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On operators Cauchy dual to 2-hyperexpansive operators: the unbounded case

by

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Abstract. The Cauchy dual operator T', given by $T(T^*T)^{-1}$, provides a bounded unitary invariant for a closed left-invertible T. Hence, in some special cases, problems in the theory of unbounded Hilbert space operators can be related to similar problems in the theory of bounded Hilbert space operators. In particular, for a closed expansive T with finite-dimensional cokernel, it is shown that T admits the Cowen–Douglas decomposition if and only if T' admits the Wold-type decomposition (see Definitions 1.1 and 1.2 below). This connection, which is new even in the bounded case, enables us to establish some interesting properties of unbounded 2-hyperexpansions and their Cauchy dual operators such as the completeness of eigenvectors, the hypercyclicity of scalar multiples, and the wandering subspace property.

In particular, certain cyclic 2-hyperexpansions can be modelled as the forward shift $\mathscr{F}$ in a reproducing kernel Hilbert space of analytic functions, where the complex polynomials form a core for $\mathscr{F}$. However, unlike unbounded subnormals, $(T^*T)^{-1}$ is never compact for unbounded 2-hyperexpansive T. It turns out that the spectral theory of unbounded 2-hyperexpansions is not as satisfactory as that of unbounded subnormal operators.

Contents

1. Preliminaries 129
2. The Cauchy dual operators: basic theory 133
  2.1. Analyticity and wandering subspace property 140
  2.2. Completeness of eigenvectors and hypercyclicity 142
3. Decompositions of unbounded 2-hyperexpansions 147
4. Operators Cauchy dual to unbounded 2-hyperexpansions 153
5. Composition operators: Examples 156
References 161

1. Preliminaries. For a subset A of the complex plane $\mathbb{C}$, let $\overline{A}$ denote the closure of A in $\mathbb{C}$. We use $\mathbb{R}$ to denote the real line, and $\Re z$ and $\Im z$ for the real and imaginary parts of a complex number z. Unless stated otherwise,

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all the Hilbert spaces occurring below are complex, infinite-dimensional, and separable.

Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$. Whenever there is no ambiguity, we suppress the subscript and simply write $\langle x, y \rangle$ and $\|x\|$. By $\mathrm{lin}\{w : w \in W\}$ (resp. $\bigvee\{w : w \in W\}$) we denote the smallest linear subspace (resp. smallest closed linear subspace) generated by the subset W of $\mathcal{H}$. If S is a densely defined linear operator in $\mathcal{H}$ with domain $\mathcal{D}(S)$, then we use $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$, $\sigma_{\mathrm{ap}}(S)$ to respectively denote the spectrum, the point spectrum and the approximate point spectrum of S. Recall that $\sigma_{\mathrm{p}}(S)$ is the set of eigenvalues of S, $\sigma_{\mathrm{ap}}(S)$ is the set of those λ in $\mathbb{C}$ for which $S - \lambda$ is not bounded below, and $\sigma(S)$ is the complement of the set of those λ in $\mathbb{C}$ for which $(T - \lambda)^{-1}$ exists as a bounded linear operator on $\mathcal{H}$.

Let T be a densely defined linear operator in $\mathcal{H}$ with domain $\mathcal{D}(T)$. We will use $\mathcal{D}_\infty(T)$ to denote the space $\bigcap_{n\geq 1} \mathcal{D}(T^n)$ of C^∞ vectors of T. The symbols $\mathrm{null}(T)$ and $\mathrm{ran}(T)$ will stand for the null-space and the range-space of T respectively. The closure (resp. adjoint) of T is denoted by $\overline{T}$ (resp. T^*). A subspace $\mathcal{D}$ of $\mathcal{H}$ is said to be a *core* of a closable linear operator T if $\mathcal{D} \subset \mathcal{D}(T)$, $\overline{\mathcal{D}} = \mathcal{H}$, and $\overline{T|_{\mathcal{D}}} = \overline{T}$. Observe that if $\mathcal{D}_1, \mathcal{D}_2$ are two subspaces of $\mathcal{H}$ such that $\mathcal{D}_1 \subset \mathcal{D}_2$ then $\mathcal{D}_2$ is a core of a closable linear operator T if so is $\mathcal{D}_1$. If S is a linear operator in $\mathcal{H}$ then we say that T *extends* S (denoted by $S \subset T$) if

$$\mathcal{D}(S) \subset \mathcal{D}(T) \quad \text{and} \quad Sh = Th \quad \text{for every } h \in \mathcal{D}(S).$$

A closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ contained in $\mathcal{D}(T)$ is said to be *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$. A closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ contained in $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ is *reducing* for T if $T\mathcal{M} \subset \mathcal{M}$ and $T^*\mathcal{M} \subset \mathcal{M}$. We say that T in $\mathcal{H}$ is *completely non-normal* (resp. *completely non-unitary*) if $\mathcal{H}$ has no non-trivial subspace $M \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$ that is reducing for T and such that $T|_M$ is normal (resp. unitary).

Let T be a densely defined, closed linear operator in $\mathcal{H}$ with domain $\mathcal{D}(T)$.

DEFINITION 1.1. We say that T admits the *Cowen–Douglas decomposition* if there exists a closed subspace $\mathcal{H}_u \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$ reducing for T such that

$$T = U \oplus A \quad \text{in } \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_a \quad \text{with } \mathcal{D}(T) = \mathcal{H}_u \oplus \mathcal{D}(A),$$

where U is unitary on $\mathcal{H}_u$, A is a densely defined, closed, completely non-unitary operator in $\mathcal{H}_a$, and $\mathcal{H}_a = \bigvee_{\mu \in \mathbb{D}_r} \mathrm{null}(A^* - \mu)$ for any positive r.

If T is a left-invertible closed operator that admits the Cowen–Douglas decomposition $U \oplus A$ as in Definition 1.1, then by standard spectral theory [14] there exists a real $r_0 > 0$ such that A^* belongs to the Cowen–Douglas class $\mathcal{B}_m(\mathbb{D}_{r_0})$, where $m = \dim \operatorname{null}(A^*)$ and $\mathbb{D}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$. This is why we refer to the decomposition of Definition 1.1 as the Cowen–Douglas decomposition. The classes $\mathcal{B}_m(\Omega)$ of bounded operators were introduced and studied by Cowen–Douglas in [9]. For unbounded operators, one may define $\mathcal{B}_m(\Omega)$ in a similar fashion.

Definition 1.2. We say that T admits the *Wold-type decomposition* if there exists a closed subspace $\mathcal{H}_u \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$ reducing for T such that

$$T = U \oplus A \quad \text{in } \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_a \quad \text{with } \mathcal{D}(T) = \mathcal{H}_u \oplus \mathcal{D}(A),$$

where U is unitary on $\mathcal{H}_u$, A is a densely defined, closed, completely non-unitary operator in $\mathcal{H}_a$, and $\mathcal{H}_a = \bigvee_{n\geq 0} A^n(\mathcal{D}_\infty(A) \cap \operatorname{null}(A^*))$.

In Definitions 1.1 and 1.2 the orthogonal direct sum $\mathcal{H}_u \oplus \mathcal{D}(A)$ is the linear subspace

$$\{x \oplus y \in \mathcal{H}_u \oplus \mathcal{H}_a : x \in \mathcal{H}_u,\, y \in \mathcal{D}(A)\}$$

of $\mathcal{H}_u \oplus \mathcal{H}_a$. In the context of bounded operators, Definition 1.2 of Wold-type decomposition is less stringent than that given in [15]. The subspace $\mathcal{H}_u$ in the Wold-type decomposition of Definition 1.1 of [15] is required to be $\bigcap_{n\geq 1} T^n\mathcal{D}(T^n)$.

In [6], we introduced and studied the so-called operators close to isometries. (Bounded 2-hyperexpansions are special operators close to isometries.) In particular, we obtained Cowen–Douglas and Wold-type decompositions of operators close to isometries by entirely different methods (see Theorems 3.2 and 4.3 of [6]). However, it turns out that the above decompositions are related to each other in the following sense:

Let T denote a densely defined, closed linear operator in $\mathcal{H}$ that is bounded below and let T' denote the Cauchy dual operator $T(T^*T)^{-1}$ (see Definition 2.1). Suppose further that T is *expansive*, that is,

$$\|Tx\| \geq \|x\| \quad \text{for every } x \in \mathcal{D}(T),$$

and that the null-space of T^* is finite-dimensional. Then T admits the Cowen–Douglas decomposition if and only if T' admits the Wold-type decomposition. Moreover, under some additional hypotheses including the absolute convergence of certain Hilbert space-valued series, T' admits the Cowen–Douglas decomposition if T admits the Wold-type decomposition. For a partial converse to the last assertion, the reader is referred to the discussion following Proposition 3.2.

In general, the Cauchy dual operator T' provides a bounded unitary invariant for unbounded left-invertible T. Hence, in some special cases, problems in the theory of unbounded Hilbert space operators can be related to similar problems in the theory of bounded Hilbert space operators. As we will see, if T is expansive then T' turns out to be a *contraction*, that is,

$$\|T'x\| \leq \|x\| \quad \text{for every } x \in \mathcal{H}$$

(Lemma 2.3), and if T is 2-hyperexpansive with invariant domain then T' is *hyponormal*, that is,

$$\|T'^*x\| \leq \|T'x\| \quad \text{for every } x \in \mathcal{H}$$

(Theorem 4.3). Recall that a densely defined linear operator T in $\mathcal{H}$ with domain $\mathcal{D}(T)$ is said to be 2-*hyperexpansive* if T is expansive and

$$\|x\|^2 - 2\|Tx\|^2 + \|T^2x\|^2 \leq 0 \quad \text{for every } x \in \mathcal{D}(T^2).$$

For the basic properties of bounded (resp. unbounded) hyperexpansions, the reader is referred to [16] (resp. [12]). We refer the reader to [8] for basic facts pertaining to bounded hyponormals.

In the present paper, we study the operators Cauchy dual to unbounded hyperexpansive operators and use their properties to derive some interesting results about unbounded hyperexpansive operators. However, there are some difficulties. Unlike the case of bounded left-invertible operators, the operator Cauchy dual to an unbounded left-invertible operator need not be left-invertible. Indeed, T' is left-invertible if and only if T is bounded on its domain (Lemma 2.3(3)). Next, even if one defines the second Cauchy dual operator T'' in a reasonable manner, the equality $(T')' = T$ is not guaranteed (Remark 2.6(2)). Note that the crucial step in the analysis of [5] was the usage of Bunce's C^*-algebraic techniques to determine the approximate point spectra of Cauchy dual operators ([8]). In the case of unbounded 2-hyperexpansions, such techniques are unavailable. Moreover, the unbounded setup brings new problems. Looking at the theory of differential operators or of unbounded subnormals, one may ask for which unbounded 2-hyperexpansive T, the operator $(T^*T)^{-1}$ is compact. Unfortunately, the answer is "for none" (Corollary 4.4). This is one of the reasons the spectral theory for unbounded 2-hyperexpansions is not so rich.

Let us mention two old problems in the theory of unbounded operators which arise very naturally in the course of our investigations. The first is to decide, for closed operators S and T, when the inclusion $S \subset T$ is trivial, that is, $S = T$. This problem has a very simple solution in terms of the Cauchy dual operators: If $S' = T'$ then the inclusion $S \subset T$ is always trivial. Another delicate issue is the density of the domain of the self-commutators of 2-hyperexpansions. The author does not know whether or not the self-commutator of a 2-hyperexpansion has dense domain (see Proposition 4.6 for a partial result).

One of the main results in [5] shows that the operator $T' \equiv T(T^*T)^{-1}$ Cauchy dual to a bounded 2-hyperexpansive T is a hyponormal contraction. Indeed, $C = TT'$ is a contraction similar to an isometry such that

$T'^* = C^*T'$. Since the second Cauchy dual operator $(T')'$ coincides with T, one can derive a lot of interesting results for 2-hyperexpansive operators from those which are known for hyponormal operators. In particular, one can ensure a rich supply of non-zero *-homomorphisms on the non-commutative C^*-algebra generated by a completely non-normal 2-hyperexpansion and a realization of an analytic finitely multicyclic 2-hyperexpansion as a compact perturbation of a unilateral shift. The present paper is a sequel to [5] and [6], and continues the study of hyperexpansive operators in almost the same spirit.

The paper is organized as follows. In the second section, the notion of the operator Cauchy dual to a bounded left-invertible linear operator, as introduced in [15], is generalized to that of the operator Cauchy dual to a closed left-invertible linear operator. We establish the basic theory of the Cauchy dual operator in the unbounded setup. In Section 2.1, we obtain unbounded counterparts of some results related to the wandering subspace problem—following [15]. The key result of Section 2.2 (Lemma 2.15) connects the wandering subspace property and the completeness of eigenvectors via the Cauchy dual operator. As an application, we exhibit a class of unbounded hypercyclic operators. We also show that certain analytic left-invertible operators and their Cauchy dual operators can be simultaneously modelled as forward shift operators and adjoint backward shift operators in reproducing kernel Hilbert spaces. Almost all main results of Sections 3 and 4 rely heavily on the properties of Cauchy dual operators deduced in Section 2. In Section 3, we discuss decompositions of certain unbounded left-invertible operators and specialize them to 2-hyperexpansions. The main result of this section is the Cowen–Douglas Decomposition Theorem for certain unbounded 2-hyperexpansions. As a corollary, we obtain a hyperexpansivity analog of Proposition 11 of [18]. This result is remarkable, for, unlike unbounded subnormals, unbounded 2-hyperexpansions do not admit functional models. In the fourth section, we establish an unbounded counterpart of Theorem 2.9 of [5] and discuss its consequences. We conclude the paper with examples of unbounded 2-hyperexpansive composition operators on discrete measure spaces illustrating the subject of the paper.

This paper may be regarded as an attempt to develop the theory of unbounded 2-hyperexpansions parallel to that of unbounded subnormals.

2. The Cauchy dual operators: basic theory. Let S be a densely defined, closable operator in $\mathcal{H}$ that is left-invertible, that is, there exists some real $c > 0$ such that $\|Sx\|_{\mathcal{H}} \geq c\|x\|_{\mathcal{H}}$ for all x in $\mathcal{D}(S)$. Note that $\overline{S}$ also satisfies $\|\overline{S}x\|_{\mathcal{H}} \geq c\|x\|_{\mathcal{H}}$ ($x \in \mathcal{D}(\overline{S})$) and that $\Gamma \equiv \mathcal{D}(\overline{S})$ is a Hilbert space with the norm $\|x\|_{\Gamma} \equiv \|\overline{S}x\|_{\mathcal{H}}$ ($x \in \Gamma$). Further, $A \equiv \overline{S}^*\overline{S}$ is an invertible self-adjoint operator, $\mathcal{D}(A^{1/2}) = \mathcal{D}(\overline{S})$, and

$$\langle \overline{S}x, \overline{S}x\rangle_{\mathcal{H}} = \langle A^{1/2}x, A^{1/2}x\rangle_{\mathcal{H}} \quad (x \in \mathcal{D}(\overline{S})), \tag{2.1}$$

where $A^{1/2}$ is the unique positive square-root of A ([13, Theorem 2.8.12]). Furthermore, it follows from Theorem 2.8.2 in [13] that $\mathcal{D}(A)$ is dense in Γ in the $\|\cdot\|_\Gamma$ norm. Throughout this paper, we will frequently use all these basic facts without mention.

Definition 2.1. Let T be a densely defined, closable linear operator in $\mathcal{H}$ that is bounded below. Then the operator T' given by $\overline{T}A^{-1}$ is said to be the *operator Cauchy dual* to T, where $A = \overline{T}^*\overline{T}$.

Remark 2.2. The following remarks are worth noting:

(1) Since $\operatorname{ran}(A^{-1}) = \mathcal{D}(A) \subset \mathcal{D}(\overline{T})$, T' is a well-defined linear operator on $\mathcal{H}$. Moreover, T' admits a densely defined, closed, linear left-inverse. Indeed, $\operatorname{ran}(T') \subset \mathcal{D}(T^*)$ and $T^*T' = I$.

(2) Let S, T be densely defined, left-invertible, closed linear operators in $\mathcal{H}$. If there exists a unitary U on $\mathcal{H}$ such that $UT = SU$ then $UT' = S'U$. In other words, the Cauchy dual operator T' is a unitary invariant for T. To see this, note that

$$UT = SU \;\Rightarrow\; T^*U^* \supset U^*S^* \;\Rightarrow\; T^*T \supset U^*S^*SU$$

([14, Section 7.7]). Since both T^*T and U^*S^*SU are self-adjoint operators, we must have $T^*T = U^*S^*SU$ ([13, Theorem 2.6.2 and Lemma 1.6.14]). It follows that $(T^*T)^{-1} = U^*(S^*S)^{-1}U$ and hence

$$UT' = UT(T^*T)^{-1} = S(S^*S)^{-1}U = S'U.$$

For a partial converse of this, see Lemma 2.3(4) below.

The following lemma records some basic properties pertaining to the Cauchy dual operator and its adjoint (cf. [15, Lemma 2.1]). Parts (4) and (5) below are of interest only in the unbounded setup.

Lemma 2.3. *Let T be a densely defined, closable linear operator in $\mathcal{H}$ such that $\|Tx\| \geq \alpha\|x\|$ $(x \in \mathcal{D}(T))$ for some positive real α. If $A \equiv \overline{T}^*\overline{T}$ and if T' is the operator Cauchy dual to T then:*

(1) *T' is a bounded linear operator with $\|T'\| \leq \alpha^{-1}$.*
(2) *A is invertible with $A^{-1} = T'^*T'$.*
(3) *T' is injective. If, in addition, T is unbounded then T' is not bounded below.*
(4) *Suppose there exists some densely defined linear operator S in $\mathcal{H}$ such that $S \subset T$. If $S' = T'$ then $\overline{S} = \overline{T}$.*
(5) *A^{-1} is compact if and only if T' is compact.*

(6) *Let n denote a positive integer. Suppose that T is closed such that $\mathcal{D}(T^n)$ is dense in $\mathcal{H}$ and that $\mathcal{D}(T^{*n})$ is a core of $(T^n)^*$. Then $(T^n)^* = \overline{T^{*n}}$ and*

$$\mathrm{null}(\overline{T^{*n}}) = \bigvee\{T'^k x : x \in \mathrm{null}(T^*),\ k = 0, \dots, n-1\}.$$

If one defines a densely defined linear operator L by

$$Lx \equiv T'^* x \quad (x \in \mathcal{D}(T^*)),$$

then:

(7) *L is a closable bounded linear operator with $\overline{L} = T'^*$.*
(8) *$\overline{L}\,\overline{T}x = x$ for any $x \in \mathcal{D}(\overline{T})$.*
(9) $\mathrm{null}(\overline{L}) = \mathrm{null}(L) = \mathrm{null}(T^*)$.
(10) *For any positive integer n, one has*

$$\mathrm{lin}\{T^k x : x \in \mathcal{D}(T^{n-1}) \cap \mathrm{null}(T^*),\ k = 0, \dots, n-1\} \subset \mathrm{null}(\overline{L}^n).$$

In particular, the operator Cauchy dual to a closable expansion is a bounded linear injective contraction defined everywhere.

Proof. (1): Note that for any $x \in \mathcal{H}$,

$$\|T'x\|^2 = \|\overline{T}A^{-1}x\|^2 = \langle \overline{T}A^{-1}x, \overline{T}A^{-1}x\rangle = \langle A^{1/2}A^{-1}x, A^{1/2}A^{-1}x\rangle = \|A^{-1/2}x\|^2$$

in view of (2.1). Thus

$$\|T'x\| = \|A^{-1/2}x\| \quad (x \in \mathcal{H}). \tag{2.2}$$

Hence T' is a bounded linear operator with $\|T'\| = \|A^{-1/2}\|$. To conclude the proof of (1), we show that $\|A^{-1/2}\| \le \alpha^{-1}$. Note that $\|\overline{T}x\| \ge \alpha\|x\|$ ($x \in \mathcal{D}(\overline{T})$). Hence, for any $x \in \mathcal{D}(\overline{T})$,

$$\langle A^{1/2}x, A^{1/2}x\rangle = \langle \overline{T}x, \overline{T}x\rangle \ge \alpha\|x\|$$

(see (2.1)). Now it is clear that $\|A^{-1/2}\| \le \alpha^{-1}$.

(2): Since $T' = \overline{T}A^{-1}$, one has $T'^* = (\overline{T}A^{-1})^* \supset A^{-1}T^*$ ([14, Section 7.7]). It follows that $T'^*T' \supseteq A^{-1}T^*\overline{T}A^{-1} = A^{-1}$. Since both T'^*T' and A^{-1} are bounded linear operators defined on $\mathcal{H}$, we must have $T'^*T' = A^{-1}$.

(3): It is clear from (2.2) that T' is injective. Suppose T' is bounded below. It follows from (2.2) that $A^{-1/2}$ is bounded below. Consequently, A is bounded on its domain, and therefore T is not unbounded.

(4): The argument is similar to that in the proof of [2, Theorem 1]. Since $S \subset T$, S is closable and $\overline{S} \subset \overline{T}$. It suffices to check that $\mathcal{D}(\overline{S}) = \mathcal{D}(\overline{T})$. Apply T^* on both sides of $S' = T'$ to obtain $T^*\overline{S}(S^*\overline{S})^{-1} = I$ (Remark 2.2(1)). Since $\overline{T}$ extends $\overline{S}$, we get $T^*\overline{T}(S^*\overline{S})^{-1} = I$. It follows that $S^*\overline{S} \subset T^*\overline{T}$. Since $S^*\overline{S}$ and $T^*\overline{T}$ are self-adjoint, we must have $S^*\overline{S} = T^*\overline{T}$. Hence

$$\mathcal{D}(\overline{S}) = \mathcal{D}((S^*\overline{S})^{1/2}) = \mathcal{D}((T^*\overline{T})^{1/2}) = \mathcal{D}(\overline{T})$$

(see the discussion preceding (2.1)).

(5): If T' is compact then the compactness of A^{-1} follows from (2) above. To see the other implication, suppose A^{-1} is compact and let $\{x_n\}_{n\geq 1}$ be a bounded sequence in $\mathcal{H}$. Since A^{-1} is compact, $\{A^{-1}x_n\}_{n\geq 1}$ admits a convergent subsequence, say $\{A^{-1}y_n\}_{n\geq 1}$. Hence, in view of $A^{-1} = T'^*T'$ (Lemma 2.3(2)),

$$\begin{aligned}\|T'y_n - T'y_m\|^2 &= \langle T'(y_n - y_m), T'(y_n - y_m)\rangle = \langle T'^*T'(y_n - y_m), y_n - y_m\rangle \\ &\leq \|T'^*T'(y_n - y_m)\|\,\|y_n - y_m\| \\ &\leq \|A^{-1}y_n - A^{-1}y_m\|\,\|y_n - y_m\|\end{aligned}$$

for all integers $n, m \geq 1$. Since $\{A^{-1}y_n\}_{n\geq 1}$ is Cauchy and $\{y_n\}_{n\geq 1}$ is bounded, $\{T'y_n\}_{n\geq 1}$ is Cauchy. This completes the verification of (5).

(6): Since T is a closed left-invertible operator with $\overline{\mathcal{D}(T^n)} = \mathcal{H}$, T^n is a densely defined closed operator. In particular, $(T^n)^*$ is a well-defined densely defined closed linear operator. Since $\overline{\mathcal{D}(T^{*n})} = \mathcal{H}$, T^{*n} is a densely defined linear operator in $\mathcal{H}$. Also, since $(T^n)^* \supset T^{*n}$ [14], T^{*n} is closable with $(T^n)^* \supset \overline{T^{*n}}$. The first part in (6) is now immediate from

$$\overline{T^{*n}} \subset (T^n)^* = \overline{(T^n)^*|_{\mathcal{D}(T^{*n})}} = \overline{T^{*n}|_{\mathcal{D}(T^{*n})}} = \overline{T^{*n}}.$$

Since $T^*T'x = x$ for any $x \in \mathcal{H}$, one has

$$\mathrm{null}(T^{*n}) \supset \mathrm{lin}\{T'^kx : x \in \mathrm{null}(T^*),\ k = 0, \ldots, n-1\}.$$

Also, since $\mathrm{null}(T^{*n}) \subset \mathrm{null}(\overline{T^{*n}})$ and $\mathrm{null}(\overline{T^{*n}})$ is closed in $\mathcal{H}$,

$$\mathrm{null}(\overline{T^{*n}}) \supset \bigvee\{T'^kx : x \in \mathrm{null}(T^*),\ k = 0, \ldots, n-1\}.$$

Next, for any $x \in \mathcal{D}(T^{*n})$,

$$\sum_{k=0}^{n-1} T'^k(I - T'T^*)T^{*k}x = x - T'^nT^{*n}x.$$

Thus the expression on the left-hand side is of the form $\sum_{k=0}^{n-1} T'^k y_k$ with $y_k = (I - T'T^*)T^{*k}x \in \mathrm{null}(T^*)$. Let $x \in \mathrm{null}(\overline{T^{*n}})$ and choose a sequence $\{x_m\}_{m\geq 1} \subset \mathcal{D}(T^{*n})$ such that $x_m \to x$ and $T^{*n}x_m \to 0$ as $m \to \infty$. Set $y_{k,m} \equiv (I - T'T^*)T^{*k}x_m \in \mathrm{null}(T^*)$. Since T' is continuous and

$$x_m - T'^nT^{*n}x_m = \sum_{k=0}^{n-1} T'^k y_{k,m} \in \bigvee\{T'^kx : x \in \mathrm{null}(T^*), k = 0, \ldots, n-1\},$$

it follows that $x \in \bigvee\{T'^kx : x \in \mathrm{null}(T^*),\ k = 0, \ldots, n-1\}$.

(7): Since L is densely defined with the bounded extension T'^*, this is obvious.

(8): Notice that the equality in (8) is trivial for any $x \in \mathcal{D}(T^*\overline{T})$. Let x be in $\mathcal{D}(\overline{T})$. Since $\mathcal{D}(T^*\overline{T})$ is dense in $\mathcal{D}(\overline{T})$ in the $\|\cdot\|_\Gamma$ norm, where $\|x\|_\Gamma \equiv \|\overline{T}x\|_{\mathcal{H}}$ $(x \in \mathcal{D}(\overline{T}))$, there exists a sequence $\{x_n\}_{n\geq 0} \subset \mathcal{D}(T^*\overline{T})$ such that $\|\overline{T}x_n - \overline{T}x\| \to 0$ as $n \to \infty$ (see the discussion following (2.1)). As $\overline{T}$ is bounded below, $\|x_n - x\| \to 0$ as $n \to \infty$. Also, since $\overline{L}$ is a bounded linear operator in view of (7),

$$\|x_n - \overline{L}\,\overline{T}x\| = \|\overline{L}\,\overline{T}x_n - \overline{L}\,\overline{T}x\| = \|\overline{L}(\overline{T}x_n - \overline{T}x)\| \to 0 \quad \text{as } n \to \infty.$$

Hence $\overline{L}\,\overline{T}x = x$ as desired.

(9): It suffices to check that $\mathrm{null}(\overline{L}) \subset \mathrm{null}(L)$. To see this, let $x \in \mathrm{null}(\overline{L})$. Write $x = \overline{T}y + z$ for some $y \in \mathcal{D}(\overline{T})$ and $z \in \mathrm{null}(T^*)$. In view of (8), one has

$$0 = \overline{L}x = \overline{L}\;\overline{T}y + \overline{L}z = y + (T^*\overline{T})^{-1}T^*z = y.$$

Thus $y = 0$ and $x = z \in \mathrm{null}(T^*) = \mathrm{null}(L)$ as required.

(10): This is immediate from (8) and (9). ■

Let T be a densely defined closed linear operator in $\mathcal{H}$ that is bounded below. Let Γ denote the Hilbert space $(\mathcal{D}(T), \|\cdot\|_\Gamma)$, where $\mathcal{D}(T)$ is the domain of T and

$$\|x\|_\Gamma \equiv \sqrt{\langle x, x\rangle_\Gamma}, \quad \langle x, y\rangle_\Gamma \equiv \langle Tx, Ty\rangle_{\mathcal{H}} \quad (x, y \in \mathcal{D}(T)).$$

Set $\|x\|_{\Gamma'} \equiv \|T'x\|_{\mathcal{H}}$ $(x \in \mathcal{H})$. Since T' is injective (Lemma 2.3(3)), $(\mathcal{H}, \|\cdot\|_{\Gamma'})$ is a pre-Hilbert space endowed with the inner product

$$\langle x, y\rangle_{\Gamma'} \equiv \langle T'x, T'y\rangle_{\mathcal{H}} \quad (x, y \in \mathcal{H}).$$

(Lemma 2.3(3) implies that $(\mathcal{H}, \|\cdot\|_{\Gamma'})$ is a Hilbert space if and only if T is bounded.) Let Γ' denote the completion of $(\mathcal{H}, \|\cdot\|_{\Gamma'})$. Thus, we have the chain $\Gamma \subseteq \mathcal{H} \subseteq \Gamma'$ of Hilbert spaces. One may refer to this as the *Hilbert rigging* of $\mathcal{H}$ by Γ and Γ' (see [4, Section 1 of Chapter 14]). One of the aspects of the duality between T and T' is the relationship between Γ and Γ'.

PROPOSITION 2.4. *Let T, T', $\mathcal{H}$, Γ, Γ' be as in the last paragraph. Then Γ' can be realized as the Hilbert space Γ^* of anti-linear, continuous functionals over Γ via the mapping $\eta_x : \Gamma \to \mathbb{C}$ defined by $\eta_x(y) = \langle x, y\rangle_{\mathcal{H}}$ $(y \in \Gamma$, $x \in \mathcal{H})$.*

Proof. It is easy to see that $\eta_x \in \Gamma^*$. We now introduce a new norm $\|\cdot\|$ on $\mathcal{H}$ by setting

$$\|x\| \equiv \|\eta_x\|_{\Gamma^*} = \sup\{|\langle x, y\rangle_{\mathcal{H}}|/\|y\|_\Gamma : 0 \neq y \in \Gamma\} \quad (x \in \mathcal{H}).$$

(That $\|\cdot\|$ is a norm follows from the density of Γ in $\mathcal{H}$.) Let $\mathcal{K}$ denote the completion of the normed linear space $(\mathcal{H}, \|\cdot\|)$. It follows from Theorem 1.1 of Chapter 14 of [4] that $\mathcal{K}$ is a Hilbert space. To complete the proof, in view of Theorem 1.2 of Chapter 14 of [4], it suffices to check that $\mathcal{K}$ is isometrically isomorphic to Γ' (see the discussion of Section 1 there). In view of Lemma 2.3(8), for any $x \in \mathcal{H}$,

$$\begin{aligned}\|x\|_{\mathcal{K}} &= \sup\Big\{\frac{|\langle x,y\rangle_{\mathcal{H}}|}{\|y\|_{\Gamma}} : 0\neq y\in\Gamma\Big\} = \sup\Big\{\frac{|\langle x,y\rangle_{\mathcal{H}}|}{\|Ty\|_{\mathcal{H}}} : 0\neq y\in\Gamma\Big\}\\ &= \sup\Big\{\frac{|\langle x,T'^{*}Ty\rangle_{\mathcal{H}}|}{\|Ty\|_{\mathcal{H}}} : 0\neq y\in\Gamma\Big\} = \sup\Big\{\frac{|\langle T'x,Ty\rangle_{\mathcal{H}}|}{\|Ty\|_{\mathcal{H}}} : 0\neq y\in\Gamma\Big\}\\ &= \sup\Big\{\frac{|\langle (T^{*}T)^{-1}x,y\rangle_{\Gamma}|}{\|y\|_{\Gamma}} : 0\neq y\in\Gamma\Big\}\\ &= \|(T^{*}T)^{-1}x\|_{\Gamma} = \|T'x\|_{\mathcal{H}} = \|x\|_{\Gamma'}.\end{aligned}$$

Since $(\mathcal{H},\|\cdot\|)$ (resp. $(\mathcal{H},\|\cdot\|_{\Gamma'})$) is dense in $\mathcal{K}$ (resp. Γ'), the proof is complete. ∎

We introduce linear maps $U:\mathcal{H}\to\Gamma$, $V:\Gamma'\to\mathcal{H}$, and $W:\Gamma'\to\Gamma$ as follows:

$$Ux = (T^{*}T)^{-1/2}x = Vx, \qquad Wx = (T^{*}T)^{-1}x \qquad (x\in\mathcal{H}).$$

Note that

$$\|Ux\|_{\Gamma} = \|x\|_{\mathcal{H}} \quad \text{and} \quad \|Vx\|_{\mathcal{H}} = \|x\|_{\Gamma'} \quad (x\in\mathcal{H}), \quad U\circ V|_{\mathcal{H}} = W|_{\mathcal{H}}.$$

Since $(\mathcal{H},\|\cdot\|_{\Gamma'})$ is dense in Γ', $V|_{\mathcal{H}}$ and $W|_{\mathcal{H}}$ can be isometrically extended to Γ' so that $U\circ V = W$. Moreover, W is surjective: For any $x\in\mathcal{H}$ and $y\in\Gamma$,

$$\langle Wx,y\rangle_{\Gamma} = 0 \;\Rightarrow\; \langle TWx,Ty\rangle_{\mathcal{H}} = 0 \;\Rightarrow\; \langle x,y\rangle_{\mathcal{H}} = 0,$$

so $\operatorname{ran}(W)$ is dense in Γ. Since $\operatorname{ran}(W)$ is closed, W is surjective (cf. [4, Theorem 1.3 of Chapter 14]).

Notice that the range of T' is a non-closed and non-dense subspace of $\mathcal{H}$ if T is unbounded and non-invertible. One can still introduce the second Cauchy dual operator T''.

Definition 2.5. Let T be a densely defined, closable linear operator such that $\|Tx\| \geq \alpha\|x\|$ ($x\in\mathcal{D}(T)$) for some positive real α. Then the *second Cauchy dual* T'' of T is defined to be the operator $T'A$, where A is equal to $\overline{T}^{*}\overline{T}$.

Remark 2.6. We make the following remarks:

(1) In view of Lemma 2.3(2), this definition is consistent with that in the bounded case.
(2) It follows from the very definition that T'' is a closable linear operator with domain $\mathcal{D}(A)$. Indeed,

$$T'' = T|_{\mathcal{D}(A)} \subset T \subset \overline{T} = \overline{T''}.$$

The last equality can be deduced from the fact that $\mathcal{D}(A)$ is dense in $\Gamma \equiv \mathcal{D}(\overline{T})$ in the $\|\cdot\|_{\Gamma}$ norm, where $\|x\|_{\Gamma} = \|\overline{T}x\|$ ($x\in\mathcal{D}(\overline{T})$). In particular, for a non-closed T, we must have $T''\neq T$.

The following shows, in particular, that any fixed point in $\mathcal{D}(T^*T)$ of a closable expansive T is also a fixed point of T^*.

Proposition 2.7. *Let T be a densely defined, closable expansion in $\mathcal{H}$ and let $A \equiv \overline{T}^*\overline{T}$. Let T' be the operator Cauchy dual to T and set $r = \|T'\|^{-1} \in [1,\infty)$* (see Lemma 2.3). *Let $x \in \mathcal{D}(A)$ and $0 \neq \mu \in \overline{\mathbb{D}}_r \equiv \{z \in \mathbb{C} : |z| \leq r\}$. If $\overline{T}x = \mu x$ then $x \in \mathcal{D}(T^*)$ and $T^*x = \overline{\mu}x$.*

Proof. Since $x \in \mathcal{D}(A)$ and $x = \mu^{-1}\overline{T}x$, it follows that $x \in \mathcal{D}(T^*)$. Also,

$$\overline{T}x = \mu x \ \Rightarrow\ T^*\overline{T}x = \mu T^*x \ \Rightarrow\ A^{-1}T^*\overline{T}x = \mu A^{-1}T^*x.$$

Since $A^{-1}T^* \subset (\overline{T}A^{-1})^* = T'^*$, one has $x = (\overline{\mu}T')^*x$. Since $|\mu| \leq \|T'\|^{-1}$, $(\overline{\mu}T')^*$ is a contraction on $\mathcal{H}$. Hence, by Proposition 3.1 of Chapter 1 in [20], one has $\overline{\mu}T'x = x$. As $T^*T'x = x$ for all $x \in \mathcal{H}$, it follows that $T^*x = \overline{\mu}x$. ∎

It is known that for any expansive T in $\mathcal{H}$, $\sigma_{\mathrm{ap}}(T) \cap \mathbb{D}_1 = \emptyset$ ([12, Lemma 3.1]); in particular, $\sigma_{\mathrm{p}}(T) \cap \mathbb{D}_1 = \emptyset$, where $\mathbb{D}_1$ is the open unit disc centred at the origin. For some special expansions, more can be said.

Corollary 2.8. *Let S be a densely defined, closable expansion in $\mathcal{H}$ such that $\mathcal{D} \equiv \mathcal{D}(S) \cap \mathcal{D}(S^*)$ is a core of S. If $T \equiv S|_{\mathcal{D}}$ is completely non-normal then the point spectrum of T is disjoint from $\overline{\mathbb{D}}_r \equiv \{z \in \mathbb{C} : |z| \leq r\}$, where $r = \|T'\|^{-1} \in [1,\infty)$. In particular, for any completely non-unitary, closable 2-hyperexpansive T with $\mathcal{D}(T) \subset \mathcal{D}(T^*)$, the point spectrum of T is empty.*

Proof. Note that

$$\mathcal{D}(T) = \mathcal{D} \subset \mathcal{D}(S^*) = \mathcal{D}(\overline{S}^*) = \mathcal{D}(\overline{T}^*) = \mathcal{D}(T^*).$$

Let $\mu \in \overline{\mathbb{D}}_r$. Since $\mathcal{D}(T) \subset \mathcal{D}(T^*)$, $\mathrm{null}(T-\mu) \subset \mathcal{D}(T) \cap \mathcal{D}(T^*) \cap \mathcal{D}(T^*T)$. Hence, by the previous proposition,

$$T(\mathrm{null}(T-\mu)) \subset \mathrm{null}(T-\mu) \quad \text{and} \quad T^*(\mathrm{null}(T-\mu)) \subset \mathrm{null}(T-\mu).$$

It follows that $\mathrm{null}(T-\mu)$ is reducing for T. Since T is completely non-normal, $\mathrm{null}(T-\mu) = \{0\}$. Also, since for any 2-hyperexpansive T, $\sigma_{\mathrm{p}}(T) \subset \partial\mathbb{D}_1$ ([12, Theorem 5.1(i)]), the remaining assertion follows. ∎

We briefly discuss here one application of Corollary 2.8. Let T be as in the hypotheses of Corollary 2.8 and assume that T is completely non-unitary. Consider the *Cayley transform* $\mathcal{C}_T : \mathrm{ran}(T+I) \to \mathrm{ran}(T-I)$ given by

$$\mathcal{C}_T(T+I)h \;=\; (T-I)h \quad (h \in \mathcal{D}(T))$$

Since T is completely non-unitary, by the last corollary, $\mathrm{null}(T+I)$ is trivial. Thus $\mathcal{C}_T$ is well-defined. Moreover, $\mathcal{C}_T$ turns out to be *accretive*, that is, the real part of $\langle \mathcal{C}_T(T+I)h, (T+I)h\rangle$ is non-negative: For every $h \in \mathcal{D}(T)$,

$$\begin{aligned}\langle \mathcal{C}_T(T+I)h, (T+I)h\rangle &+ \langle (T+I)h, \mathcal{C}_T(T+I)h\rangle \\ &= 2\Re\langle (T-I)h, (T+I)h\rangle = 2(\|Th\|^2 - \|h\|^2) \geq 0.\end{aligned}$$

Recall that a densely defined accretive operator S in $\mathcal{H}$ is *maximal accretive* if it has no proper accretive extension in $\mathcal{H}$. Suppose further that $\operatorname{ran}(T+I)$ is dense in $\mathcal{H}$. Then $\mathcal{C}_T$ admits a maximal accretive extension. If, in addition, T is invertible then $\mathcal{C}_T$ itself is maximal accretive ([20, Theorem 4.1]).

2.1. Analyticity and wandering subspace property

DEFINITION 2.9. We say that a densely defined linear operator T in $\mathcal{H}$ is

(1) *analytic* if $\bigcap_{k\geq 1} T^k\mathcal{D}(T^k) = \{0\}$,
(2) *admissible* if $\mathcal{D}_\infty(T)$ is dense in $\mathcal{H}$ and $\mathcal{D}_\infty(T^*)$ is a core of T^{n*} for every positive integer n.

REMARK 2.10. Let T be as in Definition 2.9.

(1) In Definition 2.9(2), the adjoint of T^n is well-defined since T^n is densely defined.
(2) If, in addition, T is a closed, admissible operator that is bounded below then it follows from Lemma 2.3(6) that $T^{n*} = \overline{T^{*n}}$ and
$$\operatorname{null}(\overline{T^{*n}}) = \bigvee\{T'^k x : x \in \operatorname{null}(T^*),\ k = 0, \dots, n-1\}.$$
for every positive integer n.

Obviously, bounded linear Hilbert space operators are admissible. It turns out that all closed weighted shift operators are admissible (see Example 2.11 below). In the final section, we will exhibit a class of unbounded, admissible composition operators (see Lemma 5.2(1)).

If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathcal{H}$ and S is a linear operator in $\mathcal{H}$ with domain $\operatorname{lin}\{e_n : n = 0, 1, \dots\}$ such that $Se_n = \alpha_n e_{n+1}$ for some positive numbers α_n $(n \geq 0)$, then S is called a *weighted shift operator*. We will use the notation $S : \{\alpha_n\}_{n\geq 0}$ for such an operator.

EXAMPLE 2.11. Let $S : \{\alpha_n\}_{n\geq 0}$ denote a weighted shift operator in $\mathcal{H}$ with weight sequence $\{\alpha_n\}_{n\geq 0}$ corresponding to the orthonormal basis $\{e_n\}_{n\geq 0}$ of $\mathcal{H}$. Then S is closable. Moreover,
$$\overline{S}f = \sum_{n=0}^{\infty}\langle f, e_n\rangle \alpha_n e_{n+1} \quad (f \in \mathcal{D}(\overline{S})),$$
$$S^*f = \sum_{n=0}^{\infty}\langle f, e_{n+1}\rangle \alpha_n e_n \quad (f \in \mathcal{D}(S^*))$$
(see [19]). One may refer to $\overline{S}$ as a *closed weighted shift operator*. Note that $\operatorname{lin}\{e_n : n \geq 0\} \subset \mathcal{D}_\infty(\overline{S}) \cap \mathcal{D}_\infty(S^*)$. We check that $\overline{S}$ is analytic and admissible.

Analyticity: Let $f \in \mathcal{D}_\infty(\overline{S})$. In particular, $f = \overline{S}^k g_k$ for some $g_k \in \mathcal{H}$ for every positive integer k. It follows from

$$\sum_{n=0}^{\infty}\langle f, e_n\rangle e_n = \sum_{n=0}^{\infty}\langle g_1, e_n\rangle \alpha_n e_{n+1}$$

that $\langle f, e_0\rangle = 0$. By an induction argument, we must have $\langle f, e_n\rangle = 0$ for every non-negative integer n. Hence $f = 0$ and $\overline{S}$ is analytic.

Admissibility: Set $T \equiv \overline{S}^k$ and $f_m \equiv \sum_{n=0}^{m}\langle f, e_n\rangle e_n \in \mathcal{D}_\infty(T^*)$ for $f \in \mathcal{D}(T^*)$. For any $f \in \mathcal{D}(T^*)$,

$$\begin{aligned} T^*f &= \sum_{n=0}^{\infty}\langle T^*f, e_n\rangle e_n = \sum_{n=0}^{\infty}\langle f, S^k e_n\rangle e_n = \sum_{n=0}^{\infty}\alpha_n\cdots\alpha_{n+k-1}\langle f, e_{n+k}\rangle e_n \\ &= \sum_{n=k}^{\infty}\alpha_{n-k}\cdots\alpha_{n-1}\langle f, e_n\rangle e_{n-k} = \sum_{n=k}^{\infty}\langle f, e_n\rangle {S^*}^k e_n = \lim_{m\to\infty} {S^*}^k f_m. \end{aligned}$$

Since $f_m \to f$ and $T^*f_m = {S^*}^k f_m \to T^*f$ as $m \to \infty$, we must have $\overline{T^*|_{\mathcal{D}_\infty(T^*)}} = T^*$. ∎

We include verification of the following for completeness.

LEMMA 2.12. *Let $\{M_k\}_{k\geq 1}$ be a countable collection of subspaces of $\mathcal{H}$. Then*

$$\Big\{\bigcap_{k\geq 1}\overline{M_k}\Big\}^\perp = \bigvee_{k\geq 1} M_k^\perp \subset \Big\{\bigcap_{k\geq 1} M_k\Big\}^\perp.$$

Proof. The second inclusion is a routine verification. Suppose we have the strict inclusion, $S_1 \subsetneq S_2$, of two closed subspaces $S_1 \equiv \bigvee_{k\geq 1} M_k^\perp$ and $S_2 \equiv \{\bigcap_{k\geq 1}\overline{M_k}\}^\perp$. By the Hahn–Banach Theorem, there exists $0 \neq x \in S_2$ such that $\langle y, x\rangle = 0$ for every $y \in S_1$. It follows that $\langle y, x\rangle = 0$ for every $y \in M_k^\perp$ $(k \geq 1)$. Thus $x \in \overline{M_k}$ for every integer $k \geq 1$. Since $0 \neq x \in S_2$, we arrive at a contradiction. ∎

We say that a densely defined linear operator T in $\mathcal{H}$ has the *wandering subspace property* if $\mathcal{H} = \bigvee\{T^k x : x \in \mathcal{D}_\infty(T) \cap \mathrm{null}(T^*),\ k = 0, 1, \ldots\}$.

The following result provides an unbounded counterpart of Proposition 2.7 of [15].

PROPOSITION 2.13. *Let T denote a closed linear operator in $\mathcal{H}$ that is bounded below and let T' denote the Cauchy dual operator. If T is admissible then the following duality relations hold true:*

$$\Big\{\bigcap_{n\geq 1} T'^n\mathcal{H}\Big\}^\perp \supset \bigvee_{n\geq 0} T^n(\mathcal{D}_\infty(T) \cap \mathrm{null}(T^*)),$$

$$\Big\{\bigcap_{n\geq 1} T^n\mathcal{D}(T^n)\Big\}^\perp = \bigvee_{n\geq 0} T'^n(\mathrm{null}(T^*)).$$

Proof. Note that

$$\Big\{\bigcap_{k\geq 1} T'^{k}\mathcal{H}\Big\}^{\perp} \supset \bigvee_{k\geq 1}\{T'^{k}\mathcal{H}\}^{\perp} = \bigvee_{k\geq 1} \mathrm{null}(T'^{*k}) \supset \mathrm{lin}_{k\geq 0}\{T^{k}x : x\in \mathcal{D}_{\infty}(T)\cap \mathrm{null}(T^{*})\}$$

by Lemmas 2.12 and 2.3. Hence the first part follows. Since T is bounded below, all non-negative integer powers of T are closed. Hence, the subspace $T^n\mathcal{D}(T^n)$ is closed for every integer $n \geq 1$. It now follows from Lemma 2.12 and Remark 2.10(2) that

$$\Big\{\bigcap_{k\geq 1} T^{k}\mathcal{D}(T^{k})\Big\}^{\perp} = \bigvee_{k\geq 1}\{T^{k}\mathcal{D}(T^{k})\}^{\perp} = \bigvee_{k\geq 1}\mathrm{null}(T^{k*}) = \bigvee_{k\geq 1}\mathrm{null}(\overline{T^{*k}}) = \bigvee_{k\geq 0}\{T'^{k}x : x\in \mathrm{null}(T^{*})\}. \blacksquare$$

Corollary 2.14. *Let T and T' be as in the previous proposition. Then:*

(1) *If T has the wandering subspace property then T' is analytic.*
(2) *T is analytic if and only if T' has the wandering subspace property.*

Question 1. If the operator Cauchy dual to a closed, left-invertible, admissible T is analytic, is it necessarily the case that T has the wandering subspace property?

2.2. Completeness of eigenvectors and hypercyclicity. We refer the interested reader to [7] for some interesting consequences of the bounded counterpart of the following lemma.

Lemma 2.15. *Let B be a densely defined, closed linear operator in $\mathcal{H}$ and let C denote a (possibly unbounded) closed linear operator such that $CB \subset I$. Assume that* $\mathrm{ran}(B) \subset \mathcal{D}(C)$ *and there exists a real $r_0 > 0$ such that the series*

$$e_{\mu,h} \equiv \sum_{n=0}^{\infty} \mu^n B^n h \qquad (\mu \in \mathbb{D}_{r_0},\ h\in \mathcal{D}_{\infty}(B)\cap \mathrm{null}(C))$$

is absolutely convergent in $\mathcal{H}$. Then:

(1) *$Ce_{\mu,h} = \mu e_{\mu,h}$ for every $\mu \in \mathbb{D}_{r_0}$ and every $h \in \mathcal{D}_{\infty}(B)\cap \mathrm{null}(C)$.*
(2) *Assume, in addition, that $\mathrm{null}(C) \subset \mathrm{null}(B^*)$. Then $\{e_{\mu,h_i}\}_{i=1}^{k}$ is linearly independent in $\mathrm{null}(C-\mu)$ provided $\{h_i\}_{i=1}^{k}$ is linearly independent in $\mathcal{D}_{\infty}(B)\cap \mathrm{null}(C)$ for every $\mu \in \mathbb{D}_{r_0}$. In particular, $\mathbb{D}_{r_0} \subset \sigma_{\mathrm{p}}(C)$ whenever $\mathcal{D}_{\infty}(B)\cap\mathrm{null}(C) \neq \{0\}$.*
(3) *For any positive real $r \in (0, r_0]$,*

$$\bigvee_{\mu\in\mathbb{D}_r}\{e_{\mu,h} : h \in \mathcal{D}_{\infty}(B)\cap \mathrm{null}(C)\} = \bigvee_{n\geq 0}\{B^n h : h\in \mathcal{D}_{\infty}(B)\cap \mathrm{null}(C)\}.$$

Proof. Set $e^k_{\mu,h} \equiv \sum_{n=0}^k \mu^n B^n h$ ($\mu \in \mathbb{D}_r, h \in \mathcal{D}_\infty(B) \cap \mathrm{null}(C)$). Note that $e^k_{\mu,h} \in \mathcal{D}(C)$ and $Ce^k_{\mu,h} = \mu e^{k-1}_{\mu,h}$. Since C is closed and $e^k_{\mu,h} \to e_{\mu,h}$ as $k \to \infty$, the first part is immediate. To see (2), it suffices to check that $e_{\mu,h} \neq 0$ provided $h \neq 0$. In view of $CBh = h$ and $h \in \mathrm{null}(B^*)$, one has

$$e_{\mu,h} = 0 \;\Rightarrow\; \langle e_{\mu,h}, h\rangle = 0 \;\Rightarrow\; \|h\|^2 = 0 \;\Rightarrow\; h = 0.$$

Fix $r \in (0, r_0]$. Clearly, $\mathcal{M}_1 \subset \mathcal{M}_2$, where

$$\mathcal{M}_1 \equiv \bigvee\{e_{\mu,h} : \mu \in \mathbb{D}_r,\, h \in \mathcal{D}_\infty(B) \cap \mathrm{null}(C)\},$$

$$\mathcal{M}_2 \equiv \bigvee\{B^n h : n \geq 0,\, h \in \mathcal{D}_\infty(B) \cap \mathrm{null}(C)\}.$$

Fix $x \in \mathcal{H}$ and $h \in \mathcal{D}_\infty(B) \cap \mathrm{null}(C)$. Define $f_{x,h} : \mathbb{D}_{r_0} \to \mathbb{C}$ by

$$f_{x,h}(\mu) = \sum_{n=0}^\infty \overline{\langle x, B^n h\rangle} \mu^n \quad (\mu \in \mathbb{D}_{r_0}).$$

Since $e_{\mu,h}$ is absolutely convergent, $f_{x,h}$ is a well-defined function analytic in $\mathbb{D}_{r_0}$. Now let $x \in \mathcal{M}_1^\perp$. Thus $\langle x, e_{\mu,h}\rangle = 0$ for every $\mu \in \mathbb{D}_r$ and $h \in \mathcal{D}_\infty(B) \cap \mathrm{null}(C)$. It follows that $\sum_{n=0}^k \langle x, B^n h\rangle \overline{\mu}^n \to 0$ as $k \to \infty$ for every $\mu \in \mathbb{D}_r$. Thus the analytic function $f_{x,h}$ is identically zero in $\mathbb{D}_r$. Hence $\langle x, B^n h\rangle = 0$ for all $n \geq 0$. This shows $x \in \mathcal{M}_2^\perp$, and hence $\mathcal{M}_1 = \mathcal{M}_2$. ∎

We say that a densely defined linear operator S with domain $\mathcal{D}(S)$ in $\mathcal{H}$

(1) admits a *complete set of eigenvectors* if $\mathcal{H} = \bigvee_{\mu \in \mathbb{D}_r} \mathrm{null}(S - \mu)$ for every positive real r,

(2) is *hypercyclic* if there exists an $f \in \mathcal{D}_\infty(S)$ such that $\{S^n f : n \in \mathbb{Z}_+\}$ is dense in $\mathcal{H}$, where $\mathbb{Z}_+$ denotes the set of non-negative integers.

Proposition 2.16. *Let T be a closed, admissible operator in $\mathcal{H}$ that is bounded below. If T is analytic then T^* admits a complete set of eigenvectors.*

Proof. Since T is analytic, by Corollary 2.14(2), T' has the wandering subspace property. To check that T^* admits a complete set of eigenvectors let $B \equiv T'$ and $C \equiv T^*$. Clearly, $CB = I$ and $\mathrm{ran}(B) \subset \mathcal{D}(C)$. Also, $\mathrm{null}(C) = \mathrm{null}(B^*)$ and $\mathcal{D}(B) = \mathcal{H}$ (Lemma 2.3). Since $\|T'\| \leq \alpha^{-1}$ (Lemma 2.3), it follows that

$$e_{\mu,h} \equiv \sum_{n=0}^\infty \mu^n B^n h \quad (\mu \in \mathbb{D}_\alpha,\, h \in \mathrm{null}(C))$$

is absolutely convergent in $\mathcal{H}$, where $\alpha \equiv \inf_{\|x\|=1} \|Tx\|$. Hence, by Lemma 2.15(3),

$$\bigvee\{e_{\mu,h} : \mu \in \mathbb{D}_r,\, h \in \mathrm{null}(C)\} = \bigvee_{n \geq 0}\{B^n h : h \in \mathrm{null}(C)\} = \mathcal{H}$$

for any $r \in (0, \alpha]$. ∎

The following corollary yields a class of unbounded hypercyclic operators. Since powers of adjoints of closed operators need not be closed, an unbounded counterpart of the Hypercyclicity Criterion, as established in [3], is not applicable in the present setup. Still, we have the following.

COROLLARY 2.17. *Let T be closed and admissible such that $\|Tx\| \geq \beta\|x\|$ ($x \in \mathcal{D}(T)$) for some positive real β. If T is analytic then αT^* is hypercyclic for every complex α of modulus greater than β^{-1}. In particular, for a weighted shift operator $T : \{\alpha_n\}_{n\geq 0}$ with $\beta \equiv \inf_{n\geq 0} \alpha_n > 0$, αT^* is hypercyclic for every complex α of modulus greater than β^{-1}.*

Proof. Since T is analytic, by Proposition 2.16, T^* admits a complete set of eigenvectors. In particular, the linear subspace

$$\mathcal{E}_r \equiv \mathrm{lin}\{\mathrm{null}(T^* - \mu) : \mu \in \mathbb{D}_r\}$$

is dense in $\mathcal{H}$ for every real $r > 0$. Let α denote a complex number of modulus greater than β^{-1} and let $r_0 \in (0, |\alpha|^{-1})$. Since $|\alpha| r_0 < 1$, it follows that for any $x \in \mathrm{null}(T^* - \mu)$ with $\mu \in \mathbb{D}_{r_0}$, one has

$$\|\alpha^n T^{*n} x\| = |\alpha|^n |\mu|^n \|x\| \leq (|\alpha| r_0)^n \|x\| \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

Also, since $\|T'\| \leq \beta^{-1}$ (Lemma 2.3),

$$\|\alpha^{-n} T'^n x\| = |\alpha|^{-n} \|T'^n x\| \leq |\alpha|^{-n} \beta^{-n} \|x\| \to 0 \quad \text{as } n \to \infty \tag{2.4}$$

for every $x \in \mathcal{H}$.

We adapt the proof of the Hypercyclicity Criterion [3] to the present situation. Let $\{f_k\}_{k\geq 1}$ denote a countable dense subset of $\mathcal{E}_{r_0}$ such that $\|f_k\| = 1$ for all $k \geq 1$. In view of (2.3) and (2.4), one can choose a subsequence $\{n_k\}_{k\geq 1}$ of positive integers such that

$$\|\alpha^{-n_k} T'^{n_k} f_k\| < \frac{1}{2^k}, \qquad \Big\|\alpha^{n_k} T^{*n_k}\Big(\sum_{i<k} \alpha^{-n_i} T'^{n_i} f_k\Big)\Big\| < \frac{1}{2^k}, \tag{2.5}$$

$$\|\alpha^{n_i - n_k} T'^{n_k - n_i} f_k\| < \frac{1}{2^k} \quad (1 \leq i \leq k-1) \tag{2.6}$$

([3, proof of Theorem 2.1]). Let $f = \sum_{k=1}^{\infty} \alpha^{-n_k} T'^{n_k} f_k \in \mathcal{H}$. We claim that $f \in \bigcap_{l\geq 1} \mathcal{D}(T^{*l})$. Note that $\{f_k\}_{k\geq 1} \subset \mathcal{E}_{r_0} \subset \bigcap_{l\geq 1} \mathcal{D}(T^{*l})$ and

$$T^* \sum_{k=1}^{m} \alpha^{-n_k} T'^{n_k} f_k = \sum_{k=1}^{m} \alpha^{-n_k} T'^{n_k - 1} f_k \quad (m \geq 1).$$

It follows from (2.4) that $\sum_{k=1}^{m} \alpha^{-n_k} T'^{n_k-1} f_k$ converges in $\mathcal{H}$. Since T^* is closed, we must have $f \in \mathcal{D}(T^*)$ and $T^* f = \sum_{k=1}^{\infty} \alpha^{-n_k} T'^{n_k - 1} f_k$. A simple induction argument shows $f \in \mathcal{D}(T^{*l})$ and $T^{*l} f = \sum_{k=1}^{\infty} \alpha^{-n_k} C_{k,l}$, where

$$C_{k,l} = \begin{cases} T'^{n_k - l} f_k & \text{if } n_k \geq l, \\ T^{*l - n_k} f_k & \text{otherwise.} \end{cases}$$

Hence the claim is verified. It follows that

$$\alpha^{n_k}T^{*n_k}f - f_k = \alpha^{n_k}T^{*n_k}\Big(\sum_{i<k}\alpha^{-n_i}T'^{n_i}f_i\Big) + \sum_{i>k}\alpha^{n_k-n_i}T'^{n_i-n_k}f_i \to 0$$

as $k\to\infty$, in view of (2.5) and (2.6). This completes the proof of the first part. The rest follows from the first part and Example 2.11. ∎

REMARK 2.18. Let $T:\{\alpha_n\}_{n\geq 0}$ denote a weighted shift operator such that $\beta\equiv\inf_{n\geq 0}\alpha_n>0$. The following can be deduced from Corollary 2.17:

(1) There exists an $f\in\mathcal{D}_\infty(T^*)$ such that $\{\lambda T^{*n}f : n\in\mathbb{Z}_+,\ \lambda\in\mathbb{C}\}$ is dense in $\mathcal{H}$.
(2) If, in addition, $\beta>1$ then T^* is hypercyclic.

In other words, (1) asserts that the adjoint of a weighted shift operator $T:\{\alpha_n\}_{n\geq 0}$ is *supercyclic* provided $\beta>0$. This is an unbounded counterpart of Theorem 3 of [10]. The author believes that the assumption $\beta>0$ is superfluous, as in the bounded case.

We conclude the section with another application of Proposition 2.16. Let T denote a closed, admissible, analytic operator such that $\|Tx\|\geq\alpha\|x\|$ $(x\in\mathcal{D}(T))$ for some positive real α. Suppose further that $\mathrm{null}(T^*)$ is one-dimensional and fix a non-zero $h\in\mathrm{null}(T^*)$ such that $\|h\|_{\mathcal{H}}=1$. Define $\kappa:\mathbb{D}_r\times\mathbb{D}_r\to\mathbb{C}$ by

$$\kappa(\lambda,\mu)=\langle e_{\lambda,h},e_{\mu,h}\rangle_{\mathcal{H}}\quad(\lambda,\mu\in\mathbb{D}_r),$$

where $e_{\lambda,h}\equiv\sum_{n\geq 0}\overline{\lambda}^nT'^nh\in\mathcal{H}$ for every $\lambda\in\mathbb{D}_r$ with $r\equiv\alpha$. Since κ is a positive definite kernel on $\mathbb{D}_r$, we can associate with κ a reproducing kernel Hilbert space $\mathscr{H}$ as described in [1]. Thus

$$\langle g,\kappa(\lambda,\cdot)\rangle_{\mathscr{H}}=g(\lambda)\quad(\lambda\in\mathbb{D}_r,\ g\in\mathscr{H}).$$

Set $Ue_{\lambda,h}=\kappa(\lambda,\cdot)$ $(\lambda\in\mathbb{D}_1)$ and extend U linearly to $\mathcal{E}\equiv\mathrm{lin}\{e_{\lambda,h}:\lambda\in\mathbb{D}_1\}$. Since $\overline{\mathcal{E}}=\mathcal{H}$ (proof of Proposition 2.16), U can be unitarily extended from $\mathcal{H}$ onto $\mathscr{H}$. At this point, one may be tempted to define a linear operator M_z of multiplication by the coordinate function z in $\mathscr{H}$ with the maximal domain $\{f\in\mathscr{H}: zf\in\mathscr{H}\}$. However, in that case, it is far from obvious that M_z is densely defined. Hence, we need to follow a different track. The idea is to introduce a linear operator S in $\mathscr{H}$ by $S(Ux)=UT^*x$ $(x\in\mathcal{D}(T^*))$. Since $U\mathcal{D}(T^*)$ is dense in $\mathscr{H}$, S is densely defined in $\mathscr{H}$ with domain $U\mathcal{D}(T^*)$. Since T^* is closed, so is S. Thus S^* is a densely defined closed linear operator in $\mathscr{H}$. Since $T^*e_{\lambda,h}=\overline{\lambda}e_{\lambda,h}$, for all $f\in\mathcal{D}(S^*)$ and $\lambda\in\mathbb{D}_r$,

$$\begin{aligned}S^*f(\lambda)&=\langle S^*f,\kappa(\lambda,\cdot)\rangle_{\mathscr{H}}=\langle f,S\kappa(\lambda,\cdot)\rangle_{\mathscr{H}}=\langle f,SUe_{\lambda,h}\rangle_{\mathscr{H}}\\&=\langle f,UT^*e_{\lambda,h}\rangle_{\mathscr{H}}=\lambda\langle f,Ue_{\lambda,h}\rangle_{\mathscr{H}}=\lambda\langle f,\kappa(\lambda,\cdot)\rangle_{\mathscr{H}}=\lambda f(\lambda).\end{aligned}$$

Moreover, since $SU = UT^*$, it follows that $U^*S^* \subset TU^*$. Thus $S^*U \subset UT$. Note that for all $f \in \mathcal{D}(T)$ and $g \in \mathcal{D}(T^*)$,

$$\langle Uf, SUg\rangle_{\mathscr{H}} = \langle Uf, UT^*g\rangle_{\mathscr{H}} = \langle f, T^*g\rangle_{\mathcal{H}} = \langle Tf, g\rangle_{\mathcal{H}} = \langle UTf, Ug\rangle_{\mathscr{H}}.$$

This shows that $U\mathcal{D}(T) \subset \mathcal{D}(S^*)$ and $S^*U = UT$. It follows that $\mathscr{F}U = UT$, where $\mathscr{F}$, the *forward shift operator*, is the operator of multiplication by the coordinate function z in $\mathscr{H}$ with domain $\mathcal{D}(S^*)$.

We summarize some characteristic properties of $\mathscr{H}$.

PROPOSITION 2.19. *Let $T, h, U, \mathscr{F}, \mathscr{H}$ be as in the previous discussion. Assume further that $h \in \mathcal{D}_\infty(T) \equiv \bigcap_{k\geq 0} \mathcal{D}(T^k)$. Then:*

(1) *The restriction of the vector space $\mathbb{C}[z]$ of complex polynomials to $\mathbb{D}_r$ is contained in $\mathcal{D}(\mathscr{F})$.*
(2) *For any $f \in \mathscr{H}$, there exists $g \in \mathcal{D}(\mathscr{F}) \equiv U\mathcal{D}(T)$ such that*
$$f(\mu) - f(0) = \mu g(\mu) \quad (\mu \in \mathbb{D}_r).$$
(3) *The backward shift operator $\mathscr{B} : \mathscr{H} \to \mathscr{H}$ given by*
$$(\mathscr{B}f)(z) = \frac{f(z) - f(0)}{z} \quad (z \in \mathbb{D}_r)$$
satisfies $UT'^* = \mathscr{B}U$. In particular, $\mathscr{B}$ is a bounded linear operator on $\mathscr{H}$.
(4) *For all $f \in \mathscr{H}$ and $s \in (0, r)$,*
$$\|f\|_{\infty,\overline{\mathbb{D}_s}} \leq \|f\|_{\mathscr{H}} \frac{1}{1 - r^{-1}s},$$
where $\|f\|_{\infty,\overline{\mathbb{D}_s}} \equiv \sup_{z\in\mathbb{D}_s} |f(z)|$.

Proof. (1): This follows since for any integer $n \geq 0$,

$$(UT^n h)(\lambda) = \langle UT^n h, \kappa(\lambda, \cdot)\rangle_{\mathscr{H}} = \langle T^n h, e_{\lambda,h}\rangle_{\mathcal{H}} = \lambda^n \quad (\lambda \in \mathbb{D}_r).$$

(2): This is easy since for any $\mu \in \mathbb{D}_r$ and $f \in \mathscr{H}$,

$$\begin{aligned} f(\mu) - f(0) &= \langle f, \kappa(\mu, \cdot) - \kappa(0, \cdot)\rangle_{\mathscr{H}} = \langle U^*f, e_{\mu,h} - h\rangle_{\mathcal{H}} \\ &= \langle \mu T'^* U^* f, e_{\mu,h}\rangle_{\mathcal{H}} = \mu(UT'^*U^*f)(\mu), \end{aligned}$$

in view of $e_{\mu,h} - h = \overline{\mu}T'e_{\mu,h}$.

(3): Note that $\mathscr{B}$ is well-defined because of (2). Moreover, the calculations in (2) show that $UT'^* = \mathscr{B}U$, where T' is the operator Cauchy dual to T.

(4): Notice that

$$|f(\lambda)| \leq \|f\|_{\mathscr{H}} \|\kappa(\lambda, \cdot)\|_{\mathscr{H}} \leq \|f\|_{\mathscr{H}} \|e_{\lambda,h}\|_{\mathcal{H}} \leq \|f\|_{\mathscr{H}} \frac{1}{1 - r^{-1}s}$$

for every $\lambda \in \mathbb{D}_r$ and $f \in \mathscr{H}$. ∎

Assume further that $h \in \mathcal{D}_\infty(T)$. Note that T has the wandering subspace property if and only if $\mathbb{C}[z]|_{\mathbb{D}_r}$ is dense in $\mathscr{H}$. Set $\mathscr{H}_a \equiv \overline{\mathbb{C}[z]|_{\mathbb{D}_r}}$ in $\mathscr{H}$. Then

$\mathscr{H}_a$ can be viewed as a reproducing kernel Hilbert space with the reproducing kernel $\kappa_a : \mathbb{D}_r \times \mathbb{D}_r \to \mathbb{C}$ given by

$$\kappa_a(\lambda, \mu) = \langle P_a \kappa(\lambda, \cdot), P_a \kappa(\mu, \cdot)\rangle_{\mathcal{H}} \quad (\lambda, \mu \in \mathbb{D}_r),$$

where $P_a : \mathscr{H} \to \mathscr{H}_a$ denotes the orthogonal projection [1]. Also, it follows from Proposition 2.19(4) that all functions in $\mathscr{H}_a$ are analytic in $\mathbb{D}_r$. Define an operator $\mathscr{F}_a$ in $\mathscr{H}_a$ by $\mathscr{F}_a p = \mathscr{F} p$ ($p \in \mathbb{C}[z]$). Clearly, $\mathscr{F}_a$ is a closable linear operator such that $\overline{\mathscr{F}_a} \subset \mathscr{F}$. Surprisingly, it turns out that for certain 2-hyperexpansive T, $\overline{\mathscr{F}_a} = \mathscr{F}$ if and only if T has the wandering subspace property (see Corollary 3.10).

3. Decompositions of unbounded 2-hyperexpansions. The main result of this section is the Cowen–Douglas Decomposition Theorem for certain 2-hyperexpansions. To establish it, we need several preliminary results.

LEMMA 3.1. *Let T be a densely defined, closed expansion and let T' be the Cauchy dual operator. Let $\mathcal{M} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$ be a closed subspace of $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{M}$ is reducing for T such that $T|_{\mathcal{M}}$ is unitary.*
(2) *$\mathcal{M}$ is reducing for T' such that $T'|_{\mathcal{M}}$ is unitary.*

In particular, T is completely non-unitary if and only if so is T'.

Proof. (1)⇒(2): In view of Lemma 2.3(8),

$$T\mathcal{M} = \mathcal{M} \;\Rightarrow\; T'^*T\mathcal{M} = T'^*\mathcal{M} \;\Rightarrow\; \mathcal{M} = T'^*\mathcal{M}.$$

Since T is expansive, for any $x \in \mathcal{M} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$,

$$\begin{aligned}\|Tx\| = \|x\| \;&\Rightarrow\; \langle T^*Tx - x, x\rangle = 0 \;\Rightarrow\; \|(T^*T - I)^{1/2}x\| = 0\\ &\Rightarrow\; T^*Tx = x \Rightarrow Tx = T'x.\end{aligned}$$

It is now clear that $T'\mathcal{M} = \mathcal{M}$ and that $T'|_{\mathcal{M}}$ is unitary.

(2)⇒(1): Note that

$$T'\mathcal{M} = \mathcal{M} \;\Rightarrow\; T^*T'\mathcal{M} = T^*\mathcal{M} \;\Rightarrow\; \mathcal{M} = T^*\mathcal{M}.$$

Since T' is contractive and $T'^*T' = (T^*T)^{-1}$ (Lemma 2.3), for any $x \in \mathcal{M}$,

$$\|T'x\| = \|x\| \;\Rightarrow\; \langle T'^*T'x - x, x\rangle = 0 \;\Rightarrow\; (T^*T)^{-1}x = x \;\Rightarrow\; T'x = Tx.$$

It follows that $T\mathcal{M} = \mathcal{M}$ and that $T|_{\mathcal{M}}$ is unitary. ∎

The following proposition may be regarded as the key step towards the main result of the present section, which is new even in the bounded case. We invite the interested reader to specialize it to the bounded case.

PROPOSITION 3.2. *Let T be a densely defined, closed expansion and let T' denote the Cauchy dual operator. Let* $\operatorname{nullity}(A)$ *denote the dimension of the null-space* $\operatorname{null}(A)$ *of a linear operator A. Then:*

(1) *T admits the Cowen–Douglas decomposition whenever T' admits the Wold-type decomposition.*
(2) *T' admits the Wold-type decomposition whenever T admits the Cowen–Douglas decomposition and* $\mathrm{nullity}(T^*) < \infty$.

If, in addition, there exists a real $r_0 > 0$ such that the series

$$e_{\mu,h} \equiv \sum_{n=0}^{\infty} \mu^n T^n h \quad (\mu \in \mathbb{D}_{r_0},\ h \in \mathcal{D}_\infty(T) \cap \mathrm{null}(T^*))$$

is absolutely convergent in $\mathcal{H}$ then:

(3) *T' admits the Cowen–Douglas decomposition whenever T admits the Wold-type decomposition.*

Proof. Let A denote a completely non-unitary expansion in $\mathcal{H}$. An examination of the proof of Proposition 2.16 reveals that A^* admits a complete set of eigenvectors if A' has the wandering subspace property, and also

$$\bigvee\{e_{\mu,h} : \mu \in \mathbb{D}_r, h \in \mathrm{null}(A^*)\} = \bigvee_{n\geq 0}\{A'^n h : h \in \mathrm{null}(A^*)\} \text{ for any } r \in (0,1],$$

where $e_{\mu,h} \equiv \sum_{n=0}^{\infty} \mu^n A'^n h \in \mathrm{null}(A^* - \mu)$ $(\mu \in \mathbb{D}_1)$. The desired conclusion in (1) now follows from Lemma 3.1.

To see (2), assume further that $\mathrm{nullity}(A^*) < \infty$ and that A admits a complete set of eigenvectors. Since $\mathbb{D}_1 \cap \sigma_{\mathrm{ap}}(T) = \emptyset$ ([12, Lemma 3.1]), it can be easily deduced from [14, Theorem 7.9] that there exists an $s \in (0,1)$ such that $\mathrm{nullity}(A^* - \mu) = \mathrm{nullity}(A^*)$ for every $\mu \in \mathbb{D}_s$. Since $\mathrm{nullity}(A^*) < \infty$, by Lemma 2.15, we must have

$$\bigvee\{e_{\mu,h} : \mu \in \mathbb{D}_r,\ h \in \mathrm{null}(A^*)\} = \bigvee_{\mu\in\mathbb{D}_r} \mathrm{null}(A^* - \mu) = \mathcal{H}$$

for every $r \in (0,s)$. It follows from the previous discussion that A' has the wandering subspace property. Part (2) now follows from Lemma 3.1.

Again, in view of Lemma 3.1, it suffices to check that A'^* admits a complete set of eigenvectors whenever A has the wandering subspace property. Let $B \equiv A$ and $C \equiv A'^*$. By Lemma 2.3, $CB \subset I$, $\mathrm{ran}(B) \subset \mathcal{D}(C) = \mathcal{H}$, and $\mathrm{null}(C) = \mathrm{null}(B^*)$. Now one may deduce the desired conclusion from Lemma 2.15(3). ∎

How about the converse to Proposition 3.2(3)? First, since T' need not be left-invertible, the dimension of $\mathrm{null}(T'^* - \mu)$ may depend on μ in the vicinity of the origin. Secondly, even if one assumes that $\mu \mapsto \mathrm{null}(T'^* - \mu)$ is constant in a neighbourhood of 0 and that $\mathrm{nullity}(T^*)$ is finite, it may be greater than the dimension of $\mathcal{D}_\infty(T) \cap \mathrm{null}(T^*)$. If T is a bounded expansion then it is easy to see that T admits the Wold-type decomposition (in the sense of Definition 1.2) whenever T' admits the Cowen–Douglas decomposition.

We say that $h \in \bigcap_{n\geq 1} \mathcal{D}(T^n)$ is a *bounded vector* for a densely defined linear operator T in $\mathcal{H}$ if there exist positive reals a and c such that

$$\|T^n h\| \leq ca^n \quad \text{for every integer } n \geq 0.$$

We denote by $\mathcal{B}(T)$ the set of all bounded vectors of T.

COROLLARY 3.3. *Let T be a densely defined, closed expansion in $\mathcal{H}$ and let T' denote the Cauchy dual operator. Assume further that one of the following conditions holds true:*

(1) $\mathrm{null}(T^*)$ *is finite-dimensional and contained in* $\mathcal{B}(T)$.
(2) T *is* 2*-hyperexpansive.*

If T admits the Wold-type decomposition then T' admits the Cowen–Douglas decomposition.

Proof. In view of Proposition 3.2, it suffices to check that for some positive real r,

$$e_{\mu,h} \equiv \sum_{n=0}^{\infty} \mu^n T^n h \quad (\mu \in \mathbb{D}_r,\ h \in \mathrm{null}(T^*))$$

is absolutely convergent in $\mathcal{H}$ whenever (1) or (2) holds true.

(1): Let $\{e_i\}_{i=1}^m$ denote an orthonormal basis of $\mathrm{null}(T^*)$. By hypothesis, for each $i = 1, \ldots, m$ there exist positive reals a_i and c_i such that

$$\|T^n e_i\| \leq c_i a_i^n \quad \text{for every integer } n \geq 0.$$

Let $h \in \mathrm{null}(T^*)$ be of the form $\sum_{i=1}^m \alpha_i e_i$ for some complex numbers α_i. Then for any positive integer n,

$$\|T^n h\| \leq \sum_{i=1}^m |\alpha_i| c_i a_i^n \leq \|h\| \Big(\sum_{i=1}^m c_i^2 a_i^{2n} \Big)^{1/2} \leq m\|h\| ca^n,$$

where $a = \max\{a_1, \ldots, a_n\}$ and $c = \max\{c_1, \ldots, c_n\}$. Thus one may take $r = a^{-1}$.

(2): Fix $\mu \in \mathbb{D}_1$ and choose $a > 1$ such that $|\mu|a < 1$. By hypothesis, $h \in \mathrm{null}(T^*)$ is a C^∞ vector for T. Hence, by Corollary 3.3 of [12], there exists $c > 0$ such that $\|T^n h\| \leq ca^n$ for every integer $n \geq 0$. Hence r can be chosen to be 1 and the proof is complete. ■

LEMMA 3.4. *Let T be a densely defined, closed left-invertible operator in $\mathcal{H}$ such that $\mathcal{D}_\infty(T)$ is dense in $\mathcal{H}$. Then $\mathcal{H}_u \equiv \bigcap_{n\geq 0} T^n \mathcal{D}(T^n) \subset \mathcal{D}(T)$ is a closed invariant subspace for T such that $T|_{\mathcal{H}_u}$ is an invertible bounded linear operator.*

Proof. We imitate the proof in the bounded case (see, for example, [15]). Since T^n is closed, $\mathcal{H}_u$ is a closed subspace of $\mathcal{H}$ such that $T\mathcal{H}_u \subset \mathcal{H}_u$. Indeed, $T\mathcal{H}_u = \mathcal{H}_u$. To see that, let $x \in \mathcal{H}_u$. Then there exists $y_n \in \mathcal{D}(T^n)$ such that

$x = T^n y_n$ for any $n = 0, 1, \ldots$. In particular, $x = Ty_1 = T(T^{n-1}y_n)$ for all $n = 1, 2, \ldots$. Since T is injective, we must have $y_1 = T^{n-1}y_n \in \mathcal{H}_u$ such that $x = Ty_1$. Thus $T|_{\mathcal{H}_u}$ is a bijective linear operator in $\mathcal{H}_u$. Let $\{x_n\}_{n\geq 1} \subset \mathcal{H}_u$ be such that $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$ for some $x, y \in \mathcal{H}$. Since $\mathcal{H}_u$ is closed and $T\mathcal{H}_u \subset \mathcal{H}_u$, we have $x, y \in \mathcal{H}_u$. Since T is closed, we must have $y = Tx$. This shows that $T|_{\mathcal{H}_u}$ is a closed linear operator. Hence, by the Closed Graph Theorem, $T|_{\mathcal{H}_u}$ is an invertible bounded linear operator on $\mathcal{H}_u$. ■

The following is an unbounded counterpart of Proposition 3.4 of [15].

PROPOSITION 3.5. *Let T be a densely defined, closed 2-hyperexpansion in $\mathcal{H}$ such that $\mathcal{D}_\infty(T)$ is dense in $\mathcal{H}$. Set $\mathcal{H}_u \equiv \bigcap_{n\geq 0} T^n\mathcal{D}(T^n) \subset \mathcal{D}(T)$ and let $P_{\mathcal{M}}$ denote the orthogonal projection of $\mathcal{H}$ onto the closed subspace $\mathcal{M}$ of $\mathcal{H}$. If $\mathcal{H}_u \subset \mathcal{D}(T^*)$ then $\mathcal{H}_u$ is a reducing subspace for T such that*

$$T = U \oplus A \quad \text{in } \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_a \quad \text{with} \quad \mathcal{D}(T) = \mathcal{H}_u \oplus \mathcal{D}(A),$$

where U is unitary on $\mathcal{H}_u$ and A is a densely defined, closed linear analytic 2-hyperexpansion in $\mathcal{H}_a$ with domain $\mathcal{D}(A) \equiv \{(I - P_{\mathcal{H}_u})x : x \in \mathcal{D}(T)\} = \mathcal{D}(T) \cap \mathcal{H}_a$ such that $\mathcal{D}_\infty(A)$ is dense in $\mathcal{H}_a$. Moreover, in this case:

(1) *If T is admissible then so is A.*
(2) *If $T\mathcal{D}(T) \subset \mathcal{D}(T)$ then $A\mathcal{D}(A) \subset \mathcal{D}(A)$.*

Proof. Since an invertible bounded 2-hyperexpansion is unitary (Remark 3 of [16]) and restriction of a linear 2-hyperexpansion to an invariant subspace is 2-hyperexpansive, it follows from Lemma 3.4 that $U = T|_{\mathcal{H}_u}$ is unitary. In addition, assume that $\mathcal{H}_u \subset \mathcal{D}(T^*)$. We claim that $\mathcal{H}_u$ is a reducing subspace for T. In view of Lemma 3.4, it suffices to verify that $T^*\mathcal{H}_u \subset \mathcal{H}_u$. Fix $y \in \mathcal{H}_u$. Since $T\mathcal{H}_u = \mathcal{H}_u$, there exists $x \in \mathcal{H}_u$ such that $y = Tx \in \mathcal{H}_u \subset \mathcal{D}(T^*)$. Then

$$\langle T^*Tx, x\rangle_{\mathcal{H}} = \langle Tx, Tx\rangle_{\mathcal{H}} = \langle Ux, Ux\rangle_{\mathcal{H}_u} = \langle x, x\rangle_{\mathcal{H}_u} = \langle x, x\rangle_{\mathcal{H}}.$$

Hence $\langle (T^*T - I)x, x\rangle_{\mathcal{H}} = 0$. Since T is expansive, it follows that $T^*y = T^*Tx = x \in \mathcal{H}_u$ as desired. To conclude the proof of the first part, it suffices to check that $A \equiv T|_{\mathcal{D}(A)}$ is a densely defined, closed, linear, analytic 2-hyperexpansion in $\mathcal{H}_a$ such that $\mathcal{D}_\infty(A)$ is dense in $\mathcal{H}_a$, where

$$\mathcal{D}(A) \equiv \{(I - P_{\mathcal{H}_u})x : x \in \mathcal{D}(T)\}.$$

Since $0 \oplus A \subset T$ and T is expansive, A is expansive. Since $\mathcal{D}(T) = \mathcal{H}_u \oplus \mathcal{D}(A)$ and T is a densely defined linear operator in $\mathcal{H}$, A is a densely defined linear operator in $\mathcal{H}_a$. Also, since

$$\{0\} \subset \bigcap_{n\geq 0} A^n\mathcal{D}(A^n) \subset \Big\{ \bigcap_{n\geq 0} T^n\mathcal{D}(T^n) \Big\} \cap \mathcal{H}_a = \mathcal{H}_u \cap \mathcal{H}_a = \{0\},$$

A is analytic. To check that A is closed, consider $\{x_n\} \subset \mathcal{D}(A)$ such that

$x_n \to x$ and $Ax_n \to y$ in $\mathcal{H}_a$ as $n \to \infty$. Then $\{0 \oplus x_n\} \subset \mathcal{D}(T)$ with $0 \oplus x_n \to 0 \oplus x$ and $T(0 \oplus x_n) \to 0 \oplus y$ in $\mathcal{H}$ as $n \to \infty$. Since T is closed, $0 \oplus x \in \mathcal{D}(T)$ and $T(0 \oplus x) = 0 \oplus y$. It follows that $x \in \mathcal{D}(A)$ and $Ax = y$. Since $\mathcal{D}(T^n) = \mathcal{H}_u \oplus \mathcal{D}(A^n)$ for every positive integer n, it follows that $\mathcal{D}_\infty(T) = \mathcal{H}_u \oplus \mathcal{D}_\infty(A)$. In particular, $\mathcal{D}_\infty(A)$ is dense in $\mathcal{H}_a$ provided $\mathcal{D}_\infty(T)$ is dense in $\mathcal{H}$. We leave it to the reader to check that

$$\{(I - P_{\mathcal{H}_u})x : x \in \mathcal{D}(T)\} = \mathcal{D}(T) \cap \mathcal{H}_a.$$

Finally, we verify (1) and (2).

(1): Assume that T is admissible. Since $T^{*k} = U^{*k} \oplus A^{*k}$, one has $\mathcal{D}_\infty(T^*) = \mathcal{H}_u \oplus \mathcal{D}_\infty(A^*)$. It follows that $\mathcal{D}_\infty(A^*)$ is dense in $\mathcal{H}_a$. Let $x \in \mathcal{D}(A^{n*})$ be such that $0 \oplus x \in \mathcal{D}(T^{n*})$. Then there exists a sequence $\{y_m \oplus x_m\}_{m\geq 1} \subset \mathcal{D}_\infty(T^*)$ such that $y_m \oplus x_m \to 0 \oplus x$ and $T^{n*}(y_m \oplus x_m) \to T^{n*}(0 \oplus x)$. It follows that $\{x_m\}_{m\geq 1} \subset \mathcal{D}_\infty(A^*)$, $x_m \to x$ and $A^{n*}(x_m) \to A^{n*}(x)$. Hence, $\mathcal{D}_\infty(A^*)$ is a core of A^{n*} for every integer $n \geq 0$.

(2): Assume $T\mathcal{D}(T) \subset \mathcal{D}(T)$. Let $y \in \mathcal{D}(A)$. Thus there exists $x \in \mathcal{D}(T)$ such that $y = (I - P_{\mathcal{H}_u})x$. It is easy to see that $T(I - P_{\mathcal{H}_u})x = (I - P_{\mathcal{H}_u})Tx$. Since $Tx \in \mathcal{D}(T)$, it follows that $Ay = T(I - P_{\mathcal{H}_u})x \in \mathcal{D}(A)$. ∎

COROLLARY 3.6. *Let T be an admissible, closed 2-hyperexpansion in $\mathcal{H}$ and let T' denote the operator Cauchy dual to T. Set $\mathcal{H}_u \equiv \bigcap_{n\geq 0} T^n(\mathcal{D}(T^n))$ and assume that $\mathcal{H}_u \subset \mathcal{D}(T^*)$. If T' is analytic then it has the wandering subspace property.*

Proof. It follows from Proposition 3.5 and Lemma 3.1 that $T'\mathcal{H}_u = \mathcal{H}_u$. Hence $\mathcal{H}_u = \bigcap_{n\geq 0} T'^n\mathcal{H}_u \subset \bigcap_{n\geq 0} T'^n\mathcal{H}$. Since T' is analytic, so is T. Hence, by Corollary 2.14(2), T' has the wandering subspace property. ∎

COROLLARY 3.7. *Let T be an admissible, closed 2-hyperexpansion in $\mathcal{H}$ such that*

$$\bigcap_{n\geq 0} T^n\mathcal{D}(T^n) \subset \mathcal{D}(T^*).$$

Then T' admits the Wold-type decomposition.

Proof. Apply Proposition 3.5, Lemma 3.1, and Corollary 2.14(2). ∎

The completely non-unitary part in the Wold-type decomposition of T' of the last corollary turns out to be hyponormal (see Theorem 4.3 of Section 4).

THEOREM 3.8. *If T is an admissible, closed 2-hyperexpansion in $\mathcal{H}$ such that*

$$\bigcap_{n\geq 0} T^n\mathcal{D}(T^n) \subset \mathcal{D}(T^*)$$

then T admits the Cowen–Douglas decomposition. In particular, the adjoint of a completely non-unitary, admissible, closed 2-hyperexpansion admits a complete set of eigenvectors.

Proof. The assertions follow from Corollary 3.7, Proposition 3.2(1), and Proposition 3.5. ∎

REMARK 3.9. We note the following:

(1) Suppose T is a closed, left-invertible operator in $\mathcal{H}$ that admits the Cowen–Douglas decomposition. If $\mathcal{D}_\infty(T)$ is dense in $\mathcal{H}$ then

$$\mathcal{D} \equiv \operatorname{lin}\{f \in \mathcal{H} : f \in \operatorname{null}(T^* - \mu),\ \mu \in \mathbb{C}\}$$

is dense in $\mathcal{H}$ and $\mathcal{D} \subset \mathcal{D}_\infty(T^*)$. In particular, $\mathcal{D}_\infty(T^*)$ is dense in $\mathcal{H}$. Thus the requirement in Theorem 3.8 that T is admissible is not so restrictive.

(2) Let T be an analytic, admissible, closed 2-hyperexpansion and assume that $\operatorname{nullity}(T^*) = m$. In view of the proof of Proposition 3.2, there exists $r_0 \in (0,1)$ such that $\operatorname{nullity}(T^* - \mu) = m$ for every $\mu \in \mathbb{D}_{r_0}$. Also, since $\mathbb{D}_1 \cap \sigma_{\mathrm{ap}}(T) = \emptyset$ ([12, Lemma 3.1]), it follows from Theorem 7.16 of [14] that $\operatorname{ran}(T^* - \mu)$ is closed for every $\mu \in \mathbb{D}_1$. Hence, by Theorem 3.8 above, T^* belongs to the Cowen–Douglas class $\mathcal{B}_m(\mathbb{D}_{r_0})$.

Let T be a closed, admissible, analytic expansion with one-dimensional cokernel. The discussion following Remark 2.18 shows that T can be modelled as the forward shift operator $\mathscr{F}$ in the reproducing kernel Hilbert space $\mathscr{H}$. Assume further the hypotheses of Proposition 2.19. Then $\mathcal{D}(\mathscr{F})$ contains the vector space $\mathbb{C}[z]|_{\mathbb{D}_1}$ of restrictions of complex polynomials to $\mathbb{D}_1$. Moreover, all members of $\mathscr{H}_a$ are functions analytic in $\mathbb{D}_1$, where $\mathscr{H}_a \equiv \overline{\mathbb{C}[z]|_{\mathbb{D}_1}}$ in $\mathscr{H}$. Let $\mathscr{F}_a$ denote the densely defined closable operator in $\mathscr{H}_a$ given by $\mathscr{F}_a p = \mathscr{F} p$ ($p \in \mathbb{C}[z]|_{\mathbb{D}_1}$). Notice that $\overline{\mathscr{F}_a} = \mathscr{F}$ if and only if $\mathbb{C}[z]|_{\mathbb{D}_1}$ is a core of $\mathscr{F}$.

The following result may be regarded as a hyperexpansive analog of [18, Proposition 11].

COROLLARY 3.10. *Let $T, \mathscr{F}$ be as in the preceding discussion. Assume further that T satisfies the hypotheses of Corollary* 3.3. *If $\mathcal{D} \equiv \mathbb{C}[z]|_{\mathbb{D}_1}$ is dense in $\mathscr{H}$ then $\mathbb{C}[z]|_{\mathbb{D}_1}$ is a core of $\mathscr{F}$.*

Proof. Since $\mathcal{D}$ is dense in $\mathscr{H}$, $\mathscr{F}|_{\mathcal{D}}$ is a densely defined closable linear expansion in $\mathscr{H}$. Set $S \equiv \overline{\mathscr{F}|_{\mathcal{D}}}$. Since $\mathscr{F}$ is analytic, S is analytic and hence completely non-unitary. Thus S is a densely defined, completely non-unitary, closed expansion in $\mathscr{H}$. Also, since $\mathscr{F}|_{\mathcal{D}}$ is cyclic in the sense of Stochel and Szafraniec, by Lemma 2 of [18], the dimension of $\operatorname{null}(S^*)$ is less than or equal to one. Since the constant polynomial h given by $h(\lambda) = 1$ ($\lambda \in \mathbb{D}_1$) belongs to $\operatorname{null}(S^*)$, the dimension of $\operatorname{null}(S^*)$ is one. Hence, by Corollary 3.3, S'^* admits a complete set of eigenvectors. Hence, as in the discussion following Remark 2.18, one may define $\kappa' : \mathbb{D}_1 \times \mathbb{D}_1 \to \mathbb{C}$ by

$$\kappa'(\lambda, \mu) = \langle e'_{\lambda,h}, e'_{\mu,h} \rangle_{\mathscr{H}} \qquad (\lambda, \mu \in \mathbb{D}_1),$$

where $e'_{\lambda,h} \equiv \sum_{n\geq 0} \lambda^n S^n h \in \mathscr{H}$ (see the proof of Corollary 3.3), and associate with κ' a reproducing kernel Hilbert space $\mathscr{H}'$. Moreover, the linear map $U' : \mathscr{H} \to \mathscr{H}'$ given by $U'e'_{\lambda,h} = \kappa'(\lambda, \cdot)$ $(\lambda \in \mathbb{D}_1)$ can be unitarily extended onto $\mathscr{H}'$ in such a way that $US' = M_z U$, where M_z is multiplication by z in $\mathscr{H}'$. Since S' has the wandering subspace property (Corollary 3.7), $\mathbb{C}[z]|_{\mathbb{D}_1}$ is dense in $\mathscr{H}'$.

Let $f \in \mathscr{H}'$ and choose a sequence $\{p_n\}_{n\geq 1}$ of complex polynomials such that $p_n \to f$ as $n \to \infty$ in $\mathscr{H}'$. Fix $s \in (0,1)$ and let $\lambda \in \overline{\mathbb{D}_s}$. Since $\langle g, \kappa'(\lambda,\cdot)\rangle_{\mathscr{H}'} = g(\lambda)$ for every $\lambda \in \mathbb{D}_1$ and $g \in \mathscr{H}'$, it follows that

$$\begin{aligned} |p_n(\lambda) - p_m(\lambda)| &\leq \|p_n - p_m\|_{\mathscr{H}'} \|\kappa'(\lambda,\cdot)\|_{\mathscr{H}'} = \|p_n - p_m\|_{\mathscr{H}'} \|e'_{\lambda,h}\|_{\mathscr{H}} \\ &\leq \|p_n - p_m\|_{\mathscr{H}'} \sum_{n\geq 0} |\lambda|^n \|S^n h\|_{\mathscr{H}} \end{aligned}$$

for any $m, n \geq 1$. Arguing as in Corollary 3.3, it can be seen that there exists $M_s > 0$ such that $\sum_{n\geq 0} |\lambda|^n \|S^n h\|_{\mathscr{H}} < M_s$ $(\lambda \in \overline{\mathbb{D}_s})$. It follows that every f in $\mathscr{H}'$ is analytic in $\mathbb{D}_1$. Hence every $f \in \mathscr{H}$ can be written as $\sum_{n\geq 0} a_n S'^n 1$ for some sequence $\{a_n\}$ of complex numbers. We can check that $a_n = \langle f, S^n 1\rangle$ for every $n \geq 0$. Thus every $f \in \mathscr{H}$ has the unique representation

$$\sum_{n\geq 0} \langle f, S^n 1\rangle S'^n 1.$$

To conclude the proof, in view of Lemma 2.3(4), it suffices to check that $\mathscr{F}' = S'$. This is simple since

$$\begin{aligned} \mathscr{F}' f &= \sum_{n\geq 0} \langle \mathscr{F}' f, S^n 1\rangle S'^n 1 = \sum_{n\geq 0} \langle \mathscr{F}' f, \mathscr{F}^n 1\rangle S'^n 1 = \sum_{n\geq 1} \langle f, \mathscr{F}^{n-1} 1\rangle S'^n 1 \\ &= \sum_{n\geq 1} \langle f, S^{n-1} 1\rangle S'^n 1 = S' \sum_{n\geq 0} \langle f, S^n 1\rangle S'^n 1 = S' f \quad \text{for any } f \in \mathscr{H}. \ \blacksquare \end{aligned}$$

4. Operators Cauchy dual to unbounded 2-hyperexpansions. In this short section, we prove that the operator Cauchy dual to a closable 2-hyperexpansion with invariant domain is a hyponormal contraction. We need a couple of lemmas.

LEMMA 4.1. *Let S be a densely defined, linear operator. If S^* is a bounded linear operator on $\mathcal{H}$ then $\|Sy\| \leq \|S^*\| \, \|y\|$ for every $y \in \mathcal{D}(S)$.*

Proof. Since $\langle S^* x, y\rangle = \langle x, Sy\rangle$ $(x \in \mathcal{H},\ y \in \mathcal{D}(S))$, one has $\langle S^*Sy, y\rangle = \langle Sy, Sy\rangle$ for every $y \in \mathcal{D}(S)$. Thus $\|Sy\|^2 \leq \|S^*Sy\| \, \|y\| \leq \|S^*\| \, \|Sy\| \, \|y\|$ $(y \in \mathcal{D}(S))$. Hence $\|Sy\| \leq \|S^*\| \, \|y\|$ for every $y \in \mathcal{D}(S)$. ■

LEMMA 4.2. *If T is a closable 2-hyperexpansion such that $T\mathcal{D}(T) \subset \mathcal{D}(T)$ then $\overline{T}$ is 2-hyperexpansive and $\overline{T}\mathcal{D}(\overline{T}) \subset \mathcal{D}(\overline{T})$.*

Proof. Let $x \in \mathcal{D}(\overline{T})$. Then there exists a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $x_n \to x$ and $Tx_n \to \overline{T}x$. Since $\mathcal{D}(T^2) = \mathcal{D}(T)$ and T is 2-hyperexpansive,

$$0 \le \|T^2x_n - T^2x_m\|^2 \le 2\|Tx_n - Tx_m\|^2 - \|x_n - x_m\|^2 \to 0.$$

Thus $\{T^2x_n\}$ is convergent. Since T is closable, $\overline{T}x \in \mathcal{D}(\overline{T})$ and $T^2x_n \to \overline{T}(\overline{T}x)$. Finally, letting n tend to ∞ in $\|T^2x_n\|^2 - 2\|Tx_n\|^2 + \|x_n\|^2 \le 0$ shows that $\overline{T}$ is 2-hyperexpansive. ■

THEOREM 4.3. *Let T be a densely defined, closable operator with $T\mathcal{D}(T) \subset \mathcal{D}(T)$. If T is 2-hyperexpansive then the Cauchy dual operator T' is a hyponormal contraction.*

Proof. Since $\overline{T}$ is 2-hyperexpansive and $\overline{T}\mathcal{D}(\overline{T}) \subset \mathcal{D}(\overline{T})$ (Lemma 4.2), we may assume that T is closed. Since T is expansive, it follows from Lemma 2.3 that T' is a contraction on $\mathcal{H}$. Hence it suffices to check that $\|T'^*x\| \le \|T'x\|$ for every $x \in \mathcal{H}$.

We claim that $A^{-1}\mathcal{H} \subset \mathcal{D}(T^2)$ and T^2A^{-1} is a contraction. Since $T\mathcal{D}(T) \subset \mathcal{D}(T)$, one has $A^{-1}\mathcal{H} = \mathcal{D}(A) \subset \mathcal{D}(A^{1/2}) = \mathcal{D}(T) = \mathcal{D}(T^2)$. Hence the first part of the claim follows. Also, since T is 2-hyperexpansive with $A^{-1}\mathcal{H} \subset \mathcal{D}(T^2)$, one has

$$\|A^{-1}y\|^2 - 2\|TA^{-1}y\|^2 + \|T^2A^{-1}y\|^2 \le 0 \quad \text{for every } y \in \mathcal{H}.$$

But, in view of (2.1), for any $y \in \mathcal{H}$,

$$\begin{aligned}
&\|A^{-1}y\|^2 - 2\|TA^{-1}y\|^2 + \|T^2A^{-1}y\|^2\\
&\quad = \|A^{-1}y\|^2 - 2\|TA^{-1}y\|^2 + \|y\|^2 + \|T^2A^{-1}y\|^2 - \|y\|^2\\
&\quad = \|A^{-1}y\|^2 - 2\langle A^{1/2}A^{-1}y, A^{1/2}A^{-1}y\rangle + \|y\|^2 + \|T^2A^{-1}y\|^2 - \|y\|^2\\
&\quad = \langle A^{-2}y, y\rangle - 2\langle A^{-1}y, y\rangle + \langle y, y\rangle + \|T^2A^{-1}y\|^2 - \|y\|^2\\
&\quad = \langle (A^{-1} - I)^2y, y\rangle + \|T^2A^{-1}y\|^2 - \|y\|^2.
\end{aligned}$$

Hence $\|T^2A^{-1}y\| \le \|y\|$ for every $y \in \mathcal{H}$. This proves the claim.

Since T' is a bounded linear operator, $(T'^*T^*)^* = TT' = T^2A^{-1}$. By the discussion in the previous paragraph and Lemma 4.1,

$$\|T'^*T^*y\| \le \|y\| \quad (y \in \mathcal{D}(T'^*T^*)).$$

Since $T'^* = T'^*T^*T'$, it follows that $\|T'^*x\| = \|T'^*T^*T'x\| \le \|T'x\|$ for every $x \in \mathcal{H}$. Hence T' is a hyponormal contraction. ■

A careful inspection of the proof of Theorem 4.3 reveals that its conclusion holds true for any closed 2-hyperexpansion T with $\mathcal{D}(T) = \mathcal{D}(T^2)$. It turns out that, for such T, the condition $\mathcal{D}(T) = \mathcal{D}(T^2)$ is equivalent to $T\mathcal{D}(T) \subset \mathcal{D}(T)$ ([12, Proposition 4.4 and Theorem 4.5]).

Unlike unbounded subnormals (see, for example, Corollary 3 of [2]), $(T^*T)^{-1}$ is never compact for unbounded 2-hyperexpansive T with invariant domain.

Corollary 4.4. *Let T be a densely defined, closable 2-hyperexpansive operator in $\mathcal{H}$ such that $T\mathcal{D}(T) \subset \mathcal{D}(T)$. Then $(\overline{T}^*\overline{T})^{-1}$ is not compact. In particular, there exists a sequence $\{x_n\}_{n\geq 1} \subset \mathcal{D}(\overline{T})$, without any subsequence convergent in $\mathcal{H}$, such that $\{\overline{T}x_n\}_{n\geq 1}$ is bounded.*

Proof. We may assume that T is a closed 2-hyperexpansion such that $T\mathcal{D}(T) \subset \mathcal{D}(T)$. Suppose $(T^*T)^{-1}$ is compact. By Lemma 2.3(5) the Cauchy dual operator T' is also compact. Also, by Theorem 4.3, T' is hyponormal. Since hyponormal compact operators are normal ([8, Chapter II]), T' is normal. Because T' is injective, normality of T' forces that $\mathrm{null}(T'^*) = \{0\}$. By Lemma 2.3(9), we must have $\mathrm{null}(T^*) = \{0\}$. It follows that the range-space of T is dense in $\mathcal{H}$. Hence, by Proposition 3.5 of [12], T is unitary. Thus $(T^*T)^{-1} = I$ is compact. Since $\mathcal{H}$ is infinite-dimensional, we arrive at a contradiction. This establishes the first part.

Since $(T^*T)^{-1}$ is not compact, it can be deduced from the discussion at the beginning of Section 2, and from Theorems 2.8.2(2) and 1.7.16(e) of [13], that the inclusion map $i : \Gamma \hookrightarrow \mathcal{H}$ is not compact, where $\Gamma \equiv \mathcal{D}(T)$ with inner product $\langle x, y\rangle_\Gamma \equiv \langle Tx, Ty\rangle_{\mathcal{H}}$ $(x, y \in \Gamma)$ is a Hilbert space. The remaining part of the corollary is now immediate. ∎

Recall that a bounded linear operator S on $\mathcal{H}$ is *trace class* if the series $\sum_{n\geq 0}\langle (S^*S)^{1/2}e_n, e_n\rangle$ is convergent for every orthonormal basis $\{e_n\}_{n\geq 0}$. The *trace* of such an S, given by $\mathrm{trace}(S) = \sum_{n\geq 0}\langle Se_n, e_n\rangle$, is finite and independent of the choice of $\{e_n\}_{n\geq 0}$.

For a linear operator S, let $[S^*, S]$ denote the *self-commutator* $S^*S - SS^*$ of S.

Corollary 4.5. *Let T denote an admissible 2-hyperexpansion such that $\mathrm{null}(T^*)$ is finite-dimensional with $\bigcap_{n\geq 0} T^n\mathcal{D}(T) \subset \mathcal{D}(T^*)$. Then the Cauchy dual operator T' has a trace-class self-commutator. In this case, the trace of the self-commutator of T' is at most the dimension of $\mathrm{null}(T^*)$.*

Proof. It follows from Corollary 3.7 and Theorem 4.3 that $T' = U \oplus A'$ where U is unitary and A' is a finitely multicyclic hyponormal. Hence, by the Berger–Shaw Theorem, A' has a trace-class self-commutator [8]. Since the self-commutators of T' and A' coincide, T' has a trace-class self-commutator as well. Also, since T' is a contraction, the second part follows from the Berger–Shaw inequality [8]. ∎

In view of the last corollary, it is natural to ask how the self-commutators of T and T' are related to each other.

Proposition 4.6. *Let T denote a densely defined, closed left-invertible operator and let T' denote the operator Cauchy dual to T. Then $[T'^*, T']T = \underline{-(T^*T)^{-1}}[T^*, T]T'$ if and only if $T'\mathcal{D}(T) \subset \mathcal{D}(T^*T)$. In this case,* $\operatorname{ran}(T) \subset \overline{\mathcal{D}([T^*, T])}$.

Proof. In view of Lemma 2.3(2),

$$[T'^*, T'] = T'^*T' - T'T'^* \supset (T^*T)^{-1} - T'(T^*T)^{-1}T^*.$$

We claim that $A \equiv [T'^*, T']T = (T^*T)^{-1}T - T' \equiv B$. Since $\mathcal{D}(A) = \mathcal{D}(T) = \mathcal{D}(B)$, it suffices to check that $Ax = Bx$ on $\mathcal{D}(T)$. To see that, let $x \in \mathcal{D}(T)$. Then there exists a sequence $\{x_n\}_{n\geq 1} \subset \mathcal{D}(T^*T)$ such that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Since $Ax_n = Bx_n$ for every $n \geq 1$ and since $Ax_n \to Ax, Bx_n \to Bx$ as $n \to \infty$, the claim is verified. Since $T'\mathcal{D}(T) \subset \mathcal{D}(T^*T)$, it follows from $(T^*T)^{-1}T^*T = I|_{\mathcal{D}(T^*T)}$ that $A = B = -(T^*T)^{-1}[T^*, T]T'$. Next, suppose that $[T'^*, T']T = -(T^*T)^{-1}[T^*, T]T'$. In particular,

$$\mathcal{D}(T) = \{x \in \mathcal{H} : T'x \in \mathcal{D}([T^*, T])\}.$$

Thus $T'\mathcal{D}(T) \subset \mathcal{D}([T^*, T])$.

To see the remaining part, note that $T'\mathcal{H} \subset \overline{\mathcal{D}([T^*, T])}$, since T' is continuous (Lemma 3.3) and T is densely defined. It follows that $T\mathcal{D}(T^*T) \subset \overline{\mathcal{D}([T^*, T])}$. Now, one may use the limit argument similar to that of the preceding paragraph to conclude that $\operatorname{ran}(T) \subset \overline{\mathcal{D}([T^*, T])}$. ∎

Let T be a bounded linear operator on $\mathcal{H}$. Then it follows from Proposition 4.6 that $[T'^*, T']T = -(T^*T)^{-1}[T^*, T]T'$. Since $T'T$ is the orthogonal projection of $\mathcal{H}$ onto $\operatorname{ran}(T')$, we must have

$$[T^*, T] = T^*T[T'^*, T']T^2 + [T^*, T]P_{\operatorname{null}(T^*)},$$

where $P_{\operatorname{null}(T^*)}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\operatorname{null}(T^*)$. In particular, if $\operatorname{null}(T^*)$ is finite-dimensional then T has a trace-class self-commutator whenever so does T'. If, in addition, T is 2-hyperexpansive with finite-dimensional cokernel then it follows from Corollary 4.5 that T has a trace-class self-commutator. This is a variant of the hyperexpansivity version of the Berger–Shaw Theorem (Proposition 2.21 of [5]). In view of this and Proposition 4.6, it is of interest whether the self-commutator of a closed 2-hyperexpansion is densely defined, and if it is, whether it admits a trace-class extension.

5. Composition operators: Examples. We illustrate the results of the present paper in the context of a class of composition operators defined on discrete measure spaces. The following example is borrowed from [11].

Example 5.1. Let $X = \{(n, m) : n, m \in \mathbb{Z}$ such that $n \leq m\}$ and let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of positive real numbers. Consider the measure μ on the power set of X given by

$$\mu(\{(n,m)\}) = \begin{cases} 1 & \text{if } n = m, \\ a_n & \text{if } n < m. \end{cases}$$

Consider the measurable function $\phi : X \to X$ given by

$$\phi(n,m) = \begin{cases} (n-1, m-1) & \text{if } n = m, \\ (n, m-1) & \text{if } n < m. \end{cases}$$

Define the *composition operator* C_ϕ in $L^2(X,\mu)$ (for short, $L^2(\mu)$) by

$$C_\phi f = f \circ \phi, \qquad f \in \mathcal{D}(C_\phi) \equiv \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}.$$

Let $\chi : X \to \mathbb{C}$ denote a characteristic function and let

$$e_{i,j} = \begin{cases} \chi_{\{(i,j)\}} & \text{if } i = j, \\ \dfrac{1}{\sqrt{a_i}} \chi_{\{(i,j)\}} & \text{if } i < j. \end{cases}$$

It was recorded in Example 4.4 of [11] that $\{e_{i,j} : (i,j) \in X\}$ is an orthonormal basis for $L^2(\mu)$ and

$$C_\phi e_{i,j} = \begin{cases} e_{i+1,j+1} + \sqrt{a_i}\, e_{i,j+1} & \text{if } i = j, \\ e_{i,j+1} & \text{if } i < j. \end{cases}$$

Also, it can be deduced from the discussion at the beginning of Example 4.4 of [11] that C_ϕ is a closed linear expansion. Thus the Cauchy dual operator C'_ϕ is an injective contraction (Lemma 2.3). In view of

$$C^*_\phi e_{i,j} = \begin{cases} e_{i-1,j-1} & \text{if } i = j, \\ \sqrt{a_i}\, e_{i,j-1} & \text{if } j = i+1, \\ e_{i,j-1} & \text{if } j > i+1, \end{cases}$$

it is easy to see that the Cauchy dual operator C'_ϕ is given by

$$C'_\phi e_{i,j} = \begin{cases} \dfrac{1}{1+a_i} e_{i+1,j+1} + \dfrac{\sqrt{a_i}}{1+a_i} e_{i,j+1} & \text{if } i = j, \\ e_{i,j+1} & \text{if } i < j. \end{cases}$$

The composition operator C_ϕ enjoys the following properties, which can be easily deduced from [11, Example 4.4, Remark 4.5, and Theorem 2.7]:

P1. C_ϕ is bounded if and only if $\{a_n\}_{n=-\infty}^{\infty}$ is bounded.
P2. C_ϕ is 2-hyperexpansive if and only if $\{a_n\}_{n=-\infty}^{\infty}$ is non-increasing.
P3. C_ϕ is not unitarily equivalent to any orthogonal sum of weighted shifts or isometries.
P4. $\mathcal{D} \equiv \operatorname{lin}\{e_{i,j} : (i,j) \in X\} \subset \mathcal{D}(C_\phi) \cap \mathcal{D}(C^*_\phi)$ and $\overline{C_\phi|_{\mathcal{D}}} = C_\phi$.

Lemma 5.2. *Let a_n and C_ϕ be as above. Then:*

(1) *If there exists $\alpha \geq -1/2$ such that $a_{n+1} \leq 2\alpha + 1 + \alpha a_n$ for every $n \in \mathbb{Z}$ then C_ϕ is admissible and $C_\phi \mathcal{D}(C_\phi) \subset \mathcal{D}(C_\phi)$.*
(2) *If $\{a_n\}_{n=-\infty}^{\infty}$ is non-increasing then $\mathcal{H}_u \equiv \bigcap_{k\geq 0} \mathcal{D}(C^k_\phi) \subset \mathcal{D}(C^*_\phi)$.*

Proof. (1): It can be deduced from Proposition 2.2 and the discussion at the beginning of Example 4.4 of [11] that $C_\phi \mathcal{D}(C_\phi) \subset \mathcal{D}(C_\phi)$. Set $S \equiv C_\phi$ and notice that $\mathcal{D} \subset \bigcap_{n\geq 0} \mathcal{D}(S^{*k})$, where $\mathcal{D}$ is as in P4. Let $f \in \mathcal{D}(S^{k^*})$ and let f_m denote the partial sum $\sum_{(i,j)\in X, -m\leq i<j\leq m} \langle f, e_{i,j}\rangle e_{i,j} \in \mathcal{D}$ of f. Then

$$\begin{aligned} S^{n*}f &= \sum_{(i,j)\in X} \langle S^{n*}f, e_{i,j}\rangle e_{i,j} = \sum_{(i,j)\in X} \langle f, S^n e_{i,j}\rangle e_{i,j} \\ &= \sum_{i\in\mathbb{Z}} \Big\langle f, \sum_{k=0}^{n-1} \sqrt{a_{i+k}} e_{i+k,i+n} + e_{i+n,i+n}\Big\rangle e_{i,i} + \sum_{i<j} \langle f, e_{i,j+n}\rangle e_{i,j} \\ &= \sum_{i\in\mathbb{Z}} \Big(\sum_{k=0}^{n-1} \sqrt{a_{i+k}} \langle f, e_{i+k,i+n}\rangle + \langle f, e_{i+n,i+n}\rangle\Big) e_{i,i} + \sum_{i<j} \langle f, e_{i,j+n}\rangle e_{i,j} \\ &= \sum_{(i,j)\in X} \langle f, e_{i,j}\rangle S^{*n} e_{i,j} = \lim_{m\to\infty} S^{*n} f_m \end{aligned}$$

in view of

$$S^{*n} e_{i,i+k} = \begin{cases} e_{i-n,i-n} & \text{if } k = 0, \\ \sqrt{a_i}\, e_{i-n+k,i-n+k} & \text{if } 1 \leq k \leq n, \\ e_{i,i-n+k} & \text{otherwise.} \end{cases}$$

It follows that $f \in \mathcal{D}(\overline{S^{*n}})$ and $\overline{S^{*n}} f = S^{n*} f$.

(2): Set $\mathrm{ISO}(C_\phi) \equiv \{f \in \mathcal{D}(C_\phi) : \|C_\phi f\| = \|f\|\}$. Since

$$\mathrm{ISO}(C_\phi) = \bigvee \{\chi_{\{(n,m)\}} : (n,m) \in X,\, n \neq m\}$$

([11, Remark 4.5]) and $C_\phi|_{\mathcal{H}_u}$ is unitary (Proposition 3.5 and P2), it follows that $\mathcal{H}_u \subset \mathrm{ISO}(C_\phi)$. Since $C_\phi \mathcal{H}_u \subset \mathcal{H}_u$, a routine verification shows that $\mathcal{H}_u$ is actually contained in $\bigvee\{\chi_{\{(n,m)\}} : (n,m) \in X,\, n+1 < m\}$, which can be checked to be a subspace of $\mathcal{D}(C_\phi^*)$. ■

Suppose that $\{a_n\}_{n=-\infty}^{\infty}$ is unbounded and non-increasing. Then C_ϕ is an admissible 2-hyperexpansion such that $\bigcap_{n\geq 0} \mathcal{D}(C_\phi^k) \subset \mathcal{D}(C_\phi^*)$ with invariant domain $\mathcal{D}(C_\phi)$, where

$$\mathcal{D}(C_\phi) \equiv \Big\{ \sum_{(i,j)\in X} \alpha_{i,j} e_{i,j} : \sum_{i\in\mathbb{Z}} |\alpha_{i,i}|^2 (1 + a_i) + \sum_{(i,j)\in X,\, i\neq j} |\alpha_{i,j}|^2 < \infty \Big\}$$

(see [11, Example 4.4 and Proposition 2.4]). Moreover:

(1) C'_ϕ admits a Wold-type decomposition, that is,

$$C'_\phi = U \oplus A'_\phi \quad \text{on } \mathcal{L}^2(\mu) = \mathcal{H}_u \oplus \mathcal{H}_a,$$

where U is unitary on $\mathcal{H}_u$, A'_ϕ is a completely non-unitary hyponormal contraction on $\mathcal{H}_a$, and $\mathcal{H}_a = \bigvee_{n\geq 0} C'^n_\phi(\mathrm{null}(C'^*_\phi))$ (Corollary 3.7 and Theorem 4.3).

(2) C_ϕ admits the Cowen–Douglas decomposition, that is,

$$C_\phi = U \oplus A_\phi \quad \text{in } \mathcal{L}^2(\mu) = \mathcal{H}_u \oplus \mathcal{H}_a \quad \text{with } \mathcal{D}(C_\phi) = \mathcal{H}_u \oplus \mathcal{D}(A_\phi),$$

where U is unitary on $\mathcal{H}_u$, A_ϕ is a completely non-unitary 2-hyperexpansion in $\mathcal{H}_a$ with invariant domain $\mathcal{D}(A_\phi)$ and moreover $\mathcal{H}_a = \bigvee_{\mu\in\mathbb{D}_r}(\mathrm{null}(C_\phi^* - \mu))$ for every positive real r (Theorem 3.8).

(3) αA_ϕ^* is hypercyclic for any $\alpha \in \mathbb{C}$ of modulus greater than 1 (Corollary 2.17 and Proposition 3.5).

(4) $(C_\phi^* C_\phi)^{-1}$ is not compact (Corollary 4.4).

(5) C_ϕ does not have a finite-dimensional cokernel (Corollary 4.5).

The last assertion requires justification. Suppose C_ϕ has a finite-dimensional cokernel. Since for every $n \in \mathbb{N}$,

$$\sum_{(i,j)\in X,\, -n\le i<j\le n} (\|C'_\phi e_{i,j}\|^2 - \|C'^*_\phi e_{i,j}\|^2) = \frac{1}{1+a_n} + 2n - 1 - \frac{1}{(1+a_{-n-1})^2},$$

C'_ϕ does not have a trace-class self-commutator. Hence Corollary 4.5 applies.

The following proposition gathers a few spectral properties of C_ϕ and C'_ϕ.

Proposition 5.3. *Let a_n, C_ϕ and C'_ϕ be as in Example* 5.1. *Suppose* $\inf_{n\in\mathbb{Z}} a_n > 0$. *Then:*

(1) $\sigma_{\mathrm{p}}(C_\phi^*) \supset \mathbb{D}_1$, $\sigma_{\mathrm{p}}(C'^*_\phi) = \mathbb{D}_1$.

(2) $\sigma(C_\phi) = \mathbb{C}$, $\sigma(C'_\phi) = \overline{\mathbb{D}_1}$.

(3) $\sigma_{\mathrm{ap}}(C_\phi) = \mathbb{C} \setminus \mathbb{D}_1$, $\sigma_{\mathrm{ap}}(C'_\phi) \supset \partial\mathbb{D}_1 \cup \{0\}$.

Proof. One can verify that

$$\mathrm{null}(C_\phi^*) \equiv \Big\{ \sum_{(i,j)\in X} \alpha_{i,j} e_{i,j} \in \mathcal{D}(C_\phi^*) : \alpha_{i,j} = 0 \ ((i,j)\in X,\ j > i+1), \\ \alpha_{i+1,i+1} + \alpha_{i,i+1}\sqrt{a_i} = 0 \ (i \in \mathbb{Z})\Big\}.$$

It is now easy to see that $\mathrm{null}(C_\phi^*) \cap \mathcal{D}(C_\phi) \neq \{0\}$. Thus $\mathbb{D}_1 \subset \sigma_{\mathrm{p}}(C'^*_\phi)$ in view of Lemma 2.13(2). Also, since C'_ϕ is a contraction, we have $\sigma(C'_\phi) = \overline{\mathbb{D}_1}$. Hence $\sigma_{\mathrm{ap}}(C'_\phi) \supset \partial\mathbb{D}_1 \cup \{0\}$. The remaining assertions follow from Proposition 5.2 and Corollary 5.3 of [12]. ∎

Suppose T is a densely defined, closed 2-hyperexpansion in $\mathcal{H}$ with invariant domain $\mathcal{D}(T)$ such that $\sigma_{\mathrm{ap}}(T') = \partial\mathbb{D}_1 \cup \{0\}$. (The author does not know, even in the context of Example 5.1, whether or not the inclusion $\partial\mathbb{D}_1 \cup \{0\} \subset \sigma_{\mathrm{ap}}(C'_\phi)$, as guaranteed by Proposition 5.3, is strict.) Then it can be deduced from the proof of Theorem 3.2 of [6] and from Theorem 4.3 that T' admits the Cowen–Douglas decomposition.

Question 2. Let T be an unbounded admissible 2-hyperexpansion, and T' the operator Cauchy dual to T. Is $\sigma_{\mathrm{ap}}(T')$ equal to $\partial\mathbb{D}_1 \cup \{0\}$?

Finally, we construct an unbounded, closed 2-hyperexpansion that has the wandering subspace property.

EXAMPLE 5.4. Consider a densely defined closed linear operator S in $\mathcal{K}$ with domain $\mathcal{D}(S)$ such that $S\mathcal{D}(S) \subset \mathcal{D}(S)$. Assume $\mathrm{null}(S^*) \cap \mathcal{D}(S) \neq \{0\}$ and fix a non-zero $h \in \mathrm{null}(S^*) \cap \mathcal{D}(S)$. Define

$$\begin{aligned}&\mathcal{D}(T) \equiv \mathrm{lin}\{S^k h : k = 0, 1, \ldots\} \subset \mathcal{D}(S),\\&\mathcal{H} \equiv \text{the closure of } \mathcal{D}(T) \text{ in } \mathcal{K},\\&Tx = Sx \quad \text{for every } x \in \mathcal{D}(T).\end{aligned}$$

Clearly, T is a densely defined linear operator in $\mathcal{H}$ such that $T\mathcal{D}(T) \subset \mathcal{D}(T)$. Since T admits the closed extension S, T is closable. It follows from Lemma 2 of [18] that the dimension of $\mathrm{null}(T^*)$ is less than or equal to 1. We check that $h \in \mathrm{null}(T^*)$. Since S extends T, for any $x \in \mathcal{D}(T)$ one has

$$\langle Tx, h\rangle_{\mathcal{H}} = \langle Sx, h\rangle_{\mathcal{K}} = \langle x, S^*h\rangle_{\mathcal{K}} = 0.$$

This shows that $h \in \mathcal{D}(T^*)$ and $T^*h = 0$. It follows that if S is expansive (resp. 2-hyperexpansive) then so is T. Moreover, T is always analytic. To see this, let $x \in \bigcap_{n\geq 0} T^n\mathcal{D}(T)$. In particular, $x \in \mathcal{D}(T)$. Thus there exist $\alpha_j \in \mathbb{C}$ such that $x = \sum_{j=0}^m \alpha_j T^j h$. Since $x \in T^{m+1}\mathcal{D}(T)$, there exist $\beta_j \in \mathbb{C}$ such that $x = \sum_{j=m+1}^k \beta_j T^j h$. Thus $\sum_{j=0}^k \gamma_j T^j h = 0$ for some γ_j. Since $h \in \mathrm{null}(T^*)$, by Lemma 2.3(8)&(9) we have $\overline{L}^k(\sum_{j=0}^k \gamma_j T^j h) = \gamma_k h$. Therefore $\gamma_k = 0$. By a finite induction argument, one can see that $\gamma_j = 0$ for all j. Thus $x = 0$ and T is analytic.

Let a_n, C_ϕ and C'_ϕ be as in Proposition 5.3. Choose a non-zero $h \in \mathrm{null}(C_\phi^*) \cap \mathcal{D}(C_\phi)$. Hence, as in the previous paragraph, one can associate with C_ϕ a closable, analytic 2-hyperexpansive T_ϕ. We claim that $\|T_\phi^k h\| = \|C_\phi^k h\| \to \infty$ as $k \to \infty$. To see this, note that

$$h = \sum_{i\in\mathbb{Z}} \langle h, e_{i,i+1}\rangle (e_{i,i+1} - \sqrt{a_i}\, e_{i+1,i+1}).$$

Since C_ϕ is closed and

$$\sum_{i\in\mathbb{Z}} \langle h, e_{i,i+1}\rangle C_\phi(e_{i,i+1} - \sqrt{a_i}\, e_{i+1,i+1}) \in L^2(\mu),$$

from $h \in \mathrm{null}(C_\phi^*)$ (see the proof of Proposition 5.3) it follows that $C_\phi h = \sum_{i\in\mathbb{Z}} \langle h, e_{i,i+1}\rangle C_\phi(e_{i,i+1} - \sqrt{a_i}\, e_{i+1,i+1})$. Since all non-negative integer powers of C_ϕ are closed, it follows by an induction argument that

$$C_\phi^k h = \sum_{i\in\mathbb{Z}} \langle h, e_{i,i+1}\rangle C_\phi^k(e_{i,i+1} - \sqrt{a_i}\, e_{i+1,i+1}) \quad (k \in \mathbb{N}).$$

Observe that $\|C_\phi^k h\|^2 = \sum_{i\in\mathbb{Z}} |\langle h, e_{i,i+1}\rangle|^2 (1 + \sum_{m=0}^{k-1} a_i a_{i+1+m} + a_i)$ $(k \in \mathbb{N})$.

Since $\{a_n\}_{n=-\infty}^{\infty}$ is unbounded, the claim follows. Also, since $T_\phi \mathcal{D}(T_\phi) \subset \mathcal{D}(T_\phi)$, by Lemma 4.2, $\overline{T_\phi}\mathcal{D}(\overline{T_\phi}) \subset \mathcal{D}(\overline{T_\phi})$. Thus $\overline{T_\phi}$ is an unbounded, closed 2-hyperexpansion with invariant domain that has the wandering subspace property. Now Proposition 3.5 implies that $\overline{T_\phi}$ admits the Wold-type decomposition. Hence, by Corollary 3.3, T'_ϕ admits the Cowen–Douglas decomposition.

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References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337–404.
[2] A. Athavale and S. Chavan, *Sectorial forms and unbounded subnormals*, Math. Proc. Cambridge Philos. Soc. 143 (2007), 685–702.
[3] J. Bès, K. Chan, and S. Seubert, *Chaotic unbounded differentiation operators*, Integral Equations Operator Theory 40 (2001), 257–267.
[4] Y. Berezansky, Z. Sheftel, and G. Us, *Functional Analysis Vol. II*, Oper. Theory Adv. Appl. 86, Birkhäuser, 1990.
[5] S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edinburgh Math. Soc. 50 (2007), 637–652.
[6] —, *On operators close to isometries*, Studia Math. 186 (2008), 275–293.
[7] —, *Co-analytic, right-invertible operators are supercyclic*, Colloq. Math. 119 (2010), 137–142.
[8] J. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr. 36, Amer. Math. Soc. Providence, RI, 1991.
[9] M. Cowen and R. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), 187–261.
[10] H. Hilden and L. Wallen, *Some cyclic and non-cyclic vectors of certain operators*, Indiana Univ. Math. J. 23 (1974), 557–565.
[11] Z. Jabłoński, *Hyperexpansive composition operators*, Math. Proc. Cambridge Philos. Soc. 135 (2003), 513–526.
[12] Z. Jabłoński and J. Stochel, *Unbounded 2-hyperexpansive operators*, Proc. Edinburgh Math. Soc. 44 (2001), 613–629.
[13] M. Miklavčič, *Applied Functional Analysis and Partial Differential Equations*, World Sci., Singapore, 1998.
[14] M. Schechter, *Principles of Functional Analysis*, Grad. Stud. Math. 36, Amer. Math. Soc., Providence, RI, 2002.
[15] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. 531 (2001), 147–189.
[16] V. Sholapurkar and A. Athavale, *Completely and alternatingly hyperexpansive operators*, J. Operator Theory 43 (2000), 43–68.
[17] J. Stochel and F. Szafraniec, *On normal extensions of unbounded operators. II*, Acta Sci. Math. (Szeged) 53 (1989), 153–177.
[18] —, —, *On normal extensions of unbounded operators. III, Spectral properties*, Publ. RIMS Kyoto Univ. 25 (1989), 105–139.

[19] J. Stochel and F. Szafraniec, *A peculiarity of the creation operator*, Glasgow Math. J. 44 (2002), 137–147.

[20] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland and Akadémiai Kiadó, Amsterdam and Budapest, 1970.

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