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# Approximation theorem for evolution operators

by

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**Abstract.** This paper is devoted to the study of the approximation problem for the abstract hyperbolic differential equation $u'(t) = A(t)u(t)$ for $t \in [0,T]$, where $\{A(t) : t \in [0,T]\}$ is a family of closed linear operators, without assuming the density of their domains.

**1. Introduction and the statement of the main theorem.** In this paper we discuss approximation of evolution operators associated with the initial value problem

$$\begin{cases} u'(t) = A(t)u(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in a general Banach space $X$ with norm $\|\cdot\|$. Here $\{A(t) : t \in [0,T]\}$ is a family of closed linear operators in $X$ with $D(A(t)) = Y$ for $t \in [0,T]$, where $Y$ is another Banach space with norm $\|\cdot\|_Y$, which is continuously imbedded in $X$.

Let $D$ be a subspace of $X$. By an *evolution operator on $D$ generated by* $\{A(t) : t \in [0,T]\}$ we mean the two-parameter family $\{U(t,s) : (t,s) \in \Delta\}$, where $\Delta = \{(t,s) : 0 \le s \le t \le T\}$, given by

$$U(t,s)z = \lim_{\lambda \downarrow 0} \prod_{i=[s/\lambda]+1}^{[t/\lambda]} (I - \lambda A(i\lambda))^{-1} z \quad \text{for } z \in D \text{ and } (t,s) \in \Delta,$$

which satisfies the following three conditions:

(i) $U(t,s) : D \to D$ for $(t,s) \in \Delta$.

(ii) $U(t,t)z = z$ and $U(t,r)U(r,s)z = U(t,s)z$ for $z \in D$ and for $(r,s), (t,r) \in \Delta$.

(iii) The mapping $(t,s) \mapsto U(t,s)z$ is continuous on $\Delta$, for any $z \in D$.

The class of evolution operators mentioned above provides us with mild solutions of (1.1). It should be noted that $Y$ is not assumed to be dense

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in $X$. The study of (1.1) in such situations was initiated by Da Prato and Sinestrari [1], and continued intensively by Tanaka [7].

We are interested in studying approximation of an evolution operator by a sequence $\{\prod_{k=[s/\tau_n]+1}^{[t/\tau_n]} F_n(k\tau_n)\}$ of discrete parameter evolution operators. Here $\{\tau_n\}$ is a positive sequence with $\lim_{n\to\infty} \tau_n = 0$ and $F_n(t)$ is a bounded linear operator on a Banach space $X_n$ with norm $\|\cdot\|_n$, where $\{X_n\}$ approximates $X$ in the following sense: For each $n \geq 1$ there exists a bounded linear operator $P_n$ from $X$ to $X_n$ such that

$$\lim_{n\to\infty} \|P_n u\|_n = \|u\| \quad \text{for every } u \in X. \tag{1.2}$$

The notion of approximation sequences $\{X_n, P_n\}$ is due to Trotter [8]. Such approximation problems arise when the solution of a differential equation whose coefficients depend on time is computed numerically by a finite difference method. In the case where $A(t)$ is independent of $t$ and $Y$ is dense in $X$, some interesting results for the approximation stated above were obtained by Kurtz [4]. (See also [2] and [6].) We note that property (1.2) implies the existence of a constant $K$ such that

$$\|P_n u\|_n \leq K\|u\| \quad \text{for } u \in X \text{ and } n \geq 1. \tag{1.3}$$

We also use the notation $\lim_{n\to\infty} u_n = u$, $u_n \in X_n$, $u \in X$, which means $\lim_{n\to\infty} \|u_n - P_n u\|_n = 0$.

To state the main result of this paper we need the notions of stability of $\{F_n(t) : t \in [0,T]\}$ and of convergence of a sequence of operators. The family $\{F_n(t) : t \in [0,T]\}$ is said to be *stable for time scale* $\tau_n \to 0$ if there exist $M \geq 1$ and $\omega \geq 0$, independent of $n$, such that

$$\Big\| \prod_{k=1}^{m} F_n(t_k) \Big\|_n \leq M e^{\omega \tau_n m}$$

for every finite sequence $\{t_k\}_{k=1}^m$ with $0 \leq t_1 \leq \ldots \leq t_m \leq T$ and $m = 1, 2, \ldots$ Here and below we use the conventions $\prod_{k=p}^{i+1} T_k = T_{i+1}(\prod_{k=p}^{i} T_k)$ if $i \geq p$ and $\prod_{k=p}^{i} T_k = I$ if $i < p$. We call $\{M, \omega\}$ the *stability constant.* We set

$$A_n(t) = \frac{F_n(t) - I}{\tau_n} \quad \text{for } t \in [0,T] \text{ and } n \geq 1.$$

We write $A(t) \subset \liminf_{n\to\infty} A_n(t)$ for $t \in [0,T]$ if for each $y \in Y$ there exist $y_n \in X_n$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} A_n(t) y_n = A(t) y$ for all $t \in [0,T]$.

We are now in a position to state the main result in this paper.

Main Theorem. *Assume that* $\{F_n(t) : t \in [0,T]\}$ *is stable, with stability constant* $\{M, \omega\}$, *for time scale* $\tau_n \to 0$, *and satisfies the condition*

(a) *there is a continuous function* $f : [0,T] \to X$ *which is of bounded variation on* $[0,T]$ *such that for* $t, s \in [0,T]$, $x \in X_n$ *and* $n \geq 1$,

$$\|A_n(t)x - A_n(s)x\|_n \leq \|f(t) - f(s)\|(\|x\|_n + \|A_n(s)x\|_n). \tag{1.4}$$

*Assume that for all* $t \in [0,T]$,

(b) $(\lambda_0 I - A(t))Y$ *is dense in* $X$ *for some* $\lambda_0 > \omega$.

*Then, if* $A(t) \subset \liminf_{n\to\infty} A_n(t)$ *for* $t \in [0,T]$ *then the family* $\{A(t) : t \in [0,T]\}$ *generates an evolution operator* $\{U(t,s) : (t,s) \in \Delta\}$ *on* $\overline{Y}$ *such that for each* $y \in \overline{Y}$ *and* $0 \leq s \leq t \leq T$,

$$\lim_{n\to\infty} \prod_{k=[s/\tau_n]+1}^{[t/\tau_n]} F_n(k\tau_n)P_n y = U(t,s)y, \tag{1.5}$$

*where the convergence is uniform on the triangle* $\Delta$.

Corollary. *Let* $\{h_n\}$ *be a null sequence and let* $\{T_n\}$ *be a family with* $T_n \in B(X_n)$ *satisfying the condition that there exist* $M \geq 1$ *and* $\omega \geq 0$ *such that*

$$\|T_n^k\|_n \leq M e^{\omega k h_n} \quad \text{for } k \geq 1 \text{ and } n \geq 1.$$

*Let* $A_n = (T_n - I)/h_n$ *for* $n \geq 1$, *and let* $A$ *be a closed linear operator in* $X$ *such that the range* $R(\lambda_0 I - A)$ *of* $\lambda_0 I - A$ *is dense in* $X$ *for some* $\lambda_0 > \omega$. *If* $A \subset \liminf_{n\to\infty} A_n$ *then the part of* $A$ *into* $\overline{D(A)}$ *is the infinitesimal generator of a* $(C_0)$*-semigroup* $\{T(t) : t \geq 0\}$ *on* $\overline{D(A)}$ *such that*

$$T(t)x = \lim_{n\to\infty} T_n^{[t/h_n]} P_n x \quad \text{for } x \in \overline{D(A)} \text{ and } t \geq 0,$$

*where the limit is uniform on every compact subinterval of* $[0,\infty)$.

*Proof.* By the Main Theorem, there exists a $(C_0)$-semigroup $\{T(t) : t \geq 0\}$ on $\overline{D(A)}$ given by the formula

$$T(t)x = \lim_{\lambda\downarrow 0} (I - \lambda A)^{-[t/\lambda]} x \quad \text{for } x \in \overline{D(A)} \text{ and } t \geq 0, \tag{1.6}$$

where the limit is uniform on every compact subinterval of $[0,\infty)$. We only have to show that the part of $A$ into $\overline{D(A)}$ is the infinitesimal generator of a $(C_0)$-semigroup $\{T(t) : t \geq 0\}$ on $\overline{D(A)}$. For this purpose, we denote the part of $A$ into $\overline{D(A)}$ by $\widetilde{A}$. By (1.6), we have

$$T(t)x - x = A\int_0^t T(r)x\,dr \quad \text{for } x \in \overline{D(A)} \text{ and } t \geq 0, \tag{1.7}$$

$$T(t)x - x = \int_0^t T(r)\widetilde{A}x\,dr \quad \text{for } x \in D(\widetilde{A}) \text{ and } t \geq 0. \tag{1.8}$$

Let $\widehat{A}$ be the infinitesimal generator of $\{T(t) : t \geq 0\}$ on $\overline{D(A)}$. If $x \in D(\widehat{A})$ then it follows from the closedness of $A$ that $x \in D(A)$ and $Ax = \widehat{A}x \in \overline{D(A)}$, by dividing (1.7) by $t$ and letting $t \downarrow 0$; hence $\widehat{A} \subset \widetilde{A}$. Conversely, let $x \in D(\widetilde{A})$. By the strong continuity of $T(t)$ the limit $\lim_{t\downarrow 0}(T(t)x - x)/t$ exists and equals $\widetilde{A}x$, by (1.8). This means that $x \in D(\widehat{A})$. It is thus proved that $\widetilde{A} = \widehat{A}$. ∎

Remark. If $B := \liminf_{n\to\infty} A_n$ has the property that $D(B)$ is dense in $X$ and $R(\lambda_0 I - B)$ is dense in $X$ for some $\lambda_0 > \omega$, then we can apply the Corollary with $A = B$ to prove the sufficiency of Kurtz's theorem [4, Theorem 2.13]. Kurtz's theorem improved Trotter's theorem [8, Theorem 5.3] by extending the notion of the limit of a sequence of operator used by Trotter to the notion of extended limit. Our main results give an extension of their results in this sense.

In Section 2 we prove that the family $\{A(t) : t \in [0, T]\}$ generates an evolution operator on $\overline{Y}$. Section 3 contains the proof of the convergence (1.5). For simplicity, we use the notation $N_\lambda = [T/\lambda]$ and $t_i^\lambda = i\lambda$ for $\lambda > 0$, and $J_n^\lambda(t) = (I - \lambda A_n(t))^{-1}$ for $t \in [0, T]$ and $\lambda > 0$ with $\lambda\omega_n < 1$, and $J^\lambda(t) = (I - \lambda A(t))^{-1}$ for $t \in [0, T]$ and $\lambda > 0$ with $\lambda\omega < 1$.

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**2. Existence of evolution operators.** We begin by introducing the notion of stability of the family $\{A(t) : t \in [0, T]\}$ in order to state the generation theorem for evolution operators. The family $\{A(t) : t \in [0, T]\}$ is said to be *stable* with *stability constant* $\{M, \omega\}$ if $(\omega, \infty) \subset \varrho(A(t))$ for $t \in [0, T]$ and

$$\Big\| \prod_{k=1}^{m} (\lambda I - A(t_k))^{-1} \Big\| \leq M(\lambda - \omega)^{-m} \quad \text{for } \lambda > \omega$$

and every finite sequence $\{t_k\}_{k=1}^m$ such that $0 \leq t_1 \leq \ldots \leq t_m \leq T$ and $m \geq 1$. For brevity, we then write $\{A(t) : t \in [0, T]\} \in S_\sharp(X, M, \omega)$.

Proposition 2.1. *Assume that the family* $\{A(t) : t \in [0, T]\}$ *satisfies the following two conditions*:

($a_1$) $\{A(t) : t \in [0, T]\} \in S_\sharp(X, M, \omega)$;

($a_2$) $\|A(t)x - A(s)x\| \leq \|f(t) - f(s)\|(\|x\| + \|A(s)x\|)$ *for* $t, s \in [0, T]$ *and* $x \in Y$.

*Then* $\{A(t) : t \in [0, T]\}$ *generates an evolution operator* $\{U(t, s) : (t, s) \in \Delta\}$ *on* $\overline{Y}$.

Once the following lemma is proved, Proposition 2.1 can be obtained just as in the proof of Tanaka [7, Theorem 1.5].

LEMMA 2.2. *Assume that all assumptions of Proposition* 2.1 *are satisfied. Then*

$$\Big\|A(t_j^\mu)\prod_{k=q+1}^{j} J^\mu(t_k^\mu)x\Big\| \le \overline{M}(\sup_{t\in[0,T]}\|A(t)x\| + \|x\|) \tag{2.1}$$

*for* $q \ge 0$, $\mu > 0$ *with* $\mu\omega \le 1/2$, $0 \le q \le j \le N_\mu$ *and* $x \in Y$, *where* $\overline{M} = M^2(V_f + 1)\exp(2\omega T + MV_f)$, $V_f$ *being the total variation of* $f$ *over* $[0,T]$.

*Proof.* Let $x \in Y$ and $\mu > 0$ be such that $\mu\omega \le 1/2$. Fix $q$ and $j$ arbitrarily so that $0 \le q \le j \le N_\mu$, and set $a_l^\mu = \|A(t_l^\mu)\prod_{k=q+1}^{l} J^\mu(t_k^\mu)x\|$ for $q \le l \le j$. Similarly to the proof of Tanaka [7, Lemma 1.2], we find that

$$(1-\mu\omega)^{l-q}a_l^\mu \le M\|A(t_q^\mu)x\| + \sum_{i=q}^{l-1} M\|f(t_{i+1}^\mu) - f(t_i^\mu)\|(M\|x\| + (1-\mu\omega)^{i-q}a_i^\mu)$$

for $q \le l \le j$. Denoting the right-hand side of this inequality by $b_l^\mu$, we see that

$$(1-\mu\omega)^{l-q}a_l^\mu \le b_l^\mu \quad \text{for } q \le l \le j$$

and

$$b_{l+1}^\mu \le M^2\|f(t_{l+1}^\mu) - f(t_l^\mu)\|\,\|x\| + \exp(M\|f(t_{l+1}^\mu) - f(t_l^\mu)\|)b_l^\mu \quad \text{for } q \le l \le j-1.$$

Solving this inequality with the first term $b_q^\mu = M\|A(t_q^\mu)x\|$, we find

$$b_j^\mu \le M^2(V_f + 1)\exp(MV_f)(\sup_{t\in[0,T]}\|A(t)x\| + \|x\|)$$

for $q \le j \le N_\mu$. Here we have used the following fact: If $a_i \le b_i + c_i a_{i-1}$ for $p+1 \le i \le r$, then

$$a_i \le \sum_{k=p+1}^{i}\Big(b_k\prod_{j=k+1}^{i} c_j\Big) + \Big(\prod_{k=p+1}^{i} c_k\Big)a_p \quad \text{for } p \le i \le r. \tag{2.2}$$

Since $a_j^\mu \le e^{2\omega T}b_j^\mu$, by using the fact that $(1-t)^{-1} \le e^{2t}$ for $0 \le t \le 1/2$, we obtain the desired estimate (2.1). ∎

In the rest of this section we prove that the family $\{A(t) : t \in [0,T]\}$ generates an evolution operator $\{U(t,s) : (t,s) \in \Delta\}$ on $\overline{Y}$. We first introduce a family of equivalent norms in $X_n$, depending on $t$, with respect to which each $e^{-\omega\tau_n}F_n(t)$ is a contraction on $X_n$, so that the idea of Miyadera and Kobayashi [5] can be used in our argument.

LEMMA 2.3. *Assume that* $\{F_n(t) : t \in [0,T]\}$ *is stable, with stability constant* $\{M,\omega\}$, *for time scale* $\tau_n \to 0$. *For each* $n \geq 1$, *define a family* $\{|\cdot|_t^n : t \in [0,T]\}$ *of norms in* $X_n$ *by*

$$(2.3)\qquad |x|_t^n = \sup\Big\{e^{-\omega\tau_n m}\Big\|\prod_{k=1}^m F_n(t_k)x\Big\|_n : m\geq 0,\ t\leq t_1\leq\ldots\leq t_m\leq T\Big\}.$$

*Then*

$$(2.4)\qquad \|x\|_n \leq |x|_t^n \leq M\|x\|_n \quad \text{for } x\in X_n \text{ and } t\in[0,T],$$

$$(2.5)\qquad |x|_t^n \leq |x|_s^n \quad \text{for } x\in X_n \text{ and } 0\leq s\leq t\leq T,$$

$$(2.6)\qquad |F_n(t)x|_t^n \leq e^{\omega\tau_n}|x|_t^n \quad \text{for } x\in X_n \text{ and } t\in[0,T],$$

$$(2.7)\qquad |(\lambda I - A_n(t))^{-1}x|_t^n \leq (\lambda-\omega_n)^{-1}|x|_t^n \quad \text{for } x\in X_n,\ t\in[0,T],$$
$$\text{and } \lambda>\omega_n,\ \text{where } \omega_n = (e^{\omega\tau_n}-1)/\tau_n,$$

$$(2.8)\qquad \{A_n(t) : t\in[0,T]\} \in S_\sharp(X_n, M, \omega_n).$$

*Proof.* It is obvious by the definition (2.3) that (2.4) and (2.5) hold. To prove (2.6), let $x\in X_n$ and $t\in[0,T]$. For $t\leq t_1\leq\ldots\leq t_m\leq T$ and $m\geq 1$ we have

$$e^{-\omega\tau_n m}\Big\|\prod_{k=1}^m F_n(t_k)F_n(t)x\Big\|_n = e^{\omega\tau_n}e^{-\omega\tau_n(m+1)}\Big\|\prod_{k=1}^m F_n(t_k)F_n(t)x\Big\|_n \leq e^{\omega\tau_n}|x|_t^n,$$

which implies (2.6). Since

$$\lambda I - A_n(t) = \frac{\lambda\tau_n+1}{\tau_n}\Big(I - \frac{1}{\lambda\tau_n+1}F_n(t)\Big),$$

(2.7) is a direct consequence of the Neumann series theorem, by using (2.6). To prove (2.8), let $0\leq t_1\leq\ldots\leq t_m\leq T$, $m\geq 0$, $x\in X_n$ and $\lambda>\omega_n$, and set

$$a_i = \Big|\prod_{k=1}^i(\lambda I - A_n(t_k))^{-1}x\Big|_{t_i}^n \quad \text{for } 1\leq i\leq m.$$

By (2.5) and (2.7) we have

$$a_i \leq (\lambda-\omega_n)^{-1}\Big|\prod_{k=1}^{i-1}(\lambda I - A_n(t_k))^{-1}x\Big|_{t_i}^n \leq (\lambda-\omega_n)^{-1}a_{i-1}$$

for $1\leq i\leq m$. Solving this we find

$$a_m \leq (\lambda-\omega_n)^{-m}|x|_{t_1}^n,$$

which implies (2.8), by (2.4). ∎

PROPOSITION 2.4. *Assume that the conditions of the Main Theorem are satisfied. Then* $\{A(t) : t \in [0,T]\}$ *generates an evolution operator* $\{U(t,s) : (t,s) \in \Delta\}$ *on* $\overline{Y}$.

*Proof.* Let $\omega_n$ be as in Lemma 2.3. Since $\omega_n \to \omega$ as $n \to \infty$, we have $\lambda_0 > \omega_n$ for sufficiently large $n$, and hence $\lambda_0 \in \varrho(A_n(t))$ for $t \in [0,T]$, by (2.7). As in the proof of Fattorini [2, Theorem 5.7.11] we deduce from (2.8) that $(\omega, \infty) \subset \varrho(A(t))$ for $t \in [0,T]$, and

$$\lim_{n\to\infty} \|(\lambda I - A_n(t))^{-1} P_n x - P_n(\lambda I - A(t))^{-1} x\|_n = 0 \tag{2.9}$$

for $\lambda > \omega$, $t \in [0,T]$ and $x \in X$. Using (2.8) again we find $\{A(t) : t \in [0,T]\} \in S_\sharp(X, M, \omega)$ by (2.9). Since $A(t) \subset \liminf_{n\to\infty} A_n(t)$ for $t \in [0,T]$, it follows from (1.4) that

$$\|A(t)x - A(s)x\| \le \|f(t) - f(s)\|(\|x\| + \|A(s)x\|) \tag{2.10}$$

for $t, s \in [0,T]$ and $x \in Y$. Now the assertion is a direct consequence of Proposition 2.1. ∎

**3. Appoximation of evolution operators.** In this section we assume that the conditions of the Main Theorem are satisfied.

LEMMA 3.1. *Let* $n \ge 1$. *Then*

$$\begin{aligned} &|F_n(t)x - J_n^\mu(s)y|_{t\vee s}^n \\ &\quad \le \alpha_{\tau_n,\mu} e^{\omega\tau_n} |x - J_n^\mu(s)y|_{t\vee s}^n + \beta_{\tau_n,\mu} |F_n(t)x - y|_{t\vee s}^n \\ &\qquad + M\gamma_{\tau_n,\mu}\varrho_f(|t-s|)\{(\|J_n^\mu(s)y\|_n + \|A_n(s)J_n^\mu(s)y\|_n) \\ &\qquad\qquad \vee (\|x\|_n + \|A_n(t)x\|_n)\} \end{aligned} \tag{3.1}$$

*for* $x, y \in X_n$, $t, s \in [0,T]$ *and* $\mu > 0$ *with* $\mu\omega_n < 1$ *where we set*

$$\varrho_f(r) = \sup\{\|f(t) - f(s)\| : |t-s| \le r \text{ for } t, s \in [0,T]\},$$
$$\alpha_{\tau_n,\mu} = \mu/(\tau_n + \mu), \quad \beta_{\tau_n,\mu} = \tau_n/(\tau_n + \mu), \quad \gamma_{\tau_n,\mu} = \tau_n\mu/(\tau_n + \mu).$$

*Proof.* Let $x, y \in X_n$, $t, s \in [0,T]$, and $\mu > 0$ be such that $\mu\omega_n < 1$. By the definition of $J_n^\mu(t)$ we find

$$J_n^\mu(s)y = \beta_{\tau_n,\mu} y + \alpha_{\tau_n,\mu} F_n(s) J_n^\mu(s)y,$$

which we use to obtain

$$\begin{aligned} F_n(t)x - J_n^\mu(s)y &= \beta_{\tau_n,\mu}(F_n(t)x - y) + \alpha_{\tau_n,\mu} F_n(t)(x - J_n^\mu(s)y) \\ &\quad + \alpha_{\tau_n,\mu}(F_n(t) - F_n(s)) J_n^\mu(s)y. \end{aligned}$$

The estimate (3.1) will be proved only in the case where $t \ge s$, because the other case is similar. Let $t \ge s$. We estimate the above quantity by using (2.4), (2.6) and (1.4). This yields

$$|F_n(t)x - J_n^\mu(s)y|_{t\vee s}^n \le \beta_{\tau_n,\mu}|F_n(t)x - y|_{t\vee s}^n + \alpha_{\tau_n,\mu}e^{\omega\tau_n}|x - J_n^\mu(s)y|_{t\vee s}^n$$
$$+ \gamma_{\tau_n,\mu}M\|f(t) - f(s)\|(\|J_n^\mu(s)y\|_n + \|A_n(s)J_n^\mu(s)y\|_n),$$

which proves (3.1) in the case where $t \ge s$, since $\|f(t)-f(s)\| \le \varrho_f(|t-s|)$. ■

LEMMA 3.2. *Let $x \in X_n$ and $p \ge 0$. Then, for $i$ with $p+1 \le i \le N_{\tau_n}$,*

$$\Big\|A_n(t_i^{\tau_n})\prod_{k=p+1}^{i-1}F_n(t_k^{\tau_n})x\Big\|_n \le \widehat{M}(\sup_{t\in[0,T]}\|A_n(t)x\|_n + \|x\|_n),$$

*where $\widehat{M} = M^2(V_f+1)\exp(2\widehat{\omega}T + MV_f)$ and $\widehat{\omega} = \sup\{\omega_n : n \ge 1\} \vee \omega$.*

*Proof.* Let $x \in X_n$ and $p \ge 0$ and set

$$a_i^n = \Big|A_n(t_i^{\tau_n})\prod_{k=p+1}^{i-1}F_n(t_k^{\tau_n})x\Big|_{t_i^{\tau_n}}^n \quad \text{for } p+1 \le i \le N_{\tau_n}.$$

By the triangle inequality, (2.4) and (2.5) we have

$$a_i^n \le \Big|A_n(t_{i-1}^{\tau_n})\prod_{k=p+1}^{i-1}F_n(t_k^{\tau_n})x\Big|_{t_{i-1}^{\tau_n}}^n$$
$$+ M\Big\|(A_n(t_i^{\tau_n}) - A_n(t_{i-1}^{\tau_n}))\prod_{k=p+1}^{i-1}F_n(t_k^{\tau_n})x\Big\|_n.$$

We apply (1.4) to the second term on the right-hand side, and then use the stability of $\{F_n(t) : t \in [0,T]\}$ and (2.4). This yields

$$a_i^n \le M^2e^{\omega T}\|f(t_i^{\tau_n}) - f(t_{i-1}^{\tau_n})\|\,\|x\|_n$$
$$+ (1 + M\|f(t_i^{\tau_n}) - f(t_{i-1}^{\tau_n})\|)\Big|A_n(t_{i-1}^{\tau_n})\prod_{k=p+1}^{i-1}F_n(t_k^{\tau_n})x\Big|_{t_{i-1}^{\tau_n}}^n$$

for $p+1 \le i \le N_{\tau_n}$. Since $F_n(t)$ and $A_n(t)$ commute, we have, by (2.6) and the inequality $1 + a \le e^a$ for $a \ge 0$,

$$a_i^n \le M^2e^{\omega T}\|f(t_i^{\tau_n}) - f(t_{i-1}^{\tau_n})\|\,\|x\|_n + \exp(M\|f(t_i^{\tau_n}) - f(t_{i-1}^{\tau_n})\|)e^{\omega\tau_n}a_{i-1}^n$$

for $p+2 \le i \le N_{\tau_n}$. Solving the inequality above by using (2.2) and then noting (2.4) we obtain the desired estimate. ■

LEMMA 3.3. *Let $n \ge 1$, $x \in X_n$ and $p, q \ge 0$. If $0 < \eta < \delta \le T$, $\tau_n \vee \mu < \delta - \eta$ and $\mu > 0$ with $\mu\omega_n \le 1/2$, then for $p \le i \le N_{\tau_n}$ and $q \le j \le N_\mu$ we have*

$$e^{-\omega\tau_n(i-p)}(1-\mu\omega_n)^{j-q}a_{i,j}^{\tau_n,\mu} \le d_{i,j}^{\tau_n,\mu} M \sup_{t\in[0,T]} \|A_n(t)x\|_n + (t_i^{\tau_n}-t_p^{\tau_n})\{\eta^{-1}\varrho_f(T)(d_{i,j}^{\tau_n,\mu}+|t_p^{\tau_n}-t_q^{\mu}|)+\varrho_f(\delta)\} \times 2M\widehat{M}(\sup_{t\in[0,T]}\|A_n(t)x\|_n+\|x\|_n), \tag{3.2}$$

*where*

$$a_{i,j}^{\tau_n,\mu} = \Big|\prod_{k=p+1}^{i} F_n(t_k^{\tau_n})x - \prod_{k=q+1}^{j} J_n^{\mu}(t_k^{\mu})x\Big|_{t_i^{\tau_n}\vee t_j^{\mu}}^{n},$$

$$d_{i,j}^{\tau_n,\mu} = \{((t_i^{\tau_n}-t_p^{\tau_n})-(t_j^{\mu}-t_q^{\mu}))^2+\tau_n(t_i^{\tau_n}-t_p^{\tau_n})+\mu(t_j^{\mu}-t_q^{\mu})\}^{1/2}$$

*for* $p\le i\le N_{\tau_n}$, $q\le j\le N_\mu$ *and* $x\in X_n$.

*Proof.* We use the idea of Miyadera and Kobayashi [5], applying Lemma 2.3. Let $x\in X_n$, $p,q\ge 0$ and $\mu>0$ with $\mu\omega_n\le 1/2$. For $q\le j\le N_\mu$ we have

$$x-\prod_{k=q+1}^{j} J_n^{\mu}(t_k^{\mu})x = \sum_{l=q+1}^{j}\Big(\prod_{k=l+1}^{j} J_n^{\mu}(t_k^{\mu})\Big)(x-J_n^{\mu}(t_l^{\mu})x) = -\mu\sum_{l=q+1}^{j}\Big(\prod_{k=l}^{j} J_n^{\mu}(t_k^{\mu})\Big)A_n(t_l^{\mu})x;$$

hence (2.4), (2.5) and (2.7) give

$$a_{p,j}^{\tau_n,\mu}\le \mu(j-q)M\sup_{t\in[0,T]}\|A_n(t)x\|_n(1-\mu\omega_n)^{-(j-q)},$$

which proves that $a_{p,j}^{\tau_n,\mu}$ satisfies (3.2) if $q\le j\le N_\mu$. Since

$$\prod_{k=p+1}^{i} F_n(t_k^{\tau_n})x - x = \tau_n\sum_{l=p+1}^{i}\Big(\prod_{k=l+1}^{i} F_n(t_k^{\tau_n})\Big)A_n(t_l^{\tau_n})x,$$

we find, by (2.4), (2.5) and (2.6), that

$$a_{i,q}^{\tau_n,\mu}\le \tau_n(i-p)M\sup_{t\in[0,T]}\|A_n(t)x\|_n e^{\omega\tau_n(i-p)}$$

for $p\le i\le N_{\tau_n}$, which proves that $a_{i,q}^{\tau_n,\mu}$ satisfies (3.2) if $p\le i\le N_{\tau_n}$.

Since all assumptions of Lemma 2.2 are satisfied with $A(t)$ and $\omega$ replaced by $A_n(t)$ and $\omega_n$, we have

$$\Big\|A_n(t_j^{\mu})\prod_{k=q+1}^{j} J_n^{\mu}(t_k^{\mu})x\Big\|_n \le \widehat{M}(\sup_{t\in[0,T]}\|A_n(t)x\|_n+\|x\|_n).$$

Using this estimate and Lemma 3.2 we find by (3.1) that

$$a_{i,j}^{\tau_n,\mu} \le \beta_{\tau_n,\mu} a_{i,j-1}^{\tau_n,\mu} + \alpha_{\tau_n,\mu} e^{\omega\tau_n} a_{i-1,j}^{\tau_n,\mu} + 2M\widehat{M}\gamma_{\tau_n,\mu}\varrho_f(|t_i^{\tau_n} - t_j^{\mu}|)(\sup_{t\in[0,T]} \|A_n(t)x\|_n + \|x\|_n)$$

for $p+1 \le i \le N_{\tau_n}$ and $q+1 \le j \le N_\mu$. Multiplying by $\omega_{i,j}^{\tau_n,\mu}$ ($:= e^{-\omega\tau_n(i-p)}(1-\mu\omega_n)^{j-q}$) we obtain

$$\omega_{i,j}^{\tau_n,\mu} a_{i,j}^{\tau_n,\mu} \le \beta_{\tau_n,\mu}\omega_{i,j-1}^{\tau_n,\mu} a_{i,j-1}^{\tau_n,\mu} + \alpha_{\tau_n,\mu}\omega_{i-1,j}^{\tau_n,\mu} a_{i-1,j}^{\tau_n,\mu} + 2M\widehat{M}\gamma_{\tau_n,\mu}\varrho_f(|t_i^{\tau_n} - t_j^{\mu}|)(\sup_{t\in[0,T]} \|A_n(t)x\|_n + \|x\|_n)$$

for $p+1 \le i \le N_{\tau_n}$ and $q+1 \le j \le N_\mu$. Thus we can easily modify the argument of Tanaka [7, Lemma 1.4] to obtain the desired estimate. (See also Kobayasi, Kobayashi and Oharu [3].) ∎

LEMMA 3.4. *For $y \in Y$ we have*

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \|A_n(t)y_n\|_n \le K \sup_{t\in[0,T]} \|A(t)y\| \tag{3.3}$$

*if $y_n \in X_n$ satisfy $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} A_n(t)y_n = A(t)y$ for all $t \in [0,T]$, where $K$ is the constant satisfying* (1.3).

*Proof.* Using (1.4) and the strong continuity of $A(\cdot)$ on $Y$ we can show, by an indirect proof, that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|A_n(t)y_n - P_nA(t)y\|_n = 0$$

if $y \in Y$ and $y_n \in X_n$ satisfy $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} A_n(t)y_n = A(t)y$ for all $t \in [0,T]$. The desired claim (3.3) follows from the fact above and the inequality

$$\sup_{t\in[0,T]} \|A_n(t)y_n\|_n \le \sup_{t\in[0,T]} \|A_n(t)y_n - P_nA(t)y\|_n + K \sup_{t\in[0,T]} \|A(t)y\|. \ ∎$$

*Proof of the Main Theorem.* Let $x \in \overline{Y}$, $0 < \eta < \delta \le T$ and $\mu > 0$ be such that $\mu\omega < 1/2$, and consider sufficiently large integers $n$ so that $\tau_n \vee \mu < \delta - \eta$ and $\mu\omega_n < 1/2$. Then by (1.3) we have

$$\Big\| \prod_{k=[s/\tau_n]+1}^{[t/\tau_n]} F_n(k\tau_n)P_nx - P_nU(t,s)x \Big\|_n \le \Big\| \prod_{k=[s/\tau_n]+1}^{[t/\tau_n]} F_n(k\tau_n)P_nx - \prod_{k=[s/\mu]+1}^{[t/\mu]} J_n^\mu(k\mu)P_nx \Big\|_n \tag{3.4}$$

$$+ \Big\| \prod_{k=[s/\mu]+1}^{[t/\mu]} J_n^{\mu}(k\mu) P_n x - P_n \prod_{k=[s/\mu]+1}^{[t/\mu]} J^{\mu}(k\mu) x \Big\|_n$$

$$+ K \Big\| \prod_{k=[s/\mu]+1}^{[t/\mu]} J^{\mu}(k\mu) x - U(t,s)x \Big\|.$$

Let $y \in Y$. Since $A(t) \subset \liminf_{n\to\infty} A_n(t)$ for $t \in [0,T]$, there exist $y_n \in X_n$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} A_n(t)y_n = A(t)y$ for all $t \in [0,T]$. Using (3.2) with $i = [t/\tau_n]$, $j = [t/\mu]$, $p = [s/\tau_n]$ and $q = [s/\mu]$ we see that the first term on the right-hand side of (3.4) is less than or equal to

$$\begin{aligned} &2Me^{2\omega_n T}(K\|x - y\| + \|P_n y - y_n\|_n) \\ &+ e^{4\omega_n T}((\tau_n + \mu)^2 + T(\tau_n + \mu))^{1/2} M \sup_{t\in[0,T]} \|A_n(t)y_n\|_n \\ &+ e^{4\omega_n T} T\{\eta^{-1}\varrho_f(T)((((\tau_n + \mu)^2 + T(\tau_n + \mu))^{1/2} + \tau_n + \mu) + \varrho_f(\delta)\} \\ &\times 2M\widehat{M}(\sup_{t\in[0,T]} \|A_n(t)y_n\|_n + \|y_n\|_n). \end{aligned}$$

The second term on the right-hand side of (3.4) is dominated by

$$\max_{0\le p\le i\le N_\mu} \Big\| \prod_{k=p+1}^{i} J_n^{\mu}(k\mu) P_n x - P_n \prod_{k=p+1}^{i} J^{\mu}(k\mu) x \Big\|_n$$

and we deduce from (2.9) that it converges to zero as $n \to \infty$. Taking the limit in (3.4) as $n \to \infty$, and then letting $\mu \downarrow 0$, we have, by Lemma 3.4,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{(t,s)\in\Delta} \Big\| \prod_{k=[s/\tau_n]+1}^{[t/\tau_n]} F_n(k\tau_n) P_n x - P_n U(t,s)x \Big\|_n \\ &\qquad \le 2Me^{2\omega T} K\|x - y\| + e^{4\omega T} T \varrho_f(\delta) 2M\widehat{M}(K \sup_{t\in[0,T]} \|A(t)y\| + \|y\|) \end{aligned}$$

for any $y \in Y$ and $\delta > 0$. Since $x \in \overline{Y}$ and $\varrho_f(\delta) \downarrow 0$ as $\delta \downarrow 0$, we conclude that (1.5) holds and the convergence is uniform on $\Delta$. ∎

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