Noncommutative extensions of the
Fourier transform and its logarithm

by

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Abstract. We introduce noncommutative extensions of the Fourier transform of probability measures and its logarithm to the algebra $\mathcal{A}(S)$ of complex-valued functions on the free semigroup $S = \mathrm{FS}(\{z, w\})$ on two generators. First, to given probability measures μ, ν with all moments finite, we associate states $\widehat{\mu}$, $\widehat{\nu}$ on the unital free *-bialgebra $(\mathcal{B}, \varepsilon, \Delta)$ on two self-adjoint generators X, X' and a projection P. Then we introduce and study cumulants which are additive under the convolution $\widehat{\mu} \star \widehat{\nu} = \widehat{\mu} \otimes \widehat{\nu} \circ \Delta$ when restricted to the "noncommutative plane" $\mathcal{B}_0 = \mathbb{C}\langle X, X' \rangle$. We find a combinatorial formula for the Möbius function in the inversion formula and define the moment and cumulant generating functions, $M_{\widehat{\mu}}\{z, w\}$ and $L_{\widehat{\mu}}\{z, w\}$, respectively, as elements of $\mathcal{A}(S)$. When restricted to the subsemigroups $\mathrm{FS}(\{z\})$ and $\mathrm{FS}(\{w\})$, the function $L_{\widehat{\mu}}\{z, w\}$ coincides with the logarithm of the Fourier transform and with the K-transform of μ, respectively. Moreover, $M_{\widehat{\mu}}\{z, w\}$ is a "semigroup interpolation" between the Fourier transform and the Cauchy transform of μ. If one chooses a suitable weight function W on the semigroup S, the moment and cumulant generating functions become elements of the Banach algebra $l^1(S, W)$.

1. Introduction. The main examples of noncommutative independence, like tensor, free [V1], boolean [Sp-W] and monotone [Mu1] independence, lead to different convolutions of measures on the real line. This entails existence of different cumulants which behave "nicely" with respect to these convolutions (for instance, are additive), different moment-cumulant formulas and cumulant generating functions. The latter, like for instance the R-transform and the S-transform for free probability [V1, V2], the K-transform for boolean convolution [Sp-W] or the H-transform for monotone convolution [Mu2], give noncommutative one-dimensional analogs of the logarithm of the Fourier transform of probability measures.

However, the connection between them is not clear—in fact, they arise from quite different theories, corresponding to different notions of noncommutative independence (for the latter and connections between them, see [G-S], [F], [L1], [L3], [S]). Let us mention here that certain one-parameter

interpolations between the combinatorics of classical and free convolutions have been presented in [A] and [N]. Furthermore, two different frameworks including boolean and free convolutions have also been proposed: conditional freeness [Bo-Le-Sp] and hierarchy of freeness [L1] (see also [F-L]). On the level of convolutions of measures, our motivation, outlined in [L3], can be phrased as follows: there should exist a noncommutative probability space with a convolution of states, whose restrictions to certain one-dimensional commutative subspaces ("real lines") give the known examples of convolutions of measures in noncommutative probability.

In this paper the noncommutative space of interest will be the unital free *-algebra

$$\mathcal{B}_0 = \mathbb{C}\langle X, X' \rangle$$

generated by two self-adjoint generators X, X', on which we will develop a probability theory. Intuitively, it is helpful to view $\mathcal{B}_0$ as a *noncommutative plane*. Using this geometric language, $\mathbb{C}[X]$ will correspond to the *classical real line*, whereas $\mathbb{C}[X']$ to the *boolean real line*. However, in order to make this work, a suitable convolution on $\mathcal{B}_0$ has to be introduced.

It is worth noting that most convolutions which appear in noncommutative probability are, in contrast to the classical convolution, highly nonlinear with respect to addition of measures. Therefore, in the usual formulation, one cannot expect to use *-Hopf algebras or *-bialgebras to define them. However, as we showed in [L1] and [L3], it is possible to extend a given algebra by a projection P and then introduce a *-bialgebra structure on the extended algebra in many interesting cases. Thus, let

$$\mathcal{B} = \mathbb{C}\langle X, X', P \rangle$$

be the extended free *-bialgebra endowed with the coproduct

$$\Delta(X) = X \otimes 1 + 1 \otimes X,$$
$$\Delta(X') = X' \otimes P + P \otimes X', \quad \Delta(P) = P \otimes P$$

and the convolution of states

(1.1) $$\widehat{\mu} \star \widehat{\nu} := \widehat{\mu} \otimes \widehat{\nu} \circ \Delta$$

called *filtered convolution* [L1].

To given measures μ, ν on the real line with all moments finite, one can associate states $\widehat{\mu}, \widehat{\nu}$ on $\mathcal{B}$ with P playing the role of a "separator" of words (see Definition 3.1). Then the restrictions of the convolution (1.1) to the subalgebras $\mathbb{C}[X]$ and $\mathbb{C}[X']$ of $\mathcal{B}$ give the classical and boolean convolutions of μ, ν, respectively. The nonlinearity is then "hidden" in the definition of $\widehat{\mu}, \widehat{\nu}$. Thus the projection P serves only as a tool to define appropriate convolutions. One can say that in the usual formulation of boolean convolution, where P does not appear, one "sees" the boolean real line after performing calculations involving P.

Since $\mathcal{B}_0$ is freely generated by two generators, it is quite natural to expect that our construction should lead to a more noncommutative cumulant generating function. In fact, we show that it is an element of the noncommutative semigroup algebra $\mathcal{A}(S)$ of functions on the free semigroup $S = \mathrm{FS}(2) = \mathrm{FS}(\{z, w\})$ on two letters z, w, with convolution multiplication. We allow in S the empty word denoted by 1 and write $S^+ = S \setminus \{1\}$. Let us label the variables X and X' from $\mathcal{B}$ with letters z, w, respectively, namely $X = X(z)$ and $X' = X(w)$. Thus, if $\widehat{\phi}$ is any state on $\mathcal{B}$, the mixed moments of $X(z)$ and $X(w)$ in the state $\widehat{\phi}$ can be labelled by words of S:

$$M_{\widehat{\phi}}(s) = \widehat{\phi}(X(s_1) \dots X(s_n)) \quad \text{for } s = s_1 \dots s_n,$$

where $s_i \in \{z, w\}$, $i = 1, \dots, n$, $n \geq 1$, and we set $M_{\widehat{\phi}}(1) = 1$.

The defining recurrence formula for the cumulants $L_{\widehat{\phi}}(s)$, where $s \in S^+$, corresponding to the convolution (1.1), which we call *admissible cumulants*, reads

$$M_{\widehat{\phi}}(s) = \sum_{p=1}^{l(s)} \sum_{u=(u_1, \dots, u_p) \in \mathcal{AP}(s)} L_{\widehat{\phi}}(u_1) \dots L_{\widehat{\phi}}(u_p)$$

where $\mathcal{AP}(s)$ denotes the set of *admissible partitions* of the word s, i.e. those which do not have inner w's (see Definition 2.3), and $l(s)$ denotes the length of s.

Let $\widehat{\mu}, \widehat{\nu}$ be the states on $\mathcal{B}$ associated with μ, ν and given by the moments

$$M_{\widehat{\mu}}(s) = \mu_n, \quad M_{\widehat{\nu}}(s) = \nu_n, \quad s = s_1 \dots s_n \in S,$$

on the noncommutative plane $\mathcal{B}_0$, where μ_n, ν_n are the nth moments of μ, ν, respectively, and then extended to $\mathcal{B}$ by treating P as a "separator" of words (see Definition 3.1). Then

(1.2) $$L_{\widehat{\mu} \star \widehat{\nu}}(s) = L_{\widehat{\mu}}(s) + L_{\widehat{\nu}}(s)$$

for any $s \in S^+$. Thus, admissible cumulants are additive under the convolution (1.1) of states $\widehat{\mu}, \widehat{\nu}$.

For any state $\widehat{\phi}$ on $\mathcal{B}$, the moment and cumulant generating functions are defined as

(1.3) $$M_{\widehat{\phi}}\{z, w\} = \sum_{s \in S} \frac{M_{\widehat{\phi}}(s)}{n(s)!} \, s, \quad L_{\widehat{\phi}}\{z, w\} = \sum_{s \in S^+} \frac{L_{\widehat{\phi}}(s)}{n(s)!} \, s,$$

respectively, where

(1.4) $$n(s)! = n_1! n_2! \dots n_p! \quad \text{for} \quad s = z^{n_1} w^{k_1} z^{n_2} w^{k_2} \dots w^{k_{p-1}} z^{n_p}$$

with $n_1, n_p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $k_1, n_2, k_2, \dots, n_{p-1}, k_{p-1} \in \mathbb{N}$. These formal sums should be interpreted as elements of the algebra $\mathcal{A}(S)$, a noncommutative "two-dimensional" analog of the formal power series algebra $\mathbb{C}[[z]]$.

In this framework, the moments and cumulants corresponding to the classical and boolean convolutions are now labelled not by integers but rather by elements of the infinite cyclic subsemigroups

$$S(z) = \mathrm{FS}(\{z\}), \quad S(w) = \mathrm{FS}(\{w\})$$

with z and w as their respective generators. If we restrict the supports of $M_{\widehat{\phi}}\{z,w\}$, $L_{\widehat{\phi}}\{z,w\}$ to $S(z)$ and $S(w)$, we obtain the moment and cumulant generating functions for the classical and boolean cases, respectively (the order n of the moments and cumulants in the usual formulation corresponds to z^n and w^n). In particular, on these restricted supports, (1.2) gives additivity of the classical and boolean cumulants. Thus, one can say that $M_{\widehat{\mu}}\{z,w\}$ is a "semigroup interpolation" between the Fourier transform $F_\mu\{z\} = M_{\widehat{\mu}}\{z,0\}$ and the Cauchy transform $G_\mu(1/w) = w M_{\widehat{\mu}}\{0,w\}$ (the left-hand side is treated as a formal power series in $1/w$). Moreover, $L_{\widehat{\mu}}\{z,w\}$ is a "semigroup interpolation" between the logarithm of the Fourier transform and the K-transform of the measure μ.

More generally, one can take a free *-algebra in infinitely many indeterminates and then, in this more general framework, define a similar convolution which unifies, apart from tensor and boolean convolutions, also m-free convolutions [F-L] which weakly approximate free convolution (a similar approach includes monotone convolution, see [F]). In the general case we expect to give a "universal" noncommutative transform of states on a noncommutative version of $\mathbb{R}^\infty$, whose special cases would also be the R-transform and the H-transform. Nevertheless, it follows from the construction given in [L3] that the connection between the classical case and the boolean case seems in our approach to be of main importance since the general model will be a multivariate generalization of $\mathcal{B}$, although this step is also nontrivial and will be treated in a subsequent paper.

Let us note that $L_{\widehat{\phi}}\{z,w\}$, our noncommutative extension of the logarithm of the Fourier transform, is quite different from the cumulant generating functions considered so far in noncommutative probability. In particular, $L_{\widehat{\phi}}\{z,w\}$ and $L_{\widehat{\psi}}\{z,w\}$ do not, in general, commute for $\widehat{\phi} \neq \widehat{\psi}$. Moreover, it seems interesting in its own right from the combinatorial point of view.

In principle, the paper is self-contained, although it is a continuation of the study originated in [L3] (the stochastic calculus was developed in [L2]). In Section 2 we give basic definitions on combinatorics of words and we introduce admissible partitions. A closer look at filtered convolution for $\widehat{\mathcal{B}}$ is presented in Section 3. In Section 4 we introduce admissible cumulants and prove that they are additive under filtered convolution. In Section 5 we present the inversion formula for admissible cumulants and prove a combinatorial formula for the associated Möbius function. In Section 6 we give basic facts on the semigroup algebra $\mathcal{A}(S)$ and the Banach algebra $l^1(S,W)$.

Finally, in Section 7 we derive a formula which expresses the cumulant generating function in terms of the moment generating function.

2. Combinatorics on words. Let $X = \{z, w\}$ be a two-element set. A *word* on X is a finite sequence $s = s_1 \ldots s_n$, where $s_i \in \{z, w\}$ for each $i = 1, \ldots, n$. The empty word will be denoted by 1. The length of s will be denoted by $l(s)$. The unital free semigroup $\mathrm{FS}(X) = \mathrm{FS}(2)$ generated by X is the collection of all words from X made into a semigroup by the juxtaposition product $(s_1 \ldots s_n)(t_1 \ldots t_m) = s_1 \ldots s_n t_1 \ldots t_m$. From now on we will understand that $S = \mathrm{FS}(2)$ and set $S^+ = S \setminus \{1\}$.

In what follows we will use the following terminology and notations:

(i) $s_j \in s$ if $s = s_1 \ldots s_n$, i.e. s_j is a letter in the word s;

(ii) t is a *subword* of $s = s_1 \ldots s_n \in S$ if it is a subsequence of the form

$$(2.1) \qquad t = s_{i_1} \ldots s_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq n, \ k > 0,$$

or if $t = 1$ (the empty word);

(iii) for subwords t, r of s, we write $t < r$ if $t = s_{i_1} \ldots s_{i_k}$, $r = s_{j_1} \ldots s_{j_l}$ and $i_1 < j_1$; we call t, r *disjoint* if the sets $\{i_1, \ldots, i_k\}$, $\{j_1, \ldots, j_l\}$ are disjoint;

(iv) a subword t of s is a *factor* of s if there exist words $r, r' \in S$ such that $s = rtr'$;

(v) if r, t are disjoint subwords of s, then $r \cup t$ denotes the subword of s obtained from s by deleting all letters which are not in r or t; if, in addition, $r \cup t = s$, we also write $r = s \setminus t$;

(vi) if r, t are subwords of s, then $r \cap t$ denotes the subword of s obtained from s be deleting all letters which are not in both r and t.

It should be stressed that by subwords we mean *subsequences* of the form (2.1) with encoded information not only about the letters but also about the indices $i_1, \ldots, i_k$. Thus, two distinct subwords may give the same word. For instance, in the word $zwzw$, there is only one subword equal to zwz, namely $s_1 s_2 s_3$, but there are three subwords equal to zw, namely $s_1 s_2$, $s_1 s_4$ and $s_3 s_4$. This terminology is borrowed from [Lo] and one should remember that it is not followed by all authors. Note also that the subword $r \cup t$ is an element of the shuffle of the words r and t, the latter being denoted by $r \circ t$ in [Lo]. One can say that $r \cup t$ is the only element of the shuffle $r \circ t$, where all the letters of r and t are "in the right place". For instance, if $r = s_2 s_4$, $t = s_3 s_5$ are subwords of $s = s_1 s_2 s_3 s_4 s_5$, then $r \cup t = s_2 s_3 s_4 s_5$.

DEFINITION 2.1. By a *partition* of the word $s = s_1 \ldots s_n \in S^+$ we understand any sequence $u = (u_1, \ldots, u_m)$ where $u_1, \ldots, u_m$ are disjoint subwords of s such that

$$s = u_1 \cup \ldots \cup u_m \quad \text{and} \quad u_1 < \ldots < u_m$$

where $1 \leq m \leq n$. We then write $b(u) = m$, i.e. $b(u)$ denotes the number of words in the partition u. We denote by $\mathcal{P}(s)$ the set of all partitions of the word s. By a *factorization* of s we understand a partition of s in which every subword u_k is a factor; we then write $s = u_1 \ldots u_p$.

REMARK 1. For fixed $s \in S^+$ of length n, there is a one-to-one correspondence between $\mathcal{P}(s)$ and all partitions $\mathcal{P}_n$ of the set $\{1, \ldots, n\}$: subwords correspond to blocks. Therefore, terminology which refers to partitions of $\mathcal{P}_n$ has its natural analogs in the case of $\mathcal{P}(s)$. In particular, we say that $u \in \mathcal{P}(s)$ is finer (coarser) than $v \in \mathcal{P}(s)$ if the corresponding partitions of $\mathcal{P}_n$ have this property.

REMARK 2. We adopt the convention that the one-subword partition of s consisting of s is denoted by s instead of (s).

REMARK 3. If $u' \in \mathcal{P}(s')$, $u'' \in \mathcal{P}(s'')$, where s', s'' are disjoint and $s' \cup s'' = s$, then we denote by $u' \cup u''$ the partition of s consisting of subwords of s which are in u' and subwords of s which are in u''. Finally, if u' is a (not necessarily proper) refinement of u obtained by dividing the subwords of u into perhaps smaller subwords, we write $u' \preceq u$.

DEFINITION 2.2. Let $s = s_1 \ldots s_n \in S^+$ and let $u = (u_1, \ldots, u_m) \in \mathcal{P}(s)$. We say that the letter s_j of the word u_p is *inner* with respect to the word $u_l = s_{i_1} \ldots s_{i_k}$, where $l \neq p$, if $i_1 < j < i_k$. We then also say that this letter is inner with respect to the partition u. By a *cumulant subword* of s we mean every subword r of s which does not have any inner w's in s. The set of all cumulant subwords of s is denoted by $C(s)$. All w's of a cumulant word r are called *w-legs* of r.

DEFINITION 2.3. A partition $u = (u_1, \ldots, u_m) \in \mathcal{P}(s)$, where $s \in S^+$, is called *admissible* if $u_1, \ldots, u_m$ are cumulant subwords of s. We denote by $\mathcal{AP}(s)$ the set of all admissible partitions. If $u \in \mathcal{P}(s) \setminus \mathcal{AP}(s)$, then we say that there is a *nonadmissible inversion* in the sequence $(u_{i_1}, \ldots, u_{i_p})$ of subwords, where $1 \leq i_1 < \ldots < i_p \leq m$, if there exists a w in one of these subwords which is inner with respect to another subword.

EXAMPLE 1. Let $s = z^2 w z$ and consider two partitions:

$$u = (s_1 s_3, s_2, s_4) = (zw, z, z), \quad u' = (s_1, s_2 s_4, s_3) = (z, z^2, w)$$

given by the diagrams

Then the letter $s_3 = w$ is inner with respect to the partition u', but not with respect to u. Therefore, u is admissible, but u' is not.

3. Moments and convolutions. Recall from [L1] the definition of the so-called boolean extension of a state.

DEFINITION 3.1. Let ϕ be a state on $\mathbb{C}[Y]$, where Y is self-adjoint, i.e. $Y^* = Y$. The *boolean extension* of ϕ is the state on $\mathbb{C}\langle Y, P \rangle$, where P is a projection, i.e. $P^* = P^2 = P$, given by the linear extension of

$$(3.1) \qquad \widetilde{\phi}(P^\alpha Y^{n_1} P Y^{n_2} P \ldots Y^{n_k} P^\beta) = \phi(Y^{n_1}) \phi(Y^{n_2}) \ldots \phi(Y^{n_k})$$

where $\alpha, \beta \in \{0, 1\}$ and $n_1, \ldots, n_k \in \mathbb{N}$, with $\widetilde{\phi}(P) = 1$.

The boolean extension of a state is a state since

$$\widetilde{\phi} = \phi *_{\mathrm{b}} h$$

($*_{\mathrm{b}}$ stands for the boolean product of states), i.e. $\widetilde{\phi}$ is the boolean product of the state ϕ on $\mathbb{C}[Y]$ and the unital *-homomorphism h on $\mathbb{C}[P]$ given by $h(P) = h(1) = 1$. That the boolean *product* of states is a state follows from the more general case of the so-called conditional product of states studied in [Bo-Le-Sp]. The boolean extensions of states serve as a tool to define a new type of convolution of states on the unital free *-algebra on two generators $\mathcal{B}_0$ (given below in Definition 3.3).

DEFINITION 3.2. Let $\mathcal{B} = \mathbb{C}\langle X, X', P \rangle$ be the unital *-algebra of polynomials in noncommuting self-adjoint variables X and X' and a projection P. When endowed with the coproduct $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and counit $\varepsilon : \mathcal{B} \to \mathbb{C}$ given by

$$(3.2) \qquad \Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(X') = X' \otimes P + P \otimes X',$$

$$(3.3) \qquad \Delta(P) = P \otimes P, \quad \varepsilon(X) = \varepsilon(X') = 0, \quad \varepsilon(P) = 1,$$

it becomes a unital *-bialgebra, called the *filtered bialgebra*.

DEFINITION 3.3. Let $\eta : \mathcal{B} \to \mathbb{C}\langle Y, P \rangle$, where Y is self-adjoint and P is a projection, be the unital *-homomorphism given by

$$\eta(X) = \eta(X') = Y, \quad \eta(P) = P, \quad \eta(1) = 1$$

and let ϕ, ψ be states on $\mathbb{C}[Y]$. Then $\widehat{\phi} = \widetilde{\phi} \circ \eta$ and $\widehat{\psi} = \widetilde{\psi} \circ \eta$ are states on $\mathcal{B}$. Their convolution

$$(3.4) \qquad \widehat{\phi} \star \widehat{\psi} = \widehat{\phi} \otimes \widehat{\psi} \circ \Delta$$

will be called the *filtered convolution* of $\widehat{\phi}$ and $\widehat{\psi}$.

We will use the semigroup S to label the mixed moments of the variables X and X'. Namely, we label them by z and w, respectively, to get $X = X(z)$ and $X' = X(w)$. For instance

$$M_{\widehat{\phi}}(zwz) = \widehat{\phi}(XX'X), \quad M_{\widehat{\phi}}(z^2 w) = \widehat{\phi}(X^2 X'), \quad M_{\widehat{\phi}}(w^2 zw) = \widehat{\phi}((X')^2 XX').$$

Note that in our notation, $M_{\widehat{\phi}}(s)$ (or simply $M(s)$) is a moment, not the generating function. For the latter, we will use the notation $M_{\widehat{\phi}}\{z, w\}$, or simply $M\{z, w\}$ (see Section 7).

In particular, let now ϕ and ψ be states on $\mathbb{C}[Y]$ associated with measures μ and ν on the real line with all moments finite, i.e.

$$\phi(Y^n) = \mu_n = \int_{\mathbb{R}} y^n \, d\mu(y), \quad \psi(Y^n) = \nu_n = \int_{\mathbb{R}} y^n \, d\nu(y)$$

and let $\widetilde{\phi}, \widetilde{\psi}$ be their respective boolean extensions.

Thus, with μ and ν one can associate the states

(3.5) $$\widehat{\mu} := \widehat{\phi} = \widetilde{\phi} \circ \eta, \quad \widehat{\nu} := \widehat{\psi} = \widetilde{\psi} \circ \eta$$

on $\mathcal{B}$ and thus, by restriction, on its subalgebra $\mathcal{B}_0 = \mathbb{C}\langle X, X' \rangle$, the unital free *-algebra on two generators with moments

$$M_{\widehat{\mu}}(s) = \widehat{\mu}(X(s_1) \ldots X(s_n)) = \mu_n,$$
$$M_{\widehat{\nu}}(s) = \widehat{\nu}(X(s_1) \ldots X(s_n)) = \nu_n,$$

for every word $s = s_1 \ldots s_n \in S$, i.e. the mixed moments only depend on the length of s and agree with the moments of the corresponding measures on the real line. Informally, one can view them as moments of the "two-dimensional measures" on the noncommutative plane $\mathcal{B}_0 = \mathbb{C}\langle X, X' \rangle$, canonically associated with the measures μ and ν.

The filtered convolution $\widehat{\mu} \star \widehat{\nu}$ may also be viewed as a "two-dimensional measure" on the noncommutative plane $\mathcal{B}_0$ with moments

(3.6) $$\widehat{\mu} \star \widehat{\nu}(s) := \widehat{\mu} \star \widehat{\nu}(X(s_1) \ldots X(s_n)).$$

Below we give examples of lowest order mixed moments of the filtered convolution of the states $\widehat{\mu}$ and $\widehat{\nu}$.

EXAMPLE 1. The moments of order 1 and 2 do not depend on s:

$$\widehat{\mu} \star \widehat{\nu}(s_1) = \mu_1 + \nu_1,$$
$$\widehat{\mu} \star \widehat{\nu}(s_1 s_2) = \mu_2 + \nu_2 + 2\mu_1 \nu_1.$$

There are two different expressions for moments of order 3:

$$\widehat{\mu} \star \widehat{\nu}(s_1 z s_3) = \mu_3 + 3\mu_2 \nu_1 + 3\mu_1 \nu_2 + \nu_3,$$
$$\widehat{\mu} \star \widehat{\nu}(s_1 w s_3) = \mu_3 + 2\mu_2 \nu_1 + \mu_1^2 \nu_1 + \nu_3 + 2\mu_1 \nu_2 + \mu_1 \nu_1^2,$$

for any $s_1, s_3 \in \{z, w\}$, which coincide with the moments of the classical and boolean convolutions of the measures μ, ν, respectively. In the case of moments of order 4, we get three different possibilities:

$$\widehat{\mu} \star \widehat{\nu}(s_1 z^2 s_4) = \mu_4 + 4\mu_3 \nu_1 + 6\mu_2 \nu_2 + 4\mu_1 \nu_3 + \nu_4,$$
$$\widehat{\mu} \star \widehat{\nu}(s_1 w^2 s_4) = \mu_4 + 2\mu_3 \nu_1 + 2\mu_2 \nu_2 + 2\mu_1 \nu_3 + \nu_4$$
$$+ 2\mu_2 \nu_1 \mu_1 + 2\mu_1 \nu_2 \nu_1 + \mu_2 \nu_1^2 + \nu_2 \mu_1^2 + 2\mu_1^2 \nu_1^2,$$

$$\widehat{\mu} \star \widehat{\nu}(s_1 z w s_4) = \mu_4 + 3\mu_3\nu_1 + 2\mu_2\nu_2 + 3\mu_1\nu_3 + \nu_4$$
$$+ \mu_1\nu_1\mu_2 + 2\mu_1^2\nu_2 + \nu_1\mu_1\nu_2 + 2\nu_1^2\mu_2,$$

for any $s_1, s_4 \in \{z, w\}$. It is easy to see that

$$\widehat{\mu} \star \widehat{\nu}(s_1 w z s_4) = \widehat{\mu} \star \widehat{\nu}(s_1 z w s_4),$$

se we have altogether three different cases. The first one corresponds to the classical convolution, the second to the boolean convolution, whereas the third is of a different type.

Of course, $\widehat{\mu}$, $\widehat{\nu}$ and $\widehat{\mu} \star \widehat{\nu}$ are defined on all of $\mathcal{B}$ and it is not hard to express all their moments in terms of the moments on $\mathcal{B}_0$.

PROPOSITION 3.4. *Let* $\widehat{\sigma} \in \{\widehat{\mu}, \widehat{\nu}, \widehat{\mu} \star \widehat{\nu}\}$. *Then*

$$\widehat{\sigma}(P^\alpha X(t_1) P X(t_2) P \dots P X(t_p) P^\beta) = \widehat{\sigma}(X(t_1)) \dots \widehat{\sigma}(X(t_p))$$

where $\alpha, \beta \in \{0, 1\}$ *and* $t_i \in S$, $i = 1, \dots, p$, *and the abbreviated notation*

$$X(t) = X(s_1) \dots X(s_r)$$

for $t = s_1 \dots s_r \in S$ *is used.*

Proof. This is a straightforward consequence of the definition of η, the fact that P acts as a "separator" of words in $\mathcal{B}_0$ and that it is group-like, hence convolution preserves this property. ∎

Therefore, we will restrict our attention to the moments of these states on $\mathcal{B}_0$ since P only serves as a tool to define a convolution on $\mathcal{B}_0$.

PROPOSITION 3.5. *Let* μ *and* ν *be probability measures on the real line with all moments finite. Then*

$$\widehat{\mu} \star \widehat{\nu}(s) = \begin{cases} (\mu \star \nu)_n & \text{if } s = z^n, \\ (\mu \uplus \nu)_n & \text{if } s = w^n, \end{cases}$$

where $n \geq 0$, *i.e. the moments of the filtered convolution restricted to the cyclic subsemigroups* $S(z)$ *and* $S(w)$ *agree with the moments of the classical and boolean convolutions of* μ, ν, *denoted by* $\mu \star \nu$ *and* $\mu \uplus \nu$, *respectively.*

Proof. This fact is elementary and follows directly from Definitions 3.1–3.2 (it can also serve as a definition of boolean convolution). ∎

4. Admissible cumulants. In this section we define admissible cumulants and prove that they are additive under filtered convolution on $\mathcal{B}_0$. For notational simplicity, we denote the moments and cumulants associated with the state $\widehat{\phi}$ by $M(s)$, $s \in S$, and $L(s)$, $s \in S^+$, respectively.

DEFINITION 4.1. By *admissible cumulants* associated with the moments $(M(s))_{s \in S}$ we understand the numbers $(L(s))_{s \in S^+}$ defined recursively by the

formulas

$$(4.1) \qquad M(s) = \sum_{p=1}^{l(s)} \sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)} L(u_1)\ldots L(u_p),$$

where $s \in S^+$.

It is not hard to see that (4.1) is in fact a recurrence formula, which is a noncommutative analog of similar recurrence formulas in classical and noncommutative probability. Namely, we can write

$$M(s) = L(s) + \sum_{p=2}^{l(s)} \sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)} L(u_1)\ldots L(u_p)$$

for any $s \in S^+$, and thus $L(s)$ can be expressed in terms of $M(s)$ and cumulants $L(t)$ associated with words of length $l(t) < l(s)$.

EXAMPLE 1. Let $s = s_1s_2s_3s_4 = zwz^2$. Using Definition 4.1 we get

$$\begin{aligned}
M(zwz^2) = {} & L(zwz^2) + L(z)L(wz^2) + L(zw)L(z^2) \\
& + 2L(zwz)L(z) + L(z)L(w)L(z^2) + L(zw)L(z)L(z) \\
& + 2L(z)L(wz)L(z) + L(z)L(w)L(z)L(z).
\end{aligned}$$

Note that there is no contribution to $M(zwz^2)$ from the partitions associated with the sequences (s_1s_4, s_2, s_3), (s_1s_3, s_2, s_4), $(s_1s_3s_4, s_2)$, (s_1s_3, s_2s_4), and (s_1s_4, s_2s_3) since in all of them w is inner with respect to some block as the figure below demonstrates:
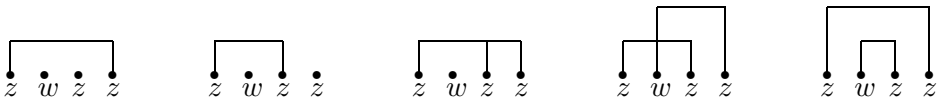
EXAMPLE 2. For comparison, take now $s = s_1s_2s_3s_4 = zw^2z$. Then

$$\begin{aligned}
M(zw^2z) = {} & L(zw^2z) + L(z)L(w^2z) + L(zw)L(wz) \\
& + L(zw^2)L(z) + L(z)L(w)L(wz) + L(zw)L(w)L(z) \\
& + L(z)L(w^2)L(z) + L(z)L(w)L(w)L(z),
\end{aligned}$$

and, in addition to five partitions which are analogous to those in Example 1, the following two give zero contribution:

The two additional partitions which do not contribute to this moment are $(s_1s_2s_4, s_3)$ and (s_1, s_2s_4, s_3) as they also have inner w's.

The restriction of Definition 4.1 to the words from $S(z)$ gives the usual expression for the classical cumulants whereas the restriction to $S(w)$ gives the boolean cumulants. This is beacause $\mathcal{AP}(z^n)$ can be put in one-to-one correspondence with all partitions of the set $\{1, \ldots, n\}$, whereas $\mathcal{AP}(w^n)$ can be put in one-to-one correspondence with the interval partitions of $\{1, \ldots, n\}$. And it is well known that these two classes of partitions give classical and boolean cumulants, respectively.

In what follows we will need a notation for the summands of $\Delta(X(z))$ and $\Delta(X(w))$ given by (3.2)–(3.3):

$$(4.2) \qquad j_1(X(z)) = X(z) \otimes 1, \quad j_1(X(w)) = X(w) \otimes P,$$

$$(4.3) \qquad j_2(X(z)) = 1 \otimes X(z), \quad j_2(X(w)) = P \otimes X(w)$$

(recall that $X = X(z)$ and $X' = X(w)$ and compare with the coproduct of Definition 3.2).

Also, for given s of the form (1.4) and given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_k \in \{1, 2\}$, $k = 1, \ldots, n$, let

$$s(1, \varepsilon) = \overrightarrow{\prod_{j:\, \varepsilon_j = 1}} s_j, \quad s(2, \varepsilon) = \overrightarrow{\prod_{j:\, \varepsilon_j = 2}} s_j,$$

the arrow indicating that the product is taken with the increasing order of indices. Clearly, $s(1, \varepsilon) \cup s(2, \varepsilon)$, where $s(1, \varepsilon)$ and $s(2, \varepsilon)$ are treated as partitions, is a partition of s.

Below we will give an explicit formula for the mixed moments

$$(4.4) \qquad \widehat{\mu} \otimes \widehat{\nu}(s, \varepsilon) := \widehat{\mu} \otimes \widehat{\nu}(j_{\varepsilon_1}(X(s_1)) \ldots j_{\varepsilon_n}(X(s_n)))$$

where $\widehat{\mu}, \widehat{\nu}$ are the states on $\mathcal{B}$ given by (3.5) and $\varepsilon_1, \ldots, \varepsilon_n \in \{1, 2\}$.

PROPOSITION 4.2. *Let μ, ν be probability measures on the real line with all moments finite and let $\varepsilon_1, \ldots, \varepsilon_n \in \{1, 2\}$. Then the mixed moments* (4.4) *are given by the formula*

$$(4.5) \qquad \widehat{\mu} \otimes \widehat{\nu}(s, \varepsilon) = \widehat{\mu}(u')\widehat{\nu}(u'')$$

where

$$(4.6) \qquad \widehat{\mu}(u') = \widehat{\mu}(u'_1) \ldots \widehat{\mu}(u'_p),$$

$$(4.7) \qquad \widehat{\nu}(u'') = \widehat{\nu}(u''_1) \ldots \widehat{\nu}(u''_q),$$

and $u' = (u'_1, \ldots, u'_p)$, $u'' = (u''_1, \ldots, u''_q)$ are the unique coarsest partitions of $s(1, \varepsilon)$ and $s(2, \varepsilon)$, respectively, which define an admissible partition of s (their dependence on ε is suppressed in the notation).

Proof. By substituting (4.2)–(4.3) into (4.4) and using Proposition 3.4, we get

$$\widehat{\mu} \otimes \widehat{\nu}(s, \varepsilon) = \widehat{\mu} \circ \tau'(s) \times \widehat{\nu} \circ \tau''(s)$$

where

$$\tau'(s) = \tau'(s_1)\ldots\tau'(s_n), \quad \tau''(s) = \tau''(s_1)\ldots\tau''(s_n),$$

and

$$\tau'(s_j) = \begin{cases} X(s_j) & \text{if } \varepsilon_j = 1, \\ P & \text{if } (s_j, \varepsilon_j) = (w, 2), \\ 1 & \text{if } (s_j, \varepsilon_j) = (z, 2), \end{cases}$$

$$\tau''(s_j) = \begin{cases} X(s_j) & \text{if } \varepsilon_j = 2, \\ P & \text{if } (s_j, \varepsilon_j) = (w, 1), \\ 1 & \text{if } (s_j, \varepsilon_j) = (z, 1), \end{cases}$$

and therefore the P's define interval partitions of $s(1,\varepsilon)$ and $s(2,\varepsilon)$, respectively, denoted by $u' = (u'_1, \ldots, u'_p)$ and $u'' = (u''_1, \ldots, u''_q)$, where $1 \leq p+q \leq n$. The words of these partitions are the longest subwords of $s(1,\varepsilon)$ and $s(2,\varepsilon)$ for which the corresponding products of $X(s_j)$'s are not separated by a P. The pair (u', u'') defines an admissible partition $u = u' \cup u'' = (u_1, \ldots, u_{p+q})$ of s. In fact, each of its subwords, say u_k, belongs to either u' or u''—without loss of generality we can suppose that $u_k = u'_r$ for some r. Then between the letters of u_k, say s_j and s_l, there can only be letters of the same block u_k or letters of the blocks of u''. The latter have to be z's since any w would produce a P between $X(s_j)$ and $X(s_l)$ at the first tensor site as the mapping τ' indicates, but then s_j and s_l would not belong to the same block, which is a contradiction. ∎

PROPOSITION 4.3. *Under the assumptions of Proposition 4.2, the mixed moments $\widehat{\mu} \otimes \widehat{\nu}(s,\varepsilon)$ can be expressed in terms of cumulants as follows*:

$$(4.8) \qquad \widehat{\mu} \otimes \widehat{\nu}(s,\varepsilon) = \sum_{p=1}^{n} \sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}_\varepsilon(s)} L_{\varepsilon(1)}(u_1)\ldots L_{\varepsilon(p)}(u_p)$$

where

$$L_i = \begin{cases} L_{\widehat{\mu}} & \text{if } i = 1, \\ L_{\widehat{\nu}} & \text{if } i = 2, \end{cases}$$

and $\mathcal{AP}_\varepsilon(s)$ denotes the set of all admissible partitions of s which are subpartitions of the partition $(s(1,\varepsilon), s(2,\varepsilon))$ and $\varepsilon(k) = i$, $i \in \{1,2\}$, if for all $s_j \in u_k$ we have $\varepsilon_j = i$.

Proof. By applying Definition 4.1 to every moment $\widehat{\mu}(u'_k)$ and $\widehat{\nu}(u''_l)$ on the RHS of (4.5), i.e. expressing these moments in terms of cumulants, we find that $\widehat{\mu} \otimes \widehat{\nu}(s,\varepsilon)$ is equal to a sum of products of type

$$(4.9) \qquad\qquad L_{\varepsilon(1)}(v_1)\ldots L_{\varepsilon(p)}(v_m),$$

where $v = (v_1, \ldots, v_m) \in \mathcal{AP}_\varepsilon(s)$ is a refinement of $u = u' \cup u'' \in \mathcal{AP}_\varepsilon(s)$. Moreover, this refinement must be admissible by the definition of admissible

cumulants (remember that a refinement of an admissible partition does not have to be admissible).

Since all products of type (4.9) which are obtained in this fashion are associated with *different* admissible refinements of the partition $u = u' \cup u''$, they give distinct elements of $\mathcal{AP}_\varepsilon(s)$. Therefore, we just need to prove that on the RHS of (4.5) we obtain products of cumulants associated with *all* $u \in \mathcal{AP}_\varepsilon(s)$. So, let $v = (v_1, \ldots, v_m) \in \mathcal{AP}_\varepsilon(s)$. Take the partition $u' \cup u''$ of Proposition 4.2. It is enough to show that letters (understood as pairs (s_k, k)) from each word v_j, $1 \le j \le m$, with say $\varepsilon(j) = 1$, cannot belong to different words of u'. Suppose that two letters, say $s_i, s_k \in v_j$, belong to different words of u'. This means that they must be separated in s by a $w = s_l$ with $\varepsilon_l = 2$ and, therefore, that v is not admissible, which is a contradiction. We conclude that v must be an admissible refinement of u. However, all admissible refinements of u give a contribution to the RHS of (4.5), which ends the proof. ∎

EXAMPLE 3. Let us give two examples of moments $\widehat{\mu} \otimes \widehat{\nu}(s, \varepsilon)$ expressed in terms of cumulants according to (4.8). For the sake of generality, we take $s = s_1 s_2 s_3 s_4$, i.e. an arbitrary word of length $l(s) = 4$. We use the notation

$$(4.10) \qquad \delta_k = \begin{cases} 1 & \text{if } s_k = z, \\ 0 & \text{if } s_k = w, \end{cases}$$

and, for simplicity, we write $L(i_1 \ldots i_n)$ instead of $L(s_{i_1} \ldots s_{i_n})$. Take, for instance, the two most interesting examples of $\varepsilon = (1, 1, 2, 1)$ and $\varepsilon' = (1, 2, 1, 2)$. We have

$$\begin{aligned}
\widehat{\mu} \otimes \widehat{\nu}(s, \varepsilon) &= L_1(12)L_2(3)L_1(4) + \delta_3 L_1(124)L_2(3) \\
&\quad + \delta_2 \delta_3 L_1(14)L_1(2)L_2(3) + \delta_3 L_1(1)L_1(24)L_2(3) \\
&\quad + L_1(1)L_1(2)L_2(3)L_1(4), \\
\widehat{\mu} \otimes \widehat{\nu}(s, \varepsilon') &= \delta_2 \delta_3 L_1(13)L_2(24) + \delta_2 L_1(13)L_2(2)L_2(4) \\
&\quad + \delta_3 L_1(1)L_2(24)L_1(3) + L_1(1)L_2(2)L_1(3)L_2(4).
\end{aligned}$$

In the special cases of $s = z^4$ (all δ's are equal to 1) and $s = w^4$ (all δ's vanish), we get mixed moments of classical and boolean variables, respectively.

Let us now show that admissible cumulants are additive under filtered convolution.

THEOREM 4.4 (Additivity of cumulants). *Let μ, ν be probability measures on the real line with all moments finite. Let $\widehat{\mu}, \widehat{\nu}$ be the associated states on $\mathcal{B}$ given by (3.5). Then*

$$(4.11) \qquad L_{\widehat{\mu} \star \widehat{\nu}}(s) = L_{\widehat{\mu}}(s) + L_{\widehat{\nu}}(s)$$

for every $s \in S^+$.

Proof. We use induction on the length of s. It is clear that if $l(s) = 1$, then

$$L_{\widehat{\mu}\star\widehat{\nu}}(s) = M_{\widehat{\mu}\star\widehat{\nu}}(s) = M_{\widehat{\mu}}(s) + M_{\widehat{\nu}}(s) = L_{\widehat{\mu}}(s) + L_{\widehat{\nu}}(s).$$

Suppose now that (4.11) holds for words s of length $l(s) \leq n-1$. We will show that then (4.11) holds for words s of length $l(s) = n$. By Definition 4.1, we have

$$L_{\widehat{\mu}\star\widehat{\nu}}(s) = M_{\widehat{\mu}\star\widehat{\nu}}(s) - \sum_{p=2}^{n}\sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)} L_{\widehat{\mu}\star\widehat{\nu}}(u_1)\ldots L_{\widehat{\mu}\star\widehat{\nu}}(u_p)$$

for $s = s_1 \ldots s_n$. We know from Section 3 that

$$M_{\widehat{\mu}\star\widehat{\nu}}(s) = \widehat{\mu}\otimes\widehat{\nu}(\Delta X(s_1)\ldots\Delta X(s_n)) = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{1,2\}} \widehat{\mu}\otimes\widehat{\nu}(s,(\varepsilon_1,\ldots,\varepsilon_n))$$

Using the inductive assumption, we have

$$L_{\widehat{\mu}\star\widehat{\nu}}(u_k) = L_{\widehat{\mu}}(u_k) + L_{\widehat{\nu}}(u_k), \quad \forall k = 1,\ldots,p,$$

since $l(u_k) < n$ for all $1 \leq k \leq n$ (recall that $p \geq 2$). Therefore,

$$L_{\widehat{\mu}\star\widehat{\nu}}(s) = M_{\widehat{\mu}}(s) + M_{\widehat{\nu}}(s) - \sum_{p=2}^{n}\sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)} L_{\widehat{\mu}}(u_1)\ldots L_{\widehat{\mu}}(u_p)$$

$$-\sum_{p=2}^{n}\sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)} L_{\widehat{\nu}}(u_1)\ldots L_{\widehat{\nu}}(u_p) + D(s)$$

$$= L_{\widehat{\mu}}(s) + L_{\widehat{\nu}}(s) + D(s)$$

where

$$D(s) = \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{1,2\}\\ \text{not all equal}}} \widehat{\mu}\otimes\widehat{\nu}(s,(\varepsilon_1,\ldots,\varepsilon_n))$$

$$-\sum_{p=2}^{n}\sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)}\sum_{\substack{\varepsilon(1),\ldots,\varepsilon(p)\in\{1,2\}\\ \text{not all equal}}} L_{\varepsilon(1)}(u_1)\ldots L_{\varepsilon(p)}(u_p)$$

where $L_1(u) = L_{\widehat{\mu}}(u)$ and $L_2(u) = L_{\widehat{\nu}}(u)$. Using Proposition 4.3 and interchanging the summations, which in this case takes the form

$$\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{1,2\}\\ \text{not all equal}}}\sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}_\varepsilon(s)} = \sum_{u=(u_1,\ldots,u_p)\in\mathcal{AP}(s)}\sum_{\substack{\varepsilon(1),\ldots,\varepsilon(p)\in\{1,2\}\\ \text{not all equal}}},$$

for every $p \geq 2$, we deduce that $D(s) = 0$, which completes the proof. ∎

5. Möbius inversion formula. In this section we apply the theory of Möbius functions to prove an inversion formula for admissible cumulants.

We also derive a combinatorial formula for the associated Möbius function. For details on the theory of Möbius functions, see [R1] and [R2].

PROPOSITION 5.1. *For every $s \in S^+$, the set $\mathcal{AP}(s)$ of admissible partitions is a lattice.*

Proof. We show that if $u, u' \in \mathcal{AP}(s)$, then $u \wedge u', u \vee u' \in \mathcal{AP}(s)$, where $\wedge$ and $\vee$ denote meet and join in the lattice $\mathcal{P}(s)$ of all partitions of s.

Let $s = s_1 \ldots s_n$, $u = (u_1, \ldots, u_r)$, $u' = (u'_1, \ldots, u'_{r'})$ and let
$$u \wedge u' = (v_1, \ldots, v_p), \qquad u \vee u' = (t_1, \ldots, t_q).$$
We have $v_j = u_k \cap u'_l$ for some k, l. Suppose that there exists $w = s_m \in v_j$ which is inner with respect to $v_{j'} = u_{k'} \cap u_{l'}$ for $j \neq j'$. Then s_m would be inner with respect to u or u' since we must have $(k, l) \neq (k', l')$, which would imply that either u or u' is not admissible, which is a contradiction.

Suppose now that there exists a $w = s_m \in t_j$ which is inner with respect to $t_{j'}$, where $j \neq j'$. Then $s_m \in u_k$ for some k. Clearly, there do not exist $s_r, s_{r'} \in u_l$ where $l \neq k$ and $r < m < r'$ since in that case s_m would be inner with respect to u_l and u would not be admissible. Therefore, there must exist $s_r \in u_l$ and $s_{r'} \in u_{l'}$ with $l \neq l'$ (of course, $l, l' \neq k$). To fix ideas, let $1 \leq r < m < r' \leq n$. Note that all letters of $u_{l'}$ must follow s_m and all letters of u_l must precede s_m in s by the argument above. In a similar manner we can show that every subword of u' must either precede or follow s_m. This implies that the partition of s obtained from $u \vee u'$ by splitting $t_{j'}$ into $t_{j'} \cap s_1 \ldots s_{m-1}$ and $t_{j'} \cap s_{m+1} \ldots s_n$ is finer than $u \vee u'$ and coarser than both u and u', which is a contradiction. ∎

Each lattice $\mathcal{AP}(s)$, where $s = s_1 \ldots s_n$, has the unique minimal element $0_s = (s_1, \ldots, s_n)$ and the unique maximal element $1_s = (s) = s$. We will often skip the index s in the first notation if it is clear which lattice is considered, whereas s will be used instead of 1_s.

In order to apply the theory of Möbius functions to the combinatorics of moments and cumulants on $\mathcal{B}_0$, we need to take the union of the lattices of admissible partitions of all nonempty words:
$$P := \bigcup_{s \in S^+} \mathcal{AP}(s),$$
on which we introduce a partial order by the condition
$$u \leq v \quad \text{iff} \quad (\exists s \in S^+ : u, v \in \mathcal{AP}(s) \text{ and } u \preceq v)$$
where $\preceq$ is the usual partial order inherited from $\mathcal{P}_n$ for $l(s) = n$. As usual, we write $u < v$ if $u \leq v$ and $u \neq v$. By the segment $[u, v]$, where $u, v \in \mathcal{AP}(s)$, we mean the set of all partitions t such that $u \leq t \leq v$.

The *incidence algebra* of the partially ordered set P, denoted by $I(P)$, is the set of functions $f : P \times P \to \mathbb{C}$, with values denoted by $f(u|v)$, such that

$f(u|v) = 0$ unless $u \preceq v$. If the second argument of $f \in I(P)$ is a one-word partition of s, we often skip it and write

$$f(u|s) = f(u) = f(u_1, \ldots, u_p)$$

for $u = (u_1, \ldots, u_p)$.

EXAMPLE 1. Note that the order relation in P is stronger than taking a refinement. For instance,

$$u = (s_1 s_3, s_2 s_5, s_4) \preceq (s_1 s_3 s_4, s_2 s_5) = v$$

for any $s = s_1 s_2 s_3 s_4 s_5$, but

$$u \leq v \quad \text{iff} \quad s_2 = s_3 = s_4 = z$$

and thus $f(u|v) = 0$ unless $s_2 = s_3 = s_4 = z$, for any $f \in I(P)$.

The sum and multiplication by scalars in $I(P)$ are defined as usual. The product is given by

$$h(u|v) = \sum_{u \leq t \leq v} f(u|t) g(t|v)$$

and the identity element of the algebra is the Kronecker delta $\delta(u|v)$. The *zeta function* of P is defined as

$$\zeta(u|v) = \begin{cases} 1 & \text{if } u \leq v, \\ 0 & \text{otherwise,} \end{cases}$$

and the function $i(u|v) = \zeta(u|v) - \delta(u|v)$ is called the *incidence function*.

It is well known that the zeta function is invertible in the incidence algebra. The inverse is called the *Möbius function* and is given by the recursion

$$(5.1) \qquad m(u|v) = \begin{cases} 1 & \text{if } u = v, \\ -\sum_{u \leq t < v} m(u|t) & \text{otherwise,} \end{cases}$$

and the *Möbius inversion formula* reads:

$$(5.2) \qquad g(u) = \sum_{v \leq u} f(v) \;\Rightarrow\; f(u) = \sum_{v \leq u} m(v|u) g(v)$$

for $f, g : P \to \mathbb{C}$.

In order to apply the theory of Möbius functions to invert formula (4.1), define multiplicative functions $M(u)$ and $L(u)$ for u ranging over $\mathcal{AP}(s)$:

$$(5.3) \qquad\qquad M(u) = M(u_1) \ldots M(u_p),$$

$$(5.4) \qquad\qquad L(u) = L(u_1) \ldots L(u_p),$$

for $u = (u_1, \ldots, u_p) \in \mathcal{AP}(s)$. For applications of the Möbius inversion formula to free probability, see [Sp].

PROPOSITION 5.2. *The partition-dependent moments and cumulants satisfy the relation*

$$(5.5) \qquad M(u) = \sum_{v \leq u} L(v)$$

where $M(u)$ and $L(v)$ are given by (5.3)–(5.4) and $u \in \mathcal{AP}(s)$, $s \in S^+$.

Proof. Clearly, formula (5.5) holds for $u = s$ by Definition 4.1. We need to justify it for $u < s$. Using (5.3) and then expressing every $M(u_k)$ on the RHS of (5.3) in terms of cumulants according to (4.1), we get

$$(5.6) \qquad M(u) = \sum_{v^1 \leq u_1} \ldots \sum_{v^p \leq u_p} L(v^1) \ldots L(v^p)$$

where $v^1, \ldots, v^p$ are partitions of $u_1, \ldots, u_p$, respectively, and we only need to show that the RHS of (5.6) can be written in the form given by (5.5). First, let us show that if $v^1 \leq u_1, \ldots, v^p \leq u_p$, then $v := v^1 \cup \ldots \cup v^p \in \mathcal{AP}(s)$. Recall that $v^k \leq u_k$ means that v^k is an admissible partition of u_k, $k = 1, \ldots, p$. Suppose there exists a $w \in v_j^k$, where v_j^k is the jth subword of v^k, which is inner with respect to $v_{j'}^{k'}$, where $v_{j'}^{k'}$ is the j'th subword of $v^{k'}$. We must have $k = k'$ since otherwise this w would be inner with respect to the subword $u_{k'}$, which would imply that u is not admissible. Thus, assume that $k = k'$ (of course, in that case we must have $j \neq j'$). But then v^k is not an admissible partition of u_k, which is a contradiction. Therefore, $v \leq u$.

Suppose now that there exists $v < u$ which is not obtained from u by taking admissible partitions of the words $u_1, \ldots, u_k$. Clearly, $v = v^1 \cup \ldots \cup v^p$, where $v^k \in \mathcal{P}(u_k)$ for $k = 1, \ldots, p$. Since $v \in \mathcal{AP}(s)$, there is no w in one subword, say v_j^k, inner with respect to another subword $v_{j'}^{k'}$, where $(j, k) \neq (j', k')$. This implies that $v^k \in \mathcal{AP}(u_k)$. ∎

More generally, one can show that

$$(5.7) \qquad [v, u] \cong [v^1, u_1] \times \ldots \times [v^p, u_p]$$

where $u = (u_1, \ldots, u_p)$, $v^k = v \cap u_k$ is the partition of u_k consisting of those subwords of v whose union gives u_k, $[v, u]$ is a segment in the lattice $\mathcal{AP}(s)$, $[v^k, u_k]$ is a segment in $\mathcal{AP}(u_k)$ with u_k being treated as a subword of s.

THEOREM 5.3 (Inversion formula for cumulants). *Let $(M(s))_{s \in S}$ be the mixed moments on the noncommutative plane $\widehat{\mathcal{B}}_0$ in some state $\widehat{\phi}$. Then the corresponding admissible cumulants $(L(s))_{s \in S^+}$ are given by*

$$(5.8) \qquad L(s) = \sum_{u \leq s} m(u) M(u)$$

where $M(u)$ is given by (5.3) and $m(u) = m(u|s)$.

Proof. This is a special case of the general Möbius inversion formula given by (5.2), which can be used in view of Proposition 5.2. ∎

In the examples given below we compute certain $m(u)$, $u \in \mathcal{AP}(s)$, using the formula

$$(5.9) \qquad m(u|v) = \zeta^{-1}(u|v) = \delta(u|v) - i(u|v) + i^2(u|v)\ldots,$$

which expresses the Möbius function in terms of the incidence function [R1].

For notational simplicity we identify u_j with the number j and use a short-hand notation $j \cup k = u_j \cup u_k$, $j \cup k \cup l = u_j \cup u_k \cup u_l$, etc. For instance

$$m(1,2,3) = m((u_1, u_2, u_3)|u_1 \cup u_2 \cup u_3),$$
$$m(1 \cup 3, 2) = m((u_1 \cup u_3, u_2)|u_1 \cup u_2 \cup u_3),$$

where we also skip some parentheses for notational simplicity.

EXAMPLE 1. Let $u = (u_1, u_2, u_3) \in \mathcal{AP}(s)$. Then

$$\begin{aligned} m(1,2,3) = &-i(1,2,3) + i(1,2,3|1 \cup 2,3)i(1 \cup 2,3) \\ &+ i(1,2,3|1 \cup 3,2)i(1 \cup 3,2) + i(1,2,3|1,2 \cup 3)i(1,2 \cup 3) \end{aligned}$$

and, in particular, if the subwords of u are one-letter words, i.e. $u_k = s_k$, $k = 1, 2, 3$, this gives

$$m(1,2,3) = -1 + 1 + \delta_2 + 1 = 1 + \delta_2 = \begin{cases} 1 & \text{if } s_2 = w, \\ 2 & \text{if } s_2 = z, \end{cases}$$

for $s = s_1 s_2 s_3$, where the notation (4.10) is used.

EXAMPLE 2. In a similar manner, if blocks are one-letter words, we get

$$\begin{aligned} m(1 \cup 3, 2, 4) = &-i(1 \cup 3, 2, 4) \\ &+ i(1 \cup 3, 2, 4|1 \cup 2 \cup 3, 4)i(1 \cup 2 \cup 3, 4) \\ &+ i(1 \cup 3, 2, 4|1 \cup 3 \cup 4, 2)i(1 \cup 3 \cup 4, 2) \\ &+ i(1 \cup 3, 2, 4|1 \cup 3, 2 \cup 4)i(1 \cup 3, 2 \cup 4) \\ = &-\delta_2 + \delta_2 + \delta_2 + \delta_2 \delta_3 = \delta_2 + \delta_2 \delta_3 \\ = &\begin{cases} 0 & \text{if } s = s_1 w s_3 s_4, \\ 1 & \text{if } s = s_1 z w s_4, \\ 2 & \text{if } s = s_1 z^2 s_4, \end{cases} \end{aligned}$$

where $s_1, s_3, s_4 \in \{z, w\}$.

Looking at these examples, it is not hard to observe that when computing the Möbius function from formula (5.9), we get cancellations (the number of these gets quite large when $b(u)$ increases). Therefore, it is important to obtain a simpler formula for the Möbius function, in which all cancellations would be taken into account. This phenomenon is a rather typical but also nontrivial part of the theory (for some classical examples, see [R1] and [R2]).

In order to do that, we will introduce the notion of *admissible shuffles* of $u \in \mathcal{AP}(s)$. It will turn out that

$$m(u) = (-1)^{b(u)-1} a(u),$$

where $a(u)$ denotes the number of admissible shuffles of u, a noncommutative analog of $(p-1)!$, the number of ways we can shuffle the $p-1$ blocks of a partition consisting of p blocks, keeping the first block fixed.

DEFINITION 5.4. Let $u = (u_1, \ldots, u_p) \in \mathcal{AP}(s)$, $s \in S^+$. By an *admissible shuffle* of u we understand a sequence of admissible partitions of s of the form

$$u = u^0 \to u^1 \to \ldots \to u^k \to \ldots \to u^{p-1} = s$$

where

$$u^k = (u_1^k, \ldots, u_{n-k}^k) = j_l(u^{k-1}), \quad 1 \le l \le n-k,$$

and

$$j_l(v_1, \ldots, v_m) = (v_1, \ldots, v_l \cup v_m, v_{l+1}, \ldots, v_{m-1}), \quad 1 \le l \le m-1,$$

i.e. each transition of the shuffle amounts to moving the last subword v_m to one of the previous subwords v_l and then forming $v_l \cup v_m$.

EXAMPLE 3. The shuffle

$$(1, 2, 3, 4) \to (1, 2 \cup 4, 3) \to (1 \cup 3, 2 \cup 4) \to (1 \cup 2 \cup 3 \cup 4)$$

is admissible if and only if $i(2,4) = i(1,3) = i(1 \cup 3, 2 \cup 4) = 1$. If $u = (1, 2, 3, 4)$ is admissible with $u_k = s_k$ for all k, then clearly $i(2,4) = i(1,3) = 1$, so we are left with the condition $i(1 \cup 3, 2 \cup 4) = 1$, which implies that we must have $\delta_2 = \delta_3 = 1$, i.e. $s_2 = s_3 = z$. The shuffle

$$(1, 2, 3, 4) \to (1, 2, 3 \cup 4) \to (1 \cup 3 \cup 4, 2) \to (1 \cup 2 \cup 3 \cup 4)$$

is admissible if and only if $i(1 \cup 3 \cup 4, 2) = 1$, which implies that we must have $\delta_2 = 1$, i.e. $s_2 = z$.

DEFINITION 5.5. Let $a(u)$ denote the number of admissible shuffles of $u \in \mathcal{AP}(s)$ (then $1 \le a(u) \le (n-1)!$). If $u \notin \mathcal{AP}(s)$, we set $a(u) = 0$.

PROPOSITION 5.6. *Let* $u = (u_1, \ldots, u_p) \in \mathcal{AP}(s)$, $p \ge 1$. *Then*

$$a(u) = \begin{cases} i(u) & \text{if } p \le 2, \\ \displaystyle\sum_{k=1}^{p-1} i(u_k, u_p) a(u_1, \ldots, u_k \cup u_p, \ldots, u_{p-1}) & \text{if } p > 2. \end{cases}$$

Proof. This recurrence formula is an easy consequence of Definitions 5.4–5.5. ∎

EXAMPLE 4. For simplicity, assume that u consists of one-letter words. We get

$$a(1, 2, 3) = i(1, 3)i(1 \cup 3, 2) + i(2, 3)i(1, 2 \cup 3) = \delta_2 + 1,$$

which can be seen to coincide with $m(1,2,3)$ (cf. Example 1). In a similar manner, we get

$$
\begin{aligned}
a(1,2,3,4) &= i(1,4)i(1\cup 4,3)i(1\cup 3\cup 4,2) + i(1,4)i(2,3)i(1\cup 4,2\cup 3) \\
&\quad + i(2,4)i(1,3)i(1\cup 4,2\cup 3) + i(2,4)i(2\cup 4,3)i(1,2\cup 3\cup 4) \\
&\quad + i(3,4)i(1,3\cup 4)i(1\cup 3\cup 4,2) + i(3,4)i(2,3\cup 4)i(1,2\cup 3\cup 4) \\
&= \delta_2\delta_3 + \delta_2\delta_3 + \delta_2\delta_3 + \delta_3 + \delta_2 + 1 \\
&= \begin{cases} 1 & \text{if } s = s_1 w^2 s_4, \\ 2 & \text{if } s \in \{s_1 wz s_4, s_1 zw s_4\}, \\ 6 & \text{if } s = s_1 z^2 s_4, \end{cases}
\end{aligned}
$$

where $s_1, s_4 \in \{z, w\}$.

In order to find a connection between the Möbius function $m(v)$ in terms of the number of admissible shuffles of $v \in \mathcal{AP}(s)$, we first need to express $m(v)$ in terms of the Möbius functions of words s' where $l(s') < l(s)$.

PROPOSITION 5.7. *If $v \in \mathcal{AP}(s)$ and $v < s$, where $s \in S^+$, then*

$$
m(v) = -\sum_{v \leq u < s} m(v|u)
$$

where $m(v|u) = m(v^1|u_1)\ldots m(v^p|u_p)$ for $u = (u_1, \ldots, u_p)$ and $v^k = v \cap u_k$ is the partition of u_k consisting of those words of v whose union gives u_k.

Proof. Applying formula (5.8) to every $L(u_k)$ on the RHS of (5.4), we get

$$
L(u) = \sum_{v^1 \leq u_1} \cdots \sum_{v^p \leq u_p} m(v^1|u_1)\ldots m(v^p|u_p) M(v^1)\ldots M(v^p),
$$

which, in view of (5.2), gives $m(v|u) = m(v^1|u_1)\ldots m(v^p|u_p)$ by arguments similar to those in the proof of Proposition 5.2. Therefore, we get

$$
\begin{aligned}
L(s) &= M(s) - \sum_{0 \leq u < s} L(u) = M(s) - \sum_{0 \leq u < s}\sum_{v \leq u} m(v|u) M(v) \\
&= M(s) - \sum_{0 \leq v < s}\sum_{v \leq u < s} m(v|u) M(v),
\end{aligned}
$$

which gives the desired formula for $m(v)$. ∎

DEFINITION 5.8. We say that u *covers* v, where $u, v \in P$, $u \leq v$, if the segment $[v, u]$ contains two elements. An *atom* in P is an element that covers 0_s (a minimal element of P) for some s, and a *dual atom* is an element that is covered by 1_s (a maximal element of P) for some $s \in S^+$ (see [R1]). Denote by $D(s)$ the set of dual atoms covered by s.

PROPOSITION 5.9. *If $v \in \mathcal{AP}(s)$ and $v < s$, where $s \in S^+$, then*

$$(5.10) \qquad m(v) = - \sum_{\substack{u \in D(s) \\ v_1, v_2 \text{ separated}}} m(v|u)$$

where the summation runs over all dual atoms $u = (u_1, u_2)$ of s in which $v_1 \subset u_1$, $v_2 \subset u_2$, i.e. the first two subwords of v are separated in u.

Proof. Clearly, (5.10) holds for $b(v) = 2$ since in that case LHS $= -i(v) = -1$ and RHS $= m(v_1|v_1)m(v_2|v_2) = -1 \cdot 1 = -1$. Suppose (5.10) holds for every $v \in \mathcal{AP}(s)$ with $2 \le b(v) \le p - 1$. We will show that it then holds for $b(v) = p$. Let $v = (v_1, \dots, v_p) \in \mathcal{AP}(s)$. We need to show that

$$E := m(v) + \sum_{\substack{u \in D(s) \\ v_1, v_2 \text{ separated}}} m(v|u) = 0.$$

We claim that

$$(5.11) \qquad E = - \sum_{\substack{u \ge v, b(u)=r \\ v_1, v_2 \subset u_1}} m(v|u) - \sum_{\substack{u \ge v \\ b(u) > r}} m(v|u)$$

for $r = 2, \dots, p - 1$. In view of Proposition 5.7, we have

$$E = - \sum_{\substack{u \ge v, b(u)=2 \\ v_1, v_2 \subset u_1}} m(v|u) - \sum_{\substack{u \ge v \\ b(u) > 2}} m(v|u),$$

thus (5.11) holds for $r = 2$. Suppose now that (5.11) holds for $r = k$, use multiplicativity of $m(v|u)$ in the first sum:

$$m(v|u) = m(v^1|u_1) \dots m(v^k|u_k),$$

and apply (5.10) for $b(v) \le p-1$ to $m(v^1|u_1)$ (note that $b(v^1) \le p-1$) to get

$$E = - \sum_{\substack{u \ge v, b(u)=k \\ v_1, v_2 \subset u_1}} \sum_{\substack{t \in D(u_1) \\ v_1, v_2 \text{ separated}}} m(v^1|t)m(v^2|u_2) \dots m(v^k|u_k) - \sum_{\substack{u \ge v \\ b(u) > k}} m(v|u)$$

$$= - \sum_{\substack{u \ge v, b(u)=k+1 \\ v_1, v_2 \subset u_1}} m(v|u) - \sum_{\substack{u \ge v \\ b(u) > k+1}} m(v|u).$$

This equality is justified as follows. If $u \ge v$, $b(u) = k$ and $v_1, v_2 \subset u_1$ and $t = (u_1', u_1'')$ is a dual atom of u_1 which separates v_1 and v_2, then the partition $(u_1', u_1'', u_2, \dots, u_k)$ is admissible. Next, if $u \ge v$, $b(u) = k+1$, then either $v_1, v_2 \subset u_1$, or $v_1 \subset u_1$ and $v_2 \subset u_2$. In the second case, there exists exactly one pair (u', t), where $u' = (u_1', \dots, u_k') \in \mathcal{AP}(s)$ and t is a dual atom of the first subword u_1' such that the resulting partition is u (take u_1 and u_2, form $u_1 \cup u_2$, keep the remaining u_l's the same and let the dual atom

of $u_1 \cup u_2$ be (u_1, u_2)). Therefore, (5.11) holds for all $2 \le r \le p - 1$. But, if $r = p - 1$, then (5.11) takes the form

$$E = -m(v_1, v_2 | v_1 \cup v_2) m(v_3 | v_3) \ldots m(v_p | v_p) - m(v_1 | v_1) \ldots m(v_p | v_p) = 0. \quad \blacksquare$$

COROLLARY 5.10. *For all* $v \in \mathcal{AP}(s)$, *where* $s \in S^+$, *we have*

$$(5.12) \qquad\qquad m(v) = (-1)^{b(v)-1} a(v).$$

Proof. Clearly, $m(s) = 1 = a(s)$ and if $v = (v_1, v_2)$, then $m(v) = -i(v) = -a(v)$. We will show that if (5.12) holds for $b(v) \le p - 1$, then it holds for $b(v) = p$. Let $b(v) = p$ and use Proposition 5.9 and the inductive assumption to get

$$m(v) = - \sum_{\substack{v = v^1 \cup v^2 \\ v_1 \in v^1, v_2 \in v^2}} (-1)^{b(v^1) + b(v^2) - 2} i(u_1, u_2) a(v^1) a(v^2)$$

$$= (-1)^{b(v)-1} \sum_{\substack{v = v^1 \cup v^2 \\ v_1 \in v^1, v_2 \in v^2}} i(u_1, u_2) a(v^1) a(v^2)$$

where (u_1, u_2)'s are the dual atoms which appear on the RHS of (5.10). Now, from Definition 5.4 we can see that in order to count all admissible shuffles of $v = (v_1, \ldots, v_p) \in \mathcal{AP}(s)$ it is enough to count all admissible shuffles of v which lead to the dual atom (u_1, u_2) after $p - 2$ transitions such that $v_1 \subset u_1$ and $v_2 \subset u_2$ (here u_1, u_2 correspond to v_1^{p-2}, v_2^{p-2} of Definition 5.4)—this computation gives the product $a(v^1) a(v^2)$—and then take into account only those of such pairs which give an admissible partition of s (this gives $i(u_1, u_2)$). $\blacksquare$

EXAMPLE 5. We apply Corollary 5.10 to compute $m(u)$ needed for the cumulants of lowest order. For simplicity, we write $L(i_1 \ldots i_n) = L(s_{i_1} \ldots s_{i_n})$ and $M(i_1 \ldots i_n) = M(s_{i_1} \ldots s_{i_n})$. We obtain

$$L(1) = M(1),$$

$$L(12) = M(12) - M(1)M(2),$$

$$L(123) = M(123) - M(12)M(3) - M(1)M(23) - \delta_2 M(13)M(2)$$
$$\qquad\qquad + (1 + \delta_2)M(1)M(2)M(3),$$

$$L(1234) = M(1234) - M(123)M(4) - \delta_2 M(134)M(2) - \delta_3 M(124)M(3)$$
$$\qquad\qquad - M(1)M(234) - M(12)M(34) - \delta_2 \delta_3 M(14)M(23)$$
$$\qquad\qquad - \delta_2 \delta_3 M(13)M(24) + (1 + \delta_3)M(12)M(3)M(4)$$
$$\qquad\qquad + \delta_2(1 + \delta_3)M(13)M(2)M(4) + 2\delta_2\delta_3 M(14)M(2)M(3)$$
$$\qquad\qquad + (1 + \delta_2\delta_3)M(1)M(23)M(4) + \delta_3(1 + \delta_2)M(1)M(24)M(3)$$
$$\qquad\qquad + (1 + \delta_2)M(1)M(2)M(34)$$
$$\qquad\qquad - (1 + \delta_2 + \delta_3 + 3\delta_2\delta_3)M(1)M(2)M(3)M(4).$$

One can recognize some coefficients computed in Examples 1, 2, 4. Also note that if all δ's are equal to 1 (i.e. $s = s_1 z^2 s_4$), we get classical cumulants, whereas if all δ's are equal to 0 (i.e. $s = s_1 w^2 s_4$), we get boolean cumulants.

6. Semigroup algebras. Let us now briefly recall some basic facts on the free semigroup algebra $\mathcal{A}(S)$ and the Banach algebra $l^1(S, W)$, where W is a weight function. For more on this subject see [P].

The *free semigroup algebra* of S, denoted by $\mathcal{A}(S)$, is the set of all functions $f : S \to \mathbb{C}$ equipped with the usual addition and convolution multiplication given by

$$(6.1) \qquad f \star g(s) = \sum_{uv=s} f(u)g(v).$$

Note that, in general, $f \star g \neq g \star f$, so the algebra $\mathcal{A}(S)$ is noncommutative.

It is convenient to identify elements f of both algebras with formal sums

$$\sum_{s \in S} f(s)s$$

with multiplication

$$(6.2) \qquad \sum_{t \in S} f(t)t \sum_{r \in S} g(r)r = \sum_{s \in S} \sum_{tr=s} f(t)g(r)s.$$

The algebra $\mathcal{A}(S)$ is our noncommutative analog of the algebra $\mathbb{C}[[z]]$ of formal power series in the variable z. The unit of the algebra is denoted by $\mathbf{1}$, where $\mathbf{1}(s) = 1$ if $s = 1$ and zero otherwise.

PROPOSITION 6.1. *Let $f \in \mathcal{A}(S)$ and assume that $f(1) = 1$. Then f is invertible in $\mathcal{A}(S)$, i.e. there exists $f^{-1} \in \mathcal{A}(S)$ such that $f^{-1} \star f = f \star f^{-1} = \mathbf{1}$, $f^{-1}(1) = 1$ and*

$$f^{-1}(s) = \sum_{p=1}^{l(s)} (-1)^p \sum_{s=u_1 \ldots u_p} f(u_1) \ldots f(u_p)$$

for any $s \in S^+$, where the second sum is taken over all factorizations of s.

We omit the proof since it is a straightforward computation.

DEFINITION 6.2. For given $f \in \mathcal{A}(S)$ and $g \in \mathcal{A}(S(z))$, we define $f_{\star g}$ to be the function from $\mathcal{A}(S)$ which agrees with f on $S(z)$ and, on $S \setminus S(z)$, is given by

$$f_{\star g}(z_1 w_1 z_2 w_2 \ldots z_{p-1} w_{p-1} z_p)$$
$$= \sum_{u_1 v_1 = z_1} \ldots \sum_{u_{p-1} v_{p-1} = z_{p-1}} f(u_1 w_1 u_2 w_2 \ldots u_{p-1} w_{p-1} z_p) g(v_1) \ldots g(v_{p-1})$$

for $w_1, \ldots, w_{p-1} \in S(w) \setminus \{1\}$, $z_1, z_p \in S(z)$, $z_2, \ldots, z_{p-1} \in S(z) \setminus \{1\}$, $p \geq 2$.

One can think of $f_{\star g}$ as a "composition" of g and f in which $w_1, \ldots, w_{p-1}$ are replaced by $gw_1, \ldots, gw_{p-1}$. Thus, we can informally write

$$f_{\star g} = \sum_{s \in S} f(s) s_{\star g}$$

where $s_{\star g}$ agrees with s for $s \in S(z)$ and

$$s_{\star g} = z_1 g w_1 z_2 g w_2 \ldots z_{p-1} g w_{p-1} z_p$$

for

$$s = z_1 w_1 z_2 w_2 \ldots z_{p-1} w_{p-1} z_p \in S \setminus S(z),$$

with the assumptions as in Definition 6.2.

Note also that if $f \in \mathcal{A}(S)$, $g \in \mathcal{A}(S(z))$, with $g(1) \neq 0$, then the following implication holds:

$$(6.3) \qquad\qquad f_{\star g} = h \;\Rightarrow\; f = h_{\star g^{-1}},$$

which is a straightforward consequence of Definition 6.2.

By a *weight function* on S we understand a (real-valued) positive function on S which is submultiplicative, i.e.

$$W(st) \le W(s)W(t), \quad \forall s, t \in S,$$

and we denote by $l^1(S, W)$ the Banach space of all functions $f : S \to \mathbb{C}$ which have finite norm

$$\|f\|_W = \sum_{s \in S} W(s)|f(s)|;$$

it becomes a Banach algebra under the convolution multiplication (6.1) (see [P]). The l^1-*semigroup algebra* of S ([B-D]), denoted by $l^1(S)$, is obtained if $W(s) = 1$ for all $s \in S$.

PROPOSITION 6.3. *Let* $f \in l^1(S, W)$ *with* $f(1) = 1$ *and* $g \in l^1(S(z), W)$, *where* $W(1) = 1$ *and* $W(st) = W(ts)$ *for all* $t, s \in S$, *and let* $Q > 1/2$. *Then the following implications hold*:

(i) *if* $\|f\|_W \le 2 - 1/Q$, *then* $\|f^{-1}\|_W \le Q$,
(ii) *if* $\|g\|_W < Q$, *then* $\|f_{\star g}\|_{\widetilde{W}} \le \|f\|_W$,

where $\widetilde{W}(s) = W(s)Q^{-m(s)}$ *and* $m(s)$ *is the number of* w's *in the word* s.

Proof. Using Proposition 6.1, the triangle inequality and submultiplicativity of W, we arrive at

$$\|f^{-1}\|_W = 1 + \sum_{s \neq 1} W(s) \left| \sum_{p=1}^{l(s)} (-1)^p \sum_{s = u_1 \ldots u_p} f(u_1) \ldots f(u_p) \right|$$

$$\le 1 + \sum_{s \neq 1} W(s)|f(s)| + \sum_{s \neq 1} W(s) \left| \sum_{s = u_1 u_2} f(u_1)f(u_2) \right| + \ldots$$

$$= \|f\|_W + (\|f\|_W - 1)^2 + \ldots = \frac{1}{2 - \|f\|_W},$$

from which (i) follows. Now,

$$\|f_{\star g}\|_W = \sum_{s\in S(z)} W(s)|f(s)| + \sum_{s\in S\setminus S(z)} W(s)|f_{\star g}(s)|,$$

and an estimate of the second sum is needed. In the sums below we will always assume that all w_k's belong to $S(w)\setminus\{1\}$ and that all z_k's belong to $S(z)$ (additional restrictions on z_k's will be given explicitly). Therefore

$$\sum_{s\in S\setminus S(z)} \widetilde{W}(s)|f_{\star g}(s)| = \sum_{p=2}^{\infty} \sum_{w_1,\ldots,w_{p-1}} \sum_{\substack{z_1,\ldots,z_p \\ z_2,\ldots,z_{p-1}\neq 1}} \widetilde{W}(z_1 w_1 z_2 \ldots z_{p-1} w_{p-1} z_p)$$

$$\times |f_{\star g}(z_1 w_1 z_2 \ldots z_{p-1} w_{p-1} z_p)|$$

$$\leq \sum_{p=2}^{\infty} \sum_{w_1,\ldots,w_{p-1}} \sum_{\substack{z_1,\ldots,z_p \\ z_2,\ldots,z_{p-1}\neq 1}} \sum_{u_1 v_1 = z_1} \cdots \sum_{u_{p-1} v_{p-1} = z_{p-1}}$$

$$\widetilde{W}(u_1 w_1 u_2 \ldots u_{p-1} w_{p-1} z_p)$$

$$\times |f(u_1 w_1 u_2 \ldots u_{p-1} w_{p-1} z_p)| \prod_{l=1}^{p-1} W(v_l)|g(v_l)|$$

$$\leq \sum_{r\in S\setminus S(z)} \widetilde{W}(r)|f(r)| \cdot \|g\|_W^{m(r)}$$

where we used submultiplicativity of W and $W(st) = W(ts)$. When we use the definition of $\widetilde{W}$ and combine the above estimate with the sum over $S(z)$, we get (ii). ∎

7. Moment and cumulant generating functions. In this section we introduce moment and cumulant generating functions associated with filtered convolution and derive a connection between them.

Let us first establish a moment-cumulant formula, which expresses the moments in terms of the cumulants *and* the moments (in contrast to the inversion formula which expresses the cumulants in terms of the moments only). This formula turns out very useful in deriving an explicit form of the cumulant generating function.

For given $s \in S \setminus S(z)$ we denote by $C_0(s)$ the subset of cumulant subwords of s which contain the first letter w in s. If $r \in C_0(s)$, we denote by $W_r(s)$ the set of subwords of $s \setminus r$ of maximal length which lie between the w-legs of r or before the first w of r. Note that they have to be z-words since otherwise r would have an inner w, which is not possible since r is a cumulant word.

LEMMA 7.1 (Moment-cumulant formula). *For each $s \in S \setminus S(z)$ we have the following moment-cumulant formula*:

$$(7.1) \quad M(s) = \sum_{r \in C_0(s)} L(r) \prod_{v \in W_r(s)} M(v) \times M\left(s \setminus \left(r \cup \bigcup_{v \in W_r(s)} v\right)\right).$$

Proof. From Definition 4.1 we have

$$M(s) = \sum_{p=1}^{l(s)} \sum_{u=(u_1,\dots,u_p) \in \mathcal{AP}(s)} L(u_1)\dots L(u_p)$$

for $s \in S \setminus S(z)$. Since s must contain at least one w, in each of the summands on the RHS of the above formula we must have one cumulant, say $L(u_j)$, such that u_j contains the first w of the word s. Denote, for fixed u, this u_j by $r = r(u)$. From the definition of admissible partitions of s, letters from $s \setminus r$ lying between two w-legs of r cannot be connected to letters lying between another pair of w-legs of r since otherwise they would be separated by a w. The same is true for the letters lying to the left of the first w. Moreover, these letters have to be z's. Therefore, the product of cumulants corresponding to such z-words $v \in W_r(s)$ have to be taken over all subpartitions of v, giving $M(v)$. The same applies to the word $v \in W_r(s)$ formed from all z's which are to the left of the first w and are not in r. Altogether, these products of cumulants give

$$\prod_{v \in W_r(s)} M(v).$$

The remaining cumulants from the product $L(u_1)\dots L(u_p)$ involve only letters lying to the right of the last w of the word r which are not in r. Their product gives

$$M\left(s \setminus \left(r \cup \bigcup_{u \in W_r(s)} v\right)\right). \quad \blacksquare$$

EXAMPLE 1. To illustrate the moment-cumulant formula, let us give a diagram corresponding to one of the summands on the RHS of (7.1). We choose the word $s = z^3wz^3wz^2wz$ (long enough to show some general features of the combinatorics involved). The diagram
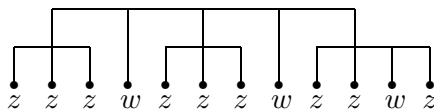
corresponds to

$$L(zwzwz) \times M(z^2)M(z^2) \times M(zwz)$$

and the upper line connects all letters associated with the cumulant, whereas the lower line connects all letters associated with the moments.

REMARK 1. If $s \in S(z)$ (the case not treated in Proposition 7.1), we get the classical moment-cumulant formula

$$(7.2) \qquad M(z^n) = \sum_{k=1}^{n} \binom{n-1}{k-1} L(z^k) M(z^{n-k}),$$

where $n \geq 1$, i.e. $M(z^n)$, $L(z^n)$ are classical moments and cumulants of order n, respectively. Now, formulas (7.1) and (7.2) cover the cases of all $s \in S$. In (7.2) we already counted the number of ways, namely $\binom{n-1}{k-1}$, of choosing all subwords r of $s = z^n$ which are equal to z^k as words and contain the first z in s—these subwords correspond to the cumulant $L(z^k)$—and the product of the remaining cumulants on the RHS of (4.1) gives $M(z^{n-k})$ since all partitions of $s \setminus r$ are allowed.

REMARK 2. On the other hand, the boolean moment-cumulant formula [Sp-W] is a special case of (7.1) and

$$(7.3) \qquad M(w^n) = \sum_{k=1}^{n} L(w^k) M(w^{n-k}),$$

where $M(w^n)$ and $L(w^n)$ are boolean moments and cumulants of order n, respectively. In this case, all cumulant subwords of $s = w^n$ containing the first w are of the form $r = s_1 \ldots s_k = w^k$ (for each k there is only one such subword), the product over $W_r(s)$ in (7.1) disappears and the remaining moment is therefore equal to $M(s \setminus r) = M(w^{n-k})$.

Note that there is a formal similarity between our formula and the moment-cumulant formula in the conditionally-free case [Bo-Le-Sp]. Nevertheless, there is a substantial difference between the two cases—our formula involves not only noncrossing partitions, which later gives rise to some classical features in the generating functions, namely they are analogs of *exponential* generating functions.

DEFINITION 7.2. Let $(M(s))_{s \in S}$ and $(L(s))_{s \in S}$ be the moments and cumulants associated with the state $\widehat{\phi}$. The corresponding *moment* and *cumulant generating functions* are defined to be the elements of the algebra $\mathcal{A}(S)$ given by the formal sums

$$M\{z, w\} = \sum_{s \in S} \frac{M(s)}{n(s)!} \, s,$$

$$L\{z, w\} = \sum_{1 \neq s \in S} \frac{L(s)}{n(s)!} \, s,$$

respectively, where

(7.4) $n(s)! = n_1! n_2! \ldots n_p!$ for $s = z^{n_1} w^{k_1} z^{n_2} w^{k_2} \ldots w^{k_{p-1}} z^{n_p}$

with $n_1, n_p \in \mathbb{N}_0$ and $k_1, n_2, k_2, \ldots, n_{p-1}, k_{p-1} \in \mathbb{N}$.

Now, let us use Definition 6.2 to introduce new notations

(7.5) $L^\star\{z, w\} = L_{\star M}\{z, w\},$

(7.6) $M_\star\{z, w\} = M_{\star M^{-1}}\{z, w\},$

where we take $f_{\star g}$ with $f = L\{z, w\}$ and $g = M\{z, 0\}$ in (7.5), and with $f = M\{z, w\}$ and $g = M^{-1}\{z, 0\}$ in (7.6). Here,

$$M\{z, 0\} = \sum_{s \in S(z)} \frac{M(s)}{l(s)!} \, s,$$

i.e. $M\{0, z\}$ is the restriction of $M\{z, w\}$ to the support $S(z)$ (then $n(s) = l(s)$) and can be treated as a formal power series in z representing the classical moment generating function. In a similar way we define $L\{z, 0\}$, the classical cumulant generating function, as well as $M\{0, w\}$ and $L\{0, w\}$ (in these two cases, by restricting the support to $S(w)$). Note that $M^{-1}\{z, 0\}$, the inverse of $M\{z, 0\}$, exists since $M(1) = 1$. Also note that $M_\star\{z, 0\} = M\{z, 0\}$ and $L^\star\{z, 0\} = L\{z, 0\}$.

Moreover, for $f = f\{z, w\} \in \mathcal{A}(S)$, we will use a special notation for the difference

(7.7) $\delta f\{z, w\} = f\{z, w\} - f\{z, 0\},$

representing the "deviation from the classical case", and apply this notation to $\delta M\{z, w\}$, $\delta M_\star\{z, w\}$, $\delta L\{z, w\}$ and $\delta L^\star\{z, w\}$.

Using these notations we can write down formulas which connect the cumulant generating function with the moment generating function.

THEOREM 7.3. *The moment and cumulant generating functions of Definition* 7.2 *satisfy the relation*

(7.8) $\delta M\{z, w\} = \delta L^\star\{z, w\} M\{z, w\}$

with the multiplication of formal sums given by (6.2).

Proof. Note that

$$\delta M\{z, w\} = \sum_{s \in S \setminus S(z)} \frac{M(s)}{n(s)!} \, s$$

and consider the RHS of the above equation. Use the moment-cumulant formula of Lemma 7.1 for each $M(s)$ with $s \in S \setminus S(z)$ (there is at least one w in each s). On the RHS of (7.1) we have to compute the number of ways in which the same *word* is obtained by taking different *subwords* of the

word s (i.e. different subsequences). Suppose s is of the form given by (7.4). If $r \in C_0(s)$, then r either ends with a z, i.e.

$$(7.9) \qquad r = z^{i_1} w^{k_1} z^{i_2} w^{k_2} \ldots z^{i_{m-1}} w^{k_{m-1}} z^{i_m} \qquad \text{with } m \leq p, \ 1 \leq i_m \leq n_m,$$

or ends with a w, i.e.

$$(7.10) \qquad r = z^{i_1} w^{k_1} z^{i_2} w^{k_2} \ldots z^{i_m} w^{q_m} \qquad \text{with } m \leq p - 1, \ 1 \leq q_m \leq k_m,$$

where all powers are assumed to be positive except perhaps i_1, which may be zero. Note that r has to assume one of these forms (i.e. the powers of w have to coincide with those in s up to some place) since otherwise the partition $u = (u_1, \ldots, u_p)$ corresponding to the product $L(u_1) \ldots L(u_p)$ of cumulants would not be admissible.

We will use the multiindex notation

$$\binom{n(s)}{n(r)} = \binom{n_1}{i_1} \ldots \binom{n_p}{i_p}$$

where

$$n(s) = (n_1, \ldots, n_p), \qquad n(r) = (i_1, \ldots, i_p)$$

for s of the form (7.4) and r given by (7.9) or (7.10), where we set $i_{m+1} = \ldots = i_p = 0$.

Thus, if we want to include in the summation only those words corresponding to $r \in C_0(s)$ which are distinct, we get

$$M(s) = \sum_{\substack{r \in C_0(s) \\ \text{distinct words}}} L(r) \binom{n(s)}{n(r)} \prod_{v \in W_r(s)} M(v) \, M\Big(s \setminus \Big(r \cup \bigcup_{v \in W_r(s)} v \Big) \Big),$$

which leads to the equation

$$\delta M\{z, w\} = \sum_{s \in S \setminus S(z)} \sum_{\substack{r \in C_0(s) \\ \text{distinct words}}} \frac{L(r)}{n(r)!} \cdot \frac{1}{(n(s) - n(r))!} \prod_{v \in W_r(s)} M(v)$$

$$\times M\Big(s \setminus \Big(r \cup \bigcup_{v \in W_r(s)} v \Big) \Big) s$$

where $(n(s) - n(r))! = (n_1 - i_1)! \ldots (n_p - i_p)!$. In order to demonstrate (7.8), we need to show that

$$\delta M\{z, w\}(s) = \delta L^\star \{z, w\} M\{z, w\}(s)$$

(treated as elements of $\mathcal{A}(S)$) for every $s \in S \setminus S(z)$. Let us write $\delta L^\star \{z, w\}$ informally as

$$\delta L^\star = \sum_{r \in S \setminus S(z)} \frac{L(r)}{n(r)!} \, r^\star$$

where

$$r^\star = z^{i_1} M\{z, 0\} w^{k_1} z^{i_2} M\{z, 0\} w^{k_2} \ldots z^{i_{m-1}} M\{z, 0\} w^{k_{m-1}} z^{i_m}$$

for r of the form (7.9) (we allow $i_m = 0$, which means that the case (7.10) is also covered). Using this notation, we can write

$$\delta L^\star\{z, w\}M\{z, w\} = \sum_{r \in S \setminus S(z)} \sum_{t \in S} \frac{L(r)}{n(r)!} \cdot \frac{M(t)}{n(t)!} r^\star t = f_1\{z, w\} + f_2\{z, w\}$$

where we have split the double sum into two sums: in the first one between the last w of $r^\star$ and the first w of t there is a z, while in the second one there are no such z's.

We have

$$f_1\{z, w\}(z^{n_1}w^{k_1}z^{n_2}w^{k_2} \ldots z^{n_{p-1}}w^{k_{p-1}}z^{n_p})$$

$$= \sum_{m=1}^{p} \sum_{i_1=0}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_{m-1}=1}^{n_{m-1}} \sum_{i_m=1}^{n_m} \frac{1}{i_1! \ldots i_m!} \cdot \frac{1}{(n_1 - i_1)! \ldots (n_{m-1} - i_{m-1})!}$$

$$\times L(z^{i_1}w^{k_1}z^{i_2}w^{k_2} \ldots z^{i_{m-1}}w^{k_{m-1}}z^{i_m})M(z^{n_1-i_1}) \ldots M(z^{n_{m-1}-i_{m-1}})$$

$$\times \frac{1}{(n_m - i_m)! n_{m+1}! \ldots n_p!} M(z^{n_m-i_m}w^{k_m}z^{n_{m+1}} \ldots z^{n_p})$$

and

$$f_2\{z, w\}(z^{n_1}w^{k_1}z^{n_2}w^{k_2} \ldots z^{n_{p-1}}w^{k_{p-1}}z^{n_p})$$

$$= \sum_{m=1}^{p-1} \sum_{i_1=0}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_{m-1}=1}^{n_{m-1}} \sum_{q_{m-1}=1}^{k_{m-1}} \frac{1}{i_1! \ldots i_{m-1}!} \cdot \frac{1}{(n_1 - i_1)! \ldots (n_{m-1} - i_{m-1})!}$$

$$\times L(z^{i_1}w^{k_1}z^{i_2}w^{k_2} \ldots z^{i_{m-1}}w^{q_{m-1}})M(z^{n_1-i_1}) \ldots M(z^{n_{m-1}-i_{m-1}})$$

$$\times \frac{1}{n_m! n_{m+1}! \ldots n_p!} M(w^{k_{m-1}-q_{m-1}}z^{n_m}w^{k_m} \ldots z^{n_p}).$$

It is not hard to see that

$$f_1\{z, w\}(s) = \sum_{\substack{r \in C_0'(s) \\ \text{distinct words}}} \frac{L(r)}{n(r)!} \cdot \frac{1}{(n(s) - n(r))!} \prod_{v \in W_r(s)} M(v)$$

$$\times M\Big(s \setminus \Big(r \cup \bigcup_{v \in W_r(s)} v\Big)\Big),$$

$$f_2\{z, w\}(s) = \sum_{\substack{r \in C_0''(s) \\ \text{distinct words}}} \frac{L(r)}{n(r)!} \cdot \frac{1}{(n(s) - n(r))!} \prod_{v \in W_r(s)} M(v)$$

$$\times M\Big(s \setminus \Big(r \cup \bigcup_{v \in W_r(s)} v\Big)\Big),$$

where $C_0'(s)$ and $C_0''(s)$ are the subsets of $C_0(s)$ which consist of those sub-words which end with z and w, respectively (summations are taken only over subwords which give distinct words).

Thus $f_1\{z,w\}(s) + f_2\{z,w\}(s) = \delta M(s)$, which finishes the proof. ∎

COROLLARY 7.4. *The cumulant generating function has the form*

$$(7.11) \qquad L\{z,w\} = L\{z,0\} + (M_\star\{z,w\} - M\{z,0\})M_\star^{-1}\{z,w\}$$

where the notation (7.5)–(7.7) *is used.*

Proof. By multiplying (7.8) from the right by the inverse of $M\{z,w\}$, which exists since $M(1) = 1$, we get

$$\delta M\{z,w\}M^{-1}\{z,w\} = \delta L^\star\{z,w\}$$

but now, using (6.3) we can get rid of $\star$ on the RHS. This leads to

$$\delta M_\star\{z,w\}M_\star^{-1}\{z,w\} = \delta L\{z,w\},$$

which is equivalent to (7.11). ∎

REMARK. The classical moment and cumulant generating functions can be identified with $M\{z,0\}$ (the Fourier transform) and $L\{z,0\}$ (its logarithm), respectively, and are related through the classical equation

$$L\{z,0\} = \log M\{z,0\},$$

which can be derived from (7.2) in the usual manner. Moreover, the boolean moment and cumulant generating functions can be identified with $M\{0,w\}$ and $L\{0,w\}$, respectively, and are related through the equation

$$L\{0,w\} = (M\{0,w\} - 1)M^{-1}\{0,w\},$$

which is a special case of (7.11) since $M_\star\{0,w\} = M\{0,w\}$, $L\{0,0\} = 0$ and $M_\star^{-1}\{0,w\} = M\{0,w\}^{-1}$.

COROLLARY 7.5. *If* $\|M\{z,w\}\|_W < 2 - 1/Q$ *and* $L\{z,0\} \in l^1(S(z), W)$, *where W and Q satisfy the assumptions of Proposition* 6.3, *then equation* (7.11) *holds in* $l^1(S, \widetilde{W})$.

Proof. First note that for any state $\widehat{\phi}$, there exists W such that the assumptions of this corollary are satisfied for $M_{\widehat{\phi}}\{z,w\}$ and $L_{\widehat{\phi}}\{z,0\}$. Now, in view of Proposition 6.3(i), these assumptions imply that

$$\|M^{-1}\{z,w\}\|_W \leq Q$$

which, by Proposition 6.3(ii), gives

$$\|M_\star^{-1}\{z,w\}\|_{\widetilde{W}} \leq \|M^{-1}\{z,w\}\|_W \leq Q.$$

Moreover, $\|M\{z,0\}\|_W < 2 - 1/Q$ implies that

$$\|M_\star\{z,w\}\|_{\widetilde{W}} \leq \|M\{z,w\}\|_W < 2 - 1/Q$$

and therefore, the RHS of (7.11) is an element of $l^1(S, \widetilde{W})$ and thus (7.11) holds in $l^1(S, \widetilde{W})$. ∎

Note that the weight function W plays a role similar to the radius of convergence of a power series. Thus, if we say that the formula for the cumulant generating function, derived on the level of the semigroup algebra $\mathcal{A}(S)$, namely (7.11), holds on the "analytic" level, i.e. there exists W such that (7.11) holds in $l^1(S, W)$, it is analogous to saying that for a generating function in the form of a formal power series from $\mathbb{C}[[z]]$, there exists $R > 0$ such that the series becomes convergent in the disc of radius R.

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