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On quasi-compactness of operator nets on Banach spaces

by

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Abstract. The paper introduces a notion of quasi-compact operator net on a Banach space. It is proved that quasi-compactness of a uniform Lotz–Räbiger net $(T_\lambda)_\lambda$ is equivalent to quasi-compactness of some operator T_λ. We prove that strong convergence of a quasi-compact uniform Lotz–Räbiger net implies uniform convergence to a finite-rank projection. Precompactness of operator nets is also investigated.

In the present paper, we study nets of bounded linear operators on Banach spaces from the point of view of uniform convergence. Throughout the paper, $\Lambda = (\Lambda, \succeq)$ is supposed to be some *fixed* directed set of indices. We do not impose any special assumption (like existence of countable cofinal subsets, etc.) on Λ.

Let X be a complex Banach space. Denote by $\mathcal{L}(X)$ the algebra of bounded linear operators on X. We call nets (indexed by Λ) of operators of $\mathcal{L}(X)$ *operator nets*. The set of all operator nets in X (indexed by Λ) is denoted by $\mathcal{NL}(X)$. The set of bounded operator nets $(T_\lambda)_\lambda$ (i.e. $\sup_{\lambda\in\Lambda} \|T_\lambda\| < \infty$) is denoted by $\mathcal{BNL}(X)$. Notice that both $\mathcal{NL}(X)$ and $\mathcal{BNL}(X)$ are algebras with respect to the operation $+$ of *pointwise addition*

$$(T_\lambda)_\lambda + (S_\lambda)_\lambda = (T_\lambda + S_\lambda)_\lambda,$$

and the operation $\bullet$ of *pointwise composition*

$$(T_\lambda)_\lambda \bullet (S_\lambda)_\lambda = (f_\lambda \circ g_\lambda)_\lambda\,,$$

also called the *product* of the nets $(T_\lambda)_\lambda$ and $(S_\lambda)_\lambda$.

The algebra $\mathcal{L}(X)$ becomes a subalgebra of $\mathcal{BNL}(X)$ (and hence of $\mathcal{NL}(X)$) after the identification of operators with nets via $T \equiv \mathbf{T} = (T)_\lambda$. Operator nets and more general nets of continuous functions have been studied recently in [Em]. There is a natural topology on $\mathcal{BNL}(X)$ generated by the following norm: $\|\cdot\| = \sup_{\lambda\in\Lambda} \|T_\lambda\|$. Obviously, $\mathcal{BNL}(X)$ is a Banach

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algebra with respect to this norm, and $\mathcal{L}(X)$ is a Banach subalgebra of $\mathcal{BNL}(X)$.

The following definition goes back to [Lo, Rä]; see also more recent papers [EE1, BE].

Definition 1. An operator net $(T_\lambda)_\lambda \in \mathcal{BNL}(X)$ is called

(a) a *Lotz–Räbiger net* (or an *LR-net*) if
$$\lim_\lambda \|T_\mu T_\lambda x - T_\lambda x\| = \lim_\lambda \|T_\lambda T_\mu x - T_\lambda x\| = 0$$
for all $\mu \in \Lambda$ and $x \in X$;

(b) a *uniform Lotz–Räbiger net* (or a *ULR-net*) if
$$\lim_\lambda \|T_\mu T_\lambda - T_\lambda\| = \lim_\lambda \|T_\lambda T_\mu - T_\lambda\| = 0$$
for all $\mu \in \Lambda$. We denote by $\mathcal{LR}(X)$ (resp. $\mathcal{ULR}(X)$) the collection of all LR-nets (resp. ULR-nets) on X indexed by Λ.

Obviously, $\mathcal{ULR}(X) \subseteq \mathcal{LR}(X)$. One of the main examples of LR-nets is any bounded sequence of Cesàro averages of an operator T such that $n^{-1}T^n \to 0$ strongly. For other examples of LR-nets we refer the reader to [Kr, Lo, Rä, EE1, EZ, Em, BE, EE2]. We shall use the following result which is a part of the convergence theorem for LR-nets (see, for example, [Em, Thm. 1]).

Theorem 2. *Let $(T_\lambda)_\lambda \in \mathcal{LR}(X)$. Then the following conditions are equivalent:*

(i) *$(T_\lambda)_\lambda$ converges strongly;*
(ii) *for every $x \in X$, the net $(T_\lambda x)_{\lambda\in\Lambda}$ has a weak cluster point;*
(iii)
$$X = \mathrm{Fix}((T_\lambda)_\lambda) \oplus \overline{\bigcup_{\lambda\in\Lambda} (I - T_\lambda)X}.$$

If these conditions are satisfied, the strong limit of $(T_\lambda)_\lambda$ is the projection onto the fixed space $\mathrm{Fix}((T_\lambda)_\lambda)$ *of* $(T_\lambda)_\lambda$.

Recall that an operator $T \in L(X)$ is called *quasi-compact* if there exist a positive integer n and a compact operator K with $\|T^n - K\| < 1$. This notion goes back to Krylov and Bogolyubov [KB1, KB2]. It is well known that an operator $T \in L(X)$ is quasi-compact if and only if there exists a sequence $(K_n)_{n=0}^\infty \subseteq L(X)$ of compact operators with $\lim_{n\to\infty} \|T^n - K_n\| = 0$ (cf. [Kr, p. 88]). This motivates the following definition.

Definition 3. An operator net $(T_\lambda)_\lambda$ is called *quasi-compact* if for every λ there is a compact operator K_λ with $\lim_\lambda \|T_\lambda - K_\lambda\| = 0$.

We emphasize that quasi-compactness in the sense of Definition 3 is a property of an operator net and not of individual operators forming the net.

We denote by $\mathcal{QNL}(X)$ the collection of all quasi-compact nets in $\mathcal{NL}(X)$ and by $\mathcal{BQNL}(X)$ the collection of all bounded nets in $\mathcal{QNL}(X)$. In general, $\mathcal{BQNL}(X)$ is a proper subset of $\mathcal{QNL}(X)$. It can be shown easily that $\mathcal{BQNL}(X)$ is a closed ideal of the Banach algebra $\mathcal{BNL}(X)$.

In the present paper we prove several results on quasi-compactness and uniform convergence of ULR-nets. We begin with the following two rather simple technical propositions.

PROPOSITION 4. *Let $\mathcal{T} = (T_\lambda)_\lambda \in \mathcal{NL}(X)$. Denote by* $\operatorname{co}\mathcal{ST}$ *the convex hull of the semigroup $\mathcal{ST}$ generated by the set $\{T_\lambda : \lambda \in \Lambda\}$.*

(a) *If $(T_\lambda)_\lambda$ is an LR-net, then*

$$\lim_\lambda \|ST_\lambda x - T_\lambda x\| = 0 = \lim_\lambda \|T_\lambda Sx - T_\lambda x\| \quad (\forall x \in X) \tag{1}$$

for every S in the closure of $\operatorname{co}\mathcal{ST}$ *in the strong operator topology.*

(b) *If $(T_\lambda)_\lambda$ is a ULR-net, then*

$$\lim_\lambda \|ST_\lambda - T_\lambda\| = 0 = \lim_\lambda \|T_\lambda S - T_\lambda\| \tag{2}$$

for every S in the norm-closure of $\operatorname{co}\mathcal{ST}$.

Proof. Since the proof is fairly elementary, we give it for the first equality of (2) only.

If $S = T_\lambda \in \mathcal{T}$, then (2) holds true by Definition 1. If $S = T_\lambda^n \in \mathcal{ST}$, then (2) follows by taking the nth iteration of the first step. If $S = \sum_{k=1}^n \alpha_k B_k$, $\alpha_k \geq 0$, $\sum_{k=1}^n \alpha_k = 1$, $B_k \in \operatorname{co}\mathcal{ST}$, then (2) follows from the second step by the linearity of the operators B_k. Finally, the passage from $\operatorname{co}\mathcal{ST}$ to $\overline{\operatorname{co}}\,\mathcal{ST}$ is justified by the uniform boundedness of $\mathcal{T}$. ∎

PROPOSITION 5. *Let $(T_\lambda)_\lambda \in \mathcal{ULR}(X)$ be such that $1 \notin \sigma(T_{\lambda_0})$ for some λ_0 and* $\lim_\lambda \|T_\lambda - P\| = 0$. *Then $P = 0$. Conversely, if* $\lim_\lambda \|T_\lambda\| = 0$ *then $1 \notin \sigma(T_{\lambda_0})$ for some λ_0.*

Proof. Let $1 \notin \sigma(T_{\lambda_0})$. Then

$$0 = \|\cdot\|\text{-}\lim_\lambda (I - T_{\lambda_0})T_\lambda = (I - T_{\lambda_0})P,$$

where the first equality is due to (2). Since $I - T_{\lambda_0}$ is invertible, $P = 0$.

Assume now that $\lim_\lambda \|T_\lambda\| = 0$. Then $\lim_\lambda r(T_\lambda) = 0$, where $r(T_\lambda)$ is the spectral radius of T_λ. Hence $1 \notin \sigma(T_{\lambda_0})$ for some λ_0. ∎

COROLLARY 6. *Let $(T_\lambda)_\lambda$ be a uniformly convergent ULR-net. Then $\|T_\lambda\| \to 0$ if and only if $r(T_\lambda) \to 0$.*

The following theorem characterizes quasi-compactness of a ULR-net through quasi-compactness of operators belonging to it.

THEOREM 7. *Let $\mathcal{T} = (T_\lambda)_\lambda \in \mathcal{ULR}(X)$. Then the following conditions are equivalent:*

(i) *the net $\mathcal{T}$ is quasi-compact;*
(ii) *there exists $\lambda_0 \in \Lambda$ such that T_λ is quasi-compact for all $\lambda \succeq \lambda_0$;*
(iii) *there exists $\lambda_0 \in \Lambda$ such that T_{λ_0} is quasi-compact;*
(iv) *there exists $S \in \overline{\mathrm{co}}\,\mathcal{S}\mathcal{T}$ such that S is quasi-compact.*

Proof. The implication (i)⇒(ii) holds true for an arbitrary operator net, by Definition 3. The implications (ii)⇒(iii)⇒(iv) are obvious.

(iv)⇒(i): Assume that $S \in \overline{\mathrm{co}}\,\mathcal{S}\mathcal{T}$ is quasi-compact. Then $S^n \in \overline{\mathrm{co}}\,\mathcal{S}\mathcal{T}$ for all $n \in \mathbb{N}$. By Proposition 4 we obtain

$$\lim_\lambda \|S^n T_\lambda - T_\lambda\| = 0 \qquad (\forall n \in \mathbb{N}).$$

Let $\varepsilon > 0$. Take a large enough n such that

$$\|S^n - K\| \leq \varepsilon$$

for some compact operator K. Then $\overline{\lim}_\lambda \|KT_\lambda - T_\lambda\| \leq \varepsilon M$, where $M = \sup_\lambda \|T_\lambda\|$. Since $\varepsilon > 0$ can be arbitrarily small and the operators KT_λ are compact, we conclude that $(T_\lambda)_\lambda$ is quasi-compact. ∎

Remark 8. The implication (iv)⇒(i) of Theorem 7 is not true if one replaces the condition $\mathcal{T} \in \mathcal{ULR}(X)$ by $\mathcal{T} \in \mathcal{LR}(X)$ (see Example 14 below).

A special case of quasi-compact nets is any operator net which converges uniformly to a compact operator. Definition 1 implies that a uniform limit of a ULR-net (if it exists) is a projection onto the fixed space of the net. The following theorem deals with existence of the uniform limit under certain additional assumptions.

Theorem 9. *A strongly convergent ULR-net $\mathcal{T} = (T_\lambda)_\lambda$ on a Banach space X is quasi-compact if and only if $\mathcal{T}$ converges uniformly to a finite-rank projection.*

Proof. If $\mathcal{T}$ converges uniformly to a finite-rank projection then the quasi-compactness of $\mathcal{T}$ follows by Definition 3.

Assume now that $\mathcal{T}$ is a quasi-compact strongly convergent ULR-net. By Definition 1, $\mathcal{T}$ converges strongly to the projection P onto the fixed space $\mathrm{Fix}(\mathcal{T})$.

First of all we prove that $\mathcal{T}$ converges uniformly to P. For this purpose it is enough to show that $\mathcal{T}$ is Cauchy in the operator norm. Let $\varepsilon > 0$. By the quasi-compactness of $\mathcal{T}$, there exists a $\mu \in \Lambda$ and a compact operator $K = K(\mu, \varepsilon)$ such that

$$\|T_\mu - K\| \leq \varepsilon. \tag{3}$$

Denote $M = \sup_\lambda \|T_\lambda\|$ and take a large enough λ_0 satisfying

$$\|T_\lambda T_\mu - T_\lambda\| \leq \varepsilon \qquad (\forall \lambda \succeq \lambda_0). \tag{4}$$

Replacing T_μ in (4) by K according to (3) gives us

$$\begin{aligned}(5)\qquad \|T_\lambda K - T_\lambda\| &= \|T_\lambda(K - T_\mu) + T_\lambda T_\mu - T_\lambda\| \\ &\le M\varepsilon + \varepsilon = (M+1)\varepsilon \quad (\forall \lambda \succeq \lambda_0).\end{aligned}$$

If $\xi, \zeta \succeq \lambda_0$, then by using (5) one gets

$$\begin{aligned}(6)\qquad \|T_\xi - T_\zeta\| &= \sup_{\|x\|\le 1} \|T_\xi x - T_\zeta x\| \\ &= \sup_{\|x\|\le 1} \|T_\xi x - T_\xi Kx + T_\xi Kx - T_\zeta Kx + T_\zeta Kx - T_\zeta x\| \\ &\le 2(M+1)\varepsilon + \sup_{\|x\|\le 1} \|T_\xi Kx - T_\zeta Kx\|.\end{aligned}$$

By the compactness of K, we can take $x_1, \dots, x_n$ in the unit ball B_X of X such that for every $x \in B_X$ there exists x_k with $\|Kx - Kx_k\| \le \varepsilon$. Then (6) gives us

$$(7)\qquad \|T_\xi - T_\zeta\| \le 2(M+1)\varepsilon + \sup_{k=1,\dots,n} \|T_\xi Kx_k - T_\zeta Kx_k\| + 2M\varepsilon$$

for $\xi, \zeta \succeq \lambda_0$. Taking into account the strong convergence of $\mathcal{T}$, we can find a large enough $\theta_0 \succeq \lambda_0$ satisfying

$$(8)\qquad \sup_{k=1,\dots,n} \|T_\xi Kx_k - T_\zeta Kx_k\| \le \varepsilon \quad (\forall \xi, \zeta \succeq \theta_0).$$

Combining (7) with (8), we obtain

$$\|T_\xi - T_\zeta\| \le 2(M+1)\varepsilon + 2M\varepsilon + \varepsilon \quad (\forall \xi, \zeta \succeq \theta_0).$$

This means that $\mathcal{T}$ is Cauchy in the operator norm, and therefore converges uniformly to P.

Now we show that P is a finite-rank projection. It is enough to prove the compactness of P. Let $\varepsilon > 0$. There exist λ_1, λ_2 such that

$$\|T_\lambda - P\| \le \varepsilon \quad (\forall \lambda \succeq \lambda_1)$$

and

$$\|T_\lambda - Q(\lambda, \varepsilon)\| \le \varepsilon \quad (\forall \lambda \succeq \lambda_2)$$

for certain compact operators $Q(\lambda, \varepsilon)$. Take a λ_3 satisfying $\lambda_3 \succeq \lambda_1, \lambda_2$; then

$$\|P - Q(\lambda_3, \varepsilon)\| \le \|T_{\lambda_3} - P\| + \|T_{\lambda_3} - Q(\lambda_3, \varepsilon)\| \le 2\varepsilon.$$

Letting $\varepsilon \to 0$, we find that P is compact as a norm-cluster point of compact operators. ∎

COROLLARY 10. *If a quasi-compact ULR-net is strongly convergent then it converges uniformly to a finite-rank projection.*

Corollary 10 is just a reformulation of Theorem 9. The next corollary follows from Theorems 2 and 9 by the weak precompactness of any bounded set in a reflexive Banach space.

Corollary 11. *A ULR-net in a reflexive Banach space X is quasi-compact if and only if it converges uniformly to a finite-rank projection.*

Theorem 12. *Let $(T_\lambda)_\lambda$ be a strongly convergent quasi-compact ULR-net. Then $\|T_\lambda\| \to 0$ if and only if $1 \notin \sigma(T_{\lambda_0})$ for some λ_0.*

Proof. Note that, by Theorem 9, $\lim_\lambda \|T_\lambda - P\| = 0$ for a projection P. If $\lim_\lambda \|T_\lambda\| = 0$ then $1 \notin \sigma(T_{\lambda_0})$ for some λ_0 according to Proposition 4.

If $1 \notin \sigma(T_{\lambda_0})$ for some λ_0 then $\lim_\lambda \|T_\lambda\| = 0$ by Corollary 6. ∎

We consider briefly a situation with LR-nets. The following example shows that it may easily happen that a quasi-compact LR-net converges strongly but does not converge uniformly.

Example 13. Define $U_n \in \mathcal{L}(\ell_2)$ for any $n \in \mathbb{N}$ by

$$(U_n x)_k := \begin{cases} 0, & k \neq n \\ x_k, & k = n \end{cases} \qquad (\forall x = (x_1, x_2, \ldots) \in \ell_2).$$

Then $\mathcal{U} = (U_n)_{n=1}^\infty$ is an LR-net, since obviously $\mathcal{U}$ converges to 0 strongly. Every operator U_n is a rank-one projection. Hence $\mathcal{U}$ is quasi-compact. It is easy to see that $\mathcal{U}$ does not converge to 0 uniformly.

It follows directly from Definition 3 that a C_0-semigroup G is quasi-compact if and only if there exists a quasi-compact operator $T \in G$ (see, for example, [Kr, Lm. 2.2.4]). It is interesting that Theorem 9 cannot be extended to arbitrary LR-nets. The following example is due to V. G. Troitsky [Tr], who showed it to the author during the Symposium on Positivity and Applications in Bolu in 2008.

Example 14. Let $X = \ell_p$ for an arbitrary $p \in [1, \infty)$. Define $T_n \in \mathcal{L}(X)$ for any $n \in \mathbb{N}$ as follows:

$$(T_n x)_k := \begin{cases} 0, & k \leq n+1, \\ x_{k-1}, & k > n{+}1 \text{ and } k \text{ is odd}, \\ 0, & k > n+1 \text{ and } k \text{ is even}, \end{cases}$$

where $x = (x_1, x_2, \ldots) \in X$. Every operator T_n is nilpotent and hence quasi-compact. Moreover $(T_n)_{n=1}^\infty$ is an LR-net since $\lim_{n\to\infty} \|T_n x\|_p \to 0$ for every $x \in X$. However, it is easy to see that the distance in the operator norm from each T_n to the space $K(X)$ of all compact operators on X is equal to 1. Hence the net $(T_n)_{n=1}^\infty$ is not quasi-compact.

We conclude the paper with a short discussion of precompactness in operator norm. The case of LR-nets of Cesàro averages of a single operator is exhaustively treated by Lyubich and Zemánek in [LZ]. It follows directly from Theorem 2 that every precompact LR-net converges strongly. In the case of a ULR-net one can say even more, namely that the net is uniformly convergent. Indeed let $(T_\lambda)_\lambda$ be a precompact strongly convergent ULR-net.

It converges strongly to a projection P onto the fixed space $\mathrm{Fix}((T_\lambda)_\lambda)$ according to Theorem 2. Since $(T_\lambda)_\lambda$ is precompact, it is enough to show that P is a unique cluster point of $(T_\lambda)_\lambda$ in the operator topology. Let $\|T_\lambda - A\| \to 0$ along some cofinal directed set $\varLambda_0 \subseteq \varLambda$. Since T_λ converges strongly along $\varLambda_0$ to P as well, we have $P = A$. It follows from (2) that for every $\mu \in \varLambda$,

$$T_\mu P = P,$$

and hence

$$(T_\mu - I)P = 0. \tag{7}$$

It follows from (7) that $1 \notin \sigma(T_\mu)$ for some $\mu \in \varLambda$ implies $P = 0$. Moreover, in this case $1 \notin \sigma(T_\lambda)$ for all large enough $\lambda \in \varLambda$ (just by the uniform convergence of $(T_\lambda)_\lambda$ to zero). Thus we have proved the following partial generalization of Proposition 5.

Theorem 15. *Let $(T_\lambda)_\lambda$ be a precompact ULR-net, and let $\mu \in \varLambda$ satisfy $1 \notin \sigma(T_\mu)$. Then $1 \notin \sigma(T_\lambda)$ for all large enough $\lambda \in \varLambda$, and moreover $\lim_\lambda \|T_\lambda\| = 0$.*

Taking into account Theorem 2 again, we obtain the following result.

Corollary 16. *If a ULR-net $(T_\lambda)_\lambda$ is precompact then it converges uniformly to a projection P, $X = \mathrm{im}(P) \oplus \ker(P)$, and $\mathrm{im}(P) = \mathrm{Fix}((T_\lambda)_\lambda)$. Moreover, there exists μ_0 such that $\ker(P) = (I - T_\mu)X$ for all $\mu \succeq \mu_0$.*

The next example (cf. [LZ, Example 3]) shows that the assumption that $(T_\lambda)_\lambda$ is a ULR-net is essential even in the finite-dimensional case.

Example 17. Let $T = \left[\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right]$, $|\alpha| = 1$, $\alpha \neq 1$. Denote by A_n^T the nth Cesàro average $n^{-1} \sum_{k=0}^{n-1} T^k$ of T. Then the operator net $(A_n^T)_n$ is precompact but it does not converge. This does not contradict the reasoning before Theorem 15, because this net is not an LR-net.

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