The power boundedness and resolvent conditions for functions of the classical Volterra operator

by

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Abstract. Let \( \phi(z) \) be an analytic function in a disk \( |z| < \rho \) (in particular, a polynomial) such that \( \phi(0) = 1 \), \( \phi(z) \not\equiv 1 \). Let \( V \) be the operator of integration in \( L_p(0,1) \), \( 1 \leq p \leq \infty \). Then \( \phi(V) \) is power bounded if and only if \( \phi'(0) < 0 \) and \( p = 2 \). In this case some explicit upper bounds are given for the norms of \( \phi(V)^n \) and subsequent differences between the powers. It is shown that \( \phi(V) \) never satisfies the Ritt condition but the Kreiss condition is satisfied if and only if \( \phi'(0) < 0 \), at least in the polynomial case.

1. Introduction and overview. The integration

\[
(Vf)(x) = \int_0^x f(t) \, dt
\]

is a traditional example of a quasinilpotent (but not nilpotent) operator in \( L_p(0,1) \), \( 1 \leq p \leq \infty \). In \( L_2(0,1) \) we have the adjoint operator

\[
(V^*f)(x) = \int_x^1 f(t) \, dt,
\]

so \( V \) is not self-adjoint. From (1.1) and (1.2) it follows that

\[
\text{Re}(Vf,f) = \frac{1}{2} \left( (V+V^*)f,f \right) = \frac{1}{2} \left( \int_0^1 f(t) \, dt \right)^2 \geq 0.
\]

Hence, \( \exp(-tV) \), \( t \geq 0 \), is a semigroup of contractions in \( L_2(0,1) \).

Recall that a bounded linear operator \( T \) in a Banach space \( X \) is called power bounded if \( \sup\{\|T^n\| : n \geq 0\} < \infty \). In particular, all contractions are power bounded, and conversely, every power bounded operator is a contraction in the equivalent norm \( \|f\|_T = \sup\{\|T^n f\| : n \geq 0\} \), \( f \in X \). Sometimes, this trick can be useful, but here we do not need it, so we will deal with a

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fixed norm in $X$. In particular, if $X = L_p(0, h), 0 < h < \infty$, then we set, as usual,

$$
\|f\|_p = \left( \int_0^h |f(t)|^p \, dt \right)^{1/p},
$$

so that, $\|f\|_p = 1$ if $f = 1$ and $h = 1$. Since all $L_p(0, h)$ are isometric, the case $h = 1$ is representative. For definiteness we can deal with $L_p(0, 1)$ and write briefly $L_p$, unless stated otherwise.

All spaces under consideration are assumed complex and all operators linear and bounded. We denote by $I$ the identity operator. Also, as usual, we denote by $\sigma(T)$ the spectrum of $T$ and by $R(\lambda; T)$ the resolvent of $T$, i.e. $R(\lambda; T) = (T - \lambda I)^{-1}, \lambda \in \mathbb{C} \setminus \sigma(T)$. If $\sigma(T)$ lies in the open unit disk $\mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$ then $T$ is power bounded. On the other hand, if $T$ is power bounded then $\sigma(T)$ lies in the closed unit disk $\overline{\mathbb{D}}$. If $\sigma(T) = \{1\}$, $T \neq I$, and $T$ is power bounded then $T^{-1}$ is not power bounded. This is a reformulation of the classical Gelfand theorem on the single-point spectrum isometries.

There is a series of resolvent conditions in the domain $|\lambda| > 1$ closely related to power boundedness. The most important are: the Ritt condition

$$
\|R(\lambda; T)\| \leq \frac{C}{|\lambda - 1|},
$$

and the Kreiss condition

$$
\|R(\lambda; T)\| \leq \frac{C}{|\lambda| - 1}.
$$

Obviously, the latter is weaker than the former. Furthermore, from the expansion

$$
R(\lambda; T) = -\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}, \quad |\lambda| > 1,
$$

it follows that every power bounded operator is a Kreiss operator, i.e. it satisfies (1.6). On the other hand, every Ritt operator is power bounded [10, 13].

The “iterated” inequality (1.6), i.e.

$$
\|R^n(\lambda; T)\| \leq \frac{C}{(|\lambda| - 1)^n}, \quad |\lambda| > 1, \ n \geq 1,
$$

is called the strong Kreiss condition. This property is intermediate between power boundedness and the Kreiss condition. All strongly Kreiss operators are uniformly Kreiss [5] in the sense that the upper bound (1.6) remains valid for all partial sums of the series (1.7). The converse is not true [12]. We refer the reader to Nevanlinna’s book [14] and to his papers [15, 16] for
some general theorems on the resolvent conditions. In particular, Theorem 4 from [16] shows that \(\|T^n\| = O(n)\) for every Kreiss operator \(T\).

In the present paper we focus on the case \(T = \phi(V)\), where \(\phi(z)\) is a polynomial or even an analytic function of the complex variable \(z\) regular at \(z = 0\). The linear and quadratic polynomials were considered in [6], [12], [18], [19]. In [11] it is proven that \(T = I - V^\alpha, \ 0 < \alpha < 1\), is power bounded (even Ritt) in any \(L_p\). However, the analytic function \(\phi(z) = 1 - z^\alpha\) is not regular at \(z = 0\).

In [6] Halmos used (1.3) to prove that \((I + V)^{-1}\) is a contraction in \(L_2\). Accordingly, \(I + V\) is not power bounded in this space. In contrast, \(I - V\) is power bounded in \(L_2\), due to the Pedersen similarity \(P^{-1}(I - V)P = (I + V)^{-1}\) where \((Pf)(x) = e^xf(x)\) (see [1] for a reference). Using these results Tsedenbayar [18] proved that the operator \(I - rV, \ r \geq 0\), is power bounded in \(L_2\). On the other hand, he showed that \(I - aV\) with \(a \in \mathbb{C}\setminus[0, \infty]\) is not Kreiss in \(L_p\) for \(p = 1, 2, \infty\), and \(I - aV^2\) with \(a \neq 0\) is not Kreiss in all \(L_p\).

In [12] Montes-Rodríguez, Sánchez-Álvarez and Zemánek proved that in \(L_p\) with \(p \neq 2\) the operator \(I - rV\) with \(r > 0\) is not power bounded. Moreover, they determined an exact order of growth of \((I - rV)^n\) and of decay of the differences between the \((n+1)\)th and the \(n\)th powers. Also they proved that \(I - rV, \ r > 0\), is uniformly Kreiss for all \(p\), but it is strongly Kreiss if \(p = 2\) only.

The quadratic polynomials \(I - aV + bV^2 \ (a \in \mathbb{R}, \ b \in \mathbb{C})\) were investigated by Tsedenbayar and Zemánek in [19], where it was proven that these operators in \(L_2\) are power bounded if \(a, b > 0\), but not Kreiss if \(a < 0\). Note that Proposition 6 from [19] should be corrected: by our Theorem 1.1 (see below) the operator \(I - aV + bV^2\) is power bounded for \(a > 0\) and all \(b \in \mathbb{C}\), not for \(b \geq 0\) only.

As mentioned before, we consider

\[
\phi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k \in \mathbb{C}, \quad |z| < \rho,
\]

where

\[
\rho = \left( \lim_{k \to \infty} k^{1/|a_k|} \right)^{-1} > 0.
\]

The latter is just the convergence radius of the power series (1.9). The series

\[
\phi(V) = \sum_{k=0}^{\infty} a_k V^k
\]

converges in the operator norm topology because of (1.10) and \(\|V^k\|^{1/k} \to 0\).
As usual, the functional calculus \( \phi \mapsto \phi(V) \) is an algebra homomorphism such that \( 1 \mapsto I \). This is injective since \( \ker(V) = 0 \) and any operator \( \phi(V) \) with \( a_0 \neq 0 \) is invertible. Indeed, the spectrum \( \sigma(\phi(V)) = \phi(\sigma(V)) \) is the singleton \( \{ \phi(0) \} = \{ a_0 \} \). If \( |a_0| < 1 \) then \( \phi(V) \) is power bounded. If \( |a_0| = 1 \) then \( \phi(V) \) is power bounded if and only if \( a_0^{-1} \phi(V) \) is power bounded. Thus, without loss of generality one can assume \( a_0 = 1 \), i.e. \( \phi(0) = 1 \). This is the only case from now on.

The operator \( \phi(V) \) can be represented in a “closed” form. Namely, since

\[
(V^k f)(x) = \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f(t) \, dt, \quad k \geq 1,
\]

we have

\[
(\phi(V)f)(x) = f(x) + \int_0^x K(x-t)f(t) \, dt, \quad 0 \leq x \leq 1,
\]

where

\[
K(u) = \sum_{k=1}^{\infty} \frac{a_k u^{k-1}}{(k-1)!}.
\]

**Theorem 1.1.** In order for the operator \( \phi(V) \neq I \) to be power bounded in \( L_p \) it is necessary and sufficient that \( p = 2 \) and \( a_1 = \phi'(0) \) is real negative.

The necessity of \( a_1 < 0 \) follows from an asymptotic formula recently obtained by a complicated complex analysis in \[2\] (1) (see Theorem 1.2 therein). Our proof of the necessity (Section 3) is elementary and rather short.

On the other hand, a comparison of the above mentioned asymptotic formula to the sufficiency in our Theorem 1.1 discovers an exponential jump in the scale of growth of \( \| \phi(V)^n \|_2 \) (2).

**Theorem 1.2.** In \( L_2(0,1) \) the following alternative holds: either \( \phi(V) \) is power bounded or

\[
\| \phi(V)^n \| \geq \exp(cn^\gamma)
\]

with some \( c > 0 \) and some \( 0 < \gamma \leq 1/2 \).

The sufficiency in Theorem 1.1 follows from the similarity between \( \phi(V) \) and \( I + a_1 V \) in \( L_2 \). The latter is a particular case (up to an obvious modification) of that of \[3\] pp. 369–370. However, our direct method (Section 4) yields some explicit upper bounds for the \( L_2 \)-norms of \( \phi(V)^n \) and of the differences \( \phi(V)^{n+1} - \phi(V)^n \). Actually, this method works in a wide class of

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(1) I thank Yu. Tomilov for providing me with this reference after my talk about the present work.

(2) I thank S. Torba for this observation.
integral convolution operators (see Theorem 4.2). This generalization does not fall under [3].

**Theorem 1.3.** If \( a_1 < 0 \) then

\[
(1.14) \quad \sup_n \| \phi(V)^n \|_2 \leq e^\mu
\]

where

\[
(1.15) \quad \mu = \frac{|a_1|}{2} + \frac{3a_1^2c + 2c^2}{|a_1|^3}
\]

and

\[
(1.16) \quad c = |a_2| + \sum_{k=3}^{\infty} \frac{a_k u^{k-3}}{(k-3)!} \left. \right|_0^1 \sum_{k=3}^{\infty} \frac{|a_k|}{(k-2)!}.
\]

Furthermore,

\[
(1.17) \quad \sup_n \sqrt{n} \| \phi(V)^{n+1} - \phi(V)^n \|_2 \leq e^{\mu_1}
\]

where

\[
(1.18) \quad \mu_1 = \frac{5a_1^2c + c^2}{|a_1|^3}.
\]

In the case \( \phi(V) = 1 - rV, r > 0 \), we have \( a_1 = -r \) and \( c = 0 \), so \( \mu = r/2 \) and \( \mu_1 = r. \) Therefore,

\[
(1.19) \quad \sup_n \| (1 - rV)^n \|_2 \leq e^{r/2},
\]

and

\[
(1.20) \quad \sup_n \sqrt{n} \| (1 - rV)^{n+1} - (1 - rV)^n \|_2 \leq e^r.
\]

The induction procedure from [18] based on Pedersen’s similarity only yields \( \exp([r] + 1) \) instead of \( \exp(r/2) \) in (1.19).

For the differences from (1.20) the rate \( \sqrt{n} \) of decay is exact [12]. In fact, this is true for every power bounded \( \phi(V) \) by the similarity from [3]. For example, the quantity \( \sqrt{n} \| \exp(-(n+1)V) - \exp(-nV) \|_2 \) stays in between some two positive constants. An upper constant is determined by (1.17) with \( \mu_1 = 1 + 5c + c^2 \) since \( a_1 = -1 \) in this case. To estimate this \( c \) we note that the series in (1.16) is of Leibniz’s type with \( a_k = (-1)^k/k! \). The sum of this series does not exceed the first term in modulus. This yields \( c \leq |a_2| + |a_3| = 2/3 \), thus \( \mu_1 \leq 43/9 \), and finally,

\[
\sqrt{n} \| \exp(-(n+1)V) - \exp(-nV) \|_2 \leq \exp(43/9) < 119.
\]

The case of alternating coefficients \( a_k \) merits a special attention since the following theorem can be proven in a very apparent way (see Section 5) that also yields an interesting upper bound.
**Theorem 1.4.** Let \( \phi \) be a polynomial,
\[
\phi(z) = 1 + \sum_{k=1}^{m} (-1)^k c_k z^k
\]
with all \( c_k > 0 \). Then
\[
(1.21) \quad \sup_{n} \|\phi(V)^n\|_2 \leq e^{1/2x_0}
\]
where
\[
(1.22) \quad x_0 = \sup\{x > 0 : \text{sign } \phi^{(k)}(x) = (-1)^k, 0 \leq k \leq m\}.
\]
Obviously, \( x_0 < \infty \) since \( \text{sign } \phi^{(k)}(\infty) = (-1)^m \), \( 0 \leq k \leq m \). If all roots of \( \phi(z) \) are real positive then \( x_0 = x_1 \), where \( x_1 \) is the smallest root, so
\[
(1.23) \quad \sup_{n} \|\phi(V)^n\|_2 \leq e^{1/2x_1},
\]
that is more concrete than (1.21).

**Example 1.5.** Let \( x_1^{(m)} \) be the smallest root of the \( m \)th Laguerre polynomial \( L_m(z) \), \( L_m(0) = 1 \). Then
\[
\sup_{n} \|L_m(V)^n\|_2 \leq e^{1/2x_1^{(m)}}.
\]
According to Theorem 6.31.3 from [17], we have
\[
x_1^{(m)} \geq \frac{j_1^2}{4m + 2}
\]
where \( j_1 \) is the smallest positive root of the Bessel function \( J_0(z) \). In its turn, \( j_1 > 3\pi/4 \) [17].

Let us emphasize that the bound (1.23) is applicable to any \( \phi \) which is a member of the system of polynomials orthogonal with a positive weight on an interval \( (0, v) \), \( 0 < v \leq \infty \). For instance, \( \phi \) can be a Jacobi polynomial modified by the linear transformation \( (-1, 1) \to (0, 1) \).

Theorem 1.1 yields a lot of remarkable corollaries, most of them simply by calculation of the corresponding derivatives at \( z = 0 \). For example, the derivatives of \( \phi(-z) \) and \( \phi(z)^{-1} \) at \( z = 0 \) are both equal to \( -\phi'(0) \) since \( \phi(0) = 1 \). This yields

**Corollary 1.6.** Each of the operators \( \phi(V)^{-1} \) and \( \phi(-V) \) is power bounded if and only if either \( p = 2 \) and \( \phi'(0) > 0 \) or \( \phi(V) \) is \( I \).

For instance, \( (I + rV)^{-1} \) in \( L_2 \) is power bounded if and only if \( r \geq 0 \).

**Corollary 1.7.** If \( \phi(V) \) is power bounded then \( \phi(rV) \) is power bounded for every \( r > 0 \).

**Corollary 1.8.** If \( \phi(V) \) is power bounded then so is \( \psi_s(V) = (1-s)I + s\phi(V) \) for all \( s \geq 0 \).
This statement can be used to immediately derive the estimate
\[ \| \phi(V)^{n+1} - \phi(V)^n \|_2 = O(1/\sqrt{n}) \]
from [14, Theorem 4.5.3] (cf. [18] where \( \phi(V) = I - V \)). In any case \( \phi(V) \) is assumed power bounded. By Theorem 1.2 the latter is necessary if the \( L_2 \)-norm of \( \phi(V)^{n+1} - \phi(V)^n \) is bounded or at least grows more slowly than every exponent \( \exp(n^\gamma), \gamma > 0 \).

The product of two commuting power bounded operators is always power bounded, though the latter may occur without power boundedness of the factors (cf. Remark 13 in [19]).

**Corollary 1.9.** For functions \( \phi_1(z) \) and \( \phi_2(z) \) such that \( \phi_1 \phi_2 \neq 1 \) the product \( \phi_1(V)\phi_2(V) \) is power bounded in \( L_p \) if and only if \( p = 2 \) and \( \phi'_1(0) + \phi'_2(0) < 0 \).

Hence, if \( \phi_1(V) \) and \( \phi_2(V) \) are not power bounded and \( \phi'_1(0) \) and \( \phi'_2(0) \) are real then either the product \( \phi_1(V)\phi_2(V) \) is not power bounded or it is \( I \).

**Corollary 1.10.** The quotient \( \phi_1(V)\phi_2(V)^{-1} \) of different functions \( \phi_1(z) \) and \( \phi_2(z) \) is power bounded in \( L_p \) if and only if \( p = 2 \) and \( \phi'_1(0) - \phi'_2(0) < 0 \).

Now we consider superpositions, the case most complicated for a direct analysis. Theorem 1.1 immediately yields

**Corollary 1.11.** Let \( \phi(V) \neq I \) be power bounded. Let \( \theta(w) \) be a non-constant analytic function in a neighborhood of \( w = 0 \) or \( w = 1 \) and \( \theta(0) = 0 \) or \( \theta(1) = 1 \), respectively. Then \( \phi(\theta(V)) \) or \( \theta(\phi(V)) \) is power bounded if and only \( \theta'(0) > 0 \) or \( \theta'(1) > 0 \), respectively.

For example, if \( \theta(0) = 0 \) then \( \exp(-\theta(V)) \) is power bounded if and only if \( \theta'(0) > 0 \), or \( \theta(w) \equiv 0 \). Another example: with \( \nu \in \mathbb{C} \) and with \( \phi(V) \) power bounded, \( \phi(V)\nu \) is power bounded if and only if \( \nu \) is real nonnegative.

**Theorem 1.12.** If in \( L_p \) the operator \( \phi(V) \neq I \) satisfies the strong Kreiss condition (1.8) at least at one point \( \lambda > 1 \) then \( p = 2 \) and \( \phi(V) \) is power bounded.

**Proof.** The inequality (1.8) just means that with \( |\lambda| > 1 \) the operator
\[ U = (1 - |\lambda|)R(\lambda; \phi(V)) \]
is power bounded. Accordingly, we set
\[ \theta(w) = (1 - \lambda)(w - \lambda)^{-1} \]
for a fixed \( \lambda > 1 \). Then \( U = \theta(\phi(V)) \) and \( \phi(V) = \chi(U) \) where \( \chi \) is the function inverse to \( \theta \). Obviously, \( \chi(1) = 1 \) and \( \chi'(1) = \lambda - 1 > 0 \). By Corollary 1.11 \( \phi(V) \) is power bounded, and by Theorem 1.1 \( p = 2 \).
Our further results related to the Kreiss and Ritt conditions are presented in the next section. In particular, we prove that the only Ritt operator \( \phi(V) \) in \( L_p \) is \( I \) (Corollary 2.5). In the polynomial case we characterize the Kreiss operators \( \phi(V) \) in \( L_p \) by the inequality \( \phi'(0) < 0 \) (Theorem 2.12).

2. The Ritt and Kreiss operators. It is convenient to reformulate the resolvent conditions as follows. For any operator \( T \) with \( \sigma(T) = \{ 1 \} \) we set \( A = T - I \) and \( \zeta = (\lambda - 1)^{-1} \). Then for \( \lambda \neq 1 \) we have

\[
R(\lambda; T) = (T - \lambda I)^{-1} = -\zeta \Phi(\zeta; A)
\]

where

\[
\Phi(\zeta; A) = (I - \zeta A)^{-1} = \sum_{n=0}^{\infty} \zeta^n A^n
\]

is the Fredholm resolvent of \( A \), an entire operator-valued function of \( \zeta \in \mathbb{C} \). Its exponential order is

\[
\omega = \omega(A) = \lim_{n \to \infty} \frac{\log n}{\log \sqrt[\zeta]{\|A^n\|}},
\]

so that

\[
\Phi(\zeta; A) = O(\exp|\zeta|^{\omega+\varepsilon})
\]

with any fixed \( \varepsilon > 0 \) (see [9, Section 1.3]).

If \( T \) is a Ritt operator then (1.5) can be extended (with another \( C \)) to a sector

\[
S_\delta = \{ \lambda \in \mathbb{C} : |\arg(\lambda - 1)| \leq \pi - \delta \}, \quad 0 < \delta < \pi/2
\]

(see [10] and [13]). In its turn, the sectorial Ritt condition implies power boundedness [8], [14]. The transformation \( \zeta = (\lambda - 1)^{-1} \) maps \( S_\delta \) onto itself. By (2.1) the sectorial Ritt condition for \( T \) becomes

\[
\|\Phi(\zeta; A)\| \leq C, \quad \zeta \in S_\delta.
\]

Lemma 2.1. If \( \sigma(A) = \{ 0 \} \), \( \omega(A) \leq 1 \), and \( T = I + A \) satisfies the Ritt condition then \( A = 0 \), i.e. \( T = I \).

Proof. The angle size of the complementary sector \( S'_\delta = \mathbb{C} \setminus S_\delta \) is \( 2\delta \), while \( \omega(A) < \pi/2\delta \). By (2.4) the Phragmén–Lindelöf Principle (see e.g. [9, Section 6.1]) yields \( \|\Phi(\zeta; A)\| \leq C \) for \( \zeta \in S'_\delta \). As a result, \( \Phi(\zeta; A) \) is bounded on the whole \( \mathbb{C} \). By the Liouville theorem \( \Phi(\zeta; A) = \text{const}. \) Then \( A = 0 \) by (2.2). 

The Kreiss condition (1.6) in terms of the Fredholm resolvent is

\[
\|\Phi(\zeta; A)\| \leq \frac{C}{|\zeta + 1| - |\zeta|}, \quad \text{Re} \zeta > -\frac{1}{2}.
\]
Lemma 2.2. If $\sigma(A) = \{0\}$, $\omega(A) < 1$ and $T = I + A$ satisfies the Kreiss condition then $A = 0$, i.e. $T = I$.

Proof. From (2.5) it follows that $\|\Phi(it; A)\| \leq O(|t|)$, $t \in \mathbb{R}$. The entire function

$$F(\zeta; A) = \frac{\Phi(\zeta; A) - I}{\zeta}$$

is of the exponential order $\omega(A) < 1$ and it is bounded on $i\mathbb{R}$. By the Phragmén–Lindelöf Principle this is bounded for $\text{Re} \zeta > 0$ and for $\text{Re} \zeta < 0$ separately. By the Liouville theorem $F(\zeta; A) = \text{const}$, i.e. $\Phi(\zeta; A)$ is a linear function of $\zeta$. However, $\|\Phi(t; A)\| \leq C$ for $t > 0$. Hence, $\Phi(\zeta; A) = \text{const}$.

Remark 2.3. In the case $\omega(A) < 1$ Lemma 2.1 follows from Lemma 2.2.

Actually, we are interested in

(2.6) \hspace{1cm} T = \phi(V) = I + aV^l + \sum_{k=l+1}^{\infty} a_k V^k,$

where $l \geq 1$, $a \neq 0$. Then

(2.7) \hspace{1cm} A = \phi(V) - I = aV^l Q,$

where

$$Q = I + \sum_{k=1}^{\infty} a^{-1} a_{k+1} V^k,$$

so that $\sigma(Q) = \{1\}$, thus the spectral radius $r(Q)$ equals 1.

Lemma 2.4. If $\phi(V) \neq I$ then in any $L_p$ the exponential order $\omega(\phi(V) - I)$ is equal to $1/l$ where $l$ is the multiplicity of the root $z = 0$ of the function $\phi(z) - 1$.

Proof. Since $\|Q^n\|^{1/n} \to 1$ as $n \to \infty$, and $Q$ commutes with $V$, we get $\omega(\phi(V) - I) \geq \omega(V^l)$ from (2.7) and (2.3). In fact, this is an equality since (2.7) can be rewritten as

$$V^l = a^{-1}(\phi(V) - I)Q^{-1}.$$ 

It remains to note that $\omega(V^l) = 1/l$ thanks to Stirling’s formula applied to the estimate

$$\frac{1}{n!(np + 1)^{1/p}} \leq \|V^n\|_p \leq \frac{1}{n!}$$

(cf. inequality (14) in [11]).

Combining this result with Lemma 2.1 we obtain

Corollary 2.5. In any $L_p$ the only Ritt operator $\phi(V)$ is $I$.

Similarly, Lemma 2.2 implies

Corollary 2.6. For $l > 1$ the operator $\phi(V) \neq I$ is not Kreiss in $L_p$. 
Remark 2.7. In particular, the operator $I - V$ is not Ritt. In contrast, $I - V^\alpha$, where

$$(V^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) \, dt,$$

is a Ritt operator in $L_p$ if $0 < \alpha < 1$. On the other hand, for $\alpha > 1$ this is not a Kreiss operator since $\omega(V^\alpha) = 1/\alpha$ for all $\alpha > 0$, hence, $\omega(V^\alpha) < 1$ if $\alpha > 1$. (For $\alpha = 2$ this was proven in \cite{18} by special considerations.)

Now we investigate the Kreiss condition in $L_p$ for

$$\phi(V) = I + \sum_{k=1}^m a_k V^k, \quad a_m \neq 0,$$

i.e. for $\phi(z)$ which is an arbitrary polynomial of degree $m \geq 1$. To this end we introduce the polynomial

$$(2.8) \quad \psi_\zeta(z) = z^m - \zeta \sum_{k=1}^m a_k z^{m-k}$$

depending on a complex parameter $\zeta$ and then consider the differential equation

$$(\psi_\zeta(D)g)(x) = f(x), \quad 0 \leq x \leq 1,$$

with $D = d/dx$ and $f \in L_p(0,1)$. Obviously, $DV = I$ and $(VDf)(x) = f(x)$ for $f$ absolutely continuous with $f(0) = 0$.

Denote by $Q(u; \zeta)$ the Cauchy function for the operator $\psi_\zeta(D)$, i.e. the solution of the differential equation

$$(2.9) \quad (\psi_\zeta(D)Q)(u; \zeta) = 0$$

under the initial conditions

$$(2.10) \quad Q^{(i)}(0; \zeta) = 0 \quad (0 \leq i \leq m - 2), \quad Q^{(m-1)}(0; \zeta) = 1.$$

Lemma 2.8. The Fredholm resolvent $\Phi(\zeta; A)$ of the operator $A = \phi(V) - I$ in $L_p(0,1)$ is the integral operator

$$(2.11) \quad (\Phi(\zeta; A)f)(x) = f(x) + \int_0^x Q^{(m)}(x - t; \zeta) f(t) \, dt, \quad 0 \leq x \leq 1.$$

Proof. One can assume $f \in C^m[0,1]$ and $f^{(i)}(0) = 0, 0 \leq i \leq m - 1$, since such functions constitute a dense subset of $L_p(0,1)$ and both sides of (2.11) are continuous operators in $L_p(0,1)$. Under this restriction formula (2.11) can be rewritten as

$$(2.12) \quad (\Phi(\zeta; A)f)(x) = \int_0^x Q(x - t; \zeta) f^{(m)}(t) \, dt, \quad 0 \leq x \leq 1,$$
by $m$ times integrating by parts. The right hand side $h(x)$ of (2.12) satisfies the equation

$$(\psi_\zeta(D)h)(x) = f^{(m)}(x), \quad 0 \leq x \leq 1,$$

i.e.

$$(2.13) \quad D^m h - \zeta \sum_{k=1}^{m} a_k D^{m-k} h = D^m f,$$

and, in addition, $h^{(i)}(0) = 0$, $0 \leq i \leq m - 1$. (This is true due to (2.9) and (2.10).) Applying $V^m$ to both sides of (2.13) we obtain

$$h - \zeta \sum_{k=1}^{m} a_k V^k h = f,$$

i.e. $h = (I - \zeta A)^{-1} f = \Phi(\zeta; A) f$. □

The polynomial $\psi_\zeta(z)$ is characteristic for the differential operator $\psi_\zeta(D)$. Let us investigate its roots $z_1, z_2, \ldots, z_m$ as $|\zeta| \to \infty$. For definiteness let $|z_1| \geq \max(|z_2|, \ldots, |z_m|)$. (If there are two or more roots with maximal modulus then $z_1$ may be any of them.) Note that all $z_i \neq 0$ since $\psi_\zeta(0) = -\zeta a_m \neq 0$.

**Lemma 2.9.** Let $a_1 \neq 0$. Then $|z_1| > \max(|z_2|, \ldots, |z_m|)$ for large $|\zeta|$, and

$$z_1 = a_1 \zeta + O(1), \quad \max(|z_2|, \ldots, |z_m|) = O(1).$$

**Proof.** According to (2.8) the equation $\psi_\zeta(z) = 0$ is equivalent to

$$(2.14) \quad \sum_{k=1}^{m} a_k w^k = \eta$$

with unknown $w = 1/z$ and parameter $\eta = 1/\zeta$. For $\zeta = \infty$ this turns into

$$(2.15) \quad \sum_{k=1}^{m} a_k w^k = 0.$$

One of the roots of (2.15) is $w_1 = 0$ and this root is simple since $a_1 \neq 0$. All other roots are separated from 0 by a circle $|w| = \delta$. By the Argument Principle all nonzero roots of (2.14) lie outside this circle as long as $|\eta| < \varepsilon$ and $\varepsilon$ is small enough. Let $|\zeta| > r \equiv 1/\varepsilon$. Then $|z_1| > 1/\delta$ but $|z_i| < 1/\delta$, $2 \leq i \leq m$. Now the relation

$$\sum_{i=1}^{m} z_i = a_1 \zeta$$

implies $|z_1 - a_1 \zeta| < (m - 1)/\delta$. □

From now on we assume $a_1 \neq 0$ and $|\zeta| > r$. Under these conditions $z_1$ is a unique root of maximal modulus, so it is a function of $\zeta$, $z_1 = z_1(\zeta)$.
**Corollary 2.10.** The coefficients of the polynomial
\[
\theta_\zeta(z) = \frac{\psi_\zeta(z)}{z - z_1(\zeta)} = \prod_{i=2}^{m}(z - z_i)
\]
are bounded functions of \(\zeta\).

The solution \(Q(u; \zeta)\) of (2.9) is of the form
\[
(2.16) \quad Q(u; \zeta) = C_1(\zeta)e^{z_1u} + R(u; \zeta)
\]
where the second term satisfies the equation
\[
(2.17) \quad (\theta_\zeta(D)R)(u; \zeta) = 0.
\]
In view of (2.10) and (2.16) the initial conditions for \(R\) are
\[
(2.18) \quad C_1(\zeta)z_1^i + R^{(i)}(0; \zeta) = 0, \quad 0 \leq i \leq m - 2,
\]
and
\[
(2.19) \quad C_1(\zeta)z_1^{m-1} + R^{(m-1)}(0; \zeta) = 1.
\]

From (2.17)–(2.19) it follows that
\[
(2.20) \quad C_1(\zeta)\theta_\zeta(z_1) = 1
\]
since the leading coefficient of \(\theta_\zeta(z)\) equals 1. However,
\[
\theta_\zeta(z_1) = \psi'_\zeta(z_1) = mz_1^{m-1} - \zeta \sum_{k=1}^{m} a_k(m - k)z_1^{m-k-1}
\]
according to (2.8). By Lemma 2.9
\[
\theta_\zeta(z_1) = (a_1\zeta)^{m-1} + O(|\zeta|^{m-2}),
\]
so (2.20) yields
\[
(2.21) \quad C_1(\zeta) = \frac{1}{(a_1\zeta)^{m-1}} + O\left(\frac{1}{|\zeta|^m}\right).
\]

**Lemma 2.11.** \(\max_{0\leq u \leq 1}|R^{(l)}(u; \zeta)| = O(|\zeta|^{-1}), \ l \geq 0.\)

**Proof.** Let \(E_\zeta\) be the evolutionary operator for the differential equation (2.17). This operator transforms the vector of initial conditions into the corresponding solution. Since the equation is linear, \(E_\zeta\) is linear. Actually, this is an isomorphism between the space of initial conditions and the space of solutions,
\[
R(\cdot; \zeta) = E_\zeta((R^{(i)}(0; \zeta))_{0}^{m-2}).
\]
Equipping these spaces with the corresponding sup-norms we get
\[
(2.22) \quad \max_{0\leq u \leq 1}|R(u; \zeta)| \leq \|E_\zeta\| \max_{0\leq i \leq m-2}|R^{(i)}(0; \zeta)|.
\]
The second factor on the right hand side of (2.22) is $O(|\zeta|^{-1})$ by (2.18), (2.21) and Lemma 2.9 while $\|E_\zeta\| = O(1)$ by Corollary 2.10. Thus,
\[
\max_{0 \leq u \leq 1} |R(u; \zeta)| = O(|\zeta|^{-1}).
\]

The same estimate is true for every derivative $R^{(l)}(u; \zeta)$, $l \geq 1$. Indeed, $R^{(l)}(u; \zeta)$ satisfies the same differential equation (2.17), which also determines its initial vector, as long as $R^{(i)}(0; \zeta)$ are given for $0 \leq i \leq m-2$. Thus,
\[
\|\left(\frac{R^{(l+i)}(0; \zeta)}{R^{(l)}(0; \zeta)}\right)_0^{m-2}\| = O(\|\left(\frac{R^{(i)}(0; \zeta)}{R^{(l)}(0; \zeta)}\right)_0^{m-2}\|), \quad l \geq 1,
\]
by Corollary 2.10 again. \(\square\)

Now we are in a position to prove our result concerning the Kreiss operators.

**Theorem 2.12.** In any $L_p$, in order for the operator
\[
(2.23) \quad \phi(V) = I + \sum_{k=1}^{m} a_k V^k, \quad m \geq 1, \quad a_m \neq 0,
\]
to be Kreiss it is necessary and sufficient that $a_1 < 0$.

**Proof of necessity.** Applying (2.11) to $f = 1$ we obtain
\[
(\Phi(\zeta; A)1)(x) = 1 + \int_0^x Q^{(m)}(x-t; \zeta) \, dt = Q^{(m-1)}(x; \zeta)
\]
since $Q^{(m-1)}(0; \zeta) = 1$. Furthermore,
\[
\int_0^1 Q^{(m-1)}(x; \zeta) \, dx = Q^{(m-2)}(1; \zeta)
\]
since $Q^{(m-2)}(0; \zeta) = 0$. Hence,
\[
|Q^{(m-2)}(1; \zeta)| \leq \int_0^1 |Q^{(m-1)}(x; \zeta)| \, dx = \int_0^1 |(\Phi(\zeta; A)1)(x)| \, dx.
\]
Using the Hölder inequality we obtain
\[
|Q^{(m-2)}(1; \zeta)| \leq \|\Phi(\zeta; A)1\|_p \leq \|\Phi(\zeta; A)\|_p.
\]
Thus, from (2.5) it follows that
\[
|Q^{(m-2)}(1; \zeta)| \leq \frac{C}{|\zeta + 1| - |\zeta|}, \quad \text{Re} \zeta > -\frac{1}{2}.
\]
This yields
\[
\exp(\text{Re}(a_1 \zeta)) = O\left(\frac{|\zeta|}{|\zeta + 1| - |\zeta|}\right), \quad \text{Re} \zeta > -\frac{1}{2},
\]
by (2.16), (2.21) and Lemmas 2.9 and 2.11. Obviously,
\[ \frac{|\zeta|}{|\zeta + 1| - |\zeta|} = \frac{\zeta(|\zeta+1|+|\zeta|)}{2\mathrm{Re}\zeta + 1} \leq \frac{|\zeta|(2|\zeta|+1)}{2\mathrm{Re}\zeta + 1}, \]
and \( \Re(a_1\zeta) = |\zeta|\Re(a_1\chi) \) where \( \chi = \frac{\zeta}{|\zeta|} \), so \( |\chi| = 1 \). Hence,
\[ \exp(|\zeta|\Re(a_1\chi)) = O\left(\frac{|\zeta|^2}{2|\zeta|\Re\chi + 1}\right). \]

Letting \( |\zeta| \to \infty \) we get \( \Re a_1 \leq 0 \) taking \( \chi = 1 \) and \( \Im a_1 = 0 \) taking \( \chi = \pm i \). Thus, \( a_1 \in \mathbb{R} \) and \( a_1 \leq 0 \). But \( a_1 \neq 0 \) by Corollary 2.6, hence, \( a_1 < 0 \). ■

Proof of sufficiency. From (2.11) it follows that
\[ \|\Phi(\zeta; A)\|_p \leq 1 + \int_0^1 |Q^{(m)}(u; \zeta)| \, du \]
in all \( L_p \), according to the well known estimate of the \( L^p \) norm of the convolution (see [20, Theorem 1.15] and [7, Lemma 23.16.1]).

By (2.16), (2.21) and Lemmas 2.9 and 2.11 again we have
\[ |Q^{(m)}(u; \zeta)| = O\left(|\zeta|e^{a_1\xi u} + \frac{1}{|\zeta|}\right), \]
where \( \xi = \Re\zeta > -1/2 \). Hence,
\[ \|\Phi(\zeta; A)\|_p = O\left(|\zeta|e^{a_1\xi} - 1\right) + 1\right). \]
On the other hand,
\[ |\zeta + 1| - |\zeta| = \frac{2\xi + 1}{|\zeta + 1| + |\zeta|} \in (0, 1]. \]
Thus, \( (|\zeta + 1| - |\zeta|)\|\Phi(\zeta; A)\|_p = O(M(\xi) + 1) \)
where
\[ M(\xi) = \frac{(2\xi + 1)(e^{a_1\xi} - 1)}{a_1\xi}. \]
Since \( a_1 < 0 \), this function is bounded on \((-1/2, \infty)\), so (2.5) follows immediately from (2.24). ■

In fact, the necessity part of Theorem 2.12 is true for all analytic \( \phi \). Indeed, if \( \phi(V) \) is a Kreiss operator then \( \|\phi(V)^n\|_p = O(n) \) and then \( a_1 \) must be real negative by Theorem 1.2 from [2].

Corollary 2.13. In \( L_2 \), if \( \phi(V) \) is a Kreiss operator then it is power bounded.
In $L_p$ with $p \neq 2$ this fails by Theorem 1.1. However, the conjecture saying that every Kreiss operator $\phi(V)$ is uniformly Kreiss seems to be plausible even if $p \neq 2$.

Perhaps, the sufficiency part of Theorem 2.12 can also be extended to the analytic situation but this requires a quite different approach.

3. The necessity in Theorem 1.1. In this section we resort to a “scaling”. All norms below are those of (1.4). For any $\epsilon$, $0 < \epsilon < 1$, the space $L_p(0, \epsilon)$ is naturally isometric to the subspace of those $f \in L_p(0, 1)$ which vanish for $x > \epsilon$. The operator $R : L_p(0, 1) \to L_p(0, \epsilon)$ such that $(Rf)(x) = f(x)$, $0 < x < \epsilon$, is the left inverse to the natural isometric embedding $E : L_p(0, \epsilon) \to L_p(0, 1)$. Denote by $V_\epsilon$ the same integration (1.1) but for $f \in L_p(0, \epsilon)$. Then $V_\epsilon R = RV$, whence $\phi(V_\epsilon)R = R\phi(V)$ for all functions $\phi$ under consideration. Hence, $\phi(V_\epsilon) = R\phi(V)E$, which yields

$$\|\phi(V_\epsilon)\| \leq \|\phi(V)\|$$

since $\|R\| = 1$ and $\|E\| = 1$.

Now let $S$ be the operator $L_p(0, \epsilon) \to L_p(0, 1)$ defined as $(Sf)(x) = f(\epsilon x)$, $0 < x < 1$. Then

$$\|Sf\|^p = \int_0^1 |f(\epsilon x)|^p \, dx = \frac{1}{\epsilon} \int_0^\epsilon |f(t)|^p \, dt = \frac{1}{\epsilon} \|f\|^p,$$

which means that $S_\epsilon = \epsilon^{1/p} S$ is an isometry. Also we have $S_\epsilon V_\epsilon = \epsilon V S_\epsilon$. Indeed,

$$(SV_\epsilon f)(x) = \int_0^x f(t) \, dt = \epsilon \int_0^x f(\epsilon s) \, ds = (\epsilon VS_\epsilon f)(x), \quad 0 \leq x \leq 1.$$ 

Now $S_\epsilon \phi(V_\epsilon) = \phi(\epsilon V)S_\epsilon$ yields

$$\|\phi(V_\epsilon)\| = \|\phi(\epsilon V)\|.$$ 

Combining (3.1) and (3.2) we obtain

$$\|\phi(\epsilon V)\| \leq \|\phi(V)\|, \quad 0 < \epsilon < 1.$$ 

This results in the following important

**Lemma 3.1.** If $\phi(V)$ is power bounded then the family $\{\phi(\epsilon V) : 0 < \epsilon < 1\}$ is uniformly power bounded, i.e.

$$\sup\{\|\phi(\epsilon V)^n\|_p : 0 < \epsilon < 1, n \geq 0\} < \infty.$$ 

Now we turn to the decomposition (2.6) with $l = 1$, i.e.

$$\phi(V) = I + aV + \sum_{k=2}^{\infty} a_k V^k.$$
Here \( a \neq 0 \), otherwise \( \phi(V) \) would not be Kreiss by Corollary 2.6, while \( \phi(V) \) is power bounded by assumption. By Lemma 3.1 with \( \varepsilon = \tau/n \), \( 0 < \tau < n \), we obtain

\[
\sup_{\tau > 0} \sup_{n > \tau} \left\| \left( I + \frac{a \tau V}{n} + O \left( \frac{\tau^2}{n^2} \right) \right)^n \right\| < \infty.
\]

Passing to the limit as \( n \to \infty \) with \( \tau \) fixed, we get

\[
(3.3) \quad \sup_{\tau > 0} \| \exp(a \tau V) \| \equiv M < \infty.
\]

By the classical resolvent criterion [7, Theorem 12.31], (3.3) implies

\[
(3.4) \quad \| R(\lambda; aV)^n \| \leq \frac{M}{(\text{Re} \lambda)^n}, \quad \text{Re} \lambda > 0, \quad n \geq 1,
\]

in particular,

\[
(3.5) \quad \| R(\lambda; aV) \| \leq \frac{M}{\text{Re} \lambda}, \quad \text{Re} \lambda > 0.
\]

However, the function \( g = R(\lambda; aV)1 \) is nothing but the solution of the integral equation

\[
a \int_0^x g(t) \, dt - \lambda g(x) = 1, \quad 0 \leq x \leq 1,
\]

or, equivalently, of the differential equation \( \lambda g'(x) - ag(x) = 0 \) with the initial condition \( g(0) = -1/\lambda \). Therefore,

\[
g(x) = -\frac{1}{\lambda} \exp \left( \frac{ax}{\lambda} \right),
\]

whence

\[
\int_0^1 g(x) \, dx = \frac{1}{a} \left( 1 - \exp \left( \frac{a}{\lambda} \right) \right),
\]

and, on the other hand,

\[
\left| \int_0^1 g(x) \, dx \right| \leq \| g \| \leq \frac{M}{\text{Re} \lambda}, \quad \text{Re} \lambda > 0,
\]

by (3.5). Thus,

\[
\left| 1 - \exp \left( \frac{a}{\lambda} \right) \right| \leq \frac{Ma}{\text{Re} \lambda}, \quad \text{Re} \lambda > 0.
\]

Setting \( \lambda = 1/\zeta \), \( \text{Re} \zeta > 0 \), we get

\[
|\exp(a\zeta)| \leq 1 + \frac{Ma|\zeta|^2}{\text{Re} \zeta}, \quad \text{Re} \zeta > 0.
\]
Letting $\zeta \in \mathbb{R}$, $\zeta \to +\infty$, we see that $\text{Re} \ a \leq 0$. On the other hand, for $\zeta = 1 + i\omega$, $\omega \in \mathbb{R}$, we have

$$\exp(\text{Re} \ a - \omega \text{Im} \ a) \leq 1 + M|a|(|\omega|^2 + 1).$$

With $\omega \to \pm \infty$ we obtain $\text{Im} \ a = 0$. Since $a \neq 0$, we conclude that $a < 0$.

Now we return to (3.4). For $\lambda = |a|$ this yields the power boundedness of $(I + V)^{-1}$ and then the power boundedness of $I - V$ by the Pedersen similarity. This implies $p = 2$ according to [12, Theorem 1.1].

The sufficiency in Theorem [1.1] is contained in Theorem [1.3] which we prove in the next section.

4. Proof of Theorem [1.3] Our main tool in this proof is the Laplace transform. To apply the latter we start with $\phi(V)$ in the form (1.11) and extend it to $x > 1$ as follows. We set

$$\phi(V)(x) = f(x) + \int_{0}^{x} k(x - t) f(t) \, dt, \quad 0 \leq x < \infty,$$

where $k(u) = K(u)$ for $0 \leq u \leq 1$ and $k(u) = K'(1)(u - 1) + K(1)$ for $u > 1$. The operator $W$ acts in the linear space $\Lambda$ of locally $L_2$-functions whose integral over $(0, x)$ grows no faster than polynomially as $x \to \infty$. Obviously, for all $n$ we have

$$\phi(V)^n(x) = (\phi(V)^n f)(x), \quad 0 \leq x \leq 1.$$

The Laplace transform of $k(u)$,

$$\tilde{k}(\lambda) = \int_{0}^{\infty} k(u)e^{-\lambda u} \, du,$$

is a regular analytic function in the half-plane $\text{Re} \ \lambda > 0$, and the same is true for all $f \in \Lambda$, thus for all $W^n f$, $n \geq 1$.

From (4.1) it follows that

$$\tilde{W}f(\lambda) = (1 + \tilde{k}(\lambda))\tilde{f}(\lambda), \quad \text{Re} \ \lambda > 0,$$

by the usual convolution rule. Now it is convenient to introduce the function

$$\psi(z) = 1 + \tilde{k}(1/z), \quad \text{Re} \ z > 0.$$

Then (4.4) takes the form

$$\tilde{W}f(\lambda) = \psi(1/\lambda)\tilde{f}(\lambda),$$

and, by iteration,

$$\tilde{W}^n f(\lambda) = \psi^n(1/\lambda)\tilde{f}(\lambda), \quad \text{Re} \ \lambda > 0, \ n \geq 1.$$
Integrating two times by parts in (4.3) and taking into account our definition of \( k(u) \) we obtain
\[
\tilde{k}(\lambda) = \frac{K(0)}{\lambda} + \frac{1}{\lambda^2} \left( K'(0) + \int_0^1 K''(u)e^{-\lambda u} \, du \right).
\]
Accordingly,
\[
\psi(z) = 1 + a_1 z + R(z)z^2
\]
where \( a_1 = K(0) < 0 \) and
\[
R(z) = K'(0) + \int_0^1 K''(u)e^{-u/z} \, du.
\]
Since \( \Re z > 0 \), we have
\[
|R(z)| \leq c = |K'(0)| + \int_0^1 |K''(u)| \, du.
\]

In the classical inversion formula for the Laplace transform the latter is a factor in the integrand when integrating along the vertical line \( \{ \lambda : \Re \lambda = \mu \} \) with any fixed \( \mu > 0 \). In view of (4.5) we have to investigate \( \psi(z) \) on the image of this line under the mapping \( z = 1/\lambda \). This is the circle
\[
Punctured at \ z = 0, \text{ but the latter "singularity" can be removed by setting } \psi(0) = 0. \text{ Since } a_1 \text{ is real, we have }
\[
|\psi(z)|^2 = 1+(2a_1 \mu + a_1^2)|z|^2 + 2 \Re(R(z)z^2) + 2a_1 \Re(R(z)z|z|^2) + |R(z)|^2|z|^4.
\]
By (4.7)
\[
|\psi(z)|^2 \leq 1 - M(\mu)|z|^2, \quad z \in C_\mu,
\]
with
\[
M(\mu) = 2|a_1|\mu - \left( a_1^2 + 2c + \frac{2c|a_1|}{\mu} + \frac{c^2}{\mu^2} \right)
\]
since \( a_1 < 0 \) and \( |z| \leq 1/\mu \) for \( z \in C_\mu \). The continuous function \( M(\mu) \), \( \mu > 0 \), is increasing and \( M(+0) = -\infty, \ M(+\infty) = +\infty \). Hence, it has a unique root \( \mu_0 \) and \( M(\mu) > 0 \) if \( \mu > \mu_0 \). By (4.8) and (4.5) we obtain the following key lemma:

**Lemma 4.1.** If \( \mu \geq \mu_0 \) then
\[
|(\hat{W}^n f)(\lambda)| \leq |\hat{f}(\lambda)|, \quad \Re \lambda = \mu.
\]
Indeed, in this case \( |\psi(z)| \leq 1 \) for \( z \in C_\mu \).
The function \((\tilde{W}^n f)(\mu + i\omega), \omega \in \mathbb{R}\), is the Fourier image of \((W^n f)(x)e^{-\mu x}\) extended by zero to \(x < 0\). By the Parseval equality and inequality (4.10),

\[
\begin{align*}
\int_0^\infty |(W^n f)(x)|^2 e^{-2\mu x} \, dx &= \frac{1}{2\pi} \int_{-\infty}^\infty |(\tilde{W}^n f)(\mu + i\omega)|^2 \, d\omega \\
&\leq \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{f}(\mu + i\omega)|^2 \, d\omega \\
&= \int_0^\infty |f(x)|^2 e^{-2\mu x} \, dx.
\end{align*}
\]

A fortiori,

\[
\int_0^1 |(W^n f)(x)|^2 e^{-2\mu x} \, dx \leq \int_0^\infty |f(x)|^2 \, dx,
\]

and finally,

\[
\int_0^1 |(W^n f)(x)|^2 dx \leq e^{2\mu} \int_0^\infty |f(x)|^2 \, dx.
\]

In particular, one can take any \(f \in L_2(0, 1)\) and extend it by zero to \(x > 1\). Then (4.12) takes the form

\[
\int_0^1 |(W^n f)(x)|^2 dx \leq e^{2\mu} \int_0^1 |f(x)|^2 \, dx.
\]

In view of (4.2) this inequality is actually

\[
\int_0^1 |(\phi(V)^n f)(x)|^2 dx \leq e^{2\mu} \int_0^1 |f(x)|^2 \, dx,
\]

i.e.

\[
\|(\phi(V)^n f)\|_2 \leq e^{\mu}.
\]

This is nothing but (1.14) with \(\mu\) determined by (1.15) and \(c\) as in (1.16). To show this, note that \(M(|a_1|/2) < 0\). Hence, every \(\mu\) such that \(M(\mu) \geq 0\) is \(\geq |a_1|/2\), i.e.

\[
\mu = |a_1|/2 + \delta, \quad \delta > 0.
\]

If \(|a_1|\delta - (3c + 2c^2/a_1^2) \geq 0\) then \(M(\mu) \geq 0\). Thus, one can take

\[
\delta = \frac{3ca_1^2 + 2c^2}{|a_1|^3}
\]

in order to get (1.15). The value \(c\) given by (1.16) appears as a result of the substitution of \(K(u)\) from (1.12) into (4.7).
The estimate (1.17) can be obtained similarly but with $\psi^n(\psi - 1)$ instead of $\psi^n$. In this case for $z \in C_\mu$ we have $|\psi(z)|^2 - 1 \leq |\psi(z)|^2$ if $\mu$ is chosen so that $|\psi(z)|^2 \leq \Re \psi(z)$. For this inequality it suffices to have $M(\mu) \geq |a_1|\mu + c$ thanks to (4.6) and (4.8). In this case we set $\mu = |a_1| + \delta$ instead of (4.13). This yields (1.18) in the same way as we obtained (1.14). It remains to note that $|\psi(z)|^{2n}|\psi(z) - 1|^2$ is bounded from above by

$$\max\{u^n(1 - u) : 0 \leq u \leq 1\} = \left(\frac{n}{n + 1}\right)^n \cdot \frac{1}{n + 1} < \frac{1}{en}.$$ 

Theorem 1.3 is proven. Moreover, literally the same proof yields the following

**Theorem 4.2.** Let $q(u)$, $0 \leq u \leq 1$, be a complex-valued function with absolutely continuous first derivative, and assume $q(0)$ is real negative. Then the integral operator

$$(Tf)(x) = f(x) + \int_0^x q(x - t)f(t) \, dt, \quad 0 \leq x \leq 1,$$

in $L^2(0, 1)$ is power bounded, and moreover,

$$\sup_n \|T^n\|_2 \leq e^\mu$$

where

$$\mu = \frac{|q(0)|}{2} + \frac{3cq(0)^2 + 2c^2}{|q(0)|^3}$$

and

$$c = |q'(0)| + \int_0^1 |q''(u)| \, du.$$ 

Furthermore,

$$\sup_n \sqrt{n} \|T^{n+1} - T^n\|_2 \leq e^{\mu_1}$$

where

$$\mu_1 = |q(0)| + \frac{5cq(0)^2 + c^2}{|q(0)|^3}.$$ 

If $q(u)$ is convex and nondecreasing then $c = q'(1)$.

Note that the conditions on the kernel $q(u)$ in Theorem 4.2 are weaker than those of [3] which provide the similarity to $I + q(0)V$.

5. Alternating coefficients. We start with the proof of Theorem 1.4. To this end we proceed to Taylor’s expansion

$$\phi(z) = \sum_{k=0}^m \frac{\phi^k(x)}{k!} (z - x)^k.$$
with $x > 0$ such that $\text{sign } \phi^{(k)}(x) = (-1)^k$, $0 \leq k \leq m$, so $x < x_0$. Then

$$\phi(z) = \sum_{k=0}^{m} p_k (1 - z/x)^k$$

where

$$p_k = \frac{\phi^{(k)}(x)(-x)^k}{k!}, \quad 0 \leq k \leq m.$$ 

Obviously, all $p_k > 0$ and

$$\sum_{k=0}^{m} p_k = \phi(0) = 1.$$ 

Furthermore,

$$\phi^n(z) = \sum_{l=0}^{mn} \left( \sum_{k_1 + \cdots + k_n = l} p_{k_1} \cdots p_{k_n} \right) (1 - z/x)^l.$$ 

Hence,

$$\sup_n \| \phi^n(V) \| \leq \sup_l \left\| \left( I - \frac{1}{x} V \right)^l \right\|$$

irrespectively of the choice of the norm. In particular,

$$\sup_n \| \phi^n(V) \|_2 \leq e^{1/(2x)}$$

by (1.19). It remains to optimize this bound by passing to $x = x_0$.

Now we denote by $A_m$ the set of real polynomials $\phi(z)$ of degree $m$ with $\phi(0) = 1$ and alternating coefficients. Obviously, $A_m$ is convex. Furthermore, the product $A_m A_s = \{ \phi_1 \phi_2 : \phi_1 \in A_m, \phi_2 \in A_s \}$ is contained in $A_{m+s}$. Indeed, $\phi \in A_m$ if and only if $\deg \phi = m$, $\phi(0) = 1$, and all coefficients of $\phi(-z)$ are positive.

If $\phi$ is real and all roots of $\phi$ lie in the open right half-plane $H_+ = \{ z : \text{Re } z > 0 \}$ then $\phi \in A_m$. (The converse is also true if $m \leq 2$.) Such $\phi$ can be called anti-Hurwitz polynomials since this is exactly the case when $\phi(-z)$ satisfies Hurwitz’s determinant condition for the roots to lie in $H_- = \{ z : \text{Re } z < 0 \}$ (see e.g. [4]). The role of the “Hurwitz polynomials” in the classical stability theory is well known.

**Corollary 5.1.** If $\phi(z)$ is an anti-Hurwitz polynomial then (1.21) holds with $x_0$ defined in (1.22).

Indeed, $\phi$ is the product of a number of polynomials, each from $A_1$ or $A_2$, thus $\phi \in A_m$, $m = \deg \phi$.

**Corollary 5.2.** Let $\phi(z)$ be a complex polynomial with real coefficient $a_1$. If all roots $z_1, \ldots, z_m$ of $\phi(z)$ lie in $H_+$ then $\phi(V)$ is power bounded in $L_2$. 
Indeed, in this case

\[ a_1 = \text{Re} \ a_1 = - \sum_{i=1}^{m} \text{Re} \left( \frac{1}{z_i} \right) = - \sum_{i=1}^{m} \frac{\text{Re} \ z_i}{|z_i|^2} < 0. \]

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