ converges in H to a solution of the equation $Bz = a$.*

Proof. Let $A = I - B$; then $A_T = A$. Since B is similar to a subnormal operator, A is also similar to a subnormal operator so that $A \in$ LME by Theorem 3.2.

By (1.1), a subsequence of $(A^n x)_{n=1}^{\infty}$ is bounded. By Lemma A, the whole sequence $(A^n x)_{n=1}^{\infty}$ is bounded. Hence $(\sum_{k=1}^{n} A^k a)_{n=1}^{\infty}$ is bounded by (1.1). By Theorem 2.11(b), $T \in$ LME so that the sequence $(n^{-1} \sum_{k=1}^{n} T^k x)_{n=1}^{\infty}$ converges to some $\overline{x} \in \text{Fix}(T)$ by Corollary 2.6. It follows that $B\overline{x} = a$. ∎

REMARK 3.4. Corollary 3.3 has an obvious counterpart for the case where $I - B$ is compact; its statement is omitted (use Theorem 2.10 and Lemma 2.4(b); in fact, it is sufficient to assume that the sequence $(T^n x)_{n=1}^{\infty}$ contains a bounded subsequence).

Finally, we have the following locally mean ergodic theorem for affine operators whose linear parts are similar to subnormal operators:

THEOREM 3.5. *Let H be a Hilbert space and $T : H \to H$ be a continuous affine operator whose linear part A_T is similar to a subnormal operator in $\mathcal{B}(H)$. Suppose any one of the following conditions is satisfied*:

(1) *The sequence $(\sum_{k=1}^{n} A_T^k a)_{n=1}^{\infty}$ is bounded where $a = T0$.*

(2) $1 \notin \sigma(A_T)$.

(3) *1 is an isolated point of $\sigma(\widehat{A}_T)$, where $\widehat{A}_T$ is the complexification of A_T when H is real, and $\widehat{A}_T = A_T$ when H is complex.*

Then T is locally mean ergodic.

Proof. When A_T satisfies condition (1), the conclusion follows from Theorems 3.2 and 2.11(b). If A_T satisfies (2), then $\text{Fix}(T) \neq \emptyset$ by Lemma 2.1 so that $T \in$ LME by Theorems 3.2 and 2.7. Finally, if A_T satisfies (3), then T has (BSP) by [ERT, Corollary 4 and the remark following its proof, p. 121] so that by Theorems 3.2 and 2.11(a), $T \in$ LME. ∎

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