STUDIA MATHEMATICA 152 (1) (2002)

Bounded and unbounded operators between Köthe spaces

by

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Abstract. We study in terms of corresponding Köthe matrices when every continuous linear operator between two Köthe spaces is bounded, the consequences of the existence of unbounded continuous linear operators, and related topics.

1. Introduction. If $a = (a_{ip})$ and $b = (b_{jq})$ are Köthe matrices we denote by $K(a)$ and $K(b)$ the Köthe spaces defined by a and b. As usual, $L(K(a), K(b))$ and $LB(K(a), K(b))$ denote, respectively, the space of all continuous linear operators and the space of all bounded continuous linear operators from $K(a)$ to $K(b)$. Our aim here is to characterize in terms of the matrices a and b when

$$L(K(a), K(b)) = LB(K(a), K(b)),$$

and to study the consequences of the existence of unbounded continuous linear operators between $K(a)$ and $K(b)$. The same questions were studied by many mathematicians; here we mention only some results that inspired our work.

Zahariuta [18, 19] discovered that if the matrices a and b satisfy conditions d_2 and d_1, respectively, then each continuous linear operator from $K(a)$ to $K(b)$ is bounded.

This phenomenon was studied later by many authors (see, e.g., [1, 4–9, 12, 14]); comprehensive results were obtained by Vogt [17] not only for Köthe spaces but also for the general case of Fréchet spaces (for further generalizations see also [2, 3]).

On the other hand Nurlu and Terzioğlu [15] proved (under some conditions) that the existence of unbounded continuous linear operators between nuclear Köthe spaces $K(a)$ and $K(b)$ implies the existence of a common basic subspace of $K(a)$ and $K(b)$.

2000 *Mathematics Subject Classification*: Primary 46A45.

Key words and phrases: bounded operators, unbounded operators, Köthe spaces.

The first author thanks the University of Michigan and B. Alan Taylor for hospitality during the academic year 98/99. His research was supported in part by NSF of Bulgaria, grant no. MM-808/98.

Our present research is highly influenced by the papers of Krone [10], Nurlu and Terzioğlu [15] and Vogt [17]. We sharpen some results of Nurlu and Terzioğlu by proving analogous statements without the nuclearity assumption. Modifying a construction of Krone [10] we consider a class of Köthe spaces that generalizes not only infinite type power series spaces, but also Dragilev spaces of infinite type, and extend Theorems 3.2 and 5.2 of [17]. It seems to us that in this more general setting it is easier to see the role of various linear topological invariants.

2. General results on Köthe spaces. We consider only ℓ_1-Köthe spaces; if $(a_{np})_{n,p=1}^{\infty}$ is a Köthe matrix we denote by $K(a_{np})$ the corresponding Köthe sequence space, that is,

$$K(a_{np}) = \Big\{x = (x_n) : \|x\|_p = \sum_n |x_n| a_{np} < \infty \ \forall p\Big\}.$$

Equipped with the system of seminorms $\|x\|_p$, $p = 1, 2, \ldots$, $K(a_{np})$ is a Fréchet space.

As usual, we denote by $\{e_n : n = 1, 2, \ldots\}$ the canonical basis of $K(a_{np})$, that is, $e_n = (\delta_{nk})_{k=1}^{\infty}$; then obviously $\|e_n\|_p = a_{np}$ for all n, p. We associate with each infinite subset $\{e_n : n \in N_1 \subset \mathbb{N}\}$ of the canonical basis the corresponding *basic* subspace, that is, the closed linear span $[e_n : n \in N_1]$. If (n_i) is a strictly increasing sequence such that $\{n_1, n_2, \ldots\} = N_1$, then we may identify the corresponding basic subspace with the Köthe space $K(a_{n_i p})$.

Recall that an operator $T : K(a_{np}) \to K(b_{mq})$ is bounded if and only if

$$\exists p_0 \ \forall q \ \exists C_q : \quad \|Te_n\|_q \le C_q \|e_n\|_{p_0} \quad \forall n. \tag{2.1}$$

An operator $T : K(a_{np}) \to K(b_{mq})$ is called *quasi-diagonal* if it satisfies

$$T(e_n) = t_n e_{m(n)}, \qquad m(\cdot) : \mathbb{N} \to \mathbb{N}. \tag{2.2}$$

Dragilev [5] and Nurlu [13] proved that if X and Y are nuclear Köthe spaces and there exists a continuous linear unbounded operator $T : X \to Y$, then there exists a continuous unbounded quasi-diagonal operator $D : X \to Y$. We sharpen that result by omitting the nuclearity condition:

PROPOSITION 1. *If $K(a_{np})$ and $K(b_{kp})$ are Köthe spaces such that there exists a continuous linear unbounded operator $T : K(a_{np}) \to K(b_{kp})$, then there exists a continuous unbounded quasi-diagonal operator $D : K(a_{np}) \to K(b_{kp})$.*

Proof. Since T is continuous and unbounded, we may assume that

$$\|Tx\|_p \le 2^{-p} \|x\|_p \quad \forall x \in K(a_{np}),\ p = 1, 2, \ldots, \tag{2.3}$$

$$\sup_n \frac{\|Te_n\|_{p+1}}{\|e_n\|_p} = \infty, \quad p = 1, 2, \ldots \tag{2.4}$$

Indeed, one may achieve (2.3) and (2.4) by using appropriate multipliers and passing to a subsequence of seminorms, if necessary.

Let (p_j) be a sequence of integers in which each $p \in \mathbb{N}$ appears infinitely many times. In view of (2.4) we may choose inductively a subsequence $n_1 < n_2 < \ldots$ such that

$$\frac{\|Te_{n_j}\|_{p_j+1}}{\|e_{n_j}\|_{p_j}} \ge 2^j \quad \forall j. \tag{2.5}$$

Let $Te_n = \sum_k t_{nk}\widetilde{e}_k$; then by (2.3) we have

$$\sum_k |t_{nk}| \sup_p \frac{\|\widetilde{e}_k\|_p}{\|e_n\|_p} \le \sum_k |t_{nk}| \sum_p \frac{\|\widetilde{e}_k\|_p}{\|e_n\|_p} = \sum_p \frac{\|Te_n\|_p}{\|e_n\|_p} \le 1.$$

Thus, for each $j = 1, 2, \ldots$, we obtain, in view of (2.5),

$$\sum_k |t_{n_jk}| \sup_p \frac{\|\widetilde{e}_k\|_p}{\|e_{n_j}\|_p} \le 1 \le 2^{-j} \sum_k |t_{n_jk}| \frac{\|\widetilde{e}_k\|_{p_j+1}}{\|e_{n_j}\|_{p_j}},$$

hence there exists a k_j such that

$$\lambda_j := \sup_p \frac{\|\widetilde{e}_{k_j}\|_p}{\|e_{n_j}\|_p} \le 2^{-j} \frac{\|\widetilde{e}_{k_j}\|_{p_j+1}}{\|e_{n_j}\|_{p_j}}.$$

Consider the quasi-diagonal operator $D : K(a_{np}) \to K(b_{kp})$ defined by

$$De_{n_j} = \lambda_j^{-1}\widetilde{e}_{k_j}, \quad j = 1, 2, \ldots, \qquad De_n = 0 \quad \text{if } n \ne n_j.$$

Since

$$\|De_{n_j}\|_p = \lambda_j^{-1}\|\widetilde{e}_{k_j}\|_p \le \|e_{n_j}\|_p \quad \forall p,$$

the operator D is continuous. In addition, D is unbounded, because if p is fixed, then for some subsequence (j_s) we have $p_{j_s} = p,\ s = 1, 2, \ldots$, so by (2.4),

$$\|De_{n_{j_s}}\|_{p+1}/\|e_{n_{j_s}}\|_p \ge 2^{j_s} \to \infty \quad \text{as } s \to \infty. \ \blacksquare$$

The next theorem gives a necessary and sufficient condition for

$$L(K(a), K(b)) = LB(K(a), K(b))$$

in terms of the Köthe matrices $a = (a_{ip})$ and $b = (b_{\nu q})$. Formally this condition coincides with the one given by Vogt (see [17], Satz 1.5), but there it is assumed that the second Köthe space is regarded with (weighted) ℓ_∞-norms instead of the usual ℓ_1-norms. So, in the case where the space $K(b)$ is nuclear the next theorem follows from that result of Vogt, since then the systems of ℓ_∞-norms and ℓ_1-norms in $K(b)$ are equivalent. But in the general case, when the Köthe space $K(b)$ is not nuclear, the theorem does not follow from Vogt's result.

THEOREM 2. *If $K(a)$ and $K(b)$ are Köthe spaces then*

$$L(K(a), K(b)) = LB(K(a), K(b))$$

if and only if

$$\forall p(q)\uparrow\infty\ \exists k\ \forall r\ \exists q_0, C: \quad \frac{b_{\nu r}}{a_{ik}} \le C \max_{1\le q\le q_0} \frac{b_{\nu q}}{a_{ip(q)}} \quad \forall i, \nu. \tag{2.6}$$

Proof. In view of Proposition 1 it is enough to prove the claim for quasi-diagonal operators. Suppose (2.6) holds and $T: K(a) \to K(b)$ is a continuous quasi-diagonal operator defined by $T(e_i) = t_i \widetilde{e}_{\nu(i)}$, $i \in \mathbb{Z}_+$. Since T is continuous there exists an increasing sequence $p(q) \uparrow \infty$ such that

$$\sup_i \frac{\|Te_i\|_q}{\|e_i\|_{p(q)}} = \sup_i \frac{|t_i| b_{\nu(i)q}}{a_{ip(q)}} = C_q < \infty.$$

Thus by (2.6) there exists a k such that for every r there exist q_0 and C with

$$\frac{|t_i| b_{\nu(i)r}}{a_{ik}} \le C \max_{1\le q\le q_0} \frac{|t_i| b_{\nu(i)q}}{a_{ip(q)}} \le C \max_{1\le q\le q_0} C_q,$$

so the operator T is bounded.

Conversely, suppose that condition (2.6) does not hold. Then

$$\exists p(q)\uparrow\infty\ \forall k\ \exists r_k\ \forall n\in\mathbb{N}\ \exists i_n, \nu_n: \quad \frac{b_{\nu_n r_k}}{a_{i_n k}} \ge n \max_{1\le q\le n} \frac{b_{\nu_n q}}{a_{i_n p(q)}},$$

where the sequences $(i_n) = (i_n(k))$ and $(\nu_n) = (\nu_n(k))$ depend on k.

There exist new sequences (for convenience we use the same notations (i_n) and (ν_n)) such that the sequence (i_n) is strictly increasing and for each k there exists a subsequence (n_j) with $i_{n_j} = i_{n_j}(k)$, $\nu_{n_j} = \nu_{n_j}(k)$ for all j. Indeed, to obtain such a sequence (i_n) one may choose an element from the first sequence $(i_n(1))$, say $i_1 = i_{k_1}(1)$, then an element from the second sequence $(i_n(2))$, say $i_2 = i_{k_2}(2) > i_1$, then again an element from the first sequence $i_3 = i_{k_3}(1) > i_2$, after that from the second sequence $i_4 = i_{k_4}(2) > i_3$, then from the third sequence $i_5 = i_{k_5}(3) > i_4$, and after that again return to choose an element from the first sequence, and so on.

So, briefly, we may describe the construction as follows. Suppose $\mathbb{N} = \bigcup_s B_s$, where each subset B_s is infinite; then we choose consecutively elements $i_n = i_{k_n}(s)$ and $\nu_n = \nu_{k_n}(s)$ for $n \in B_s$ so that $i_n > i_{n-1}$.

Consider the quasi-diagonal operator $T: K(a) \to K(b)$ defined by

$$Te_i = 0 \quad \text{for } i \ne i_n, \qquad Te_{i_n} = t_n \widetilde{e}_{\nu_n},$$

where

$$t_n^{-1} := \max_{1\le q\le n} \frac{b_{\nu_n q}}{a_{i_n p(q)}}.$$

By the choice of the constants t_n the operator T is continuous. On the other hand for each k there exists r_k such that for some subsequence (n_j) we have

$$t_{n_j} b_{\nu_{n_j} r_k} / a_{i_{n_j} k} \geq n_j,$$

hence the operator T is unbounded. ∎

Following [11] we say that an ordered pair $(K(b), K(a))$ of Köthe spaces satisfies *condition* S if

$$\forall p\ \exists q, k\ \forall s, l\ \exists r, C: \quad \frac{b_{ms}}{a_{nk}} \leq C \max\left(\frac{b_{mq}}{a_{np}}, \frac{b_{mr}}{a_{nl}}\right). \tag{2.7}$$

We say that Köthe spaces $K(a)$ and $K(b)$ have a *common basic subspace* if there exists a quasi-diagonal operator $T : X \to Y$ such that the restriction of T to some infinite-dimensional basic subspace of X is an isomorphism.

Our next proposition sharpens Proposition 3 of [15]. We prove a similar claim but without the nuclearity assumption. In addition, we require a weaker assumption on the pair of Köthe spaces: condition S instead of Apiola's splitting condition used in [15].

PROPOSITION 3. *If a pair $(K(b), K(a))$ of Köthe spaces satisfies condition S and there exists a continuous unbounded operator $T : K(a) \to K(b)$, then $K(a)$ and $K(b)$ have a common basic subspace.*

Proof. Let $(e_i)_{i\in\mathbb{N}}$ and $(\widetilde{e}_i)_{i\in\mathbb{N}}$ be, respectively, the canonical bases in $K(a)$ and $K(b)$. In view of Proposition 1 we may assume that there exists a continuous unbounded quasi-diagonal operator $T : K(a) \to K(b)$ given by

$$Te_i = t_i \widetilde{e}_{j(i)}, \quad i \in \mathbb{N}.$$

In addition we may assume that the mapping $j(i)$ is injective. Otherwise, since T is unbounded, there exists an infinite subset N_1 of indices such that the restriction of $j(i)$ to N_1 is injective and the restriction of T to the basic subspace E_1 generated by e_i, $i \in N_1$, is unbounded. So, one may consider E_1 instead of $K(a)$.

Observe that if there exist sequences (p_k) and (q_k) with $p_k \to \infty$, and infinite sets $N_1 \supset N_2 \supset \ldots$ of indices i such that

$$\|Te_i\|_{q_k} \geq \|e_i\|_{p_k} \quad \forall i \in N_k, \tag{2.8}$$

then the claim holds. Indeed, choose a sequence (i_k) with $i_k \uparrow \infty$ and $i_k \in N_k$ for all k, and let E be the basic subspace generated by $\{e_{i_k} : k = 1, 2, \ldots\}$ Then, since the operator T is quasi-diagonal, $T(E)$ is a basic subspace of $K(b)$, and (2.8) shows that T maps E isomorphically onto $T(E)$.

Let us try to construct $(p_k), (q_k)$ and N_k so that (2.7) holds with $p = p_k$ and $q = q_k$, and

$$N_k = \{i \in N_{k-1} : \|Te_n\|_{q_k} \geq \|e_n\|_{p_k}\},$$

where $N_0 = \mathbb{N}$. We can choose p_1 in an arbitrary way and assume that q_1 is chosen so that $\sup_{\mathbb{N}} \|Te_i\|_{q_1}/\|e_i\|_{p_1} = \infty$ and (2.7) holds with $p = p_1$ and $q = q_1$. Then the set N_1 is obviously infinite.

If the restriction of T to the basic subspace $E_{N_1} = [e_i : i \in N_1]$ is unbounded, then we can take any $p_2 > p_1$ and choose a q_2 so that $\sup_{N_1} \|Te_i\|_{q_2}/\|e_i\|_{p_2} = \infty$ and (2.7) holds with $p = p_2$ and $q = q_2$. Then the set

$$N_2 = \{i \in N_1 : \|Te_i\|_{q_2} \geq \|e_i\|_{p_2}\}$$

will be infinite.

Obviously one can proceed by induction, provided at each step the restriction of the operator T to the corresponding basic subspace $E_{N_k} = [e_i : i \in N_k]$ is unbounded.

Otherwise, there exists $k_0 \geq 1$ such that the restriction of T to $E_{N_{k_0}}$ is bounded, while the restriction of T to $E_{N_{k_0-1}}$ is unbounded. Consider the set

$$M_0 = N_{k_0-1} \setminus N_{k_0} = \{i \in N_{k_0-1} : \|Te_i\|_{q_0} < \|e_i\|_{p_0}\}, \tag{2.9}$$

where $p_0 = p_{k_0}$, $q_0 = q_{k_0}$. Then the restriction of the operator T to $E_{M_0} = [e_i : i \in M_0]$ is unbounded.

Since $(K(b), K(a)) \in S$ we have

$$\exists r_1\ \forall s_1, r\ \exists s, C : \quad \frac{\|Te_i\|_{s_1}}{\|e_i\|_{r_1}} \leq C \max\left(\frac{\|Te_i\|_{q_0}}{\|e_i\|_{p_0}}, \frac{\|Te_i\|_s}{\|e_i\|_r}\right). \tag{2.10}$$

Choose s_1 so that

$$\sup\{\|Te_i\|_{s_1}/\|e_i\|_{r_1} : i \in M_0\} = \infty.$$

Then there exists a sequence $(i_\nu)_{\nu=1}^\infty$, $i_\nu \in M_0$, such that

$$\frac{\|Te_{i_\nu}\|_{s_1}}{\|e_{i_\nu}\|_{r_1}} > \nu, \quad \nu = 1, 2, \ldots \tag{2.11}$$

Consider the basic subspace $E = [e_{i_\nu} : \nu = 1, 2, \ldots]$. Take an arbitrary r and choose s so that (2.10) holds. By (2.9) we have $\|Te_{i_\nu}\|_{q_0}/\|e_{i_\nu}\|_{p_0} < 1$, so from (2.10) and (2.11) it follows that

$$\forall r\ \exists s : \quad \|e_{i_\nu}\|_r \leq \|Te_{i_\nu}\|_s \quad \text{for } \nu \geq C.$$

This means that the operator T^{-1} maps continuously $T(E)$ onto E, hence the basic subspaces E and $T(E)$ are isomorphic. ■

3. Properties of Köthe spaces $K_F(\alpha)$

1. In this section we consider a wide class of Köthe spaces that includes, in particular, infinite type power series spaces and Dragilev spaces of infinite type. We define that class by slightly modifying a construction due to Krone [10].

Suppose

$$F = (F_p)_{p=1}^{\infty}, \quad F_p : (0,\infty) \to (0,\infty),$$

is a family of increasing unbounded functions such that $F_p(t) \le F_{p+1}(t)$.

For any increasing sequence $\alpha = (\alpha_i)_{i=1}^{\infty}$ of positive numbers with $\alpha_i \uparrow \infty$ we denote by $K_F(\alpha)$ the Köthe space defined by the matrix

$$a_{jp} = F_p(\alpha_j), \quad j,p \in \mathbb{N}.$$

In addition, we always require the following growth conditions:

$$\forall p\ \exists p_1, C_1 : \quad tF_p(t) \le C_1 F_{p_1}(t), \qquad t \ge 1, \tag{3.1}$$

$$\forall p, L\ \exists p_2, C_2 : \quad F_p(F_L(t)) \le C_2 F_{p_2}(t), \quad t \ge 1. \tag{3.2}$$

If $F = (F_p)$ and $G = (G_q)$ are two families of functions we say that F *dominates* G and write $G \prec F$ if

$$\forall q\ \exists p, C : \quad G_q(t) \le CF_p(t), \quad t \ge 1.$$

The families F and G are said to be *equivalent* if $F \prec G$ and $G \prec F$. Obviously, if F and G are equivalent then $K_F(\alpha) \simeq K_G(\alpha)$.

It is easy to see that for every sequence of functions with (3.1) and (3.2) one may obtain (by using appropriate multipliers and passing to a subsequence) an equivalent family satisfying the following conditions:

$$\forall p : \quad 1 \le tF_p(t) \le F_{p+1}(t), \quad t \ge 1, \tag{3.3}$$

$$\forall p, L\ \exists p_2 : \quad F_p(F_L(t)) \le F_{p_2}(t), \qquad t \ge 1. \tag{3.4}$$

Since in the following we are interested in some relations involving F that are invariant under equivalence we may think that (3.3) and (3.4) hold.

Remark. We can always think, if necessary, that the family $F = (F_p)$ is defined for $p \in [1,\infty)$ by setting in case p is not a whole number

$$F_p(t) = F_{[p]}^{1-\alpha}(t) \cdot F_{[p]+1}^{\alpha}(t),$$

where $[p]$ means the integer part of p and $\alpha = p - [p]$. Let us mention, however, that in the most interesting examples F_p is "naturally" defined for $p \in [1,\infty)$. For example, consider the family of power functions $F_p(t) = t^p$.

2. *Characterization of* $L(K_F(\alpha), K(a)) = LB(K_F(\alpha), K(a))$. The space $K_F(\alpha)$ is called *shift-stable* if

$$\forall p\ \exists p_1, \widetilde{j} : \quad F_p(\alpha_{j+1}) \le F_{p_1}(\alpha_j), \quad j \ge \widetilde{j}. \tag{3.5}$$

We say that a Köthe space $K(a)$ has *property* $LB_F(K)$ and write $K(a) \in LB_F(K)$ if

$$\forall \tau(p)\uparrow\infty\ \exists k\ \forall p_0\ \exists P_0, D > 0\ \forall i\ \exists p \in (p_0, P_0) : \tag{3.6}$$

$$F_{\tau(p)}\left(\frac{a_{ip_0}}{a_{ik}}\right) \le D\,\frac{a_{ip}}{a_{ip_0}}.$$

Theorem 4. *If $K_F(\alpha)$ is shift-stable, then the following conditions are equivalent*:

(i) *Each continuous linear operator from $K_F(\alpha)$ into $K(a)$ is bounded.*
(ii) $K(a) \in LB_F(K)$.

Proof. One can easily see that if $F \prec G$ then $LB_G(K) \Rightarrow LB_F(K)$. Therefore if F and G are equivalent then properties $LB_F(K)$ and $LB_G(K)$ are also equivalent, so we can assume that (3.3) and (3.4) hold.

By Theorem 2 condition (i) is equivalent to

$$\forall \sigma(p)\uparrow\infty\ \exists k\ \forall p_0\ \exists P_0, C: \quad \frac{a_{ip_0}}{F_{\sigma(k)}(\alpha_j)} \le C \max_{1\le p\le P_0} \frac{a_{ip}}{F_{\sigma(p)}(\alpha_j)}. \tag{3.7}$$

So, it is enough to prove that (3.6) and (3.7) are equivalent. Observe that we may consider in (3.7) only "large" indices j such that $F_1(\alpha_j) \ge 1$.

(3.7)⇒(3.6). Fix $\tau(p) \uparrow \infty$ and choose $\sigma(p) \uparrow \infty$ so that

$$\forall L\ \exists p_L: \quad F_L(t) \cdot F_{\tau(p)}(F_L(t)) \le F_{\sigma(p)}(t) \quad \forall p \ge p_L. \tag{3.8}$$

Such a choice is possible. Indeed, by the growth conditions (3.3) and (3.4),

$$\forall L\ \exists \tau_1(p) \uparrow \infty, \sigma_L(p) \uparrow \infty:$$
$$F_L(t) \cdot F_{\tau(p)}(F_L(t)) \le F_{\tau_1(p)}(F_L(t)) \le F_{\sigma_L(p)}(t) \quad \forall p.$$

In addition, we may assume that if $L_1 < L_2$ then $\sigma_{L_1}(p) \le \sigma_{L_2}(p)$. Take $\sigma(p) = \sigma_p(p)$; then

$$\forall L: \quad F_{\sigma_L(p)}(t) \le F_{\sigma(p)}(t) \quad \text{for } p \ge L.$$

Now for $\sigma(p)$ as in (3.8) there exists a k such that (3.7) holds. We shall show that (3.6) holds with that k. By shift stability (3.5),

$$\exists L_1, \widetilde{j}_1: \quad F_{\sigma(k)}(\alpha_{j+1}) \le F_{L_1}(\alpha_j) \quad \text{for } j \ge \widetilde{j}_1. \tag{3.9}$$

Fix by (3.3) a constant L so that

$$tF_{L_1}(t) \le F_L(t), \quad t > 1. \tag{3.10}$$

Obviously, it is enough to prove (3.6) for "large" p_0. Fix a $p_0 > \max(k, p_L)$, where p_L is the constant from (3.8); then (3.7) holds with some P_0 and $C > 1$. Choose $\widetilde{j} > \widetilde{j}_1$ so that $c_{\widetilde{j}} > C$. Then from (3.9) and (3.10) it follows that

$$CF_{\sigma(k)}(\alpha_{j+1}) \le F_L(\alpha_j) \quad \text{for } j \ge \widetilde{j}. \tag{3.11}$$

Observe that there exists a j_0 such that if $j \ge j_0$ then the maximum in (3.7) occurs for $p \in [1, k] \cup (p_0, P_0]$. Indeed, otherwise there exist sequences $(j_\nu), (i_\nu), (p_\nu)$ with $p_\nu \in (k, p_0]$ and $j_\nu \to \infty$ such that (in view of (3.3))

$$1 \le \frac{a_{i_\nu p_0}}{a_{i_\nu p_\nu}} \le \frac{F_{\sigma(k)}(\alpha_{j_\nu})}{F_{\sigma(p_\nu)}(\alpha_{j_\nu})} \to 0 \quad \text{as } \nu \to \infty,$$

which is impossible.

Choose a j_0 with the above property so that $j_0 > \widetilde{j}$ and $F_1(\alpha_{j_0}) \geq 1$. Fix an i; then one of the following two cases can occur:

Case 1: $a_{ip_0}/a_{ik} \leq CF_{\sigma(k)}(\alpha_{j_0})$. Then

$$F_{\tau(p)}\left(\frac{a_{ip_0}}{a_{ik}}\right) \leq F_{\tau(p)}(CF_{\sigma(k)}(\alpha_{j_0}))$$

and (3.6) holds with

$$D = \max_{p_0 \leq p \leq P_0} F_{\tau(p)}(CF_{\sigma(k)}(\alpha_{j_0})).$$

Case 2: $a_{ip_0}/a_{ik} > CF_{\sigma(k)}(\alpha_{j_0})$. Then, since $F_{\sigma(k)}(\alpha_j) \uparrow \infty$, there exists $j \geq j_0$ such that

$$CF_{\sigma(k)}(\alpha_j) < a_{ip_0}/a_{ik} \leq CF_{\sigma(k)}(\alpha_{j+1}). \tag{3.12}$$

Observe that the maximum in (3.7) does not occur for $p \in [1, k]$ because otherwise

$$CF_{\sigma(k)}(\alpha_j) < \frac{a_{ip_0}}{a_{ik}} \leq \frac{a_{ip_0}}{a_{ip}} \leq C\,\frac{F_{\sigma(k)}(\alpha_j)}{F_{\sigma(p)}(\alpha_j)} < CF_{\sigma(k)}(\alpha_j),$$

which is impossible. Therefore the maximum occurs for some $p \in (p_0, P_0]$, that is,

$$\frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \leq C\,\frac{a_{ip}}{a_{ip_0}}. \tag{3.13}$$

By (3.11) the right inequality in (3.12) implies

$$\frac{a_{ip_0}}{a_{ik}} \leq F_L(\alpha_j). \tag{3.14}$$

Therefore (since $j > j_0 > \widetilde{j}$), from (3.8), (3.11) and (3.13) it follows that

$$F_{\tau(p)}\left(\frac{a_{ip_0}}{a_{ik}}\right) \leq F_{\tau(p)}(F_L(\alpha_j)) \leq \frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \leq C\,\frac{a_{ip}}{a_{ip_0}},$$

hence (3.6) holds with $D = C$.

(3.6)⇒(3.7). Take $\tau(\cdot) = \sigma(\cdot)$; then there exists a k such that (3.6) holds. Choose $\widetilde{k}$ by (3.3) so that

$$tF_{\sigma(k)}(t) \leq F_{\sigma(\widetilde{k})}(t), \qquad t > 1.$$

Fix any p_0 and choose $P_0 \geq k$ and D so that (3.6) holds; then (3.7) also holds with $\widetilde{k}, p_0, P_0$. Indeed, for every pair (i, j) one of the following two cases can occur:

$$\text{(a)}\ \frac{F_{\sigma(\widetilde{k})}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} > \frac{a_{ip_0}}{a_{ik}}; \qquad \text{(b)}\ \frac{F_{\sigma(\widetilde{k})}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \leq \frac{a_{ip_0}}{a_{ik}}.$$

In case (a),

$$\frac{a_{ip_0}}{F_{\sigma(\widetilde{k})}(\alpha_j)} < \frac{a_{ik}}{F_{\sigma(k)}(\alpha_j)} \le \max_{1\le p\le P_0} \frac{a_{ip}}{F_{\sigma(p)}(\alpha_j)},$$

thus (3.7) holds with $C=1$.

If (b) occurs we have, by the choice of $\widetilde{k}$,

$$\alpha_j \le \frac{F_{\sigma(\widetilde{k})}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \le \frac{a_{ip_0}}{a_{ik}}.$$

So from (3.6) it follows that, for some $p\in[p_0,P_0]$,

$$\frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(\widetilde{k})}(\alpha_j)} \le F_{\sigma(p)}(\alpha_j) \le F_{\sigma(p)}\left(\frac{a_{ip_0}}{a_{ik}}\right) \le D\frac{a_{ip}}{a_{ip_0}},$$

hence (3.7) holds with $C=D$. ∎

3. Let $F=(F_n)$ and $G=(G_m)$ be two families of increasing functions that satisfy the growth conditions (3.1) and (3.2). Krone [10] showed (assuming nuclearity) that if two spaces of the kind $K_F(\alpha)$ and $K_G(\beta)$ have no common basic subspace then the relations $(K_F(\alpha),K_G(\beta))\in S$ and $L(K_G(\beta),K_F(\alpha)) = LB(K_G(\beta),K_F(\alpha))$ are equivalent. Using Theorem 4 we provide another approach to Krone's result that allows us to remove the nuclearity assumption.

We say that a Köthe space $K(a)$ has *property* $DN_F(K)$ and write $K(a)\in DN_F(K)$ if

$$\exists p_0\ \forall p_1\ \forall F_n\ \exists p_2, C_1>0: \quad F_n\left(\frac{a_{ip_1}}{a_{ip_0}}\right) \le C_1\,\frac{a_{ip_2}}{a_{ip_1}}. \tag{3.15}$$

It is easy to see by (3.1) and (3.2) that $K_F(\alpha)\in DN_F(K)$ and that

$$LB_F(K) \ \Rightarrow\ DN_F(K). \tag{3.16}$$

We say that a Köthe space $K(b)$ has *property* $\Omega_G(K)$ and write $K(b)\in \Omega_G(K)$ if

$$\forall q_0\ \exists q_1\ \forall q_2\ \exists G_m, C_2>0: \quad \frac{b_{jq_2}}{b_{jq_1}} \le C_2 G_m\left(\frac{b_{jq_1}}{b_{jq_0}}\right). \tag{3.17}$$

It is easy to see by (3.1) and (3.2) that $K_G(\beta)\in\Omega_G(K)$.

PROPOSITION 5. *If $K(a)\in DN_F(K)$ and $K(b)\in\Omega_G(K)$, where $G\prec F$, then $(K(a),K(b))\in S$.*

Proof. We are to prove

$$\forall q_0\ \exists p_0,q_1\ \forall p_1,q_2\ \exists p_2,C: \quad \frac{a_{ip_1}}{b_{jq_1}} \le C\max\left(\frac{a_{ip_0}}{b_{jq_0}},\ \frac{a_{ip_2}}{b_{jq_2}}\right). \tag{3.18}$$

Fix an arbitrary index q_0; then choose p_0 from (3.15) and q_1 from (3.17); after that fix arbitrary indices p_1,q_2.

Consider an arbitrary pair (i,j). If $a_{ip_1}/b_{jq_1} \le a_{ip_0}/b_{jq_0}$, then (3.18) holds with $C=1$. Otherwise we have

$$\frac{b_{jq_1}}{b_{jq_0}} < \frac{a_{ip_1}}{a_{ip_0}} \cdot$$

As $K(b) \in \Omega_G(K)$ there exists a G_m such that (3.17) holds. Since the family F dominates G there exists an n such that $G_m(t) \le C_3 F_n(t)$, and since $K(a) \in DN_F$ there exists a p_2 such that (3.15) holds. Therefore

$$\frac{1}{C_2}\frac{b_{jq_2}}{b_{jq_1}} \le G_m\left(\frac{b_{jq_1}}{b_{jq_0}}\right) \le G_m\left(\frac{a_{ip_1}}{a_{ip_0}}\right) \le C_3 F_n\left(\frac{a_{ip_1}}{a_{ip_0}}\right) \le C_3 C_1 \frac{a_{ip_2}}{a_{ip_1}}.$$

Hence $a_{ip_1}/b_{jq_1} \le C a_{ip_2}/b_{jq_2}$ with $C = C_1C_2C_3$, that is, (3.18) holds. ■

Remark. Observe that the assertion of the proposition holds when $F=G$ since each family dominates itself. Moreover, it was enough to prove the proposition only when $F=G$ because if $G \prec F$ then $K(a) \in DN_F(K)$ implies $K(a) \in DN_G(K)$.

Proposition 6. *If we have* $L(K_G(\beta), K_F(\alpha)) = LB(K_G(\beta), K_F(\alpha))$, *then* $(K_F(\alpha), K_G(\beta)) \in S$.

Proof. If each continuous linear operator from $K_G(\beta)$ into $K_F(\alpha)$ is bounded then by Theorem 4 the space $K_F(\alpha)$ has property $LB_G(K)$, so $K_F(\alpha) \in DN_G(K)$ by (3.16). Now $(K_F(\alpha), K_G(\beta)) \in S$ by Proposition 5 because the first space satisfies $DN_G(K)$ and the second satisfies $\Omega_G(K)$. ■

Krone [10] obtained Proposition 6 with the additional assumption of nuclearity. Combining Propositions 6 and 3 results in the following statement (also due to Krone [10] in the nuclear case):

Proposition 7. *If the spaces* $K_F(\alpha)$ *and* $K_G(\beta)$ *have no common basic subspaces then*

$$L(K_G(\beta), K_F(\alpha)) = LB(K_G(\beta), K_F(\alpha))$$

if and only if

$$(K_F(\alpha), K_G(\beta)) \in S.$$

4. *Characterization of* $L(K(a), K_F(\alpha)) = LB(K(a), K_F(\alpha))$. We say that a Köthe space $K(a)$ has *property* $LB^F(K)$ and write $K(a) \in LB^F(K)$ if

$$\forall \tau(p) \uparrow \infty\ \forall p_1\ \exists k\ \forall p_0\ \exists P_0, D>0\ \forall i\ \exists p \in (p_0, P_0) : \tag{3.19}$$
$$\frac{a_{ip}}{a_{ik}} \le D F_{\tau(p)}\left(\frac{a_{ik}}{a_{ip_1}}\right).$$

Theorem 8. *If* $K_F(\alpha)$ *is shift-stable, then the following conditions are equivalent*:

(i) $L(K(a), K_F(\alpha)) = LB(K(a), K_F(\alpha))$.
(ii) $K(a) \in LB^F(K)$.

Proof. We may assume that the family F satisfies the growth conditions (3.3) and (3.4) because (i) and (ii) are invariant conditions and would not change if we replace F by an equivalent family.

(i)⇒(ii). By Theorem 2 condition (i) is equivalent to

$$\forall \sigma(p)\uparrow\infty \ \exists k \ \forall p_0 \ \exists P_0, C > 1 : \quad \frac{F_{p_0}(\alpha_j)}{a_{i\sigma(k)}} \le C \max_{1\le p\le P_0} \frac{F_p(\alpha_j)}{a_{i\sigma(p)}}. \tag{3.20}$$

It is easy to see that (3.20) is equivalent to

$$\forall \tau(p) \uparrow \infty \ \exists k \ \forall p_0 \ \exists P_0, C > 1 : \quad \frac{F_{\tau(p_0)}(\alpha_j)}{a_{ik}} \le C \max_{1\le p\le P_0} \frac{F_{\tau(p)}(\alpha_j)}{a_{ip}}. \tag{3.21}$$

Therefore it is enough to prove that (3.21) implies (3.19) with the same $\tau(\cdot)$. By shift stability and the growth conditions (3.3) and (3.4) there exist $L > 0$ and $\widetilde{j} > 1$ such that

$$\alpha_j \le F_R(\alpha_j) \le F_L(\alpha_{j-1}), \qquad j \ge \widetilde{j}. \tag{3.22}$$

We can always think that $\widetilde{j}$ is so large that $\alpha_{j-1} > 1$ for $j \ge \widetilde{j}$.

It is enough to show that (3.19) holds with $p_1 = 1$ and $k > 1$ coming from (3.21). One can easily see by (3.3) and (3.4) that there exists $\widetilde{p} > k$ such that

$$C F_L(t) F_{\tau(k-1)}(t) \le F_{\tau(\widetilde{p})}(t), \qquad t > 1, \tag{3.23}$$

where C is the constant from (3.21).

It is enough to prove (3.19) for "large" p_0 such that $\tau(p_0-1) < \tau(p_0)$. Fix $p_0 > \widetilde{p}$. Next choose a $j_0 > \widetilde{j}$ so that if $j \ge j_0$ then the maximum in (3.21) occurs for $p \in [1, k-1] \cup [p_0, P_0]$. Such an index j_0 exists because otherwise there would exist sequences $(j_\nu), (i_\nu), (p_\nu)$ such that $j_\nu \uparrow \infty$, $p_\nu \in [k, p_0)$, and

$$1 \le \frac{a_{i_\nu p_\nu}}{a_{i_\nu k}} \le C \frac{F_{\tau(p_\nu)}(\alpha_{j_\nu})}{F_{\tau(p_0)}(\alpha_{j_\nu})} \to 0,$$

which is impossible.

If for some j the maximum in (3.21) occurs for $p \in [1, k-1]$ then

$$\frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(k-1)}(\alpha_j)} \le \frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(p)}(\alpha_j)} \le C \frac{a_{ik}}{a_{ip}} \le C \frac{a_{ik}}{a_{i1}}. \tag{3.24}$$

Therefore, if for some $j \ge j_0$ we have

$$\frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(k-1)}(\alpha_j)} > C \frac{a_{ik}}{a_{i1}} \tag{3.25}$$

then the maximum in (3.21) occurs for $p \in [p_0, P_0]$.

Fix an i; then we have the following two cases:

Case 1. The inequality (3.25) holds with $j = j_0$. Then the maximum in (3.21) (with $j = j_0$ and the fixed i) occurs for some $p \in [p_0, P_0]$, so

$$\frac{a_{ip}}{a_{ik}} \le C\,\frac{F_{\tau(p)}(\alpha_{j_0})}{F_{\tau(p_0)}(\alpha_{j_0})} \le C\overline{C} \le C\overline{C}F_{\tau(p)}\left(\frac{a_{ik}}{a_{i1}}\right),$$

where

$$\overline{C} = \max_{p_0 \le p \le P_0} \frac{F_{\tau(p)}(\alpha_{j_0})}{F_{\tau(p_0)}(\alpha_{j_0})}.$$

Thus (3.19) holds with $D = C\overline{C}$.

Case 2. If (3.25) fails for $j = j_0$ then (since $F_{\tau(p_0)}(\alpha_j)/F_{\tau(k-1)}(\alpha_j) \to \infty$ as $j \to \infty$) there exists $j > j_0$ such that

$$\frac{F_{\tau(p_0)}(\alpha_{j-1})}{F_{\tau(k-1)}(\alpha_{j-1})} \le C\,\frac{a_{ik}}{a_{i1}} < \frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(k-1)}(\alpha_j)}. \tag{3.26}$$

Since the right inequality in (3.26) coincides with (3.25) the maximum in (3.21) occurs for some $p \in [p_0, P_0]$.

On the other hand by (3.22), (3.23) and the left inequality in (3.26) we obtain (since $j > j_0 \ge \widetilde{j}$)

$$\alpha_j \le \frac{a_{ik}}{a_{i1}},$$

therefore from (3.21) it follows that for some $p \in [p_0, P_0]$,

$$\frac{a_{ip}}{a_{ik}} \le C\,\frac{F_{\tau(p)}(\alpha_j)}{F_{\tau(p_0)}(\alpha_j)} \le CF_{\tau(p)}(\alpha_j) \le CF_{\tau(p)}\left(\frac{a_{ik}}{a_{i1}}\right),$$

so (3.19) holds with $D = C$.

In order to prove that (ii)⇒(i) we shall check that (3.19) implies

$$\forall \sigma(p) \uparrow \infty\ \exists k\ \forall p_0\ \exists P_0, C > 1 : \quad \frac{F_{\sigma(p_0)}(\alpha_j)}{a_{ik}} \le C \max_{1 \le p \le P_0} \frac{F_{\sigma(p)}(\alpha_j)}{a_{ip}}. \tag{3.27}$$

Fix $\sigma(p) \uparrow \infty$ and take an arbitrary p_0. Then choose $\tau(p)$ so that

$$F_{\sigma(p_0)}(t) \cdot F_{\tau(p)}[F_{\sigma(p_0)}(t)] \le F_{\sigma(p)}(t), \quad t > 1. \tag{3.28}$$

Such a choice is possible. In order to see that, we can regard $\sigma(\cdot)$ and $\tau(\cdot)$ as bijections mapping the interval $[1, \infty)$ into itself. We can also think that the family $f = (F_p)$ is defined for the "continuous" parameter $p \in [1, \infty)$ so that the growth conditions (3.3) and (3.4) hold.

Now put $L = \sigma(p_0)$ and $q = \tau(p)$. Then we have to find τ such that

$$F_L(t) \cdot F_q(F_L(t)) \le F_{\sigma(\tau^{-1}(q))}(t).$$

By (3.3) and (3.4) there exists $\mu(q)$ such that

$$F_L(t) \cdot F_q(F_L(t)) \le F_{\mu(q)}(t),$$

so it is enough to have

$$F_{\mu(q)}(t) \le F_{\sigma(\tau^{-1}(q))}(t).$$

That inequality will hold if $\mu(q) \le \sigma(\tau^{-1}(q))$, or equivalently $\sigma^{-1}(\mu(q)) \le \tau^{-1}(q)$. Thus, choosing τ^{-1} so that the latter condition holds we obtain (3.28).

Fix a pair (i,j). Either

$$a_{ik}/a_{i1} > F_{\sigma(p_0)}(\alpha_j)/F_{\sigma(1)}(\alpha_j),$$

then obviously (3.27) holds, or

$$a_{ik}/a_{i1} \le F_{\sigma(p_0)}(\alpha_j)/F_{\sigma(1)}(\alpha_j).$$

Then by (3.19) with $p_1 = 1$ and (3.28) we obtain, with some $p \in [p_0, P_0]$,

$$\frac{a_{ip}}{a_{ik}} \le DF_{\tau(p)}[F_{\sigma(p_0)}(\alpha_j)] \le D\,\frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(p_0)}(\alpha_j)},$$

that is, (3.27) holds with $C = D$. ∎

4. Examples and comments

1. Consider the family of power functions

$$F_p(t) = t^p, \quad p \ge 1.$$

Obviously the growth conditions (3.3) and (3.4) hold. For each sequence $\alpha = (\alpha_j)$, $\alpha_j > 0$, $\alpha_j \to \infty$, we have

$$a_{jp} = F_p(\alpha_j) = \alpha_j^p = e^{p\beta_j}, \qquad \beta_j = \log \alpha_j,$$

so the corresponding Köthe space $K_F(\alpha)$ coincides with an infinite type power series space, namely

$$K_F(\alpha) = \Lambda_\infty(\beta), \quad \beta = (\beta_j).$$

2. More generally, let $f : (0,\infty) \to (0,\infty)$ be a continuous strictly increasing function; then $f^{-1} : (0,\infty) \to (0,\infty)$ is also continuous and strictly increasing. Consider the family

$$F_p(t) = \exp\circ f \circ p \circ f^{-1} \circ \log(t), \qquad p \ge 1,\ t > 1.$$

(One can define $F_p(t)$ for $t \in (0,1]$ in an arbitrary way because the values of $F_p(t)$ on that interval determine only the norms on some finite-dimensional subspace.)

Since $F_{p_1} \circ F_{p_2} = F_{p_1 p_2}$, the growth condition (3.4) holds. It is easy to see that (3.3) would hold if the function f satisfies

$$\exists c > 1, p > 1 : \quad cf(t) \le f(pt). \tag{4.1}$$

Indeed, (4.1) implies immediately

$$\exists \widetilde{p} : \quad 2f(t) \le f(\widetilde{p}t), \tag{4.2}$$

thus

$$F_{\widetilde{p}}(t) = \exp\{f[\widetilde{p}f^{-1}(\log t)]\} \geq \exp\{2f[f^{-1}(\log t)]\} = t^2.$$

Since $F_1(t) = t$ we obtain, for any p,

$$tF_p(t) = F_1(t)F_p(t) \leq (F_p(t))^2 \leq F_{\widetilde{p}}(F_p(t)) = F_{p\widetilde{p}}(t),$$

so (3.3) holds.

For each sequence $\alpha = (\alpha_j)$, $\alpha_j > 1$, $\alpha_j \to \infty$, we have

$$a_{jp} = F_p(\alpha_j) = e^{f(p\beta_j)}, \quad \beta_j = f^{-1}(\log \alpha_j),$$

so $K_F(\alpha)$ coincides with the Dragilev space of infinite type generated by the function f and the sequence β. Usually in the definition of Dragilev spaces it is supposed that f is a logarithmically convex function (that is, $\log f(e^x)$ is a convex function). Then condition (4.1) holds because the logarithmic convexity of f implies that $f(pt)/f(t)$ is an increasing function of t.

3. For each family of increasing functions $\Phi = (\varphi_k(t))$, $t > 0$, such that

$$t^2 \leq \varphi_1(t) \leq \varphi_2(t) \leq \ldots, \quad t \geq t_0, \tag{4.3}$$

and each sequence $\alpha = (\alpha_i)$ of positive numbers with $\alpha_i \uparrow \infty$ Krone [10] considers the Köthe space $\Lambda_\Phi(\alpha) = K(a_{ip})$, where

$$a_{i1} = \varphi_1(\alpha_i), \quad a_{ip} = \varphi_p(a_{i,p-1}), \quad p > 1.$$

It is easy to see that the construction of the spaces $K_F(\alpha)$ is equivalent to Krone's construction in the following sense:

(a) Set

$$F_p = \varphi_p \circ \varphi_{p-1} \circ \ldots \circ \varphi_1;$$

then the growth conditions (3.1) and (3.2) hold and $K_F(\alpha) = \Lambda_\Phi(\alpha)$.

(b) Conversely, if F is a family of functions such that the growth conditions (3.1) and (3.2) hold, then there exists a subsequence (p_k) such that the family Φ of functions $\varphi_1 = F_{p_1}$, $\varphi_k = F_{p_k} \circ F^{-1}_{p_{k-1}}$, $k > 1$, satisfies (4.3). Obviously the Köthe spaces $\Lambda_\Phi(\alpha)$ and $K_F(\alpha)$ are isomorphic.

4. In the previous sections we consider only Köthe spaces, but many of the theorems proved there have "versions" for general Fréchet spaces. Of course, the theorems for Köthe spaces are formulated in terms of Köthe matrices, while the corresponding claims for Fréchet spaces have to be formulated in terms of seminorms. In general, these Fréchet space theorems do not generalize the corresponding Köthe space versions, because a condition given in terms of a Köthe matrix is less restrictive than the corresponding condition formulated in terms of seminorms of arbitrary elements. Moreover, it often seems easier to prove separately a Köthe space version than to derive it from the corresponding Fréchet space theorem.

For example, Theorem 4 characterizes the Köthe spaces $K(a)$ such that

$$L(K_F(\alpha), K(a)) = LB(K_F(\alpha), K(a)).$$

In an analogous way it is possible to characterize the Fréchet spaces X with

$$L(K_F(\alpha), X) = LB(K_F(\alpha), X).$$

We say that a Fréchet space X has *property* LB_F and write $X \in LB_F$ if

$$\text{(4.4)} \qquad \forall \tau(p) \uparrow \infty \ \exists k \ \forall p_0 \ \exists P_0, D > 0 \ \forall x \in X \ \exists p \in (p_0, P_0): \quad F_{\tau(p)}\left(\frac{\|x\|_{p_0}}{\|x\|_k}\right) \le D \, \frac{\|x\|_p}{\|x\|_{p_0}}.$$

In case F is the family of power functions $F_p(t) = t^p$ property LB_F coincides with property LB_∞ introduced by Vogt [17]. Our next theorem extends Theorem 3.2 of [17].

THEOREM 9. *If* $K_F(\alpha)$ *is shift-stable*, *then the following conditions are equivalent*:

(i) *Each linear continuous operator from* $K_F(\alpha)$ *into* X *is bounded.*
(ii) $X \in LB_F$.

Proof. By Proposition 1.3 of [17] condition (i) is equivalent to

$$\text{(iii)} \ \forall \sigma(p) \uparrow \infty \ \exists k \ \forall p_0 \ \exists P_0, C: \quad \frac{\|x\|_{p_0}}{F_{\sigma(k)}(\alpha_j)} \le C \max_{1 \le p \le P_0} \frac{\|x\|_p}{F_{\sigma(p)}(\alpha_j)}.$$

So, it is enough to prove that (ii) and (iii) are equivalent. As in the proof of Theorem 4 we may assume that F satisfies the growth conditions (3.3) and (3.4), and it is enough to consider in (iii) only indices j such that $F_1(\alpha_j) \ge 1$.

(iii)⇒(ii). Fix a sequence $\tau(p) \uparrow \infty$. There are several steps repeating parts of the proof of Theorem 4:

Step 1. Choose $\sigma(p) \uparrow \infty$ so that

$$\text{(4.5)} \qquad \forall L \ \exists p_L: \quad F_L(t) \cdot F_{\tau(p)}(F_L(t)) \le F_{\sigma(p)}(t) \quad \forall p \ge p_L.$$

Step 2. Choose $L > 0$ and $\widetilde{j}_1$ (see (3.9) and (3.10)) so that

$$\text{(4.6)} \qquad \alpha_j F_{\sigma(k)}(\alpha_{j+1}) \le F_L(\alpha_j) \quad \text{for } j \ge \widetilde{j}_1.$$

Obviously, it is enough to prove (4.4) for “large” p_0. Fix $p_0 > \max(k, p_L)$; then (iii) holds with some P_0 and $C > 1$. Choose $\widetilde{j}$ so that $\alpha_{\widetilde{j}} > C$. Then

$$\text{(4.7)} \qquad C F_{\sigma(k)}(\alpha_{j+1}) \le F_L(\alpha_j) \quad \text{for} \quad j \ge \widetilde{j}.$$

Step 3. Choose a $j_0 \ge \widetilde{j}$ so that for $j \ge j_0$ the maximum in (iii) occurs for $p \in [1, k] \cup (p_0, P_0]$. Fix an $x \in F$; then one of the following two cases can occur:

Case 1: $\|x\|_{p_0}/\|x\|_k \le C F_{\sigma(k)}(\alpha_{j_0})$. Then

$$F_{\tau(p)}(\|x\|_{p_0}/\|x\|_k) \le F_{\tau(p)}(C F_{\sigma(k)}(\alpha_{j_0}))$$

and (4.4) holds with

$$D = \max_{p_0 \le p \le P_0} F_{\tau(p)}(C F_{\sigma(k)}(\alpha_{j_0})).$$

Case 2: $\|x\|_{p_0}/\|x\|_k > CF_{\sigma(k)}(\alpha_{j_0})$. Then, since $F_{\sigma(k)}(\alpha_j) \uparrow \infty$, there exists $j > j_0$ such that

$$CF_{\sigma(k)}(\alpha_j) < \|x\|_{p_0}/\|x\|_k \le CF_{\sigma(k)}(\alpha_{j+1}). \tag{4.8}$$

Observe that the maximum in (iii) does not occur for $p \in [1, k]$ because otherwise

$$CF_{\sigma(k)}(\alpha_j) < \frac{\|x\|_{p_0}}{\|x\|_k} \le \frac{\|x\|_{p_0}}{\|x\|_p} \le C\,\frac{F_{\sigma(k)}(\alpha_j)}{F_{\sigma(p)}(\alpha_j)} < CF_{\sigma(k)}(\alpha_j),$$

which is impossible. Therefore the maximum occurs for some $p \in (p_0, P_0]$, that is,

$$\frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \le C\,\frac{\|x\|_p}{\|x\|_{p_0}}. \tag{4.9}$$

By (4.7) the right inequality in (4.8) implies

$$\frac{\|x\|_{p_0}}{\|x\|_k} \le F_L(\alpha_j). \tag{4.10}$$

Therefore (since $j \ge j_0 > \widetilde{j}$), from (4.5), (4.7) and (4.9) it follows that

$$F_{\tau(p)}\left(\frac{\|x\|_{p_0}}{\|x\|_k}\right) \le F_{\tau(p)}(F_L(\alpha_j)) \le \frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \le C\,\frac{\|x\|_p}{\|x\|_{p_0}},$$

hence (4.4) holds with $D = C$.

(ii)⇒(iii). Take $\tau(\cdot) = \sigma(\cdot)$; then there exists a k such that (4.4) holds. Choose $\widetilde{k}$ by (3.3) so that

$$tF_{\sigma(k)}(t) \le F_{\sigma(\widetilde{k})}(t), \quad t > 1.$$

Fix any p_0 and choose P_0 and D so that (4.4) holds. We shall show that (iii) holds with $\widetilde{k}, p_0, P_0$. Indeed, for every pair (x, j) one of the following two cases can occur:

$$\text{(a)}\ \frac{F_{\sigma(\widetilde{k})}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} > \frac{\|x\|_{p_0}}{\|x\|_k}; \qquad \text{(b)}\ \frac{F_{\sigma(\widetilde{k})}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \le \frac{\|x\|_{p_0}}{\|x\|_k}.$$

In case (a),

$$\frac{\|x\|_{p_0}}{F_{\sigma(\widetilde{k})}(\alpha_j)} < \frac{\|x\|_k}{F_{\sigma(k)}(\alpha_j)} \le \max_{1\le p\le P_0} \frac{\|x\|_p}{F_{\sigma(p)}(\alpha_j)},$$

thus (iii) holds with $C = 1$.

If (b) occurs we have, by the choice of $\widetilde{k}$,

$$\alpha_j \le \frac{F_{\sigma(\widetilde{k})}(\alpha_j)}{F_{\sigma(k)}(\alpha_j)} \le \frac{\|x\|_{p_0}}{\|x\|_k}.$$

Therefore from (4.4) it follows that for some $p \in [p_0, P_0]$,

$$\frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(\widetilde{k})}(\alpha_j)} \le F_{\sigma(p)}(\alpha_j) \le F_{\sigma(p)}\left(\frac{\|x\|_{p_0}}{\|x\|_k}\right) \le D\,\frac{\|x\|_p}{\|x\|_{p_0}},$$

hence (iii) holds with $C = D$. ∎

5. Theorem 8 gives a characterization of Köthe spaces with

$$L(K(a), K_F(\alpha)) = LB(K(a), K_F(\alpha)).$$

In order to obtain a similar result for Fréchet spaces we say that a Fréchet space X has *property* LB^F and write $X \in LB^F$ if

$$\forall \tau(p) \uparrow \infty\ \forall p_1\ \exists k\ \forall p_0\ \exists P_0, D > 0\ \forall y \in X'\ \exists p \in (p_0, P_0): \quad \|y\|_k^*/\|y\|_p^* \le DF_{\tau(p)}(\|y\|_{p_1}^*/\|y\|_k^*). \tag{4.11}$$

In case F is the family of power functions $F_p(t) = t^p$ property LB^F coincides with property LB^∞ introduced by Vogt [17]. Our next theorem extends Theorem 5.2 of [17].

Let $K_F^\infty(\alpha)$ denote the ℓ_∞-Köthe space defined by a matrix $a_{jp} = F_p(\alpha_j)$, that is,

$$K_F^\infty(\alpha) = \{x = (x_j) : |x|_p = \sup_j a_{jp}|x_j| < \infty\ \forall p\}.$$

THEOREM 10. *If $K_F(\alpha)$ is shift-stable and nuclear, then the following conditions are equivalent*:

(i) $L(X, K_F^\infty(\alpha)) = LB(X, K_F^\infty(\alpha))$.
(ii) $X \in LB^F$.

Proof. As in the proof of Theorem 8 we may assume that F satisfies the growth conditions (3.3) and (3.4).

(i)⇒(ii). By Proposition 1.4 of [17] condition (i) is equivalent to

$$\forall \sigma(p) \uparrow \infty\ \exists k\ \forall p_0\ \exists P_0, C > 1: \quad F_{p_0}(\alpha_j)\|y\|_{\sigma(k)}^* \le C \max_{1\le p\le P_0} F_p(\alpha_j)\|y\|_{\sigma(p)}^*. \tag{4.12}$$

It is easy to see that (4.12) is equivalent to

$$\forall \tau(p) \uparrow \infty\ \exists k\ \forall p_0\ \exists P_0, C > 1: \quad F_{\tau(p_0)}(\alpha_j)\|y\|_k^* \le C \max_{1\le p\le P_0} F_{\tau(p)}(\alpha_j)\|y\|_p^*. \tag{4.13}$$

We show that (4.13) implies (4.11) with the same $\tau(\cdot)$.

As in the proof of Theorem 8 we can choose L and $\widetilde{j}$ so that (3.22) holds. Now, if $p_1 = 1$ and $k > 1$ comes from (4.13) then there exists $\widetilde{p} > k$ such that (3.23) holds with the constant C from (4.13).

It is enough to check (4.11) for "large" p_0. Fix $p_0 > \widetilde{p}$. Next choose a $j_0 > \widetilde{j}$ such that if $j \geq j_0$ then the maximum in (4.13) occurs for $p \in [1, k-1] \cup [p_0, P_0]$. Such an index j_0 exists, because otherwise there would exist sequences $(j_\nu), (y_\nu), (p_\nu)$ such that $j_\nu \uparrow \infty$, $p_\nu \in [k, p_0)$, and

$$1 \leq \frac{\|y_\nu\|_k^*}{\|y_\nu\|_{p_\nu}^*} \leq C\, \frac{F_{\tau(p_\nu)}(\alpha_{j_\nu})}{F_{\tau(p_0)}(\alpha_{j_\nu})} \to 0,$$

which is impossible.

If for some j the maximum in (4.13) occurs for $p \in [1, k-1]$ then

$$\frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(k-1)}(\alpha_j)} \leq \frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(p)}(\alpha_j)} \leq C\, \frac{\|y\|_p^*}{\|y\|_k^*} \leq C\, \frac{\|y\|_1^*}{\|y\|_k^*}. \tag{4.14}$$

Therefore, if for some $j \geq j_0$ we have

$$\frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(k-1)}(\alpha_j)} > C\, \frac{\|y\|_1^*}{\|y\|_k^*} \tag{4.15}$$

then the maximum in (4.13) will occur for $p \in [p_0, P_0]$.

Fix a $y \in E'$; then we have the following two cases:

Case 1. The inequality (4.15) holds with $j = j_0$. Then the maximum in (4.13) (with $j = j_0$ and the fixed y) occurs for some $p \in [p_0, P_0]$, so

$$\frac{\|y\|_k^*}{\|y\|_p^*} \leq C\, \frac{F_{\tau(p)}(\alpha_{j_0})}{F_{\tau(p_0)}(\alpha_{j_0})} \leq C\overline{C} \leq C\overline{C} F_{\tau(p)}\left(\frac{\|y\|_1^*}{\|y\|_k^*}\right),$$

where

$$\overline{C} = \max_{p_0 \leq p \leq P_0} \frac{F_{\tau(p)}(\alpha_{j_0})}{F_{\tau(p_0)}(\alpha_{j_0})}.$$

Thus (4.11) holds with $D = C\overline{C}$.

Case 2. If (4.15) fails for $j = j_0$ then (since $F_{\tau(p_0)}(\alpha_j)/F_{\tau(k-1)}(\alpha_j) \to \infty$ as $j \to \infty$) there exists $j > j_0$ such that

$$\frac{F_{\tau(p_0)}(\alpha_{j-1})}{F_{\tau(k-1)}(\alpha_{j-1})} \leq C\, \frac{\|y\|_1^*}{\|y\|_k^*} < \frac{F_{\tau(p_0)}(\alpha_j)}{F_{\tau(k-1)}(\alpha_j)}. \tag{4.16}$$

Since the right inequality in (4.16) coincides with (4.15) the maximum in (4.13) occurs for some $p \in [p_0, P_0]$.

On the other hand by (3.22), (3.23) and the left inequality in (4.16) we obtain (since $j > j_0 \geq \widetilde{j}$)

$$\alpha_j \leq \|y\|_1^* / \|y\|_k^*,$$

therefore from (4.13) it follows that for some $p \in [p_0, P_0]$,

$$\frac{\|y\|_k^*}{\|y\|_p^*} \leq C\, \frac{F_{\tau(p)}(\alpha_j)}{F_{\tau(p_0)}(\alpha_j)} \leq C F_{\tau(p)}(\alpha_j) \leq C F_{\tau(p)}\left(\frac{\|y\|_1^*}{\|y\|_k^*}\right),$$

which proves (4.11) with $D = C$.

In order to prove that (ii)⇒(i) we shall check that (3.19) implies

$$\text{(4.17)}\qquad \forall\sigma(p)\uparrow\infty\ \exists k\ \forall p_0\ \exists P_0, C>1:\quad F_{\sigma(p_0)}(\alpha_j)\|y\|_k^*\le C\max_{1\le p\le P_0}F_{\sigma(p)}(\alpha_j)\|y\|_p^*.$$

Fix $\sigma(p)\uparrow\infty$ and take an arbitrary p_0. Then choose $\tau(p)$ as in the proof of Theorem 8 so that (3.28) holds.

Fix a pair (y,j), $y\in E'$, $j\in\mathbb{N}$. Either

$$\|y\|_1^*/\|y\|_k^*>F_{\sigma(p_0)}(\alpha_j)/F_{\sigma(1)}(\alpha_j),$$

then obviously (4.17) holds, or

$$\|y\|_1^*/\|y\|_k^*\le F_{\sigma(p_0)}(\alpha_j)/F_{\sigma(1)}(\alpha_j).$$

Then by (4.11) with $p_1=1$ and (3.28) we obtain, for some $p\in[p_0,P_0]$,

$$\frac{\|y\|_k^*}{\|y\|_p^*}\le DF_{\tau(p)}[F_{\sigma(p_0)}(\alpha_j)]\le D\,\frac{F_{\sigma(p)}(\alpha_j)}{F_{\sigma(p_0)}(\alpha_j)},$$

that is, (4.17) holds with $C=D$. ∎

References

[1] M. Alpseymen, M. S. Ramanujan and T. Terzioğlu, *Subspaces of some nuclear sequence spaces*, Proc. Koninkl. Nederl. Akad. Wetensch. A 82 (1979), 217–224.

[2] J. Bonet, *On the identity $L(E,F)=LB(E,F)$ for pairs of locally convex spaces E and F*, Proc. Amer. Math. Soc. 99 (1987), 249–255.

[3] J. Bonet and A. Galbis, *The identity $L(E,F)=LB(E,F)$, tensor products and inductive limits*, Note Mat. 9 (1989), 195–216.

[4] L. Crone and W. Robinson, *Diagonal maps and diameters in Köthe spaces*, Israel J. Math. 20 (1975), 13–21.

[5] M. M. Dragilev, *Riesz classes and multi-regular bases*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 15 (1972), 512–525 (in Russian).

[6] —, *Binary relations between Köthe spaces*, in: Math. Analysis and Applications 6, Rostov Univ., 1974, 112–135 (in Russian).

[7] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Springer, 1979.

[8] V. Kashirin, *Compact operators in generalized power series spaces*, Izv. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly 1980, no. 4, 13–16 (in Russian).

[9] V. P. Kondakov, *On the structure of unconditional bases in some Köthe spaces*, Studia Math. 76 (1983), 137–151 (in Russian).

[10] J. Krone, *Zur topologischen Charakterisierung von Unter- und Quotientenräumen spezieller nuklearer Kötheräume mit der Splittungmethode*, Diplomarbeit, Wuppertal, 1984.

[11] J. Krone and D. Vogt, *The splitting relation for Köthe spaces*, Math. Z. 190 (1985), 387–400.

[12] B. S. Mityagin and G. M. Henkin, *Linear problems of complex analysis*, Uspekhi Mat. Nauk 26 (1971), no. 4, 93–152 (in Russian); English transl.: Russian Math. Surveys 26 (1971), no. 4, 99–164.

[13] Z. Nurlu, *On basic sequences in some Köthe spaces and existence of non-compact operators*, Ph.D. thesis, Clarkson College of Technology, Potsdam, NY, 1981.

[14] Z. Nurlu, *On pairs of Köthe spaces between which all operators are compact*, Math. Nachr. 122 (1985), 277–287.
[15] Z. Nurlu and T. Terzioğlu, *Consequences of the existence of a non-compact operator*, Manuscripta Math. 47 (1984), 1–12.
[16] T. Terzioğlu, *A note on unbounded operators and quotient spaces*, Doğa Mat. 10 (1986), 338–344.
[17] D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.
[18] V. P. Zahariuta, *On the isomorphism of Cartesian products of linear topological spaces*, Funktsional. Anal. i Prilozhen. 4 (1970), no. 2, 87–88 (in Russian).
[19] —, *On the isomorphism of Cartesian products of locally convex spaces*, Studia Math. 46 (1973), 201–221.

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Received June 12, 2000
Revised version January 21, 2002 (4546)