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Quasiconformal mappings and exponentially integrable functions

by

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Abstract. We prove that a K-quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ which maps the unit disk $\mathbb{D}$ onto itself preserves the space $\mathrm{EXP}(\mathbb{D})$ of exponentially integrable functions over $\mathbb{D}$, in the sense that $u \in \mathrm{EXP}(\mathbb{D})$ if and only if $u \circ f^{-1} \in \mathrm{EXP}(\mathbb{D})$. Moreover, if f is assumed to be conformal outside the unit disk and principal, we provide the estimate

$$\frac{1}{1 + K \log K} \le \frac{\|u \circ f^{-1}\|_{\mathrm{EXP}(\mathbb{D})}}{\|u\|_{\mathrm{EXP}(\mathbb{D})}} \le 1 + K \log K$$

for every $u \in \mathrm{EXP}(\mathbb{D})$. Similarly, we consider the distance from L^∞ in EXP and we prove that if $f : \Omega \to \Omega'$ is a K-quasiconformal mapping and $G \subset\subset \Omega$, then

$$\frac{1}{K} \le \frac{\mathrm{dist}_{\mathrm{EXP}(f(G))}(u \circ f^{-1}, L^\infty(f(G)))}{\mathrm{dist}_{\mathrm{EXP}(f(G))}(u, L^\infty(G))} \le K$$

for every $u \in \mathrm{EXP}(\mathbb{G})$. We also prove that the last estimate is sharp, in the sense that there exist a quasiconformal mapping $f : \mathbb{D} \to \mathbb{D}$, a domain $G \subset\subset \mathbb{D}$ and a function $u \in \mathrm{EXP}(G)$ such that

$$\mathrm{dist}_{\mathrm{EXP}(f(G))}(u \circ f^{-1}, L^\infty(f(G))) = K\, \mathrm{dist}_{\mathrm{EXP}(f(G))}(u, L^\infty(G)).$$

1. Introduction and main results. Let Ω and Ω' be domains in $\mathbb{R}^n$. A homeomorphism $f : \Omega \to \Omega'$ is a K-*quasiconformal mapping* for a constant $K \ge 1$ if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \Omega')$ and

$$|Df(x)|^n \le K J_f(x) \quad \text{ a.e. } x \in \Omega,$$

where Df stands for the *differential* of f, the norm $|Df|$ of Df is defined as

$$|Df(x)| = \sup_{\xi \in \mathbb{R}^n,\, |\xi|=1} |Df(x)\xi|,$$

and J_f denotes the *jacobian determinant* of f,

$$J_f(x) = \det Df(x).$$

When $K = 1$ we say that f is *conformal* in Ω.

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If G is a bounded domain in $\mathbb{R}^n$ with measure $|G|$ the space $\mathrm{EXP}(G)$ is the set of measurable functions $u : G \to \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$\fint_G \exp \frac{|u(x)|}{\lambda}\, dx < \infty,$$

where the mean value notation $\fint_G = |G|^{-1} \int_G$ is used. We recall (see e.g. [3]) that $\mathrm{EXP}(G)$ is a Banach space equipped with the norm

$$\|u\|_{\mathrm{EXP}(G)} = \sup_{0<t<|G|} \left(1 + \log \frac{|G|}{t}\right)^{-1} u^*(t), \tag{1.1}$$

where u^* is the *non-increasing rearrangement* of u,

$$u^*(t) = \sup\{\tau \geq 0 : \mu_u(\tau) > t\} \qquad \forall t \in (0, |G|), \tag{1.2}$$

and μ_u is the *distribution function* of u,

$$\mu_u(\tau) = |\{x \in G : |u(x)| > \tau\}| \qquad \forall \tau \geq 0.$$

In this paper we consider the problem of composing functions in $\mathrm{EXP}(G)$ with quasiconformal mappings and we deal with the case of dimension $n = 2$.

The results of this paper are in the spirit of the following theorem of H. M. Reimann [12], featuring the class of functions of bounded mean oscillation.

THEOREM 1.1 ([12]). *Let Ω and Ω' be domains in $\mathbb{R}^n$ and let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping. Then there exists a constant C which depends only on n and K such that*

$$\frac{1}{C}\|u\|_{\mathrm{BMO}(G)} \leq \|u \circ f^{-1}\|_{\mathrm{BMO}(G')} \leq C\|u\|_{\mathrm{BMO}(G)},$$

for every subdomain G of Ω and for every $u \in \mathrm{BMO}(G)$, with $G' = f(G)$.

We recall that a locally integrable function $u : G \to \mathbb{R}$ has *bounded mean oscillation*, $u \in \mathrm{BMO}(G)$, if

$$\|u\|_{\mathrm{BMO}(G)} = \sup_Q \fint_Q |u(x) - u_Q|\, dx < \infty. \tag{1.3}$$

The supremum in (1.3) is taken over all open cubes Q of G with sides parallel to the axes, and the notation

$$u_Q = \fint_Q u(x)\, dx$$

is used.

We also recall a similar result which holds for the space $W^{1,n}_{\mathrm{loc}}$: if Ω and Ω' are bounded domains in $\mathbb{R}^n$ and $f : \Omega \to \Omega'$ is a K-quasiconformal mapping, then

$$\frac{1}{K}\|\nabla u\|_{L^n(G)} \leq \|\nabla(u \circ f^{-1})\|_{L^n(G')} \leq K\|\nabla u\|_{L^n(G)}$$

for every subdomain G of Ω and for every $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$, with $G' = f(G)$. The proof of this result can be found in [4, 8, 10, 13, 14].

We denote by $\mathbb{D}$ the unit disk $\{x \in \mathbb{R}^2 : |x| < 1\}$ and we prove the following result.

THEOREM 1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a K-quasiconformal principal mapping that is conformal outside $\mathbb{D}$ and maps $\mathbb{D}$ onto itself. Then*

$$\begin{aligned}(1.4)\qquad \frac{1}{1+K\log K}\|u\|_{\mathrm{EXP}(\mathbb{D})} &\le \|u\circ f^{-1}\|_{\mathrm{EXP}(\mathbb{D})}\\ &\le (1+K\log K)\|u\|_{\mathrm{EXP}(\mathbb{D})}\end{aligned}$$

for every $u \in \mathrm{EXP}(\mathbb{D})$.

Here and in what follows we call a quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ *principal* if it is conformal outside $\mathbb{D}$ and satisfies the following normalization:

$$|f(x) - x| = \mathcal{O}(1/|x|) \quad \text{if } |x| > 1.$$

Observe that our result gives that if f is a conformal mapping, then (1.4) reduces to the equality

$$\|u\circ f^{-1}\|_{\mathrm{EXP}(\mathbb{D})} = \|u\|_{\mathrm{EXP}(\mathbb{D})} \quad \text{for every } u \in \mathrm{EXP}(\mathbb{D}).$$

The *Luxemburg norm* of a function $u \in \mathrm{EXP}(G)$ is defined as

$$(1.5)\qquad \|u\|_{\mathcal{EXP}(G)} = \inf\Big\{\lambda > 0 : \fint_G \exp\frac{|u(x)|}{\lambda}\,dx \le 2\Big\}.$$

We recall that (see e.g. [3] and [11]) the Luxemburg norm is equivalent to the norm defined in (1.1). We also remark that $L^\infty(G)$ is not a dense subspace of $\mathrm{EXP}(G)$ (see e.g. [11]). Appealing to the results in [5] and [7], we find that the distance to $L^\infty(G)$ in $\mathrm{EXP}(G)$ evaluated with respect to the Luxemburg norm (1.5) is given by

$$\mathrm{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G)) = \inf\Big\{\lambda > 0 : \fint_G \exp\frac{|u(x)|}{\lambda}\,dx < \infty\Big\}$$

for every $u \in \mathrm{EXP}(G)$.

Our next result compares the distances from L^∞ of u and $u\circ f^{-1}$. We note that the estimates we provide are sharp (see Example 3.3 below).

THEOREM 1.3. *Let Ω and Ω' be bounded domains in $\mathbb{R}^2$ and let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping. Then*

$$(1.6)\qquad \mathrm{dist}_{\mathrm{EXP}(G')}(u\circ f^{-1}, L^\infty(G')) \le K\,\mathrm{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G))$$

and

$$(1.7)\qquad \frac{1}{K}\,\mathrm{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G)) \le \mathrm{dist}_{\mathrm{EXP}(G')}(u\circ f^{-1}, L^\infty(G')),$$

for every subdomain G of Ω and for every $u \in \mathrm{EXP}(G)$, with $G' = f(G)$.

As for Theorem 1.2, if f is a conformal mapping then (1.6) and (1.7) reduce to the equality

$$\mathrm{dist}_{\mathrm{EXP}(G')}(u \circ f^{-1}, L^\infty(G')) = \mathrm{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G))$$

for every $u \in \mathrm{EXP}(G)$.

2. Preliminary results. We review some of the standard facts on quasiconformal mappings in dimension $n = 2$. Our main sources are [2, 10, 13].

From now on Ω and Ω' are domains in $\mathbb{R}^2$. It is well-known that if $f : \Omega \to \Omega'$ is a K-quasiconformal mapping then it is differentiable a.e., the inverse f^{-1} is a K-quasiconformal mapping and for a.e. $x \in \Omega$,

$$Df^{-1}(f(x)) = (Df(x))^{-1},$$

and

$$J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)}. \tag{2.1}$$

It will be convenient to recall the following version of the change of variables formula.

Lemma 2.1. *Let Ω and Ω' be domains in $\mathbb{R}^2$ and let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping. If $w \in L^1(\Omega')$ then $(w \circ f)J_f \in L^1(\Omega)$ and*

$$\int_\Omega w(f(z))J_f(z)\,dz = \int_{\Omega'} w(y)\,dy. \tag{2.2}$$

For later use, we recall K. Astala's theorem on the distortion of area under a quasiconformal mapping (see [1]), in the form appropriate for our purposes (see [6]).

Theorem 2.2 ([1, 6]). *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a K-quasiconformal principal mapping, that is, conformal outside the unit disk $\mathbb{D}$. Then, for every measurable subset $E \subset \mathbb{D}$,*

$$|f(E)| \le K\pi^{1-1/K}|E|^{1/K}. \tag{2.3}$$

All constants in (2.3) are sharp. We also recall that if f is a quasiconformal mapping defined in a planar domain Ω then

$$J_f \in L^p_{\mathrm{loc}}(\Omega) \quad \text{if } p < p_K = \frac{K}{K-1}, \tag{2.4}$$

and the exponent $p_K = K/(K-1)$ is the best possible. This is a direct consequence of the area distortion estimate (see again [1]).

3. Proofs of Theorems 1.2 and 1.3. Before we give the proofs of Theorems 1.2 and 1.3 we recall the following fundamental lemma which provides a precise connection between the spaces $\mathrm{BMO}(G)$ and $\mathrm{EXP}(G)$ for G a bounded domain in $\mathbb{R}^n$.

LEMMA 3.1 ([9]). *Let G be a bounded domain in $\mathbb{R}^n$ and let $u : G \to \mathbb{R}$ be a measurable function. Then $u \in \mathrm{EXP}(G)$ if and only there exists $v \in \mathrm{BMO}(G)$ such that*

$$|u(x)| \le v(x) \qquad a.e.\ x \in G.$$

Moreover, there exists a constant C which depends only on n such that

$$\|v\|_{\mathrm{BMO}(G)} \le C \operatorname{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G)).$$

Theorem 1.1 and Lemma 3.1 are the key ingredients in the proof of the following result, which is the starting point of our study.

LEMMA 3.2. *Let Ω be a domain in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be a quasiconformal mapping. Let G be any bounded subdomain of Ω and let $G' = f(G)$. Then $u \in \mathrm{EXP}(G)$ if and only if $u \circ f^{-1} \in \mathrm{EXP}(G')$.*

Proof. Since both f and f^{-1} are quasiconformal mappings it is sufficient to prove that $u \circ f^{-1} \in \mathrm{EXP}(G')$ if $u \in \mathrm{EXP}(G)$. From Lemma 3.1, to the function $u \in \mathrm{EXP}(G)$ there corresponds a function $v \in \mathrm{BMO}(G)$ such that $|u(x)| \le v(x)$ for a.e. $x \in G$. As a consequence of Theorem 1.1 the function $v \circ f^{-1}$ belongs to $\mathrm{BMO}(G')$. Clearly $|u(f^{-1}(y))| \le v(f^{-1}(y))$ for a.e. $y \in G'$. The result immediately follows from Lemma 3.1. ■

Proof of Theorem 1.2. The proof is based on Theorem 2.2. Let $u \in \mathrm{EXP}(\mathbb{D})$. First, we notice that for every $\tau > 0$,

$$\{y \in \mathbb{D} : |u(f^{-1}(y))| > \tau\} = f(\{x \in \mathbb{D} : |u(x)| > \tau\}).$$

We compare the distribution functions of u and $u \circ f^{-1}$ by means of the area distortion estimate (2.3) and we obtain

$$\begin{aligned} \mu_{u \circ f^{-1}}(\tau) &= |\{y \in \mathbb{D} : |u(f^{-1}(y))| > \tau\}| \\ &= |f(\{x \in \mathbb{D} : |u(x)| > \tau\})| \\ &\le K\pi^{1-1/K}\mu_u(\tau)^{1/K}. \end{aligned}$$

Since for every $t \in (0, \pi)$,

$$\{\tau \ge 0 : \mu_{u \circ f^{-1}}(\tau) > t\} \subset \left\{\tau \ge 0 : \mu_u(\tau) > \frac{t^K}{K^K \pi^{K-1}}\right\},$$

it follows from the definition of non-increasing rearrangement (1.2) that

$$(u \circ f^{-1})^*(t) \le u^*\left(\frac{t^K}{K^K \pi^{K-1}}\right). \tag{3.1}$$

We deduce directly from the definition of the norm (1.1) that

$$\begin{aligned} u^*\left(\frac{t^K}{K^K\pi^{K-1}}\right) &\le \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1+\log\frac{\pi}{\frac{t^K}{K^K\pi^{K-1}}}\right) \\ &= \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1+K\log K\frac{\pi}{t}\right) \\ &= \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1+K\log K+K\log\frac{\pi}{t}\right). \end{aligned}$$

Thus, from (3.1) we get

$$(u\circ f^{-1})^*(t)\le \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1+K\log K+K\log\frac{\pi}{t}\right).$$

Our aim is to prove that there exists a constant $c=c(K)$ which depends on K such that

$$1+K\log K+K\log\frac{\pi}{t}\le c(K)\left(1+\log\frac{\pi}{t}\right)\qquad \forall t\in(0,\pi). \tag{3.2}$$

It will be sufficient to prove that the function

$$\gamma(t)=\frac{1+K\log K+K\log\frac{\pi}{t}}{1+\log\frac{\pi}{t}}\qquad \forall t\in(0,\pi),$$

is bounded in the interval $(0,\pi)$ by some constant which only depends on K. To this end, we observe that

$$\gamma'(t)=\frac{1+K\log K-K}{t\big(1+\log\frac{\pi}{t}\big)^2}\qquad \forall t\in(0,\pi).$$

We define

$$\psi(K)=1+K\log K-K\qquad \forall K\in[1,\infty).$$

Since

$$\psi'(K)=\log K\ge 0\qquad \forall K\in[1,\infty),$$

we have

$$\psi(K)\ge\psi(1)=0\qquad \forall K\in[1,\infty),$$

and therefore γ is increasing in $(0,\pi)$. Then

$$\gamma(t)\le\gamma(\pi)=1+K\log K\qquad \forall t\in(0,\pi),$$

and inequality (3.2) holds with

$$c(K)=1+K\log K.$$

Therefore (3.1) gives

$$(u\circ f^{-1})^*(t)\le(1+K\log K)\|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1+\log\frac{\pi}{t}\right)\qquad \forall t\in(0,\pi).$$

Hence, the inequality

$$\|u \circ f^{-1}\|_{\mathrm{EXP}(\mathbb{D})} \leq (1 + K \log K)\|u\|_{\mathrm{EXP}(\mathbb{D})} \quad \forall u \in \mathrm{EXP}(\mathbb{D}) \tag{3.3}$$

holds if f is a K-quasiconformal principal mapping. Recalling that the inverse of a K-quasiconformal principal mapping is also a K-quasiconformal principal mapping, it follows that

$$\|v \circ f\|_{\mathrm{EXP}(\mathbb{D})} \leq (1 + K \log K)\|v\|_{\mathrm{EXP}(\mathbb{D})} \quad \forall v \in \mathrm{EXP}(\mathbb{D}). \tag{3.4}$$

If we substitute $v = u \circ f^{-1}$ with $u \in \mathrm{EXP}(\mathbb{D})$ into (3.4) (observe that v belongs to $\mathrm{EXP}(\mathbb{D})$ by Lemma 3.2), we have

$$\|u\|_{\mathrm{EXP}(\mathbb{D})} \leq (1 + K \log K)\|u \circ f^{-1}\|_{\mathrm{EXP}(\mathbb{D})} \quad \forall u \in \mathrm{EXP}(\mathbb{D}). \tag{3.5}$$

Inequalities (3.3) and (3.5) show that (1.4) holds, completing the proof. ∎

Proof of Theorem 1.3. Let λ be such that

$$\lambda > p' \,\mathrm{dist}_{\mathrm{EXP(G)}}(u, L^\infty(G)), \tag{3.6}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad 1 < p < \frac{K}{K-1}.$$

Since

$$\left(\exp \frac{|u(x)|}{\lambda}\right)^{p'} = \exp \frac{|u(x)|}{\lambda/p'},$$

from (3.6) it follows that

$$\exp \frac{|u|}{\lambda} \in L^{p'}(G). \tag{3.7}$$

Recalling that $J_f \in L^p(G)$ (see (2.4)), we deduce from (3.7) that

$$\exp \frac{|u|}{\lambda} J_f \in L^1(G).$$

It follows directly from the change of variables formula (2.2) and also from the identity (2.1) that

$$\int_{G'} \exp \frac{|u(f^{-1}(y))|}{\lambda} dy = \int_G \exp \frac{|u(x)|}{\lambda} J_f(x)dx < \infty.$$

Therefore

$$\mathrm{dist}_{\mathrm{EXP}(G')}(u \circ f^{-1}, L^\infty(G')) \leq p' \,\mathrm{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G)). \tag{3.8}$$

Passing to the limit in (3.8) for p approaching $K/(K-1)$ we finally get (1.6). Recalling that the inverse of a K-quasiconformal mapping is also a K-quasiconformal mapping, it follows that

$$\mathrm{dist}_{\mathrm{EXP}(G)}(v \circ f, L^\infty(G)) \leq K \,\mathrm{dist}_{\mathrm{EXP}(G')}(v, L^\infty(G')) \quad \forall v \in \mathrm{EXP}(G'). \tag{3.9}$$

If we substitute the function $v = u \circ f^{-1}$ with $u \in \mathrm{EXP}(G)$ into (3.9) (observe that $v \in \mathrm{EXP}(G')$ by Lemma 3.2), we have

$$\mathrm{dist}_{\mathrm{EXP}(G)}(u, L^\infty(G)) \le K\, \mathrm{dist}_{\mathrm{EXP}(G')}(u \circ f^{-1}, L^\infty(G')) \qquad \forall u \in \mathrm{EXP}(G),$$

and this proves (1.7). ∎

Now we prove, by means of an example, that equality can occur in inequality (1.6).

Example 3.3. Here and in what follows let $0 < R \le 1$ and

$$\mathbb{D}_R = \{x \in \mathbb{R}^2 : |x| < R\}.$$

For every $K \ge 1$ we show that there exist a K-quasiconformal mapping $f : \mathbb{D} \to \mathbb{D}$ and a function $u \in \mathrm{EXP}(\mathbb{D}_R)$ such that

$$\mathrm{dist}_{\mathrm{EXP}(f(\mathbb{D}_R))}(u \circ f^{-1}, L^\infty(f(\mathbb{D}_R))) = K\, \mathrm{dist}_{\mathrm{EXP}(\mathbb{D}_R)}(u, L^\infty(\mathbb{D}_R)). \tag{3.10}$$

Let $f : \mathbb{D} \to \mathbb{D}$ be the K-quasiconformal mapping defined as

$$f(z) = \frac{z}{|z|^{1-1/K}},$$

and let

$$u(x) = -2\log|x|.$$

Then $u \in \mathrm{EXP}(\mathbb{D}_R)$ and

$$\mathrm{dist}_{\mathrm{EXP}(\mathbb{D}_R)}(u, L^\infty(\mathbb{D}_R)) = 1.$$

This follows from the fact that if $\lambda > 1$ then

$$\fint_{\mathbb{D}_R} e^{|u(x)|/\lambda}\, dx = \frac{\lambda}{(\lambda - 1)R^{2/\lambda}} < \infty,$$

while $e^{|u|/\lambda} \notin L^1(\mathbb{D}_R)$ for $0 < \lambda \le 1$. We notice that the inverse of f is given by

$$f^{-1}(y) = |y|^{K-1} y.$$

Therefore, the function $v = u \circ f^{-1}$ is given by

$$v(y) = -2K\log|y|.$$

Then $v \in \mathrm{EXP}(\mathbb{D}_R)$ and arguing as for u one has

$$\mathrm{dist}_{\mathrm{EXP}(\mathbb{D}_R)}(v, L^\infty(f(\mathbb{D}_R))) = K.$$

This proves (3.10).

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