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The power boundedness and resolvent conditions for functions of the classical Volterra operator

by

YURI LYUBICH (Warszawa and Haifa)

Abstract. Let $\phi(z)$ be an analytic function in a disk $|z| < \rho$ (in particular, a polynomial) such that $\phi(0) = 1$, $\phi(z) \not\equiv 1$. Let V be the operator of integration in $L_p(0,1)$, $1 \le p \le \infty$. Then $\phi(V)$ is power bounded if and only if $\phi'(0) < 0$ and $p = 2$. In this case some explicit upper bounds are given for the norms of $\phi(V)^n$ and subsequent differences between the powers. It is shown that $\phi(V)$ never satisfies the Ritt condition but the Kreiss condition is satisfied if and only if $\phi'(0) < 0$, at least in the polynomial case.

1. Introduction and overview. The integration

$$(1.1)\qquad (Vf)(x) = \int_0^x f(t)\,dt$$

is a traditional example of a quasinilpotent (but not nilpotent) operator in $L_p(0,1)$, $1 \le p \le \infty$. In $L_2(0,1)$ we have the adjoint operator

$$(1.2)\qquad (V^*f)(x) = \int_x^1 f(t)\,dt,$$

so V is not self-adjoint. From (1.1) and (1.2) it follows that

$$(1.3)\qquad \mathrm{Re}(Vf,f) = \frac{1}{2}\,((V+V^*)f,f) = \frac{1}{2}\Big(\int_0^1 f(t)\,dt\Big)^2 \ge 0.$$

Hence, $\exp(-tV)$, $t \ge 0$, is a semigroup of contractions in $L_2(0,1)$.

Recall that a bounded linear operator T in a Banach space X is called *power bounded* if $\sup\{\|T^n\| : n \ge 0\} < \infty$. In particular, all contractions are power bounded, and conversely, every power bounded operator is a contraction in the equivalent norm $\|f\|_T = \sup\{\|T^n f\| : n \ge 0\}$, $f \in X$. Sometimes, this trick can be useful, but here we do not need it, so we will deal with a

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fixed norm in X. In particular, if $X = L_p(0,h), 0 < h < \infty$, then we set, as usual,

$$\|f\|_p = \Big(\int_0^h |f(t)|^p \, dt\Big)^{1/p}, \tag{1.4}$$

so that, $\|f\|_p = 1$ if $f = \mathbf{1}$ and $h = 1$. Since all $L_p(0,h)$ are isometric, the case $h = 1$ is representative. For definiteness we can deal with $L_p(0,1)$ and write briefly L_p, unless stated otherwise.

All spaces under consideration are assumed complex and all operators linear and bounded. We denote by I the identity operator. Also, as usual, we denote by $\sigma(T)$ the spectrum of T and by $R(\lambda;T)$ the resolvent of T, i.e. $R(\lambda;T) = (T - \lambda I)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(T)$. If $\sigma(T)$ lies in the open unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ then T is power bounded. On the other hand, if T is power bounded then $\sigma(T)$ lies in the closed unit disk $\overline{\mathbb{D}}$. *If $\sigma(T) = \{1\}$, $T \neq I$, and T is power bounded then T^{-1} is not power bounded.* This is a reformulation of the classical Gelfand theorem on the single-point spectrum isometries.

There is a series of resolvent conditions in the domain $|\lambda| > 1$ closely related to power boundedness. The most important are: the *Ritt condition*

$$\|R(\lambda;T)\| \le \frac{C}{|\lambda - 1|}, \tag{1.5}$$

and the *Kreiss condition*

$$\|R(\lambda;T)\| \le \frac{C}{|\lambda| - 1}. \tag{1.6}$$

Obviously, the latter is weaker than the former. Furthermore, from the expansion

$$R(\lambda;T) = -\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}, \quad |\lambda| > 1, \tag{1.7}$$

it follows that every power bounded operator is a *Kreiss operator*, i.e. it satisfies (1.6). On the other hand, every Ritt operator is power bounded [10, 13].

The "iterated" inequality (1.6), i.e.

$$\|R^n(\lambda;T)\| \le \frac{C}{(|\lambda| - 1)^n}, \quad |\lambda| > 1,\, n \ge 1, \tag{1.8}$$

is called the *strong Kreiss condition*. This property is intermediate between power boundedness and the Kreiss condition. All strongly Kreiss operators are uniformly Kreiss [5] in the sense that the upper bound (1.6) remains valid for all partial sums of the series (1.7). The converse is not true [12]. We refer the reader to Nevanlinna's book [14] and to his papers [15], [16] for some general theorems on the resolvent conditions. In particular, Theorem 4 from [16] shows that $\|T^n\| = O(n)$ for every Kreiss operator T.

In the present paper we focus on the case $T = \phi(V)$, where $\phi(z)$ is a polynomial or even an analytic function of the complex variable z regular at $z = 0$. The linear and quadratic polynomials were considered in [6], [12], [18], [19]. In [11] it is proven that $T = I - V^\alpha$, $0 < \alpha < 1$, is power bounded (even Ritt) in any L_p. However, the analytic function $\phi(z) = 1 - z^\alpha$ is not regular at $z = 0$.

In [6] Halmos used (1.3) to prove that $(I + V)^{-1}$ is a contraction in L_2. Accordingly, $I + V$ is not power bounded in this space. In contrast, $I - V$ is power bounded in L_2, due to the Pedersen similarity $P^{-1}(I - V)P = (I + V)^{-1}$ where $(Pf)(x) = e^x f(x)$ (see [1] for a reference). Using these results Tsedenbayar [18] proved that the operator $I - rV$, $r \geq 0$, is power bounded in L_2. On the other hand, he showed that $I - aV$ with $a \in \mathbb{C} \setminus [0, \infty]$ is not Kreiss in L_p for $p = 1, 2, \infty$, and $I - aV^2$ with $a \neq 0$ is not Kreiss in all L_p.

In [12] Montes-Rodríguez, Sánchez-Álvarez and Zemánek proved that in L_p with $p \neq 2$ the operator $I - rV$ with $r > 0$ is not power bounded. Moreover, they determined an exact order of growth of $(I - rV)^n$ and of decay of the differences between the $(n+1)$th and the nth powers. Also they proved that $I - rV$, $r > 0$, is uniformly Kreiss for all p, but it is strongly Kreiss if $p = 2$ only.

The quadratic polynomials $I - aV + bV^2$ ($a \in \mathbb{R}$, $b \in \mathbb{C}$) were investigated by Tsedenbayar and Zemánek in [19], where it was proven that these operators in L_2 are power bounded if $a, b > 0$, but not Kreiss if $a < 0$. Note that Proposition 6 from [19] should be corrected: by our Theorem 1.1 (see below) the operator $I - aV + bV^2$ is power bounded for $a > 0$ and all $b \in \mathbb{C}$, not for $b \geq 0$ only.

As mentioned before, we consider

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k \in \mathbb{C}, \quad |z| < \rho, \tag{1.9}$$

where

$$\rho = (\varlimsup_{k\to\infty} \sqrt[k]{|a_k|})^{-1} > 0. \tag{1.10}$$

The latter is just the convergence radius of the power series (1.9). The series

$$\phi(V) = \sum_{k=0}^{\infty} a_k V^k$$

converges in the operator norm topology because of (1.10) and $\|V^k\|^{1/k} \to 0$.

As usual, the functional calculus $\phi \mapsto \phi(V)$ is an algebra homomorphism such that $\mathbf{1} \mapsto I$. This is injective since $\ker(V) = 0$ and any operator $\phi(V)$ with $a_0 \neq 0$ is invertible. Indeed, the spectrum $\sigma(\phi(V)) = \phi(\sigma(V))$ is the singleton $\{\phi(0)\} = \{a_0\}$. If $|a_0| < 1$ then $\phi(V)$ is power bounded. If $|a_0| = 1$ then $\phi(V)$ is power bounded if and only if $a_0^{-1}\phi(V)$ is power bounded. Thus, without loss of generality one can assume $a_0 = 1$, i.e. $\phi(0) = 1$. *This is the only case from now on.*

The operator $\phi(V)$ can be represented in a "closed" form. Namely, since

$$(V^k f)(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)\,dt, \quad k \geq 1,$$

we have

$$(\phi(V)f)(x) = f(x) + \int_0^x K(x-t)f(t)\,dt, \quad 0 \leq x \leq 1, \tag{1.11}$$

where

$$K(u) = \sum_{k=1}^{\infty} \frac{a_k u^{k-1}}{(k-1)!}. \tag{1.12}$$

THEOREM 1.1. *In order for the operator $\phi(V) \neq I$ to be power bounded in L_p it is necessary and sufficient that $p = 2$ and $a_1 = \phi'(0)$ is real negative.*

The necessity of $a_1 < 0$ follows from an asymptotic formula recently obtained by a complicated complex analysis in [2] ([1]) (see Theorem 1.2 therein). Our proof of the necessity (Section 3) is elementary and rather short.

On the other hand, a comparison of the above mentioned asymptotic formula to the sufficiency in our Theorem 1.1 discovers an exponential jump in the scale of growth of $\|\phi(V)^n\|_2$ ([2]).

THEOREM 1.2. *In $L_2(0,1)$ the following alternative holds: either $\phi(V)$ is power bounded or*

$$\|\phi(V)^n\| \geq \exp(cn^\gamma) \tag{1.13}$$

with some $c > 0$ and some $0 < \gamma \leq 1/2$.

The sufficiency in Theorem 1.1 follows from the similarity between $\phi(V)$ and $I + a_1 V$ in L_2. The latter is a particular case (up to an obvious modification) of that of [3, pp. 369–370]. However, our direct method (Section 4) yields some explicit upper bounds for the L_2-norms of $\phi(V)^n$ and of the differences $\phi(V)^{n+1} - \phi(V)^n$. Actually, this method works in a wide class of

([1]) I thank Yu. Tomilov for providing me with this reference after my talk about the present work.

([2]) I thank S. Torba for this observation.

integral convolution operators (see Theorem 4.2). This generalization does not fall under [3].

THEOREM 1.3. *If* $a_1 < 0$ *then*

$$\sup_n \|\phi(V)^n\|_2 \le e^{\mu} \tag{1.14}$$

where

$$\mu = \frac{|a_1|}{2} + \frac{3a_1^2 c + 2c^2}{|a_1|^3} \tag{1.15}$$

and

$$c = |a_2| + \int_0^1 \bigg|\sum_{k=3}^{\infty} \frac{a_k u^{k-3}}{(k-3)!}\bigg|\, du \le \sum_{k=2}^{\infty} \frac{|a_k|}{(k-2)!}. \tag{1.16}$$

Furthermore,

$$\sup_n \sqrt{n}\, \|\phi(V)^{n+1} - \phi(V)^n\|_2 \le e^{\mu_1} \tag{1.17}$$

where

$$\mu_1 = |a_1| + \frac{5a_1^2 c + c^2}{|a_1|^3}. \tag{1.18}$$

In the case $\phi(V) = 1 - rV$, $r > 0$, we have $a_1 = -r$ and $c = 0$, so $\mu = r/2$ and $\mu_1 = r$. Therefore,

$$\sup_n \|(1 - rV)^n\|_2 \le e^{r/2}, \tag{1.19}$$

and

$$\sup_n \sqrt{n}\, \|(1 - rV)^{n+1} - (1 - rV)^n\|_2 \le e^{r}. \tag{1.20}$$

The induction procedure from [18] based on Pedersen's similarity only yields $\exp([r] + 1)$ instead of $\exp(r/2)$ in (1.19).

For the differences from (1.20) the rate $\sqrt{n}$ of decay is exact [12]. In fact, this is true for every power bounded $\phi(V)$ by the similarity from [3]. For example, the quantity $\sqrt{n}\, \|\exp(-(n+1)V) - \exp(-nV)\|_2$ stays in between some two positive constants. An upper constant is determined by (1.17) with $\mu_1 = 1 + 5c + c^2$ since $a_1 = -1$ in this case. To estimate this c we note that the series in (1.16) is of Leibniz's type with $a_k = (-1)^k/k!$. The sum of this series does not exceed the first term in modulus. This yields $c \le |a_2| + |a_3| = 2/3$, thus $\mu_1 \le 43/9$, and finally,

$$\sqrt{n}\, \|\exp(-(n+1)V) - \exp(-nV)\|_2 \le \exp(43/9) < 119.$$

The case of alternating coefficients a_k merits a special attention since the following theorem can be proven in a very apparent way (see Section 5) that also yields an interesting upper bound.

Theorem 1.4. *Let ϕ be a polynomial,*

$$\phi(z) = 1 + \sum_{k=1}^{m} (-1)^k c_k z^k$$

with all $c_k > 0$. Then

$$\sup_n \|\phi(V)^n\|_2 \le e^{1/2x_0} \tag{1.21}$$

where

$$x_0 = \sup\{x > 0 : \operatorname{sign} \phi^{(k)}(x) = (-1)^k,\ 0 \le k \le m\}. \tag{1.22}$$

Obviously, $x_0 < \infty$ since $\operatorname{sign} \phi^{(k)}(\infty) = (-1)^m$, $0 \le k \le m$. If all roots of $\phi(z)$ are real positive then $x_0 = x_1$, where x_1 is the smallest root, so

$$\sup_n \|\phi(V)^n\|_2 \le e^{1/2x_1}, \tag{1.23}$$

that is more concrete than (1.21).

Example 1.5. Let $x_1^{(m)}$ be the smallest root of the mth Laguerre polynomial $L_m(z)$, $L_m(0) = 1$. Then

$$\sup_n \|L_m(V)^n\|_2 \le e^{1/2x_1^{(m)}}.$$

According to Theorem 6.31.3 from [17], we have

$$x_1^{(m)} \ge \frac{j_1^2}{4m+2}$$

where j_1 is the smallest positive root of the Bessel function $J_0(z)$. In its turn, $j_1 > 3\pi/4$ [17].

Let us emphasize that the bound (1.23) is applicable to any ϕ which is a member of the system of polynomials orthogonal with a positive weight on an interval $(0, v)$, $0 < v \le \infty$. For instance, ϕ can be a Jacobi polynomial modified by the linear transformation $(-1, 1) \to (0, 1)$.

Theorem 1.1 yields a lot of remarkable corollaries, most of them simply by calculation of the corresponding derivatives at $z = 0$. For example, the derivatives of $\phi(-z)$ and $\phi(z)^{-1}$ at $z = 0$ are both equal to $-\phi'(0)$ since $\phi(0) = 1$. This yields

Corollary 1.6. *Each of the operators $\phi(V)^{-1}$ and $\phi(-V)$ is power bounded if and only if either $p = 2$ and $\phi'(0) > 0$ or $\phi(V)$ is I.*

For instance, $(I + rV)^{-1}$ in L_2 is power bounded if and only if $r \ge 0$.

Corollary 1.7. *If $\phi(V)$ is power bounded then $\phi(rV)$ is power bounded for every $r > 0$.*

Corollary 1.8. *If $\phi(V)$ is power bounded then so is $\psi_s(V) = (1-s)I + s\phi(V)$ for all $s \ge 0$.*

This statement can be used to immediately derive the estimate

$$\|\phi(V)^{n+1} - \phi(V)^n\|_2 = O(1/\sqrt{n}) \tag{1.24}$$

from [14, Theorem 4.5.3] (cf. [18] where $\phi(V) = I - V$). In any case $\phi(V)$ is assumed power bounded. By Theorem 1.2 the latter is necessary if the L_2-norm of $\phi(V)^{n+1} - \phi(V)^n$ is bounded or at least grows more slowly than every exponent $\exp(n^\gamma)$, $\gamma > 0$.

The product of two commuting power bounded operators is always power bounded, though the latter may occur without power boundedness of the factors (cf. Remark 13 in [19]).

Corollary 1.9. *For functions $\phi_1(z)$ and $\phi_2(z)$ such that $\phi_1\phi_2 \neq \mathbf{1}$ the product $\phi_1(V)\phi_2(V)$ is power bounded in L_p if and only if $p = 2$ and $\phi_1'(0) + \phi_2'(0) < 0$.*

Hence, *if $\phi_1(V)$ and $\phi_2(V)$ are not power bounded and $\phi_1'(0)$ and $\phi_2'(0)$ are real then either the product $\phi_1(V)\phi_2(V)$ is not power bounded or it is I.*

Corollary 1.10. *The quotient $\phi_1(V)\phi_2(V)^{-1}$ of different functions $\phi_1(z)$ and $\phi_2(z)$ is power bounded in L_p if and only if $p = 2$ and $\phi_1'(0)-\phi_2'(0) < 0$.*

Now we consider superpositions, the case most complicated for a direct analysis. Theorem 1.1 immediately yields

Corollary 1.11. *Let $\phi(V) \neq I$ be power bounded. Let $\theta(w)$ be a non-constant analytic function in a neighborhood of $w = 0$ or $w = 1$ and $\theta(0) = 0$ or $\theta(1) = 1$, respectively. Then $\phi(\theta(V))$ or $\theta(\phi(V))$ is power bounded if and only $\theta'(0) > 0$ or $\theta'(1) > 0$, respectively.*

For example, if $\theta(0) = 0$ then $\exp(-\theta(V))$ is power bounded if and only if $\theta'(0) > 0$, or $\theta(w) \equiv 0$. Another example: with $\nu \in \mathbb{C}$ and with $\phi(V)$ power bounded, $\phi(V)^\nu$ is power bounded if and only if ν is real nonnegative.

Theorem 1.12. *If in L_p the operator $\phi(V) \neq I$ satisfies the strong Kreiss condition* (1.8) *at least at one point $\lambda > 1$ then $p = 2$ and $\phi(V)$ is power bounded.*

Proof. The inequality (1.8) just means that with $|\lambda| > 1$ the operator

$$U = (1 - |\lambda|)R(\lambda; \phi(V))$$

is power bounded. Accordingly, we set

$$\theta(w) = (1 - \lambda)(w - \lambda)^{-1}$$

for a fixed $\lambda > 1$. Then $U = \theta(\phi(V))$ and $\phi(V) = \chi(U)$ where χ is the function inverse to θ. Obviously, $\chi(1) = 1$ and $\chi'(1) = \lambda - 1 > 0$. By Corollary 1.11, $\phi(V)$ is power bounded, and by Theorem 1.1, $p = 2$. ∎

Our further results related to the Kreiss and Ritt conditions are presented in the next section. In particular, we prove that *the only Ritt operator* $\phi(V)$ *in* L_p *is* I (Corollary 2.5). In the polynomial case we characterize the Kreiss operators $\phi(V)$ in L_p by the inequality $\phi'(0) < 0$ (Theorem 2.12).

2. The Ritt and Kreiss operators. It is convenient to reformulate the resolvent conditions as follows. For any operator T with $\sigma(T) = \{1\}$ we set $A = T - I$ and $\zeta = (\lambda - 1)^{-1}$. Then for $\lambda \neq 1$ we have

$$R(\lambda; T) = (T - \lambda I)^{-1} = -\zeta\Phi(\zeta; A) \tag{2.1}$$

where

$$\Phi(\zeta; A) = (I - \zeta A)^{-1} = \sum_{n=0}^{\infty} \zeta^n A^n \tag{2.2}$$

is the *Fredholm resolvent* of A, an entire operator-valued function of $\zeta \in \mathbb{C}$. Its *exponential order* is

$$\omega = \omega(A) = \varlimsup_{n\to\infty} \frac{\log n}{|\log \sqrt[n]{\|A^n\|}|}, \tag{2.3}$$

so that

$$\Phi(\zeta; A) = O(\exp |\zeta|^{\omega+\varepsilon})$$

with any fixed $\varepsilon > 0$ (see [9, Section 1.3]).

If T is a Ritt operator then (1.5) can be extended (with another C) to a sector

$$S_\delta = \{\lambda \in \mathbb{C} : |\arg(\lambda - 1)| \leq \pi - \delta\}, \quad 0 < \delta < \pi/2$$

(see [10] and [13]). In its turn, the sectorial Ritt condition implies power boundedness [8], [14]. The transformation $\zeta = (\lambda - 1)^{-1}$ maps S_δ onto itself. By (2.1) the sectorial Ritt condition for T becomes

$$\|\Phi(\zeta; A)\| \leq C, \quad \zeta \in S_\delta. \tag{2.4}$$

Lemma 2.1. *If* $\sigma(A) = \{0\}$, $\omega(A) \leq 1$, *and* $T = I + A$ *satisfies the Ritt condition then* $A = 0$, *i.e.* $T = I$.

Proof. The angle size of the complementary sector $S'_\delta = \mathbb{C} \setminus S_\delta$ is 2δ, while $\omega(A) < \pi/2\delta$. By (2.4) the Phragmén–Lindelöf Principle (see e.g. [9, Section 6.1]) yields $\|\Phi(\zeta; A)\| \leq C$ for $\zeta \in S'_\delta$. As a result, $\Phi(\zeta; A)$ is bounded on the whole $\mathbb{C}$. By the Liouville theorem $\Phi(\zeta; A) = \text{const}$. Then $A = 0$ by (2.2). ∎

The Kreiss condition (1.6) in terms of the Fredholm resolvent is

$$\|\Phi(\zeta; A)\| \leq \frac{C}{|\zeta + 1| - |\zeta|}, \quad \operatorname{Re} \zeta > -\frac{1}{2}. \tag{2.5}$$

LEMMA 2.2. *If $\sigma(A) = \{0\}$, $\omega(A) < 1$ and $T = I + A$ satisfies the Kreiss condition then $A = 0$, i.e. $T = I$.*

Proof. From (2.5) it follows that $\|\Phi(it;A)\| \le O(|t|)$, $t \in \mathbb{R}$. The entire function

$$F(\zeta;A) = \frac{\Phi(\zeta;A) - I}{\zeta}$$

is of the exponential order $\omega(A) < 1$ and it is bounded on $i\mathbb{R}$. By the Phragmén–Lindelöf Principle this is bounded for $\operatorname{Re}\zeta > 0$ and for $\operatorname{Re}\zeta < 0$ separately. By the Liouville theorem $F(\zeta;A) = \text{const}$, i.e. $\Phi(\zeta;A)$ is a linear function of ζ. However, $\|\Phi(t;A)\| \le C$ for $t > 0$. Hence, $\Phi(\zeta;A) = \text{const}$. ∎

REMARK 2.3. In the case $\omega(A) < 1$ Lemma 2.1 follows from Lemma 2.2.

Actually, we are interested in

$$T = \phi(V) = I + aV^l + \sum_{k=l+1}^{\infty} a_k V^k, \tag{2.6}$$

where $l \ge 1$, $a \ne 0$. Then

$$A = \phi(V) - I = aV^l Q, \tag{2.7}$$

where

$$Q = I + \sum_{k=1}^{\infty} a^{-1} a_{k+l} V^k,$$

so that $\sigma(Q) = \{1\}$, thus the spectral radius $r(Q)$ equals 1.

LEMMA 2.4. *If $\phi(V) \ne I$ then in any L_p the exponential order $\omega(\phi(V) - I)$ is equal to $1/l$ where l is the multiplicity of the root $z = 0$ of the function $\phi(z) - 1$.*

Proof. Since $\|Q^n\|^{1/n} \to 1$ as $n \to \infty$, and Q commutes with V, we get $\omega(\phi(V) - I) \ge \omega(V^l)$ from (2.7) and (2.3). In fact, this is an equality since (2.7) can be rewritten as

$$V^l = a^{-1}(\phi(V) - I)Q^{-1}.$$

It remains to note that $\omega(V^l) = 1/l$ thanks to Stirling's formula applied to the estimate

$$\frac{1}{n!(np+1)^{1/p}} \le \|V^n\|_p \le \frac{1}{n!}$$

(cf. inequality (14) in [11]). ∎

Combining this result with Lemma 2.1 we obtain

COROLLARY 2.5. *In any L_p the only Ritt operator $\phi(V)$ is I.*

Similarly, Lemma 2.2 implies

COROLLARY 2.6. *For $l > 1$ the operator $\phi(V) \ne I$ is not Kreiss in L_p.*

Remark 2.7. In particular, the operator $I - V$ is not Ritt. In contrast, $I - V^\alpha$, where

$$(V^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt,$$

is a Ritt operator in L_p if $0 < \alpha < 1$ [11]. On the other hand, for $\alpha > 1$ this is not a Kreiss operator since $\omega(V^\alpha) = 1/\alpha$ for all $\alpha > 0$, hence, $\omega(V^\alpha) < 1$ if $\alpha > 1$. (For $\alpha = 2$ this was proven in [18] by special considerations.)

Now we investigate the Kreiss condition in L_p for

$$\phi(V) = I + \sum_{k=1}^{m} a_k V^k, \quad a_m \neq 0,$$

i.e. for $\phi(z)$ which is an arbitrary polynomial of degree $m \geq 1$. To this end we introduce the polynomial

$$\psi_\zeta(z) = z^m - \zeta \sum_{k=1}^{m} a_k z^{m-k} \tag{2.8}$$

depending on a complex parameter ζ and then consider the differential equation

$$(\psi_\zeta(D)g)(x) = f(x), \quad 0 \leq x \leq 1,$$

with $D = d/dx$ and $f \in L_p(0,1)$. Obviously, $DV = I$ and $(VDf)(x) = f(x)$ for f absolutely continuous with $f(0) = 0$.

Denote by $Q(u;\zeta)$ the *Cauchy function* for the operator $\psi_\zeta(D)$, i.e. the solution of the differential equation

$$(\psi_\zeta(D)Q)(u;\zeta) = 0 \tag{2.9}$$

under the initial conditions

$$Q^{(i)}(0;\zeta) = 0 \quad (0 \leq i \leq m-2), \quad Q^{(m-1)}(0;\zeta) = 1. \tag{2.10}$$

Lemma 2.8. *The Fredholm resolvent $\Phi(\zeta; A)$ of the operator $A = \phi(V) - I$ in $L_p(0,1)$ is the integral operator*

$$(\Phi(\zeta;A)f)(x) = f(x) + \int_0^x Q^{(m)}(x-t;\zeta) f(t)\,dt, \quad 0 \leq x \leq 1. \tag{2.11}$$

Proof. One can assume $f \in C^m[0,1]$ and $f^{(i)}(0) = 0$, $0 \leq i \leq m-1$, since such functions constitute a dense subset of $L_p(0,1)$ and both sides of (2.11) are continuous operators in $L_p(0,1)$. Under this restriction formula (2.11) can be rewritten as

$$(\Phi(\zeta;A)f)(x) = \int_0^x Q(x-t;\zeta) f^{(m)}(t)\,dt, \quad 0 \leq x \leq 1, \tag{2.12}$$

by m times integrating by parts. The right hand side $h(x)$ of (2.12) satisfies the equation

$$(\psi_\zeta(D)h)(x) = f^{(m)}(x), \quad 0 \le x \le 1,$$

i.e.

$$D^m h - \zeta \sum_{k=1}^{m} a_k D^{m-k} h = D^m f, \tag{2.13}$$

and, in addition, $h^{(i)}(0) = 0$, $0 \le i \le m-1$. (This is true due to (2.9) and (2.10).) Applying V^m to both sides of (2.13) we obtain

$$h - \zeta \sum_{k=1}^{m} a_k V^k h = f,$$

i.e. $h = (I - \zeta A)^{-1} f = \Phi(\zeta; A)f$. ∎

The polynomial $\psi_\zeta(z)$ is characteristic for the differential operator $\psi_\zeta(D)$. Let us investigate its roots $z_1, z_2, \ldots, z_m$ as $|\zeta| \to \infty$. For definiteness let $|z_1| \ge \max(|z_2|, \ldots, |z_m|)$. (If there are two or more roots with maximal modulus then z_1 may be any of them.) Note that all $z_i \ne 0$ since $\psi_\zeta(0) = -\zeta a_m \ne 0$.

Lemma 2.9. *Let $a_1 \ne 0$. Then $|z_1| > \max(|z_2|, \ldots, |z_m|)$ for large $|\zeta|$, and*

$$z_1 = a_1\zeta + O(1), \quad \max(|z_2|, \ldots, |z_m|) = O(1).$$

Proof. According to (2.8) the equation $\psi_\zeta(z) = 0$ is equivalent to

$$\sum_{k=1}^{m} a_k w^k = \eta \tag{2.14}$$

with unknown $w = 1/z$ and parameter $\eta = 1/\zeta$. For $\zeta = \infty$ this turns into

$$\sum_{k=1}^{m} a_k w^k = 0. \tag{2.15}$$

One of the roots of (2.15) is $w_1 = 0$ and this root is simple since $a_1 \ne 0$. All other roots are separated from 0 by a circle $|w| = \delta$. By the Argument Principle all nonzero roots of (2.14) lie outside this circle as long as $|\eta| < \varepsilon$ and ε is small enough. Let $|\zeta| > r \equiv 1/\varepsilon$. Then $|z_1| > 1/\delta$ but $|z_i| < 1/\delta$, $2 \le i \le m$. Now the relation

$$\sum_{i=1}^{m} z_i = a_1\zeta$$

implies $|z_1 - a_1\zeta| < (m-1)/\delta$. ∎

From now on we assume $a_1 \ne 0$ and $|\zeta| > r$. Under these conditions z_1 is a unique root of maximal modulus, so it is a function of ζ, $z_1 = z_1(\zeta)$.

Corollary 2.10. *The coefficients of the polynomial*

$$\theta_\zeta(z) = \frac{\psi_\zeta(z)}{z - z_1(\zeta)} = \prod_{i=2}^{m}(z - z_i)$$

are bounded functions of ζ.

The solution $Q(u;\zeta)$ of (2.9) is of the form

$$Q(u;\zeta) = C_1(\zeta)e^{z_1 u} + R(u;\zeta) \tag{2.16}$$

where the second term satisfies the equation

$$(\theta_\zeta(D)R)(u;\zeta) = 0. \tag{2.17}$$

In view of (2.10) and (2.16) the initial conditions for R are

$$C_1(\zeta)z_1^i + R^{(i)}(0;\zeta) = 0, \quad 0 \le i \le m-2, \tag{2.18}$$

and

$$C_1(\zeta)z_1^{m-1} + R^{(m-1)}(0;\zeta) = 1. \tag{2.19}$$

From (2.17)–(2.19) it follows that

$$C_1(\zeta)\theta_\zeta(z_1) = 1 \tag{2.20}$$

since the leading coefficient of $\theta_\zeta(z)$ equals 1. However,

$$\theta_\zeta(z_1) = \psi'_\zeta(z_1) = mz_1^{m-1} - \zeta\sum_{k=1}^{m} a_k(m-k)z_1^{m-k-1}$$

according to (2.8). By Lemma 2.9,

$$\theta_\zeta(z_1) = (a_1\zeta)^{m-1} + O(|\zeta|^{m-2}),$$

so (2.20) yields

$$C_1(\zeta) = \frac{1}{(a_1\zeta)^{m-1}} + O\left(\frac{1}{|\zeta|^m}\right). \tag{2.21}$$

Lemma 2.11. $\max_{0\le u\le 1}|R^{(l)}(u;\zeta)| = O(|\zeta|^{-1})$, $l \ge 0$.

Proof. Let E_ζ be the evolutionary operator for the differential equation (2.17). This operator transforms the vector of initial conditions into the corresponding solution. Since the equation is linear, E_ζ is linear. Actually, this is an isomorphism between the space of initial conditions and the space of solutions,

$$R(\cdot;\zeta) = E_\zeta((R^{(i)}(0;\zeta))_0^{m-2}).$$

Equipping these spaces with the corresponding sup-norms we get

$$\max_{0\le u\le 1}|R(u;\zeta)| \le \|E_\zeta\| \max_{0\le i\le m-2}|R^{(i)}(0;\zeta)|. \tag{2.22}$$

The second factor on the right hand side of (2.22) is $O(|\zeta|^{-1})$ by (2.18), (2.21) and Lemma 2.9, while $\|E_\zeta\| = O(1)$ by Corollary 2.10. Thus,

$$\max_{0\le u\le 1} |R(u;\zeta)| = O(|\zeta|^{-1}).$$

The same estimate is true for every derivative $R^{(l)}(u;\zeta)$, $l \ge 1$. Indeed, $R^{(l)}(u;\zeta)$ satisfies the same differential equation (2.17), which also determines its initial vector, as long as $R^{(i)}(0;\zeta)$ are given for $0 \le i \le m-2$. Thus,

$$\|(R^{(l+i)}(0;\zeta))_0^{m-2}\| = O(\|(R^{(i)}(0;\zeta))_0^{m-2}\|), \qquad l \ge 1,$$

by Corollary 2.10 again. ∎

Now we are in a position to prove our result concerning the Kreiss operators.

Theorem 2.12. *In any L_p, in order for the operator*

$$\phi(V) = I + \sum_{k=1}^{m} a_k V^k, \quad m \ge 1, \quad a_m \ne 0, \tag{2.23}$$

to be Kreiss it is necessary and sufficient that $a_1 < 0$.

Proof of necessity. Applying (2.11) to $f = \mathbf{1}$ we obtain

$$(\Phi(\zeta;A)\mathbf{1})(x) = 1 + \int_0^x Q^{(m)}(x-t;\zeta)\,dt = Q^{(m-1)}(x;\zeta)$$

since $Q^{(m-1)}(0;\zeta) = 1$. Furthermore,

$$\int_0^1 Q^{(m-1)}(x;\zeta)\,dx = Q^{(m-2)}(1;\zeta)$$

since $Q^{(m-2)}(0;\zeta) = 0$. Hence,

$$|Q^{(m-2)}(1;\zeta)| \le \int_0^1 |Q^{(m-1)}(x;\zeta)|\,dx = \int_0^1 |(\Phi(\zeta;A)\mathbf{1})(x)|\,dx.$$

Using the Hölder inequality we obtain

$$|Q^{(m-2)}(1;\zeta)| \le \|\Phi(\zeta;A)\mathbf{1}\|_p \le \|\Phi(\zeta;A)\|_p.$$

Thus, from (2.5) it follows that

$$|Q^{(m-2)}(1;\zeta)| \le \frac{C}{|\zeta+1| - |\zeta|}, \quad \operatorname{Re}\zeta > -\frac{1}{2}.$$

This yields

$$\exp(\operatorname{Re}(a_1\zeta)) = O\left(\frac{|\zeta|}{|\zeta+1| - |\zeta|}\right), \quad \operatorname{Re}\zeta > -\frac{1}{2},$$

by (2.16), (2.21) and Lemmas 2.9 and 2.11. Obviously,

$$\frac{|\zeta|}{|\zeta+1|-|\zeta|} = \frac{|\zeta|(|\zeta+1|+|\zeta|)}{2\operatorname{Re}\zeta+1} \le \frac{|\zeta|(2|\zeta|+1)}{2\operatorname{Re}\zeta+1},$$

and $\operatorname{Re}(a_1\zeta) = |\zeta|\operatorname{Re}(a_1\chi)$ where $\chi = \zeta/|\zeta|$, so $|\chi| = 1$. Hence,

$$\exp(|\zeta|\operatorname{Re}(a_1\chi)) = O\biggl(\frac{|\zeta|^2}{2|\zeta|\operatorname{Re}\chi+1}\biggr).$$

Letting $|\zeta| \to \infty$ we get $\operatorname{Re} a_1 \le 0$ taking $\chi = 1$ and $\operatorname{Im} a_1 = 0$ taking $\chi = \pm i$. Thus, $a_1 \in \mathbb{R}$ and $a_1 \le 0$. But $a_1 \ne 0$ by Corollary 2.6, hence, $a_1 < 0$. ∎

Proof of sufficiency. From (2.11) it follows that

$$\|\Phi(\zeta;A)\|_p \le 1 + \int_0^1 |Q^{(m)}(u;\zeta)|\,du$$

in all L_p, according to the well known estimate of the L_p-norm of the convolution (see [20, Theorem 1.15] and [7, Lemma 23.16.1]).

By (2.16), (2.21) and Lemmas 2.9 and 2.11 again we have

$$|Q^{(m)}(u;\zeta)| = O\biggl(|\zeta|e^{a_1\xi u} + \frac{1}{|\zeta|}\biggr),$$

where $\xi = \operatorname{Re}\zeta > -1/2$. Hence,

$$\|\Phi(\zeta;A)\|_p = O\biggl(|\zeta|\,\frac{e^{a_1\xi}-1}{a_1\xi} + 1\biggr).$$

On the other hand,

$$|\zeta+1|-|\zeta| = \frac{2\xi+1}{|\zeta+1|+|\zeta|} \in (0,1].$$

Thus,

$$(|\zeta+1|-|\zeta|)\|\Phi(\zeta;A)\|_p = O(M(\xi)+1) \tag{2.24}$$

where

$$M(\xi) = \frac{(2\xi+1)(e^{a_1\xi}-1)}{a_1\xi}.$$

Since $a_1 < 0$, this function is bounded on $(-1/2,\infty)$, so (2.5) follows immediately from (2.24). ∎

In fact, the necessity part of Theorem 2.12 is true for all analytic ϕ. Indeed, if $\phi(V)$ is a Kreiss operator then $\|\phi(V)^n\|_p = O(n)$ and then a_1 must be real negative by Theorem 1.2 from [2].

COROLLARY 2.13. *In L_2, if $\phi(V)$ is a Kreiss operator then it is power bounded.*

In L_p with $p \neq 2$ this fails by Theorem 1.1. However, the conjecture saying that *every Kreiss operator* $\phi(V)$ *is uniformly Kreiss* seems to be plausible even if $p \neq 2$.

Perhaps, the sufficiency part of Theorem 2.12 can also be extended to the analytic situation but this requires a quite different approach.

3. The necessity in Theorem 1.1. In this section we resort to a "scaling". All norms below are those of (1.4). For any ε, $0 < \varepsilon < 1$, the space $L_p(0,\varepsilon)$ is naturally isometric to the subspace of those $f \in L_p(0,1)$ which vanish for $x > \varepsilon$. The operator $R : L_p(0,1) \to L_p(0,\varepsilon)$ such that $(Rf)(x) = f(x)$, $0 < x < \varepsilon$, is the left inverse to the natural isometric embedding $E : L_p(0,\varepsilon) \to L_p(0,1)$. Denote by V_ε the same integration (1.1) but for $f \in L_p(0,\varepsilon)$. Then $V_\varepsilon R = RV$, whence $\phi(V_\varepsilon)R = R\phi(V)$ for all functions ϕ under consideration. Hence, $\phi(V_\varepsilon) = R\phi(V)E$, which yields

$$\|\phi(V_\varepsilon)\| \leq \|\phi(V)\| \tag{3.1}$$

since $\|R\| = 1$ and $\|E\| = 1$.

Now let S be the operator $L_p(0,\varepsilon) \to L_p(0,1)$ defined as $(Sf)(x) = f(\varepsilon x)$, $0 < x < 1$. Then

$$\|Sf\|^p = \int_0^1 |f(\varepsilon x)|^p\,dx = \frac{1}{\varepsilon}\int_0^\varepsilon |f(t)|^p\,dt = \frac{1}{\varepsilon}\,\|f\|^p,$$

which means that $S_\varepsilon = \varepsilon^{1/p}S$ is an isometry. Also we have $S_\varepsilon V_\varepsilon = \varepsilon V S_\varepsilon$. Indeed,

$$((SV_\varepsilon)f)(x) = \int_0^{\varepsilon x} f(t)\,dt = \varepsilon\int_0^x f(\varepsilon s)\,ds = ((\varepsilon VS)f)(x), \quad 0 \leq x \leq 1.$$

Now $S_\varepsilon\phi(V_\varepsilon) = \phi(\varepsilon V)S_\varepsilon$ yields

$$\|\phi(V_\varepsilon)\| = \|\phi(\varepsilon V)\|. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$\|\phi(\varepsilon V)\| \leq \|\phi(V)\|, \quad 0 < \varepsilon < 1.$$

This results in the following important

Lemma 3.1. *If* $\phi(V)$ *is power bounded then the family* $\{\phi(\varepsilon V) : 0 < \varepsilon < 1\}$ *is uniformly power bounded, i.e.*

$$\sup\{\|\phi(\varepsilon V)^n\|_p : 0 < \varepsilon < 1,\ n \geq 0\} < \infty.$$

Now we turn to the decomposition (2.6) with $l = 1$, i.e.

$$\phi(V) = I + aV + \sum_{k=2}^\infty a_k V^k.$$

Here $a \neq 0$, otherwise $\phi(V)$ would not be Kreiss by Corollary 2.6, while $\phi(V)$ is power bounded by assumption. By Lemma 3.1 with $\varepsilon = \tau/n$, $0 < \tau < n$, we obtain

$$\sup_{\tau>0}\sup_{n>\tau}\left\|\left(I + \frac{a\tau V}{n} + O\left(\frac{\tau^2}{n^2}\right)\right)^n\right\| < \infty.$$

Passing to the limit as $n \to \infty$ with τ fixed, we get

$$\sup_{\tau>0}\|\exp(a\tau V)\| \equiv M < \infty. \tag{3.3}$$

By the classical resolvent criterion [7, Theorem 12.31], (3.3) implies

$$\|R(\lambda; aV)^n\| \leq \frac{M}{(\operatorname{Re}\lambda)^n}, \quad \operatorname{Re}\lambda > 0, \quad n \geq 1, \tag{3.4}$$

in particular,

$$\|R(\lambda; aV)\| \leq \frac{M}{\operatorname{Re}\lambda}, \quad \operatorname{Re}\lambda > 0. \tag{3.5}$$

However, the function $g = R(\lambda; aV)\mathbf{1}$ is nothing but the solution of the integral equation

$$a\int_0^x g(t)\,dt - \lambda g(x) = 1, \quad 0 \leq x \leq 1,$$

or, equivalently, of the differential equation $\lambda g'(x) - ag(x) = 0$ with the initial condition $g(0) = -1/\lambda$. Therefore,

$$g(x) = -\frac{1}{\lambda}\exp\left(\frac{ax}{\lambda}\right),$$

whence

$$\int_0^1 g(x)\,dx = \frac{1}{a}\left(1 - \exp\left(\frac{a}{\lambda}\right)\right),$$

and, on the other hand,

$$\left|\int_0^1 g(x)\,dx\right| \leq \|g\| \leq \frac{M}{\operatorname{Re}\lambda}, \quad \operatorname{Re}\lambda > 0,$$

by (3.5). Thus,

$$\left|1 - \exp\left(\frac{a}{\lambda}\right)\right| \leq \frac{M|a|}{\operatorname{Re}\lambda}, \quad \operatorname{Re}\lambda > 0.$$

Setting $\lambda = 1/\zeta$, $\operatorname{Re}\zeta > 0$, we get

$$|\exp(a\zeta)| \leq 1 + \frac{M|a|\,|\zeta|^2}{\operatorname{Re}\zeta}, \quad \operatorname{Re}\zeta > 0.$$

Letting $\zeta \in \mathbb{R}$, $\zeta \to +\infty$, we see that $\operatorname{Re} a \le 0$. On the other hand, for $\zeta = 1 + i\omega$, $\omega \in \mathbb{R}$, we have

$$\exp(\operatorname{Re} a - \omega \operatorname{Im} a) \le 1 + M|a|(\omega^2 + 1).$$

With $\omega \to \pm\infty$ we obtain $\operatorname{Im} a = 0$. Since $a \ne 0$, we conclude that $a < 0$.

Now we return to (3.4). For $\lambda = |a|$ this yields the power boundedness of $(I + V)^{-1}$ and then the power boundedness of $I - V$ by the Pedersen similarity. This implies $p = 2$ according to [12, Theorem 1.1].

The sufficiency in Theorem 1.1 is contained in Theorem 1.3 which we prove in the next section.

4. Proof of Theorem 1.3. Our main tool in this proof is the Laplace transform. To apply the latter we start with $\phi(V)$ in the form (1.11) and extend it to $x > 1$ as follows. We set

$$(Wf)(x) = f(x) + \int_0^x k(x-t) f(t)\, dt, \quad 0 \le x < \infty, \tag{4.1}$$

where $k(u) = K(u)$ for $0 \le u \le 1$ and $k(u) = K'(1)(u-1) + K(1)$ for $u > 1$. The operator W acts in the linear space Λ of locally L_2-functions whose integral over $(0, x)$ grows no faster than polynomially as $x \to \infty$. Obviously, for all n we have

$$(W^n f)(x) = (\phi(V)^n f)(x), \quad 0 \le x \le 1. \tag{4.2}$$

The Laplace transform of $k(u)$,

$$\tilde{k}(\lambda) = \int_0^\infty k(u) e^{-\lambda u}\, du, \tag{4.3}$$

is a regular analytic function in the half-plane $\operatorname{Re} \lambda > 0$, and the same is true for all $f \in \Lambda$, thus for all $W^n f$, $n \ge 1$.

From (4.1) it follows that

$$(\widetilde{Wf})(\lambda) = (1 + \tilde{k}(\lambda))\tilde{f}(\lambda), \quad \operatorname{Re} \lambda > 0, \tag{4.4}$$

by the usual convolution rule. Now it is convenient to introduce the function

$$\psi(z) = 1 + \tilde{k}(1/z), \quad \operatorname{Re} z > 0.$$

Then (4.4) takes the form

$$(\widetilde{Wf})(\lambda) = \psi(1/\lambda)\tilde{f}(\lambda),$$

and, by iteration,

$$(\widetilde{W^n f})(\lambda) = \psi^n(1/\lambda)\tilde{f}(\lambda), \quad \operatorname{Re} \lambda > 0,\ n \ge 1. \tag{4.5}$$

Integrating two times by parts in (4.3) and taking into account our definition of $k(u)$ we obtain

$$\tilde{k}(\lambda) = \frac{K(0)}{\lambda} + \frac{1}{\lambda^2}\Big(K'(0) + \int_0^1 K''(u)e^{-\lambda u}\,du\Big).$$

Accordingly,

$$\psi(z) = 1 + a_1 z + R(z)z^2 \tag{4.6}$$

where $a_1 = K(0) < 0$ and

$$R(z) = K'(0) + \int_0^1 K''(u)e^{-u/z}\,du.$$

Since $\operatorname{Re} z > 0$, we have

$$|R(z)| \le c = |K'(0)| + \int_0^1 |K''(u)|\,du. \tag{4.7}$$

In the classical inversion formula for the Laplace transform the latter is a factor in the integrand when integrating along the vertical line $\{\lambda : \operatorname{Re}\lambda = \mu\}$ with any fixed $\mu > 0$. In view of (4.5) we have to investigate $\psi(z)$ on the image of this line under the mapping $z = 1/\lambda$. This is the circle

$$C_\mu = \left\{z : \left|z - \frac{1}{2\mu}\right| = \frac{1}{2\mu}\right\} = \{z : \operatorname{Re} z = \mu|z|^2\}$$

punctured at $z = 0$, but the latter "singularity" can be removed by setting $\psi(0) = 0$. Since a_1 is real, we have

$$|\psi(z)|^2 = 1+(2a_1\mu+a_1^2)|z|^2+2\operatorname{Re}(R(z)z^2)+2a_1\operatorname{Re}(R(z)z|z|^2)+|R(z)|^2|z|^4.$$

By (4.7)

$$|\psi(z)|^2 \le 1 - M(\mu)|z|^2, \quad z \in C_\mu, \tag{4.8}$$

with

$$M(\mu) = 2|a_1|\mu - \left(a_1^2 + 2c + \frac{2c|a_1|}{\mu} + \frac{c^2}{\mu^2}\right) \tag{4.9}$$

since $a_1 < 0$ and $|z| \le 1/\mu$ for $z \in C_\mu$. The continuous function $M(\mu)$, $\mu > 0$, is increasing and $M(+0) = -\infty$, $M(+\infty) = +\infty$. Hence, it has a unique root μ_0 and $M(\mu) > 0$ if $\mu > \mu_0$. By (4.8) and (4.5) we obtain the following key lemma:

Lemma 4.1. *If $\mu \ge \mu_0$ then*

$$|(\widetilde{W^n f})(\lambda)| \le |\tilde{f}(\lambda)|, \quad \operatorname{Re}\lambda = \mu. \tag{4.10}$$

Indeed, in this case $|\psi(z)| \le 1$ for $z \in C_\mu$.

The function $(\widetilde{W^n f})(\mu+i\omega)$, $\omega\in\mathbb{R}$, is the Fourier image of $(W^n f)(x)e^{-\mu x}$ extended by zero to $x<0$. By the Parseval equality and inequality (4.10),

$$\int_0^\infty |(W^n f)(x)|^2 e^{-2\mu x}\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty |(\widetilde{W^n f})(\mu+i\omega)|^2\,d\omega \tag{4.11}$$
$$\le \frac{1}{2\pi}\int_{-\infty}^\infty |\tilde{f}(\mu+i\omega)|^2\,d\omega$$
$$= \int_0^\infty |f(x)|^2 e^{-2\mu x}\,dx.$$

A fortiori,

$$\int_0^1 |(W^n f)(x)|^2 e^{-2\mu x}\,dx \le \int_0^\infty |f(x)|^2\,dx,$$

and finally,

$$\int_0^1 |(W^n f)(x)|^2\,dx \le e^{2\mu}\int_0^\infty |f(x)|^2\,dx. \tag{4.12}$$

In particular, one can take any $f\in L_2(0,1)$ and extend it by zero to $x>1$. Then (4.12) takes the form

$$\int_0^1 |(W^n f)(x)|^2\,dx \le e^{2\mu}\int_0^1 |f(x)|^2\,dx.$$

In view of (4.2) this inequality is actually

$$\int_0^1 |(\phi(V)^n f)(x)|^2\,dx \le e^{2\mu}\int_0^1 |f(x)|^2\,dx,$$

i.e.

$$\|\phi(V)^n\|_2 \le e^{\mu}.$$

This is nothing but (1.14) with μ determined by (1.15) and c as in (1.16). To show this, note that $M(|a_1|/2)<0$. Hence, every μ such that $M(\mu)\ge 0$ is $\ge |a_1|/2$, i.e.

$$\mu = |a_1|/2+\delta,\quad \delta>0. \tag{4.13}$$

If $|a_1|\delta-(3c+2c^2/a_1^2)\ge 0$ then $M(\mu)\ge 0$. Thus, one can take

$$\delta = \frac{3ca_1^2+2c^2}{|a_1|^3}$$

in order to get (1.15). The value c given by (1.16) appears as a result of the substitution of $K(u)$ from (1.12) into (4.7).

The estimate (1.17) can be obtained similarly but with $\psi^n(\psi-1)$ instead of ψ^n. In this case for $z \in C_\mu$ we have $|\psi(z)-1|^2 \le 1-|\psi(z)|^2$ if μ is chosen so that $|\psi(z)|^2 \le \operatorname{Re}\psi(z)$. For this inequality it suffices to have $M(\mu) \ge |a_1|\mu+c$ thanks to (4.6) and (4.8). In this case we set $\mu = |a_1| + \delta$ instead of (4.13). This yields (1.18) in the same way as we obtained (1.14). It remains to note that $|\psi(z)|^{2n}|\psi(z)-1|^2$ is bounded from above by

$$\max\{u^n(1-u) : 0 \le u \le 1\} = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} < \frac{1}{en}.$$

Theorem 1.3 is proven. Moreover, literally the same proof yields the following

THEOREM 4.2. *Let $q(u)$, $0 \le u \le 1$, be a complex-valued function with absolutely continuous first derivative, and assume $q(0)$ is real negative. Then the integral operator*

$$(Tf)(x) = f(x) + \int_0^x q(x-t)f(t)\,dt, \quad 0 \le x \le 1,$$

in $L_2(0,1)$ is power bounded, and moreover,

$$\sup_n \|T^n\|_2 \le e^\mu$$

where

$$\mu = \frac{|q(0)|}{2} + \frac{3cq(0)^2 + 2c^2}{|q(0)|^3}$$

and

$$c = |q'(0)| + \int_0^1 |q''(u)|\,du.$$

Furthermore,

$$\sup_n \sqrt{n}\,\|T^{n+1} - T^n\|_2 \le e^{\mu_1}$$

where

$$\mu_1 = |q(0)| + \frac{5cq(0)^2 + c^2}{|q(0)|^3}.$$

If $q(u)$ is convex and nondecreasing then $c = q'(1)$.

Note that the conditions on the kernel $q(u)$ in Theorem 4.2 are weaker than those of [3] which provide the similarity to $I + q(0)V$.

5. Alternating coefficients. We start with the proof of Theorem 1.4. To this end we proceed to Taylor's expansion

$$\phi(z) = \sum_{k=0}^{m} \frac{\phi^k(x)}{k!}\,(z-x)^k$$

with $x > 0$ such that $\operatorname{sign} \phi^{(k)}(x) = (-1)^k$, $0 \le k \le m$, so $x < x_0$. Then

$$\phi(z) = \sum_{k=0}^{m} p_k (1 - z/x)^k$$

where

$$p_k = \frac{\phi^{(k)}(x)(-x)^k}{k!}, \quad 0 \le k \le m.$$

Obviously, all $p_k > 0$ and

$$\sum_{k=0}^{m} p_k = \phi(0) = 1.$$

Furthermore,

$$\phi^n(z) = \sum_{l=0}^{mn} \Big(\sum_{k_1 + \cdots + k_n = l} p_{k_1} \ldots p_{k_n} \Big) (1 - z/x)^l.$$

Hence,

$$\sup_n \|\phi^n(V)\| \le \sup_l \left\| \left(I - \frac{1}{x} V \right)^l \right\|$$

irrespective of the choice of the norm. In particular,

$$\sup_n \|\phi^n(V)\|_2 \le e^{1/2x}$$

by (1.19). It remains to optimize this bound by passing to $x = x_0$.

Now we denote by $\mathcal{A}_m$ the set of real polynomials $\phi(z)$ of degree m with $\phi(0) = 1$ and alternating coefficients. Obviously, $\mathcal{A}_m$ is convex. Furthermore, the product $\mathcal{A}_m \mathcal{A}_s = \{\phi_1 \phi_2 : \phi_1 \in \mathcal{A}_m, \phi_2 \in \mathcal{A}_s\}$ is contained in $\mathcal{A}_{m+s}$. Indeed, $\phi \in \mathcal{A}_m$ if and only if $\deg \phi = m$, $\phi(0) = 1$, and all coefficients of $\phi(-z)$ are positive.

If ϕ is real and all roots of ϕ lie in the open right half-plane $H_+ = \{z : \operatorname{Re} z > 0\}$ then $\phi \in \mathcal{A}_m$. (The converse is also true if $m \le 2$.) Such ϕ can be called *anti-Hurwitz polynomials* since this is exactly the case when $\phi(-z)$ satisfies Hurwitz's determinant condition for the roots to lie in $H_- = \{z : \operatorname{Re} z < 0\}$ (see e.g. [4]). The role of the "Hurwitz polynomials" in the classical stability theory is well known.

COROLLARY 5.1. *If $\phi(z)$ is an anti-Hurwitz polynomial then* (1.21) *holds with x_0 defined in* (1.22).

Indeed, ϕ is the product of a number of polynomials, each from $\mathcal{A}_1$ or $\mathcal{A}_2$, thus $\phi \in \mathcal{A}_m$, $m = \deg \phi$.

COROLLARY 5.2. *Let $\phi(z)$ be a complex polynomial with real coefficient a_1. If all roots $z_1, \ldots, z_m$ of $\phi(z)$ lie in H_+ then $\phi(V)$ is power bounded in L_2.*

Indeed, in this case

$$a_1 = \operatorname{Re} a_1 = -\sum_{i=1}^{m} \operatorname{Re}\left(\frac{1}{z_i}\right) = -\sum_{i=1}^{m} \frac{\operatorname{Re} z_i}{|z_i|^2} < 0.$$

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Yuri Lyubich
Institute of Mathematics
Polish Academy of Sciences
Warszawa, Poland
and
Technion, Haifa, Israel
E-mail: lyubich@tx.technion.ac.il

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