Square functions associated to Schrödinger operators

by

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Abstract. We characterize geometric properties of Banach spaces in terms of boundedness of square functions associated to general Schrödinger operators of the form $\mathcal{L} = -\Delta + V$, where the nonnegative potential V satisfies a reverse Hölder inequality. The main idea is to sharpen the well known localization method introduced by Z. Shen. Our results can be regarded as alternative proofs of the boundedness in H^1, L^p and BMO of classical $\mathcal{L}$-square functions.

1. Introduction. Consider the time independent Schrödinger operator in $\mathbb{R}^d$, $d \geq 3$,

$$(1.1) \qquad \mathcal{L} := -\Delta + V,$$

where the nonnegative potential V satisfies a reverse Hölder inequality for some $s > d/2$ (see (2.1)).

Let X be a Banach space and let $\{\mathcal{P}_t\}_{t>0} = \{e^{-t\sqrt{\mathcal{L}}}\}_{t>0}$ be the (subordinated) Poisson semigroup associated to $\mathcal{L}$ (see (2.8)). For $2 \leq q < \infty$ consider the generalized square function

$$(1.2) \qquad g^{\mathcal{L},q}f(x) = \left(\int_0^\infty \left\| t\frac{\partial \mathcal{P}_t f(x)}{\partial t} \right\|_X^q \frac{dt}{t} \right)^{1/q}$$
$$= \| t\partial_t \mathcal{P}_t f(x) \|_{L^q_X((0,\infty),dt/t)}, \qquad x \in \mathbb{R}^d.$$

By using the method described below we prove the following theorem.

THEOREM A. *Let X be a Banach space and $2 \leq q < \infty$. The following statements are equivalent.*

(i) *X admits an equivalent norm for which it is q-uniformly convex.*
(ii) *The operator $g^{\mathcal{L},q}$ maps $\mathrm{BMO}_{\mathcal{L},X}$ into $\mathrm{BMO}_{\mathcal{L}}$.*

[171]

(iii) *The operator $g^{\mathcal{L},q}$ maps $L_X^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for any p in the range $1 < p < \infty$.*

(iv) *The operator $g^{\mathcal{L},q}$ maps $L_X^1(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$.*

(v) *The operator $g^{\mathcal{L},q}$ maps $H^1_{\mathcal{L},X}$ into $L^1(\mathbb{R}^d)$.*

(vi) *For every $f \in L_X^1(\mathbb{R}^d)$, $g^{\mathcal{L},q}f(x) < \infty$ for almost every $x \in \mathbb{R}^d$.*

In 1995 Z. Shen proved L^p-boundedness of the Riesz transforms associated to the operator $\mathcal{L}$ (see [7]). The main idea in that paper is to break the kernels of the operators into "local" and "global" parts (close to the diagonal and far from the diagonal according to a certain distance $\rho(x)$ related to $\mathcal{L}$). The paper, a nice and exhaustive piece of mathematics, has become a classic and it has been a source of inspiration for a lot of manuscripts regarding harmonic analysis of operators associated to (1.1). However, when these operators are defined by some formula involving the heat semigroup (as in the case of the maximal operator $\sup_{t>0} |e^{-t\mathcal{L}}f|$ and the square function $(\int_0^\infty |t\partial_t e^{-t\mathcal{L}}f|^2\,dt/t)^{1/2})$ the word "local" usually refers to the parameter t of $e^{-t\mathcal{L}}$ being small and controlled in some sense by $\rho(x)$ (see [1]).

Beyond the characterization of q-uniformly convex Banach spaces through boundedness properties of $\mathcal{L}$-square functions, we have another purpose: to enlighten the "localization" technique by sharpening the method introduced in [7] in order to avoid manipulations with the parameter t. At the same time, we get a unified approach to proving H^1, L^p and BMO boundedness results for classical harmonic analysis operators associated to $\mathcal{L}$. Observe that, in particular, Theorem A gives an alternative proof of the boundedness of $g^{\mathcal{L},2}$ in the scalar case.

Let us briefly describe the procedure that is later developed in detail for the case of the square function $g^{\mathcal{L},q}$ acting on vector-valued functions.

Description of the method. Let $\rho(x)$ be an auxiliary critical radii function determined by the potential V (see (2.2)), and N be the region consisting of points $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $|x-y| \leq \rho(x)$. Given a linear operator associated to $\mathcal{L}$, that we denote by $T^{\mathcal{L}}$, let T^{Δ} be the parallel operator associated to the classical Laplacian $-\Delta$. Define the localized operator $T^{\mathcal{L}}_{\text{loc}}f(x) := T^{\mathcal{L}}(\chi_N(x,\cdot)f(\cdot))(x)$ and analogously $T^{\Delta}_{\text{loc}}f(x)$. Then T^{Δ}_{loc} inherits the L^p-boundedness properties from T^{Δ}. Even more, if T^{Δ}_{loc} is bounded in L^p then it is also bounded in $\text{BMO}_{\mathcal{L}}$. In other words, the operator T^{Δ}_{loc} behaves as a natural operator associated to $\mathcal{L}$. Now the method finishes by observing that the difference operators $T^{\Delta}_{\text{loc}} - T^{\mathcal{L}}_{\text{loc}}$ and $T^{\mathcal{L}}_{\text{loc}} - T^{\mathcal{L}}$ are bounded from L^p into L^p for $1 \leq p \leq \infty$ and from $\text{BMO}_{\mathcal{L}}$ into L^{∞}.

In order to make the method uniform, we consider a "local" part, defined through ρ, where the cutting acts on the heat kernel. This idea allows us to handle any operator defined via a formula involving the heat kernel, like Riesz transforms, square functions, etc.

Besides the uniformization, we believe that our main contribution is to show that the local part (given in terms of ρ) of an operator associated to the standard Laplacian $-\Delta$ shares the natural boundedness properties with the corresponding operators associated to $\mathcal{L}$. We emphasize that this phenomenon is quite surprising in the case of boundedness in BMO. See Theorems 3.7 and 3.9 in Section 3 for the case of the g-function. The general ideas are summarized in Remarks 3.8 and 3.10. Observe that the localized operator always has a rough kernel (see Remark 3.10), and it is not clear a priori how to prove the necessary smoothness properties in order to get the desired boundedness in BMO.

In this paper $L^p_X(\mathbb{R}^d)$ denotes the usual L^p-space of Bochner–Lebesgue p-integrable functions on $\mathbb{R}^d$ with values in X. The spaces $H^1_{\mathcal{L},X}$ and $\mathrm{BMO}_{\mathcal{L},X}$ are defined in the same way as in the scalar case just by replacing the absolute value of $\mathbb{C}$ by the norm of X (see (2.12) and (2.9)). For the definition of q-uniform convexity we refer to Section 4. Throughout the paper the letter C denotes a positive constant that may change in each appearance and does not depend on the significant quantities.

The paper is organized as follows. We collect in Section 2 the preliminary results already known in the context of Schrödinger operators. Section 3 contains the technical results needed for the application of the method. Finally Section 4 is devoted to the proof of Theorem A.

2. Preliminaries. The nonnegative potential V in (1.1) satisfies a reverse Hölder inequality for some $s > d/2$; that is, there exists a constant $C = C(s, V)$ such that

$$(2.1) \qquad \left(\frac{1}{|B|} \int_B V(y)^s \, dy \right)^{1/s} \le \frac{C}{|B|} \int_B V(y) \, dy,$$

for all balls $B \subset \mathbb{R}^d$. Associated to this potential, Shen defines in [7] the *critical radii function* as

$$(2.2) \qquad \rho(x) := \sup\left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y) \, dy \le 1 \right\}, \quad x \in \mathbb{R}^d.$$

Some properties of this function ρ are well known. We are particularly interested in the following.

LEMMA 2.1 (see Lemma 1.4 in [7]). *There exist $c > 0$ and $k_0 \ge 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$(2.3) \qquad c^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \le \rho(y) \le c \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{k_0/(k_0+1)}.$$

In particular, there exists a positive constant $C_1 < 1$ such that

$$\text{if} \quad |x - y| \le \rho(x) \quad \text{then} \quad C_1\rho(x) < \rho(y) < C_1^{-1}\rho(x).$$

LEMMA 2.2 (see Lemma 2.3 in [2]). *There exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$ such that the balls $\{Q_k\}_{k=1}^{\infty}$ defined by $Q_k := B(x_k, \rho(x_k))$ satisfy*

- $\bigcup_k Q_k = \mathbb{R}^d$;
- *there exists $N = N(\rho)$ such that, for every $k \ge 1$,*

$$\text{card}\{j : 2Q_j \cap 2Q_k \ne \emptyset\} \le N,$$

where for a ball B and a positive number c we denote by cB the ball with the same center as B and radius c times the radius of B.

Let $\{\mathcal{T}_t\}_{t>0}$ be the heat-diffusion semigroup associated to $\mathcal{L}$ acting on X-valued functions:

$$(2.4) \quad \mathcal{T}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^d} k_t(x, y) f(y) \, dy, \quad f \in L_X^2(\mathbb{R}^d), \, x \in \mathbb{R}^d, \, t > 0.$$

The following lemmas are known.

LEMMA 2.3 (see [3, 4]). *For every $\alpha > 0$ there exists a constant C_α such that*

$$(2.5) \qquad 0 \le k_t(x, y) \le C_\alpha \frac{1}{t^{d/2}} e^{-|x-y|^2/(5t)} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-\alpha}$$

for all $x, y \in \mathbb{R}^d$, $t > 0$.

Let

$$h_t(x) := \frac{1}{(4\pi t)^{d/2}} \, e^{-|x|^2/(4t)}, \qquad x \in \mathbb{R}^d, \, t > 0,$$

be the kernel of the classical heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$ in $\mathbb{R}^d$.

LEMMA 2.4 (see Proposition 2.16 in [3]). *There exists a nonnegative Schwartz class function ω in $\mathbb{R}^d$ such that*

$$(2.6) \qquad |k_t(x, y) - h_t(x - y)| \le \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} \omega_t(x - y), \qquad x, y \in \mathbb{R}^d, \, t > 0,$$

where $\omega_t(x - y) := t^{-d/2}\omega((x - y)/\sqrt{t})$ and

$$(2.7) \qquad\qquad\qquad\qquad \delta := 2 - d/s > 0.$$

Given the heat semigroup (2.4), the Poisson semigroup associated to $\mathcal{L}$ is obtained through Bochner's subordination formula (see [8]):

$$(2.8) \quad \mathcal{P}_t f(x) \equiv e^{-t\sqrt{\mathcal{L}}} f(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} \mathcal{T}_u f(x) \, du, \quad x \in \mathbb{R}^d, \, t > 0.$$

With this we define, for $2 \le q < \infty$, the square function related to $\mathcal{L}$ as in (1.2).

REMARK 2.5 (notational convention). The Poisson semigroup associated to the classical Laplace operator in $\mathbb{R}^d$ will be denoted by $\{P_t\}_{t>0} = \{e^{-t\sqrt{-\Delta}}\}_{t>0}$. Recall that $P_t f(x) = P_t * f(x)$, where

$$P_t(x) = c_d \frac{t}{(t^2 + |x|^2)^{(d+1)/2}}, \quad x \in \mathbb{R}^d, \, t > 0.$$

The square function considered in (1.2) will be denoted by $g^{\Delta,q} f$ when we replace $\mathcal{P}_t f$ by $P_t f$.

A locally integrable function $f : \mathbb{R}^d \to X$ is in $\mathrm{BMO}_{\mathcal{L},X}$ whenever there exists a constant C such that

(i) $|B|^{-1} \int_B \|f(x) - f_B\|_X \, dx \leq C$ for every ball B in $\mathbb{R}^d$,
(ii) $|B|^{-1} \int_B \|f(x)\|_X \, dx \leq C$ for every $B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^d$ and $r_0 \geq \rho(x_0)$.

As usual, $f_B := |B|^{-1} \int_B f(x) \, dx$ for every ball B in $\mathbb{R}^d$. The norm $\|f\|_{\mathrm{BMO}_{\mathcal{L},X}}$ is defined as

(2.9) $$\|f\|_{\mathrm{BMO}_{\mathcal{L},X}} = \inf\{C \geq 0 : \text{(i) and (ii) hold}\}.$$

Let us note that if (ii) is true for some ball B then (i) holds true for the same ball, so we might require (i) only for balls with radii smaller than $\rho(x_0)$. By using the classical John–Nirenberg inequality it can be seen that if in (i) and (ii) the L_X^1-norms are replaced by L_X^p-norms, for $1 < p < \infty$, then the space $\mathrm{BMO}_{\mathcal{L},X}$ does not change and equivalent norms appear (see [1, Corollary 3]).

We define the vector-valued atomic Hardy space related to $\mathcal{L}$ following the scalar-valued definition in [2]. A function $a : \mathbb{R}^d \to X$ is an $H_{\mathcal{L},X}^1$-*atom* associated with a ball $B(x_0, r)$ when $\operatorname{supp} a \subset B(x_0, r)$ and

(2.10) $$\|a\|_{L_X^\infty(\mathbb{R}^d)} \leq \frac{1}{|B(x_0, r)|},$$

and, in addition,

(2.11) $$\int_{\mathbb{R}^d} a(x) \, dx = 0 \quad \text{whenever } 0 < r < \rho(x_0).$$

An X-valued integrable function f in $\mathbb{R}^d$ belongs to $H_{\mathcal{L},X}^1$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H_{\mathcal{L},X}^1$-atoms and $\sum_j |\lambda_j| < \infty$. The norm is given by

(2.12) $$\|f\|_{H_{\mathcal{L},X}^1} = \inf\left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}.$$

In [1] it is shown that $\mathrm{BMO}_{\mathcal{L}}$ is the dual space of $H_{\mathcal{L}}^1$.

3. Technical lemmas. As we said in the description of our method, the following region N will play a fundamental role:

$$N := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq \rho(x)\}.$$

Given N we define the "global" and "local" parts of the square function defined in (1.2) as

$$(3.1) \qquad g_{\text{glob}}^{\mathcal{L},q} f(x) = g^{\mathcal{L},q}(\chi_{N^c}(x, \cdot) f(\cdot))(x),$$
$$g_{\text{loc}}^{\mathcal{L},q} f(x) = g^{\mathcal{L},q} f(x) - g_{\text{glob}}^{\mathcal{L},q} f(x).$$

Note that

$$(3.2) \qquad g_{\text{loc}}^{\mathcal{L},q} f(x) \leq g^{\mathcal{L},q}(\chi_N(x, \cdot) f(\cdot))(x) \leq g^{\mathcal{L},q} f(x) + g_{\text{glob}}^{\mathcal{L},q} f(x), \quad \text{a.e. } x \in \mathbb{R}^d,$$

or equivalently,

$$(3.3) \qquad |g^{\mathcal{L},q} f(x) - g^{\mathcal{L},q}(\chi_N(x, \cdot) f(\cdot))(x)| \leq g_{\text{glob}}^{\mathcal{L},q} f(x), \quad \text{a.e. } x \in \mathbb{R}^d.$$

LEMMA 3.1. *Let X be any Banach space and $\alpha > 0$. Then for any $f \in \bigcup_{1 \leq p \leq \infty} L_X^p(\mathbb{R}^d)$ we have*

$$g_{\text{glob}}^{\mathcal{L},q} f(x) \leq C \int_{\mathbb{R}^d} L(x, y) \chi_{N^c}(x, y) \|f(y)\|_X \, dy, \qquad x \in \mathbb{R}^d,$$

where $L(x, y) = \rho(x)^\alpha / |x - y|^{d+\alpha}$ for $x, y \in \mathbb{R}^d$.

Proof. Using Bochner's subordination formula (2.8) it can be checked that for any function h,

$$\|\partial_t \mathcal{P}_t h(x)\|_X \leq C \int_0^\infty \frac{e^{-t^2/(8u)}}{u^{3/2}} \|\mathcal{T}_u h(x)\|_X \, du,$$

where we applied the inequality $r^\eta e^{-r} \leq C_\eta e^{-r/2}$, valid for $\eta \geq 0$, $r > 0$. Hence, by Minkowski's inequality,

$$g_{\text{glob}}^{\mathcal{L},q} f(x) \leq C \int_0^\infty \left\| t \frac{e^{-t^2/(8u)}}{u^{3/2}} \right\|_{L^q((0,\infty), dt/t)} \|\mathcal{T}_u(\chi_{N^c}(x, \cdot) f(\cdot))(x)\|_X \, du$$

$$= C \int_0^\infty \|\mathcal{T}_u(\chi_{N^c}(x, \cdot) f(\cdot))(x)\|_X \frac{du}{u}$$

$$\leq C \int_0^\infty \int_{\mathbb{R}^d} k_u(x, y) \chi_{N^c}(x, y) \|f(y)\|_X \, dy \, \frac{du}{u}.$$

From (2.5) and the change of variables $r = |x - y|^2 / (cu)$ we get

$$\int\limits_0^\infty k_u(x,y)\,\frac{du}{u} \le C \int\limits_0^\infty \frac{1}{u^{d/2}} e^{-|x-y|^2/(cu)} \left(\frac{\rho(x)}{\sqrt{u}}\right)^\alpha \frac{du}{u}$$

$$= C\frac{\rho(x)^\alpha}{|x-y|^{d+\alpha}} \int\limits_0^\infty r^{(d+\alpha)/2} e^{-r}\,\frac{dr}{r}. \quad \blacksquare$$

LEMMA 3.2. *Let X be any Banach space. Then the global operator $g_{\mathrm{glob}}^{\mathcal{L},q}$ maps*

(a) $L_X^p(\mathbb{R}^d)$ *into* $L^p(\mathbb{R}^d)$ *for any* p, $1 \le p \le \infty$,
(b) $\mathrm{BMO}_{\mathcal{L},X}$ *into* $L^\infty(\mathbb{R}^d)$,
(c) $H_{\mathcal{L},X}^1$ *into* $L^1(\mathbb{R}^d)$.

Proof. Let $L(x,y)$, $x,y \in \mathbb{R}^d$, be as in Lemma 3.1. Observe that

$$\int\limits_{\mathbb{R}^d} L(x,y)\chi_{N^c}(x,y)\,dy = \rho(x)^\alpha \int\limits_{|x-y|>\rho(x)} \frac{1}{|x-y|^{d+\alpha}}\,dy = C$$

for all $x \in \mathbb{R}^d$. On the other hand, by Lemma 2.1, there exists a positive number $\varepsilon < 1$ such that

$$(3.4) \qquad L(x,y) \le C\frac{\rho(y)^\alpha}{|x-y|^{d+\alpha}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{\varepsilon\alpha}$$

$$\le C\left(\frac{\rho(y)^\alpha}{|x-y|^{d+\alpha}} + \frac{\rho(y)^{(1-\varepsilon)\alpha}}{|x-y|^{d+(1-\varepsilon)\alpha}}\right).$$

Assume that $|x-y| > \rho(x)$. Then we claim that $|x-y| \ge C\rho(y)$ for some positive constant C depending on the constants c and k_0 that appear in Lemma 2.1. Indeed, by Lemma 2.1 and the fact that $|x-y|/\rho(x) \ge 1$ and $k_0/(k_0+1) \le 1$, we have

$$\rho(y) \le C\rho(x)\left(1 + \left(\frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}\right) \le C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)$$

$$\le C(\rho(x) + |x-y|) \le 2C|x-y|.$$

This together with (3.4) gives $\int_{\mathbb{R}^d} L(x,y)\chi_{N^c}(x,y)\,dx \le C$. Hence the operator given by the kernel $L(x,y)\chi_{N^c}(x,y)$ maps $L_X^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for every p, $1 \le p \le \infty$. Using Lemma 3.1 we get (a).

To see (b) we observe that for f in $\mathrm{BMO}_{\mathcal{L},X}$, by Lemma 3.1,

$$g_{\mathrm{glob}}^{\mathcal{L},q} f(x) \le C\rho(x)^\alpha \sum_{j=0}^\infty \int\limits_{2^j\rho(x)<|x-y|\le 2^{j+1}\rho(x)} \frac{1}{|x-y|^{d+\alpha}} \|f(y)\|_X\,dy$$

$$\le C\rho(x)^\alpha \sum_{j=0}^\infty \frac{1}{(2^j\rho(x))^{d+\alpha}} \int\limits_{|x-y|\le 2^{j+1}\rho(x)} \|f(y)\|_X\,dy$$

$$= C \sum_{j=0}^{\infty} \frac{1}{2^{j\alpha}} \frac{1}{(2^{j+1}\rho(x))^d} \int\limits_{|x-y|\leq 2^{j+1}\rho(x)} \|f(y)\|_X \, dy$$

$$\leq C\|f\|_{\mathrm{BMO}_{\mathcal{L},X}} \sum_{j=0}^{\infty} \frac{1}{2^{j\alpha}} = C\|f\|_{\mathrm{BMO}_{\mathcal{L},X}} \quad \text{for all } x \in \mathbb{R}^d.$$

For (c) just note that $H^1_{\mathcal{L},X} \subset L^1_X(\mathbb{R}^d)$ and then apply (a). ■

LEMMA 3.3. *Let X be any Banach space. Then, for any strongly measurable X-valued function f,*

$$|g^{\mathcal{L},q}_{\mathrm{loc}} f(x) - g^{\Delta,q}_{\mathrm{loc}} f(x)| \leq C \int_{\mathbb{R}^d} M(x,y)\chi_N(x,y)\|f(y)\|_X \, dy, \quad x \in \mathbb{R}^d,$$

where $M(x,y) = \rho(x)^{-\delta}/|x-y|^{d-\delta}$ for $x,y \in \mathbb{R}^d$, and $\delta > 0$ is given in (2.7).

Proof. Proceeding as in the proof of Lemma 3.1 it is easy to check that

$$|g^{\mathcal{L},q}_{\mathrm{loc}} f(x) - g^{\Delta,q}_{\mathrm{loc}} f(x)|$$

$$\leq C \int_0^{\infty} \int_{\mathbb{R}^d} |k_u(x,y) - h_u(x-y)|\chi_N(x,y)\|f(y)\|_X \, dy \, \frac{du}{u}.$$

Using (2.6) and the fact that ω is rapidly decreasing we get

$$\int_0^{\infty} |k_u(x,y) - h_u(x-y)| \frac{du}{u}$$

$$\leq C\rho(x)^{-\delta} \int_0^{\infty} \frac{1}{u^{(d-\delta)/2}} \omega((x-y)/\sqrt{u}) \frac{du}{u}$$

$$\leq C\rho(x)^{-\delta} \left[\frac{1}{|x-y|^{d-\delta+\varepsilon}} \int_0^{|x-y|^2} \left(\frac{|x-y|}{\sqrt{u}} \right)^{d-\delta+\varepsilon} \omega((x-y)/\sqrt{u}) \frac{du}{u^{1-\varepsilon/2}} \right.$$

$$\left. + \int_{|x-y|^2}^{\infty} \frac{1}{u^{(d-\delta)/2}} \frac{du}{u} \right]$$

$$\leq C \frac{\rho(x)^{-\delta}}{|x-y|^{d-\delta}} \left[\frac{1}{|x-y|^{\varepsilon}} \int_0^{|x-y|^2} \frac{du}{u^{1-\varepsilon/2}} + 1 \right] = C \frac{\rho(x)^{-\delta}}{|x-y|^{d-\delta}}. \quad ■$$

LEMMA 3.4. *Let X be any Banach space. Then the difference operator $g^{\mathcal{L},q}_{\mathrm{loc}} - g^{\Delta,q}_{\mathrm{loc}}$ maps*

(a) $L^p_X(\mathbb{R}^d)$ *into* $L^p(\mathbb{R}^d)$ *for any p, $1 \leq p \leq \infty$,*
(b) $\mathrm{BMO}_{\mathcal{L},X}$ *into* $L^{\infty}(\mathbb{R}^d)$,
(c) $H^1_{\mathcal{L},X}$ *into* $L^1(\mathbb{R}^d)$.

Proof. Let $M(x, y)$, $x, y \in \mathbb{R}^d$, be as in Lemma 3.3. First note that

$$\int\limits_{\mathbb{R}^d} M(x, y)\chi_N(x, y)\, dy = \rho(x)^{-\delta} \int\limits_{|x-y| \leq \rho(x)} \frac{1}{|x - y|^{d-\delta}}\, dy = C, \quad x \in \mathbb{R}^d.$$

On the other hand, by Lemma 2.1,

$$M(x, y) \leq \frac{C}{|x - y|^{d-\delta}}\rho(y)^{-\delta}\left(1 + \frac{|x - y|}{\rho(y)}\right)^{k_0\delta}$$

$$\leq C\left(\frac{\rho(y)^{-\delta}}{|x - y|^{d-\delta}} + \frac{\rho(y)^{-(1+k_0)\delta}}{|x - y|^{d-(1+k_0)\delta}}\right),$$

where $k_0 \geq 1$. This, and the fact that $|x - y| > \rho(x)$ implies $|x - y| > C\rho(y)$ (see the proof of Lemma 3.2), give $\int_{\mathbb{R}^d} M(x, y)\chi_N(x, y)\, dx \leq C$ for all $y \in \mathbb{R}^d$. Applying Lemma 3.3 we deduce (a) and as a consequence also (c).

We shall prove (b). Let $f \in \mathrm{BMO}_{\mathcal{L}, X}$. Then

$$\int\limits_{\mathbb{R}^d} M(x, y)\chi_N(x, y)\|f(y)\|_X\, dy$$

$$= \sum_{j=0}^{\infty} \int\limits_{2^{-(j+1)}\rho(x) < |x-y| \leq 2^{-j}\rho(x)} \frac{\rho(x)^{-\delta}}{|x - y|^{d-\delta}}\|f(y)\|_X\, dy$$

$$\leq C\sum_{j=0}^{\infty} \frac{1}{2^{j\delta}} \frac{1}{(2^{-j}\rho(x))^d} \int\limits_{|x-y| \leq 2^{-j}\rho(x)} \|f(y)\|_X\, dy$$

$$\leq C\sum_{j=0}^{\infty} \frac{1}{2^{j\delta}}\left[\frac{1}{(2^{-j}\rho(x))^d} \int\limits_{|x-y| \leq 2^{-j}\rho(x)} \|f(y) - f_{B(x, 2^{-j}\rho(x))}\|_X\, dy\right.$$

$$\left.+ \sum_{k=0}^{j-1}(\|f_{B(x, 2^{-k}\rho(x))} - f_{B(x, 2^{-(k+1)}\rho(x))}\|_X) + \|f_{B(x, \rho(x))}\|_X\right]$$

$$\leq C\sum_{j=0}^{\infty} \frac{1}{2^{j\delta}}[\|f\|_{\mathrm{BMO}_X} + j\|f\|_{\mathrm{BMO}_X} + \|f\|_{\mathrm{BMO}_{X, \mathcal{L}}}]$$

$$\leq C\|f\|_{\mathrm{BMO}_{\mathcal{L}, X}} \sum_{j=0}^{\infty} \frac{j + 2}{2^{j\delta}} = C\|f\|_{\mathrm{BMO}_{\mathcal{L}, X}}$$

for all $x \in \mathbb{R}^d$. To finish use Lemma 3.3. ∎

LEMMA 3.5. *Let C_1 be the constant that appears in Lemma 2.1 and $\gamma > 0$. Take $x, y \in \mathbb{R}^d$ such that $|x| < \gamma$ and $|y| < (C_1^2/2)\rho(0)$. Then there exists a sufficiently large $R = R_\gamma > 0$ for which $|x/R - y| < \rho(x/R)$.*

Proof. Lemma 2.1 ensures that $C_1\rho(0) < \rho(y) < C_1^{-1}\rho(0)$. Let $R > 0$ be such that $|x/R - y| < C_1^2\rho(0)$ (it is enough to take $R > 2\gamma/(C_1^2\rho(0))$).

Hence $|x/R - y| < C_1\rho(y) < \rho(y)$. Once more using Lemma 2.1 we obtain $\rho(y) < C_1^{-1}\rho(x/R)$ and therefore $|x/R - y| < C_1 C_1^{-1}\rho(x/R) = \rho(x/R)$. ■

LEMMA 3.6. *Let f be a function with compact support. For a real number r denote by f^r the dilation of f defined by $f^r(x) := f(rx)$, $x \in \mathbb{R}^d$. Then for any given $\gamma > 0$ there exists $R > 0$, depending on γ and the support of f, such that*

$$g^{\Delta,q}f(x) = g^{\Delta,q}(\chi_N(x/R, \cdot)f^R(\cdot))(x/R) \quad \text{for all } |x| < \gamma.$$

Proof. The scaling of the classical Poisson semigroup $P_t f^R(x/R) = P_{tR}f(x)$, $R > 0$ (see Remark 2.5), implies that the square function satisfies $g^{\Delta,q}f(x) = g^{\Delta,q}f^R(x/R)$ for all $R > 0$. In order to get the conclusion it is enough to take a sufficiently large R such that the support of f^R is contained in $B(0, (C_1^2/2)\rho(0))$ and such that Lemma 3.5 can be applied. ■

The following result establishes that the boundedness in L^p of the square function $g^{\Delta,q}$ related to the Laplacian $-\Delta$ implies the same type of boundedness for the ρ-localized operator $g_{\text{loc}}^{\Delta,q}$. In fact this is a fairly general property: see Remark 3.8 below.

THEOREM 3.7. *Assume that $g^{\Delta,q}$ maps $L_X^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for some p, $1 < p < \infty$ (resp. $L_X^1(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$). Then the operator $f \mapsto t\partial_t P_t(\chi_N(x, \cdot)f(\cdot))(x)$, $x \in \mathbb{R}^d$, $t > 0$, maps $L_X^p(\mathbb{R}^d)$ into $L_{L_X^q((0,\infty),dt/t)}^p(\mathbb{R}^d)$ (resp. $L_X^1(\mathbb{R}^d)$ into weak-$L_{L_X^q((0,\infty),dt/t)}^1(\mathbb{R}^d)$). In particular $g_{\text{loc}}^{\Delta,q}$ maps $L_X^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ (resp. $L_X^1(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$).*

Moreover, if for every $f \in L_X^1(\mathbb{R}^d)$ we have $g^{\Delta,q}f(x) < \infty$ for almost all $x \in \mathbb{R}^d$, then $\|t\partial_t P_t(\chi_N(x, \cdot)f(\cdot))(x)\|_{L_X^q((0,\infty),dt/t)} < \infty$ for almost all $x \in \mathbb{R}^d$.

Proof. We shall prove only the boundedness in L^p. We leave to the reader the details of the rest of the proof.

Let $\{Q_k\}_{k=1}^\infty$ be the covering of $\mathbb{R}^d$ by critical balls whose existence is guaranteed by Lemma 2.2. Consider the auxiliary operator given by

$$f \mapsto Sf(x) = \sum_{k \geq 1} \chi_{Q_k}(x) t\partial_t P_t(\chi_{2Q_k}f)(x), \quad x \in \mathbb{R}^d, \, t > 0.$$

Then S is a bounded operator from $L_X^p(\mathbb{R}^d)$ into $L_{L_X^q((0,\infty),dt/t)}^p(\mathbb{R}^d)$. Indeed, by using Minkowski's inequality, the finite overlapping of the balls Q_k, the boundedness of $g^{\Delta,q}$ in L^p and once more the finite overlapping of $2Q_k$ we get

$$\|Sf\|_{L^p_{L^q_X((0,\infty),dt/t)}(\mathbb{R}^d)}$$

$$\leq \Big(\int_{\mathbb{R}^d} \Big| \sum_{k\geq 1} \chi_{Q_k}(x) \|t\partial_t P_t(\chi_{2Q_k}f)(x)\|_{L^q_X((0,\infty),dt/t)} \Big|^p dx \Big)^{1/p}$$

$$\leq \sum_{k\geq 1} \|\chi_{Q_k} g^{\Delta,q}(\chi_{2Q_k}f)\|_{L^p_X(\mathbb{R}^d)} \leq C \sum_{k\geq 1} \|g^{\Delta,q}(\chi_{2Q_k}f)\|_{L^p_X(\mathbb{R}^d)}$$

$$\leq C \sum_{k\geq 1} \|\chi_{2Q_k}f\|_{L^p_X(\mathbb{R}^d)} \leq C\|f\|_{L^p_X(\mathbb{R}^d)}.$$

Recall that for a compactly supported function f in $L^\infty_X(\mathbb{R}^d)$ we have, as in (3.2),

$$g^{\Delta,q}_{\mathrm{loc}}f(x) \leq g^{\Delta,q}(\chi_N(x,\cdot)f(\cdot))(x) = \|t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)\|_{L^q_X((0,\infty),dt/t)},$$

a.e. $x \in \mathbb{R}^d$. Our idea is to compare the operators $f \mapsto t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)$ and S. In order to do that we need some geometrical considerations. Let C_1 be the constant that appears in Lemma 2.1. Consider the set

$$\widetilde{N} = \Big\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < \frac{C_1}{1+C_1}\rho(x) \Big\}.$$

It is an exercise to prove that if $(x,y) \in \widetilde{N}$ then, since $\{Q_k\}_{k=1}^\infty$ is a covering of $\mathbb{R}^d$, there exists a positive integer k such that $(x,y) \in Q_k \times 2Q_k$. On the other hand, if $(x,y) \in Q_k \times 2Q_k$, then by using Lemma 2.1 we get $|x-y| \leq |x-x_k| + |x_k-y| \leq 3C_1^{-1}\rho(x)$. Observe that it follows from the finite overlapping property of the balls Q_k that

$$\|t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)\|_X \sim \Big\| \sum_{k\geq 1} \chi_{Q_k}(x) t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x) \Big\|_X$$

for all $x \in \mathbb{R}^d$, $t > 0$. The geometrical comments just made ensure that the kernel of the difference operator

$$(3.5) \quad f \mapsto \sum_{k\geq 1} \chi_{Q_k}(x) t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x) - Sf(x), \quad x \in \mathbb{R}^d, \, t > 0,$$

is supported in the region

$$A := \Big\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \frac{C_1}{1+C_1}\rho(x) \leq |x-y| \leq 3C_1^{-1}\rho(x) \Big\}.$$

Consequently, as

$$(3.6) \qquad \|t\partial_t P_t(x-y)\|_{L^q((0,\infty),dt/t)} = \frac{C}{|x-y|^d}, \quad x,y \in \mathbb{R}^d,$$

we have

$$\left\| \sum_{k \geq 1} \chi_{Q_k}(x) t \partial_t P_t(\chi_N(x, \cdot) f(\cdot))(x) - S f(x) \right\|_{L^q((0,\infty), dt/t)}$$

$$\leq C \int_{\mathbb{R}^d} \frac{\chi_A(x, y)}{|x - y|^d} \|f(y)\|_X \, dy.$$

Note that

$$\int_{\mathbb{R}^d} \frac{\chi_A(x, y)}{|x - y|^d} \, dy = \int_{\frac{C_1}{1+C_1} \rho(x) \leq |x - y| \leq 3 C_1^{-1} \rho(x)} \frac{1}{|x - y|^d} \, dy = C \log \frac{3(1 + C_1)}{C_1^2}$$

for all $x \in \mathbb{R}^d$, and, by Lemma 2.1,

$$\int_{\mathbb{R}^d} \frac{\chi_A(x, y)}{|x - y|^d} \, dx \leq \int_{\alpha_1 \rho(y) \leq |x - y| \leq \alpha_2 \rho(y)} \frac{1}{|x - y|^d} \, dx = C \log \frac{\alpha_2}{\alpha_1}, \quad y \in \mathbb{R}^d,$$

for some constants α_1 and α_2 independent of y. Therefore the operator

$$f \mapsto \int_{\mathbb{R}^d} \frac{\chi_A(x, y)}{|x - y|^d} \|f(y)\|_X \, dy$$

is bounded from L_X^p into L^p for every p, $1 \leq p < \infty$. Hence we get the conclusion. ∎

REMARK 3.8. Consider two Banach spaces X_1 and X_2. Let T be a linear operator that maps $C_c^\infty(\mathbb{R}^d; X_1)$ into X_2-valued strongly measurable functions. Suppose T has an associated kernel which satisfies the standard Calderón–Zygmund estimates. Define the "ρ-localized" operator

$$T_{\mathrm{loc}} f(x) = T(\chi_N(x, \cdot) f(\cdot))(x), \quad x \in \mathbb{R}^d,$$

where N is the region determined by $|x - y| \leq \rho(x)$ as above. Then:

- Assume T has a bounded extension from $L_{X_1}^p(\mathbb{R}^d)$ into $L_{X_2}^p(\mathbb{R}^d)$ for some p, $1 < p < \infty$. Then T_{loc} has a bounded extension from $L_{X_1}^p(\mathbb{R}^d)$ into $L_{X_2}^p(\mathbb{R}^d)$.
- Assume T has a bounded extension from $L_{X_1}^1(\mathbb{R}^d)$ into weak-$L_{X_2}^1(\mathbb{R}^d)$. Then T_{loc} has a bounded extension from $L_{X_1}^1(\mathbb{R}^d)$ into weak-$L_{X_2}^1(\mathbb{R}^d)$.
- Assume that for every $f \in L_{X_1}^1(\mathbb{R}^d)$ we have $\|Tf(x)\|_{X_2} < \infty$ for almost all $x \in \mathbb{R}^d$. Then the same is true for T_{loc}.

The reader can check the validity of this remark just by replacing X by X_1, $L_X^q((0,\infty), dt/t)$ by X_2 and $f \mapsto t \partial_t P_t f(x)$ by $f \mapsto Tf(x)$ along the lines of the proof of Theorem 3.7 above.

The next theorem permits us to pass, for ρ-localized operators related to $-\Delta$, from L^p-boundedness to $\mathrm{BMO}_{\mathcal{L}}$ and $H_{\mathcal{L}}^1$-L^1 boundedness.

THEOREM 3.9. *Let X be a Banach space such that the operator*

$$f \mapsto Tf(x) = t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x), \quad x \in \mathbb{R}^d, \, t > 0,$$

is bounded from $L^p_X(\mathbb{R}^d)$ into $L^p_{L^q_X((0,\infty),dt/t)}(\mathbb{R}^d)$ for some p, $1 < p < \infty$. Then T maps

$$\mathrm{BMO}_{\mathcal{L},X} \text{ into } \mathrm{BMO}_{\mathcal{L},L^q_X((0,\infty),dt/t)} \quad and \quad H^1_{\mathcal{L},X} \text{ into } L^1_{L^q_X((0,\infty),dt/t)}(\mathbb{R}^d).$$

Proof. Boundedness from $\mathrm{BMO}_{\mathcal{L},X}$ into $\mathrm{BMO}_{\mathcal{L},L^q_X((0,\infty),dt/t)}$. We first analyze the behavior over "small" balls. Consider a ball $B = B(x_0, r_0)$ such that $5r_0 < C_1\rho(x_0)$, where $C_1 < 1$ is the constant that appears in Lemma 2.1. Given a function f we decompose it as

$$f = (f - f_B)\chi_{4B} + (f - f_B)\chi_{(4B)^c} + f_B =: f_1 + f_2 + f_3.$$

Before entering the proof proper, we need a preparation. For $x, z \in B$,

$$Tf(x) - Tf(z) = Tf_1(x) - Tf_1(z) + Tf_2(x) - Tf_2(z) + Tf_3(x) - Tf_3(z).$$

We begin by observing that

$$
\begin{aligned}
&Tf_2(x) - Tf_2(z) + Tf_3(x) - Tf_3(z) \\
&\quad = \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|\leq\rho(x)}(y)f_2(y)\,dy \\
&\qquad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(x)}(y))f_2(y)\,dy \\
&\qquad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|z-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(z)}(y))f_2(y)\,dy \\
&\qquad + f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|\leq\rho(x)}(y)\,dy \\
&\qquad + f_B \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(x)}(y))\,dy \\
&\qquad + f_B \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|z-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(z)}(y))\,dy.
\end{aligned}
$$

By Lemma 2.1,

$$\chi_{(4B)^c}(y)(\chi_{|z-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(z)}(y)) = \chi_{|z-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(z)}(y).$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|z-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(z)}(y))f_2(y)\,dy \\
&\quad = \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|z-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(z)}(y))(f(y) - f_B)\,dy.
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
Tf_2(x) &- Tf_2(z) + Tf_3(x) - Tf_3(z) \\
&= \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|\le\rho(x)}(y) f_2(y)\, dy \\
&\quad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|x-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(x)}(y)) f_2(y)\, dy \\
&\quad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)) f(y)\, dy \\
&\quad + f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|\le\rho(x)}(y)\, dy \\
&\quad + f_B \int_{\mathbb{R}^d} t\partial_t P_t(z-y)(\chi_{|x-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(x)}(y))\, dy \\
&=: A_1(x,z) + A_2(x,z) + A_3(x,z) + A_4(x,z) + A_5(x,z).
\end{aligned}
$$

After these remarks, we can start the actual proof of the boundedness in BMO. We have

$$
\frac{1}{|B|} \int_B \|Tf(x) - (Tf)_B\|_{L_X^q((0,\infty),dt/t)}\, dx
$$

$$
\le \frac{2}{|B|} \int_B \|Tf_1(x)\|_{L_X^q((0,\infty),dt/t)}\, dx
$$

$$
+ \sum_{i=1}^{5} \frac{1}{|B|^2} \int_B \int_B \|A_i(x,z)\|_{L_X^q((0,\infty),dt/t)}\, dx\, dz.
$$

By hypothesis, T is bounded from $L_X^p(\mathbb{R}^d)$ into $L_{L_X^q((0,\infty),dt/t)}^p(\mathbb{R}^d)$, so

$$
\frac{1}{|B|} \int_B \|Tf_1(x)\|_{L_X^q((0,\infty),dt/t)} dx \le C\left(\frac{1}{|B|}\int_B \|Tf_1(x)\|_{L_X^q((0,\infty),dt/t)}^p dx\right)^{1/p}
$$

$$
\le C\left(\frac{1}{|B|}\int_{\mathbb{R}^d} \|f_1(x)\|_X^p\, dx\right)^{1/p}
$$

$$
= C\left(\frac{1}{|B|}\int_{4B} \|f(x) - f_B\|_X^p\, dx\right)^{1/p}
$$

$$
\le C\|f\|_{\mathrm{BMO}_X} \le C\|f\|_{\mathrm{BMO}_{\mathcal{L},X}},
$$

where in the penultimate inequality we applied an argument as in (3.7) below. Let us now estimate all the $A_i(x,z)$, $i = 1,\ldots,5$, for $x, z \in B = B(x_0, r_0)$. By the Mean Value Theorem and (3.6),

$\|A_1(x,z)\|_{L_X^q((0,\infty),dt/t)}$

$$\leq C \int_{\mathbb{R}^d} \frac{|x-z|}{|x-y|^{d+1}} \|f_2(y)\|_X \, dy$$

$$\leq C r_0 \int_{|x_0-y|>4r_0} \frac{1}{|x_0-y|^{d+1}} \|f(y) - f_B\|_X \, dy$$

$$= C r_0 \sum_{j=2}^{\infty} \int_{2^j r_0 < |x_0-y| \leq 2^{j+1} r_0} \frac{1}{|x_0-y|^{d+1}} \|f(y) - f_B\|_X \, dy$$

$$\leq C \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{1}{(2^{j+1}r_0)^d} \int_{|x_0-y|\leq 2^{j+1}r_0} \|f(y) - f_B\|_X \, dy$$

$$\leq C\|f\|_{\mathrm{BMO}_X} \leq C\|f\|_{\mathrm{BMO}_{X,\mathcal{L}}}.$$

Again by (3.6),

$\|A_2(x,z)\|_{L_X^q((0,\infty),dt/t)}$

$$\leq C \int_{\mathbb{R}^d} \frac{1}{|z-y|^d} |\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|z-y|\leq\rho(x)}(y)| \, \|f_2(y)\|_X \, dy.$$

Observe that A_2 will be nonzero in the following cases:

(i) $|x-y| \leq \rho(x)$ and $|z-y| > \rho(x)$,
(ii) $|x-y| > \rho(x)$ and $|z-y| \leq \rho(x)$.

In the first case $\rho(x) < |z-y| \leq |z-x| + |x-y| < 2r_0 + |x-y|$ and then $\rho(x) - 2r_0 < |x-y| \leq \rho(x)$. While in (ii) we have $\rho(x) < |x-y| \leq |x-z| + |z-y| < 2r_0 + \rho(x)$. On the other hand $|x-y| \sim |z-y|$. Let j_0 and j_1 be nonnegative integers such that $2^{j_0}r_0 \leq \rho(x)/2 < 2^{j_0+1}r_0$ and $2^{j_1}r_0 \leq 2\rho(x) < 2^{j_1+1}r_0$. Observe that, since $5r_0 < \rho(x)$ for all $x \in B(x_0, r_0)$, we have $j_0 \geq 1$. The Mean Value Theorem gives $(\rho(x)-2r_0)^d - (\rho(x)+2r_0)^d \leq C\rho(x)^{d-1}r_0$, hence applying Hölder's inequality with some $r \in (1,\infty)$ we get

$\|A_2(x,z)\|_{L_X^q((0,\infty),dt/t)}$

$$\leq C \int_{\rho(x)-2r_0<|x-y|<\rho(x)+2r_0} \frac{1}{|x-y|^d} \|f_2(y)\|_X \, dy$$

$$\leq C \left(\int_{\rho(x)-2r_0<|x-y|<\rho(x)+2r_0} \frac{1}{|x-y|^{dr}} \|f(y) - f_B\|_X^r \, dy \right)^{1/r} \rho(x)^{(d-1)/r'} r_0^{1/r'}$$

$$\leq C \left(\int_{\rho(x)/2<|x-y|<2\rho(x)} \frac{1}{|x-y|^{dr}} \|f(y) - f_B\|_X^r \, dy \right)^{1/r} \rho(x)^{(d-1)/r'} r_0^{1/r'}$$

$$\leq C\bigg(\sum_{j=j_0}^{j_1} \frac{1}{(2^j r_0)^{(d-1)(r-1)}} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy\bigg)^{1/r}$$

$$\times \, \rho(x)^{(d-1)/r'}$$

$$\leq C\bigg(\frac{1}{(2^{j_0} r_0)^{(d-1)(r-1)}} \sum_{j=j_0}^{j_0+2} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy\bigg)^{1/r}$$

$$\times \, \rho(x)^{(d-1)/r'}$$

$$\leq C\bigg(\sum_{j=0}^{\infty} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy\bigg)^{1/r} \leq C\|f\|_{\mathrm{BMO}_X}.$$

Observe that in the penultimate inequality above we pass to the infinite series since j_0 depends on $\rho(x)$ and we want an estimate independent of it. For the last inequality above we first note that, by the triangle inequality and Minkowski's integral inequality,

$$(3.7) \quad \bigg(\frac{1}{(2^j r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy\bigg)^{1/r}$$

$$\leq \bigg(\frac{1}{(2^j r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \bigg(\|f(y)-f_{2^{j+2}B}\|_X + \sum_{k=0}^{j+1} \|f_{2^{k+1}B}-f_{2^k B}\|_X\bigg)^r \, dy\bigg)^{1/r}$$

$$\leq \bigg(\frac{4^d}{(2^{j+2} r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_{2^{j+2}B}\|_X^r \, dy\bigg)^{1/r}$$

$$+ \, C\sum_{k=0}^{j+1} \|f_{2^{k+1}B}-f_{2^k B}\|_X$$

$$\leq C(j+3)\|f\|_{\mathrm{BMO}_X}.$$

Hence,

$$\sum_{j=0}^{\infty} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int\limits_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy$$

$$\leq C\sum_{j=0}^{\infty} \frac{(j+3)^r}{2^{j(r-1)}} \|f\|_{\mathrm{BMO}_X}^r = C\|f\|_{\mathrm{BMO}_X}^r.$$

Since $x, z \in B$,

$$\|A_3(x,z)\|_{L_X^q((0,\infty),dt/t)} \leq \int\limits_{C_1\rho(x_0)<|z-y|<C_1^{-1}\rho(x_0)} \frac{1}{|z-y|^d} \|f(y)\|_X \, dy$$

$$\leq \frac{C}{\rho(x_0)^d} \int\limits_{|z-y|<C_1^{-1}\rho(x_0)} \|f(y)\|_X \, dy$$

$$\leq \frac{C}{\rho(z)^d} \int\limits_{|z-y|<C\rho(z)} \|f(y)\|_X \, dy \leq C\|f\|_{\mathrm{BMO}_{\mathcal{L},X}}.$$

By dominated convergence,

$$\int\limits_{\mathbb{R}^d} \partial_t P_t(x-y) \, dy = \partial_t \int\limits_{\mathbb{R}^d} P_t(x-y) \, dy = \partial_t 1 = 0, \quad x \in \mathbb{R}^d.$$

Therefore,

$$\|A_4(x,z)\|_{L_X^q((0,\infty),dt/t)}$$

$$= \left\| f_B \int\limits_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|\leq\rho(x)}(y) \, dy \right\|_{L_X^q((0,\infty),dt/t)}$$

$$= \left\| f_B \int\limits_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|\leq\rho(x)}(y) \, dy \right.$$

$$\left. - f_B \int\limits_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y)) \, dy \right\|_{L_X^q((0,\infty),dt/t)}$$

$$= \left\| f_B \int\limits_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\chi_{|x-y|>\rho(x)}(y) \, dy \right\|_{L_X^q((0,\infty),dt/t)}$$

$$\leq C\|f_B\|_X \int\limits_{\mathbb{R}^d} \|t\partial_t P_t(x-y) - t\partial_t P_t(z-y)\|_{L^q((0,\infty),dt/t)}\chi_{|x-y|>\rho(x)}(y) \, dy$$

$$\leq C\|f_B\|_X \int\limits_{\mathbb{R}^d} \frac{|x-z|}{|x-y|^{d+1}}\chi_{|x-y|>\rho(x)}(y) \, dy$$

$$\leq C\|f_B\|_X \frac{r_0}{\rho(x)} \leq C\|f_B\|_X \frac{r_0}{\rho(x_0)}.$$

As $\|f_B\|_X \leq C(1 + \log(\rho(x_0)/r_0))\|f\|_{\mathrm{BMO}_{\mathcal{L},X}}$ (see [1, Lemma 2]) we get the appropriate bound for A_4. Finally, by using the arguments for A_2,

$$\|A_5(x,z)\|_{L_X^q((0,\infty),dt/t)} \leq C\|f_B\|_X \int\limits_{\rho(x)-2r_0<|x-y|<\rho(x)+2r_0} \frac{1}{|x-y|^d} \, dy$$

$$\leq C\|f\|_{\mathrm{BMO}_{\mathcal{L},X}} \left(1 + \log \frac{\rho(x_0)}{r_0}\right) \log \frac{\rho(x)+2r_0}{\rho(x)-2r_0}.$$

Since $r_0/\rho(x) < 1/5$, we have $\log \frac{\rho(x)+2r_0}{\rho(x)-2r_0} \sim r_0/\rho(x) \sim r_0/\rho(x_0)$, which gives the desired bound for A_5.

Let us now analyze the behavior over "big" balls. Let $B_1 = B(x_0, k\rho(x_0))$ with $k \geq C_1/5$. Given a function f we decompose it as $f = f_1 + f_2$, where $f_1 = f\chi_{2B_1}$. By Hölder's inequality and the hypothesis,

$$\frac{1}{|B_1|} \int_{B_1} \|t\partial_t P_t(\chi_N(x, \cdot)f_1(\cdot))(x)\|_{L_X^q((0,\infty), dt/t)} \, dx$$

$$\leq C\left(\frac{1}{|B_1|} \int_{2B_1} \|f(x)\|_X^p \, dx\right)^{1/p} \leq C\|f\|_{\mathrm{BMO}_{\mathcal{L}, X}}.$$

On the other hand, by Lemma 2.1 and (3.6),

$$\|t\partial_t P_t(\chi_N(x, \cdot)f_2(\cdot))(x)\|_{L_X^q((0,\infty), dt/t)}$$

$$\leq \int_{\mathbb{R}^d} \frac{1}{|x-y|^d} \chi_{2k\rho(x_0) \leq |x-y| \leq \rho(x)}(y)\|f(y)\|_X \, dy$$

$$\leq \frac{C}{\rho(x)^d} \int_{|x-y| \leq \rho(x)} \|f(y)\|_X dy \leq C\|f\|_{\mathrm{BMO}_{\mathcal{L}, X}}, \quad x \in B_1.$$

This finishes the proof of the BMO boundedness.

Boundedness from $H_{\mathcal{L}, X}^1$ *into* $L_{L_X^q((0,\infty), dt/t)}^1(\mathbb{R}^d)$. We begin with the analysis over atoms supported on "small" balls. Let a be an atom with support contained in a ball $\tilde{B} = B(y_0, r_0)$, with $r_0 < \rho(y_0)$. Then

$$\int_{\mathbb{R}^d} \|Ta(x)\|_{L_X^q((0,\infty), dt/t)} \, dx$$

$$= \int_{4\tilde{B}} \|Ta(x)\|_{L_X^q((0,\infty), dt/t)} \, dx + \int_{(4\tilde{B})^c} \|Ta(x)\|_{L_X^q((0,\infty), dt/t)} \, dx$$

$$=: A_1 + A_2.$$

Since T is bounded in L^p, by (2.10) we have

$$A_1 \leq C\left(\int_{4\tilde{B}} \|Ta(x)\|_{L_X^q((0,\infty), dt/t)}^p \, dx\right)^{1/p} |\tilde{B}|^{1/p'}$$

$$\leq C\left(\int_{\tilde{B}} \|a(x)\|_X^p \, dx\right)^{1/p} |\tilde{B}|^{1/p'} \leq C.$$

Applying the fact that the atom a has mean zero (see (2.11)), we get

$$A_2 = \int_{(4\tilde{B})^c} \left\| \int_{\mathbb{R}^d} \left(t\partial_t P_t(x-y)\chi_{|x-y| \leq \rho(x)}(y) - t\partial_t P_t(x-y_0)\chi_{|x-y_0| \leq \rho(x)}(x)\right) \right.$$

$$\left. \times a(y) \, dy \right\|_{L_X^q((0,\infty), dt/t)} dx$$

$$\leq \int\limits_{(4\tilde{B})^c} \left\| \int\limits_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(x-y_0))\chi_{|x-y|\leq\rho(x)}(y) \right.$$
$$\left. \times a(y)\,dy \right\|_{L^q_X((0,\infty),dt/t)} dx$$

$$+ \int\limits_{(4\tilde{B})^c} \left\| \int\limits_{\mathbb{R}^d} t\partial_t P_t(x-y_0)(\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|x-y_0|\leq\rho(x)}(x)) \right.$$
$$\left. \times a(y)\,dy \right\|_{L^q_X((0,\infty),dt/t)} dx$$

$$\leq C \int\limits_{(4\tilde{B})^c} \int\limits_{\mathbb{R}^d} \frac{|y-y_0|}{|x-y_0|^{d+1}} \|a(y)\|_X \, dy\, dx$$

$$+ C \int\limits_{(4\tilde{B})^c} \int\limits_{\mathbb{R}^d} \frac{1}{|x-y_0|^d} |\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|x-y_0|\leq\rho(x)}(x)| \, \|a(y)\|_X \, dy\, dx$$

$$=: C(A_{21} + A_{22}).$$

Fubini's Theorem and (2.10) give

$$A_{21} = \int\limits_{\mathbb{R}^d} |y-y_0| \, \|a(y)\|_X \left[\int\limits_{\mathbb{R}^d} \chi_{|x-y_0|\geq 4r_0}(x) \frac{1}{|x-y_0|^{d+1}} \, dx \right] dy$$
$$= \frac{C}{r_0} \int\limits_{|y-y_0|<r_0} |y-y_0| \, \|a(y)\|_X dy \leq C.$$

A geometric reasoning parallel to the one developed above for the BMO case gives that in order to have $A_{22} \neq 0$ we must have $3r_0 < \rho(x)$, $\rho(x) - r_0 < |x-y| < \rho(x) + r_0$ and, in addition, $|x-y_0| \sim \rho(x) \sim \rho(y_0)$. Therefore, since the atom a is supported in $\tilde{B} = B(y_0, r_0)$ and is controlled in L^∞ norm by Cr_0^{-d},

$$A_{22} \leq \frac{C}{\rho(y_0)^d} \int\limits_{|x-y_0|\leq C\rho(y_0)} \int\limits_{\rho(x)-r_0<|x-y|<\rho(x)+r_0} \|a(y)\|_X \, dy\, dx$$
$$\leq \frac{C}{\rho(y_0)^d} \int\limits_{|x-y_0|\leq C\rho(y_0)} \int\limits_{|y-y_0|<r_0} \|a(y)\|_X \, dy\, dx$$
$$\leq \frac{C}{\rho(y_0)^d} \int\limits_{|x-y_0|\leq C\rho(y_0)} dx \leq C.$$

We continue with the analysis over atoms supported on "big" balls. Let a be an atom supported in a ball $\bar{B}(y_0, \gamma\rho(y_0))$ with $\gamma > 1$. We begin by proceeding as in the previous case for A_1. For A_2, since we do not have the cancelation property (2.11), we estimate its size as follows:

$$A_2 = \int\limits_{(4\bar{B})^c} \|Ta(x)\|_{L^q_X((0,\infty),dt/t)}\, dx$$

$$\leq C \int\limits_{(4\bar{B})^c} \int\limits_{\bar{B}} \frac{1}{|x-y|^d} \|a(y)\|_X \chi_{|x-y|\leq\rho(x)}(y)\, dy\, dx.$$

The domain of integration above is contained in the set defined by the conditions $|x-y_0| \geq 4\gamma\rho(y_0)$, $|y-y_0| < \gamma\rho(y_0)$ and $|x-y| \leq \rho(x)$. These conditions imply that $4\gamma\rho(y_0) \leq |x-y_0| \leq |x-y| + |y-y_0| < |x-y| + \gamma\rho(y_0)$, hence $3\gamma\rho(y_0) \leq |x-y|$. Note that, by Lemma 2.1, $\rho(x) \leq C\rho(y) \leq \bar{C}\gamma\rho(y_0)$. Therefore $3\gamma\rho(y_0) \leq |x-y| \leq \bar{C}\gamma\rho(y_0)$ and we get

$$A_2 \leq C \int\limits_{\bar{B}} \|a(y)\|_X \int\limits_{3\gamma\rho(y_0)\leq|x-y|\leq\bar{C}\gamma\rho(y_0)} \frac{1}{|x-y|^d}\, dx\, dy \leq C. \quad \blacksquare$$

REMARK 3.10. Consider two Banach spaces X_1 and X_2. Let T be a linear operator that maps $L^p_{X_1}(\mathbb{R}^d)$ into $L^p_{X_2}(\mathbb{R}^d)$ for some p, $1 < p < \infty$, such that $T1$ can be defined and $T1 = 0$. Assume T has an associated kernel which satisfies the standard estimates of Calderón–Zygmund operators. Define the operator

$$T_{\mathrm{loc}}f(x) = T(\chi_N(x,\cdot)f(\cdot))(x), \qquad x \in \mathbb{R}^d.$$

Then:

- T_{loc} is bounded from $\mathrm{BMO}_{\mathcal{L},X_1}$ into $\mathrm{BMO}_{\mathcal{L},X_2}$,
- T_{loc} is bounded from $H^1_{\mathcal{L},X_1}$ into $L^1_{X_2}(\mathbb{R}^d)$.

Parallel to Remark 3.8, the reader can check the validity of these claims just by replacing, along the lines of the proof of Theorem 3.9, X by X_1, $L^q_X((0,\infty),dt/t)$ by X_2 and $f \mapsto t\partial_t P_t f(x)$ by $f \mapsto Tf(x)$.

4. Proof of Theorem A. Given a Banach space X, define its *modulus of convexity* by

$$\delta_X(\varepsilon) = \inf\left\{ 1 - \left\|\frac{x+y}{2}\right\| : x,y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon \right\}$$

for $0 < \varepsilon < 2$. The Banach space X is called *q-uniformly convex*, $2 \leq q < \infty$, if $\delta_X(\varepsilon) \geq c\varepsilon^q$ for some positive constant c. By Pisier's Renorming Theorem [6], X is q-uniformly convex if and only if X is of martingale cotype q. For martingale cotype the following theorem holds (see [9] and [5]).

THEOREM 4.1. *Let X be a Banach space and $2 \leq q < \infty$. The following statements are equivalent:*

(1) *X is of martingale cotype q.*
(2) *The operator $g^{\Delta,q}$ maps $\mathrm{BMO}_{c,X}$ into BMO.*
(3) *The operator $g^{\Delta,q}$ maps $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for any p in the range $1 < p < \infty$.*

(4) *The operator $g^{\Delta,q}$ maps $L^1_X(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$.*

(5) *The operator $g^{\Delta,q}$ maps H^1_X into $L^1(\mathbb{R}^d)$.*

(6) *For every $f \in L^1_X(\mathbb{R}^d)$, $g^{\Delta,q}f(x) < \infty$ for almost every $x \in \mathbb{R}^d$.*

Here H^1_X denotes the atomic Hardy space in $\mathbb{R}^d$, and $\mathrm{BMO}_{c,X}$ is the set of functions that belong to the classical BMO with values in X and have compact support.

Proof of Theorem A. Observe that hypothesis (i) is equivalent to one of the statements in Theorem 4.1.

(i)$\Rightarrow$(ii). We can apply Theorems 3.7 and 3.9 to infer that the operator $f \mapsto t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))$ maps $\mathrm{BMO}_{\mathcal{L},X}$ into $\mathrm{BMO}_{\mathcal{L},L^q_X((0,\infty),dt/t)}$. By using Lemma 3.4 we obtain the boundedness of $g^{\mathcal{L},q}_{\mathrm{loc}}$ from $\mathrm{BMO}_{\mathcal{L},X}$ into $\mathrm{BMO}_{\mathcal{L}}$. Finally, by Lemma 3.2(b) we arrive at (ii).

(i)$\Rightarrow$(iii). By Theorem 3.7 and Lemma 3.4(a) the local operator $g^{\mathcal{L},q}_{\mathrm{loc}}$ is bounded in L^p. Boundedness of the global part follows from Lemma 3.2(a).

(i)$\Rightarrow$(iv). Theorem 3.7 and Lemma 3.4(a), together with Lemma 3.2(a), give the conclusion.

(i)$\Rightarrow$(v). By using Theorems 3.7, 3.9 and 3.4(c) we see that $g^{\mathcal{L},q}_{\mathrm{loc}}$ maps $H^1_{\mathcal{L},X}$ into $L^1(\mathbb{R}^d)$. Then Lemma 3.2(c) gives the result.

(i)$\Rightarrow$(vi). Apply Theorem 3.7 and Lemmas 3.4(a) and 3.2(a).

(ii)$\Rightarrow$(i). Theorem 4.1 tells us that it is enough to prove the boundedness of $g^{\Delta,q}$ from $\mathrm{BMO}_{c,X}$ into BMO. From the hypothesis, Lemma 3.2(b) and (3.3) we can deduce that the operator $f(x) \mapsto g^{\mathcal{L},q}(\chi_N(x,\cdot)f(\cdot))(x)$, $x \in \mathbb{R}^d$, is bounded from $\mathrm{BMO}_{\mathcal{L},X}$ into $\mathrm{BMO}_{\mathcal{L}}$. On the other hand, the proof of Lemma 3.4 shows that the difference operator $f(x) \mapsto g^{\mathcal{L},q}(\chi_N(x,\cdot)f(\cdot))(x) - g^{\Delta,q}(\chi_N(x,\cdot)f(\cdot))(x)$ is bounded from $\mathrm{BMO}_{\mathcal{L},X}$ into L^∞. Thus the operator $f(x) \mapsto g^{\Delta,q}(\chi_N(x,\cdot)f(\cdot))(x)$ is bounded from $\mathrm{BMO}_{\mathcal{L},X}$ into $\mathrm{BMO}_{\mathcal{L}} \subset \mathrm{BMO}$. Let f be a function in $\mathrm{BMO}_{c,X}$. Given a ball $B(x_0, s)$, by Lemma 3.6 there exists $R > 0$ depending on s and the support of f such that $\mathrm{supp}\, f^R \subset B(0, \rho(0)/2)$ (see the proof of Lemma 3.6) and

$$\frac{1}{|B(x_0,s)|} \int\limits_{B(x_0,s)} g^{\Delta,q}f(x)\,dx$$

$$= \frac{1}{|B(x_0,s)|} \int\limits_{B(x_0,s)} g^{\Delta,q}(\chi_N(x/R,\cdot)f^R(\cdot))(x/R)\,dx$$

$$= \frac{1}{|B(x_0/R,s/R)|} \int\limits_{B(x_0/R,s/R)} g^{\Delta,q}(\chi_N(z,\cdot)f^R(\cdot))(z)\,dz.$$

Since R can be arbitrarily large, we fix it so that $(R\rho(0))^{-d}\|f\|_{L^1_X(\mathbb{R}^d)} \leq \|f\|_{\mathrm{BMO}_X}$. Therefore,

$$\frac{1}{|B(x_0,s)|} \int\limits_{B(x_0,s)} |g^{\Delta,q} f(x) - (g^{\Delta,q}(\chi_N(z,\cdot)f^R(\cdot))(z))_{B(x_0/R,s/R)}| \, dx$$

$$= \frac{1}{|B(x_0/R,s/R)|} \int\limits_{B(x_0/R,s/R)} |g^{\Delta,q}(\chi_N(x,\cdot)f^R(\cdot))(x)$$

$$- (g^{\Delta,q}(\chi_N(z,\cdot)f^R(\cdot))(z))_{B(x_0/R,s/R)}| \, dx$$

$$\leq C\|f^R\|_{\mathrm{BMO}_{\mathcal{L},X}} \leq C\|f\|_{\mathrm{BMO}_X},$$

where for the last inequality above the following argument is applied. Note that to have such an inequality we only have to compare the integral means of f^R with the BMO_X-norm of f. Let $\alpha \geq 1$. If $B(x, \alpha\rho(x))$ does not intersect $B(0, \rho(0)/2)$ then $\int_{B(x,\alpha\rho(x))} \|f^R(y)\|_X \, dy = 0$ and there is nothing to prove. In case $B(x, \alpha\rho(x)) \cap B(0, \rho(0)/2) \neq \emptyset$, by Lemma 2.1, $\rho(x) \sim \rho(0)$ and, by the choice of R,

$$\frac{1}{|B(x,\alpha\rho(x))|} \int\limits_{B(x,\alpha\rho(x))} \|f^R(y)\|_X \, dy$$

$$\leq \frac{C_n}{(R\alpha\rho(x))^d} \int\limits_{B(0,R\rho(0)/2)} \|f(z)\|_X \, dz$$

$$\leq \frac{C}{(R\rho(0))^d} \|f\|_{L^1_X(\mathbb{R}^d)} \leq C\|f\|_{\mathrm{BMO}_X};$$

here the constant C is independent of f.

(iii)$\Rightarrow$(i). Lemmas 3.2(a) and 3.4(a) show that $g^{\Delta,q}(\chi_N f)$ is bounded from $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$. Let $f \in L^p_X(\mathbb{R}^d)$ with support in a ball $B_M = B(0, M)$, $M > 0$. By Lemma 3.6 we can find $R > 0$ such that $g^{\Delta,q} f(x) = g^{\Delta,q}(\chi_N(x/R,\cdot)f^R(\cdot))(x/R)$ for all $|x| < M$. Hence

$$\|\chi_{B_M} g^{\Delta,q} f\|^p_{L^p(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} |\chi_{B_M}(x) g^{\Delta,q}(\chi_N(x/R,\cdot)f^R(\cdot))(x/R)|^p \, dx$$

$$\leq R^d \int\limits_{\mathbb{R}^d} |g^{\Delta,q}(\chi_N(x/R,\cdot)f^R(\cdot))(x/R)|^p \, dx$$

$$\leq CR^d \int\limits_{\mathbb{R}^d} \|f^R(x)\|^p_X \, dx = C\|f\|^p_{L^p_X(\mathbb{R}^d)}.$$

As the constant C does not depend on M we can let $M \to \infty$ to conclude that $\|g^{\Delta,q} f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p_X(\mathbb{R}^d)}$.

(iv)$\Rightarrow$(i). We leave this case to the reader.

(v)$\Rightarrow$(i). By Theorem 4.1 it is enough to prove the boundedness of $g^{\Delta,q}$ from H^1_X into $L^1(\mathbb{R}^d)$. Lemmas 3.2(c) and 3.4(c) imply that the localized operator $f \mapsto \|t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)\|_{L^q_X((0,\infty),dt/t)}$ is bounded from $H^1_{\mathcal{L},X}$ into $L^1(\mathbb{R}^d)$. Therefore we only have to prove the boundedness over H^1-atoms

with cancelation but supported in big balls. Let a be such an atom, namely a function supported in a ball $B(y_0, \gamma\rho(y_0))$ with $\gamma > 1$ and $\int_{\mathbb{R}^d} a(y)\, dy = 0$. Consider the function $\widetilde{a}^R(x) := R^d a^R(x) = R^d a(Rx)$, $x \in \mathbb{R}^d$, $R > 0$. The function $\widetilde{a}^R$ is an atom with support contained in the ball $B(y_0/R, \gamma y_0/R)$. Given $M > 0$, Lemma 3.6 allows us to choose a sufficiently large R such that $g^{\Delta,q}a(x) = g^{\Delta,q}(\chi_N(x/R, \cdot)a^R(\cdot))(x/R)$ for $|x| < M$. Hence

$$\int\limits_{|x|<M} |g^{\Delta,q}a(x)|\, dx = \int\limits_{|x|<M} |g^{\Delta,q}(\chi_N(x/R, \cdot)a^R(\cdot))(x/R)|\, dx$$

$$= \int\limits_{|z|<M/R} |g^{\Delta,q}(\chi_N(z, \cdot)\widetilde{a}^R(\cdot))(z)|\, dz$$

$$\leq C\|\widetilde{a}^R\|_{H^1_{\mathcal{L},X}} = C\|\widetilde{a}^R\|_{H^1_X} \leq C,$$

where C does not depend on M. To conclude, let $M \to \infty$.

(vi)$\Rightarrow$(i). We will prove that $g^{\Delta,q}f(x) < \infty$ for almost every $x \in \mathbb{R}^d$ (see Theorem 4.1). By Lemma 3.2(a) we have $g^{\mathcal{L},q}_{\mathrm{loc}}f(x) < \infty$ for almost all $x \in \mathbb{R}^d$. Hence by Lemma 3.4(a) we have $g^{\Delta,q}_{\mathrm{loc}}f(x) < \infty$ for almost all $x \in \mathbb{R}^d$. In fact, from the proof of Lemma 3.4 it can be deduced that

$$\|t\partial_t P_t(\chi_N(x, \cdot)f(\cdot))(x)\|_{L^q_X((0,\infty),dt/t)} < \infty,$$

for almost all $x \in \mathbb{R}^d$. The arguments in the proof of Theorem 3.7 can be used to conclude that

$$\left\| \sum_{k \geq 1} \chi_{Q_k}(x)t\partial_t P_t(\chi_{2Q_k}f)(x) \right\|_{L^q_X((0,\infty),dt/t)}$$

is finite for almost all $x \in \mathbb{R}^d$. By the finite overlapping property of the balls Q_k we get the finiteness for almost every x of each term

$$\|\chi_{Q_k}(x)t\partial_t P_t(\chi_{2Q_k}f)(x)\|_{L^q_X((0,\infty),dt/t)}.$$

On the other hand, observe that

$$\|\chi_{Q_k}(x)t\partial_t P_t((1 - \chi_{2Q_k})f)(x)\|^q_{L^q_X((0,\infty),dt/t)}$$

$$\leq C\int\limits_0^\infty \left(\int\limits_{|x-y|>\rho(x_k)} \frac{t\|f(y)\|_X}{(t + |x-y|)^{d+1}}\, dy \right)^q \frac{dt}{t}$$

$$\leq C\|f\|^q_{L^1_X(\mathbb{R}^d)} \int\limits_0^\infty \frac{t^q}{(t + \rho(x_k))^{(d+1)q}} \frac{dt}{t} \leq C_k\|f\|^q_{L^1_X(\mathbb{R}^d)}.$$

Putting together the last two observations we find that for every k and almost every $x \in \mathbb{R}^d$ the norm $\|\chi_{Q_k}(x)t\partial_t P_t f(x)\|_{L^q_X((0,\infty),dt/t)}$ is finite. Hence $\|t\partial_t P_t f(x)\|_{L^q_X((0,\infty),dt/t)} = g^{\Delta,q}f(x)$ is finite for almost all x. ∎

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