STUDIA MATHEMATICA 154 (3) (2003)

The complete hyperexpansivity analog of the Embry conditions

by

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Abstract. The Embry conditions are a set of positivity conditions that characterize subnormal operators (on Hilbert spaces) whose theory is closely related to the theory of positive definite functions on the additive semigroup $\mathbb{N}$ of non-negative integers. Completely hyperexpansive operators are the negative definite counterpart of subnormal operators. We show that completely hyperexpansive operators are characterized by a set of negativity conditions, which are the natural analog of the Embry conditions for subnormality. While the genesis of the Embry conditions can be traced to the Hausdorff Moment Problem, the genesis of the conditions to be established here lies in the Lévy–Khinchin representation as holding in the context of abelian semigroups. We actually establish the desired negativity criteria for completely hyperexpansive operator tuples.

If $\mathcal{H}$ is a complex infinite-dimensional separable Hilbert space, we let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$, with I and 0 standing respectively for the identity operator and zero operator on $\mathcal{H}$. An m-tuple $S = (S_1, \ldots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an m-tuple $N = (N_1, \ldots, N_m)$ of commuting normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that $N_i\mathcal{H} \subset \mathcal{H}$ and $N_i|\mathcal{H} = S_i$ for $1 \le i \le m$ (refer to [Lu]). The most comprehensive account of the theory of a single subnormal operator can be found in [Co]. There exist a number of equivalent conditions (see [Co, Ch. 2, Thm. 1.9]) that characterize subnormal operators (with their appropriate generalizations carrying over to subnormal tuples). The genesis of all of those conditions can be traced to the theory of positive definite functions on abelian semigroups, and in particular to the theory of the Hausdorff Moment Problem. In [At2], the author introduced *completely hyperexpansive* operators (refer to Definition 1 below with $m = 1$) which are

2000 *Mathematics Subject Classification*: Primary 47B20; Secondary 47B39.

Key words and phrases: positive definite, negative definite, completely monotone, completely alternating, subnormal, completely hyperexpansive.

antithetical to subnormals in the sense that their defining properties and behavior are directly related to the theory of negative definite functions on abelian semigroups. In addition to [At2], completely hyperexpansive operators and related classes of operators have been studied in [At3], [At-Ar1], [At-Ar2], [At-Sh], [Sh-At], [Ja] and [Ja-St].

The original Halmos characterization of subnormality (see [Ha]) consisted of two parts: a "positivity" part and a "boundedness" part (refer to conditions (1.10) and (1.11) as given in (c) of Theorem 1.9 in Chapter II of [Co]). It was shown by Bram (see [Br]) that the positivity part by itself characterizes subnormal operators, and by Szymański (see [Sz]) that the boundedness part by itself serves the same purpose. While a number of "negativity" conditions (of a somewhat abstract nature) characterizing completely hyperexpansive operators were presented in [Sh-At] (see also [At-Sh]), in [Ja] Jabłoński established a few "inverted boundedness conditions" characterizing completely hyperexpansive operators that, like their subnormality counterparts, make the underlying geometry of the Hilbert space more pronounced.

Now, among the host of positivity conditions characterizing subnormal operators are a set of particularly elegant conditions that are due to Embry (refer to [Em], [Lu] and part (f) of Theorem 1.9 in Chapter II of [Co]): An operator T in $\mathcal{B}(\mathcal{H})$ is subnormal if and only if, for any finite set of vectors f_i ($i \in \mathbb{N}$, $0 \le i$) in $\mathcal{H}$, one has $\sum_{i,j\ge 0}\langle T^{i+j} f_i, T^{i+j} f_j\rangle \ge 0$. It is the purpose of the present paper to establish the complete hyperexpansivity analog of the Embry conditions (see Theorem 2 below) by exploiting the Lévy–Khinchin representation as obtaining in the context of abelian semigroups (refer to [B-C-R2]). We will actually be treating the case of completely hyperexpansive operator tuples.

For any set A, A^m will denote the cartesian product of A with itself m times. If $\mathbb{N}$ is the set of natural numbers, then $\mathbb{N}^m$ is an abelian semigroup under coordinatewise addition with the identity element $0 = (0,\dots,0)$. For $p = (p_1,\dots,p_m)$ and $n = (n_1,\dots,n_m)$ in $\mathbb{N}^m$ we write $p \le n$ if $p_i \le n_i$ for $1 \le i \le m$, and use $|p|$ to denote $p_1 + \dots + p_m$. For $p \le n$, $\binom{n}{p}$ is understood to be the product $\binom{n_1}{p_1}\dots\binom{n_m}{p_m}$. For $x = (x_1,\dots,x_m)$ in the Euclidean space $\mathbb{R}^m$ and n in $\mathbb{N}^m$, x^n is the product $x_1^{n_1}\dots x_m^{n_m}$ and $\langle x, n\rangle$ is $x_1n_1 + \dots + x_mn_m$.

A real map φ on $\mathbb{N}^m$ is said to be *positive definite* if $\sum_{1\le i,j\le n} c_ic_j\varphi(s_i+s_j) \ge 0$ for all $n \ge 1, \{s_1,\dots,s_n\} \subset \mathbb{N}^m$, and $\{c_1,\dots,c_n\} \subset \mathbb{R}$. A real map ψ on $\mathbb{N}^m$ is said to be *negative definite* if $\sum_{1\le i,j\le n} c_ic_j\psi(s_i+s_j) \le 0$ for all $n \ge 2, \{s_1,\dots,s_n\} \subset \mathbb{N}^m$, and $\{c_1,\dots,c_n\} \subset \mathbb{R}$ such that $\sum_{1\le i\le n} c_i = 0$. A map $\psi : \mathbb{N}^m \to \mathbb{R}$ is *completely negative definite* if $n \mapsto \psi(n+k)$ is negative definite for every k in $\mathbb{N}^m$.

We let the *backward difference operators* $\nabla_i\,(1 \le i \le m)$ act on $\varphi : \mathbb{N}^m \to \mathbb{R}$ through $(\nabla_i \varphi)(n) = \varphi(n) - \varphi(n + \varepsilon(i))$ where $\varepsilon(i)$ is the m-tuple with 1 in the ith coordinate place and 0's elsewhere. The relations $\nabla_i^0 \varphi = \varphi$ and $(\nabla_i \nabla^j \varphi =)\, \nabla_i \nabla_1^{j_1} \ldots \nabla_m^{j_m} \varphi = \nabla_1^{j_1} \ldots \nabla_i^{j_i+1} \ldots \nabla_m^{j_m} \varphi$ inductively define ∇^j for any $j = (j_1, \ldots, j_m)$ in $\mathbb{N}^m$. The *forward difference operators* $\triangle_i$ are given by $(\triangle_i \varphi)(n) = \varphi(n + \varepsilon(i)) - \varphi(n)$.

A non-negative map φ on $\mathbb{N}^m$ is said to be *completely monotone* if $(\nabla^j \varphi)(n) \ge 0$ for all n, j in $\mathbb{N}^m$. A real map ψ on $\mathbb{N}^m$ is said to be *completely alternating* if $(\nabla^j \psi)(n) \le 0$ for all $n \in \mathbb{N}^m$ and $j \in \mathbb{N}^m \setminus \{0\}$. We remark that completely monotone functions form an extreme subset of the set of positive definite functions on $\mathbb{N}^m$ that are bounded above, while completely alternating functions are an extreme subset of the set of negative definite functions on $\mathbb{N}^m$ that are bounded below (refer to [B-C-R2]).

A *Radon measure* μ on a subset X of $\mathbb{R}^m$ will be understood to be a Borel measure μ satisfying (i) $\mu(C) < \infty$ for every compact subset C of X, (ii) $\mu(B) = \sup\{\mu(C) : C \subset B,\, C \text{ compact}\}$ for each Borel set B in X. A finite Radon measure can be verified to be a regular Borel measure. Propositions 1 and 2 below are respectively Propositions 6.11 and 6.12 in Chapter 4 of [B-C-R2]. We use the symbol $\widetilde{1}$ to denote $(1, \ldots, 1)$ in $\mathbb{R}^m$.

PROPOSITION 1. *For $\varphi : \mathbb{N}^m \to \mathbb{R}$, the following are equivalent.*

(i) *φ is completely monotone.*

(ii) $\displaystyle\sum_{p \in \mathbb{N}^m,\, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} \varphi(p+q) \ge 0$ *for all q, n in $\mathbb{N}^m$.*

(iii) *There exists a positive Radon measure μ on $[0,1]^m$ such that*

$$\varphi(n) = \int_{[0,1]^m} x^n \, d\mu(x) \qquad (n \in \mathbb{N}^m). \tag{A}$$

PROPOSITION 2. *For $\psi : \mathbb{N}^m \to \mathbb{R}$, the following are equivalent.*

(i) *ψ is completely alternating.*

(ii) $\displaystyle\sum_{p \in \mathbb{N}^m,\, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} \psi(p+q) \le 0$ *for all q, n in $\mathbb{N}^m$, $n \ne 0$.*

(iii) *There exist an m-tuple $b = (b_1, \ldots, b_m)$ of non-negative reals and a positive Radon measure ν on $[0,1]^m \setminus \{\widetilde{1}\}$ such that*

$$\psi(n) = \psi(0) + \langle b, n\rangle + \int_{[0,1]^m \setminus \{\widetilde{1}\}} (1 - x^n)\, d\nu(x) \qquad (n \in \mathbb{N}^m \setminus \{0\}). \tag{B}$$

REMARK 1. The result in Proposition 1 is the solution of the multivariable Hausdorff Moment Problem and was arrived at in [H-S]. The result in Proposition 2 was derived in [B-C-R1] as a special case of the Lévy–Khinchin representation on abelian semigroups. It was observed in [At2] that, for the case $m = 1$, one has in (B) $b = \mu(\{1\})$ and $d\nu(x) = d\mu(x)/(1-x)$, where μ is a positive regular Borel measure on $[0,1]$. The corresponding situation in the multivariable case is more intriguing. For our purposes, *and unlike the presentation in* [B-C-R2], it will be useful and instructive to derive the Lévy–Khinchin representation (of Proposition 2) from the solution of the multi-dimensional Hausdorff Moment Problem (as given by Proposition 1). While the proofs of parts (1), (2) and (3) of Theorem 1 below are present in some disguise or other in [B-C-R2], the observations in parts (4) and (5) of Theorem 1 are going to be the key ingredients in our derivation of the complete hyperexpansivity analog of the Embry conditions; in particular, the observation in (5) shows that the assumption regarding the Lévy measure ν in Proposition 4 of [At-Sh] is redundant.

Note that $\psi : \mathbb{N}^m \to \mathbb{R}$ is completely alternating if and only if $\triangle_i\psi$ is completely monotone for all $i = 1, \ldots, m$ (this can be deduced from [B-C-R2, Ch. 4, Lemma 6.3]).

THEOREM 1. *Let $\psi : \mathbb{N}^m \to \mathbb{R}$ be completely alternating. If the completely monotone functions $\varphi_i(n) = \triangle_i\psi(n)\,(i = 1, \ldots, m)$ have the measures μ_i associated with them as in Proposition* 1, *then the following are true.*

(1) *$(1-x_i)d\mu_j(x) = (1-x_j)d\mu_i(x)$ for all $i, j = 1, \ldots m$.*

(2) *The measure ν defined on $[0,1]^m \setminus \{\widetilde{1}\}$ by*

$$d\nu(x) = \frac{d\mu_i(x)}{1-x_i}, \qquad x \in [0,1]^m \setminus A_i,$$

where $A_i = [0,1] \times \ldots \times [0,1] \times \overbrace{\{1\}}^{i} \times [0,1] \times \ldots \times [0,1]$, *is well defined.*

(3) *$d\mu_i(x) = (1-x_i)d\nu(x)$ on $[0,1]^m \setminus \{\widetilde{1}\}$ for all $i = 1, \ldots, m$.*

(4) *The measure ν and the m-tuple $(\mu_1(\{\widetilde{1}\}), \ldots, \mu_m(\{\widetilde{1}\}))$ satisfy the condition* (B) *of Proposition* 2.

(5) $d\nu(x) = \dfrac{d\mu_1(x) + \ldots + d\mu_m(x)}{m - (x_1 + \ldots + x_m)}$.

Proof. (1) We employ the argument of Lemma 3.12 in Chapter 4 of [B-C-R2]. Since $\triangle_i\triangle_j\psi(n) = \triangle_j\triangle_i\psi(n)$, one has $\triangle_i\varphi_j(n) = \triangle_j\varphi_i(n)$, and consequently

$$\int\limits_{[0,1]^m} x^n(x_i-1)\,d\mu_j(x) = \int\limits_{[0,1]^m} x^n(x_j-1)\,d\mu_i(x).$$

The Weierstrass Theorem and the Riesz Representation Theorem now complete the proof of (1).

(2) The result here follows from Theorem 1.18 in Chapter 2 of [B-C-R2].

(3) In view of the definition of the measure ν in (2), it suffices to prove that $\mu_i(A_i \setminus \{\widetilde{1}\}) = 0$ for $i = 1, \ldots, m$. One may write

$$\begin{aligned} A_i &= [0,1) \times [0,1] \times \ldots \times [0,1] \times \overbrace{\{1\}}^{i} \times [0,1] \times \ldots \times [0,1] \\ &\quad \cup \{1\} \times [0,1) \times \ldots \times [0,1] \times \overbrace{\{1\}}^{i} \times [0,1] \times \ldots \times [0,1] \cup \ldots \\ &\quad \cup \{1\} \times \{1\} \times \ldots \times \{1\} \times \overbrace{\{1\}}^{i} \times \ldots \times [0,1) \cup \{\widetilde{1}\}. \end{aligned}$$

The desired assertion now follows from the conditions in (1).

(4) Notice that

$$\begin{aligned} &\psi(n_1, \ldots, n_m) \\ &\quad = \psi(n_1, \ldots, n_{m-1}, 0) + \sum_{0 \le p_m \le n_m - 1} \varphi_m(n_1, \ldots, n_{m-1}, p_m) = \ldots \\ &\quad = \psi(0, \ldots, 0) + \sum_{0 \le p_1 \le n_1 - 1} \varphi_1(p_1, 0, \ldots, 0) + \ldots \\ &\qquad + \sum_{0 \le p_i \le n_i - 1} \varphi_i(n_1, \ldots, n_{i-1}, p_i, 0, \ldots, 0) + \ldots \\ &\qquad + \sum_{0 \le p_m \le n_m - 1} \varphi_m(n_1, \ldots, n_{m-1}, p_m) \\ &\quad = \psi(0) + \sum_{1 \le j \le m} \mu_j(\{\widetilde{1}\}) n_j + \sum_{0 \le p_1 \le n_1 - 1} \int_{[0,1]^m \setminus \{\widetilde{1}\}} x_1^{p_1}\, d\mu_1(x) + \ldots \\ &\qquad + \sum_{0 \le p_i \le n_i - 1} \int_{[0,1]^m \setminus \{\widetilde{1}\}} x_1^{n_1} \ldots x_{i-1}^{n_{i-1}} x_i^{p_i}\, d\mu_i(x) + \ldots \\ &\qquad + \sum_{0 \le p_m \le n_m - 1} \int_{[0,1]^m \setminus \{\widetilde{1}\}} x_1^{n_1} \ldots x_{m-1}^{n_{m-1}} x_m^{p_m}\, d\mu_m(x). \end{aligned}$$

Using (3), one further has

$$\begin{aligned} \psi(n) &= \psi(0) + \sum_{1 \le j \le m} \mu_j(\{\widetilde{1}\}) n_j + \int_{[0,1]^m \setminus \{\widetilde{1}\}} (1 - x_1^{n_1})\, d\nu(x) + \ldots \\ &\quad + \int_{[0,1]^m \setminus \{\widetilde{1}\}} x_1^{n_1} \ldots x_{i-1}^{n_{i-1}} (1 - x_i^{n_i})\, d\nu(x) + \ldots \\ &\quad + \int_{[0,1]^m \setminus \{\widetilde{1}\}} x_1^{n_1} \ldots x_{m-1}^{n_{m-1}} (1 - x_m^{n_m})\, d\nu(x) \\ &= \psi(0) + \sum_{1 \le j \le m} \mu_j(\{\widetilde{1}\}) n_j + \int_{[0,1]^m \setminus \{\widetilde{1}\}} (1 - x^n)\, d\nu(x). \end{aligned}$$

This establishes (4).

(5) The desired result follows from (3). ■

If $T = (T_1, \dots, T_m)$ is a tuple of commuting operators T_i in $\mathcal{B}(\mathcal{H})$, then we interpret T^* to be $(T_1^*, \dots, T_m^*)$ and T^n to be $T_1^{n_1} \dots T_m^{n_m}$. The operator theoretic significance of Proposition 1 is revealed by the following result in [At1], which is an m-variable generalization of J. Agler's criterion in [Ag] for the subnormality of a contraction (that is, a member of $\mathcal{B}(\mathcal{H})$ whose norm does not exceed 1): An m-tuple T of commuting operators in $\mathcal{B}(\mathcal{H})$ is a subnormal tuple of contractions if and only if

$$\text{(G)} \qquad B_n(T) \equiv \sum_{p \in \mathbb{N}^m,\, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} T^{*p} T^p \ge 0 \quad \text{for all } n \text{ in } \mathbb{N}^m.$$

It was highlighted in [At1] that (G) is equivalent to requiring $n \mapsto \|T^n h\|^2$ to be completely monotone for every h in $\mathcal{H}$.

Definition 1. An m-tuple $T = (T_1, \dots, T_m)$ of commuting operators in $\mathcal{B}(\mathcal{H})$ is said to be *completely hyperexpansive* if

$$\text{(H)} \quad B_n(T) \equiv \sum_{p \in \mathbb{N}^m,\, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} T^{*p} T^p \le 0 \quad \text{for all } n \text{ in } \mathbb{N}^m \setminus \{0\}.$$

It follows from the discussion in Remark 2 of [At-Sh] that (H) is equivalent to requiring $n \mapsto \|T^n h\|^2$ to be completely alternating for every h in $\mathcal{H}$.

Theorem 2. *Let $T = (T_1, \dots, T_m)$ be a tuple of commuting operators in $\mathcal{B}(\mathcal{H})$. Then* (1), (2) *and* (3) *below are equivalent.*

(1) *T is completely hyperexpansive.*

(2) *For any integer tuple n in $\mathbb{N}^m \setminus \{0\}$ and any finite set of vectors f_i $(i \in \mathbb{N}^m,\ 0 \le i \le n)$ in $\mathcal{H}$ satisfying $\sum_i f_i = 0$, one has*

$$\sum_{0 \le i,j \le n} \langle T^{i+j} f_i, T^{i+j} f_j \rangle \le 0.$$

(3) *For any integer tuple n in $\mathbb{N}^m \setminus \{0\}$, any f in $\mathcal{H}$, and any real scalars λ_i $(i \in \mathbb{N}^m,\ 0 \le i \le n)$ satisfying $\sum_i \lambda_i = 0$, one has*

$$\sum_{0 \le i,j \le n} \lambda_i \lambda_j \|T^{i+j} f\|^2 \le 0.$$

Proof. Suppose the conditions in (2) hold. For any tuple $k = (k_1, \dots, k_m)$ of non-negative integers, a fixed vector h in $\mathcal{H}$ and reals c_i $(i \in \mathbb{N}^m,\ 0 \le i \le n)$ satisfying $\sum_i c_i = 0$, consider $f_i = c_i T^k h$. Clearly, $\sum_i f_i = 0$ so that

$$\sum_{0 \le i,\, j \le n} \langle T^{i+j} f_i, T^{i+j} f_j \rangle = \sum_{0 \le i,\, j \le n} c_i c_j \|T^{i+j+2k} h\|^2 \le 0.$$

This shows that the map $n \mapsto \|T^n h\|^2$ is completely negative definite on $\mathbb{N}^m$. It follows from Theorem 1.10 in Chapter 7 of [B-C-R2] that $n \mapsto \|T^n h\|^2$ is completely alternating on $\mathbb{N}^m$. In view of our comment following Definition 1, the operator tuple T is then completely hyperexpansive. Thus (2) implies (1), and the argument here also makes it clear that (3) implies (1). Further, (2) obviously implies (3).

We now show that (1) implies (2). Suppose then that T is completely hyperexpansive. It will suffice to treat the case $m = 2$. Now, for any h in $\mathcal{H}$, the map $n \mapsto \psi_h(n) = \|T^n h\|^2$ is completely alternating. In view of our observation preceding Theorem 1, the maps $n \mapsto \varphi_1(n) = \triangle_1 \psi_h(n)$ and $n \mapsto \varphi_2(n) = \triangle_2 \psi_h(n)$ are completely monotone on $\mathbb{N}^2$, so that there exist two (regular) positive Borel measures μ_{1h} and μ_{2h} on $[0,1]^2$ satisfying, for $n = (n_1, n_2)$ in $\mathbb{N}^2$,

$$\|T_1^{n_1+1} T_2^{n_2} h\|^2 - \|T_1^{n_1} T_2^{n_2} h\|^2 = \int_{[0,1]^m} x_1^{n_1} x_2^{n_2} \, d\mu_{1h}(x_1, x_2),$$

$$\|T_1^{n_1} T_2^{n_2+1} h\|^2 - \|T_1^{n_1} T_2^{n_2} h\|^2 = \int_{[0,1]^m} x_1^{n_1} x_2^{n_2} \, d\mu_{2h}(x_1, x_2).$$

A standard polarization argument now leads to positive $\mathcal{B}(\mathcal{H})$-valued measures $F_1(\cdot)$ and $F_2(\cdot)$ on $[0,1]^2$ such that, for any Borel subset σ of $[0,1]^2$, one has $\mu_{1h}(\sigma) = \langle F_1(\sigma)h, h\rangle$ and $\mu_{2h}(\sigma) = \langle F_2(\sigma)h, h\rangle$.

One can now see that, for $n = (n_1, n_2)$ in $\mathbb{N}^2 \setminus \{0\}$, the Lévy–Khinchin representation

$$\begin{aligned}\|T_1^{n_1} T_2^{n_2} h\|^2 &= \|h\|^2 + n_1 \mu_{1h}(\{\widetilde{1}\}) + n_2 \mu_{2h}(\{\widetilde{1}\}) \\ &\quad + \int_{[0,1]^2 \setminus \{\widetilde{1}\}} (1 - x_1^{n_1} x_2^{n_2}) \, \frac{d(\mu_{1h} + \mu_{2h})(x_1, x_2)}{2 - x_1 - x_2}\end{aligned}$$

is equivalent to the operator representation

$$\begin{aligned}T_1^{*n_1} T_2^{*n_2} T_1^{n_1} T_2^{n_2} &= I + n_1 F_1(\{\widetilde{1}\}) + n_2 F_2(\{\widetilde{1}\}) \\ &\quad + \int_{[0,1]^2 \setminus \{\widetilde{1}\}} (1 - x_1^{n_1} x_2^{n_2}) \, \frac{d(F_1 + F_2)(x_1, x_2)}{2 - x_1 - x_2}.\end{aligned}$$

Thus, for any integer tuple n in $\mathbb{N}^2 \setminus \{0\}$ and any finite set of vectors f_i ($i \in \mathbb{N}^2$, $0 \le i \le n$) in $\mathcal{H}$ satisfying $\sum_i f_i = 0$, one has

$$\begin{aligned}\sum_{0 \le i,\, j \le n} \langle T^{i+j} f_i, T^{i+j} f_j\rangle &= \|f_0\|^2 + \sum_{i+j \ne 0} \langle T^{*\,i+j} T^{i+j} f_i, f_j\rangle \\ &= \|f_0\|^2 + \sum_{i+j \ne 0} \langle f_i, f_j\rangle + \sum_{i+j \ne 0} (i_1 + j_1) \langle F_1(\{\widetilde{1}\}) f_i, f_j\rangle\end{aligned}$$

$$+\sum_{i+j\neq 0}(i_2+j_2)\langle F_2(\{\widetilde{1}\})f_i,f_j\rangle$$

$$+\sum_{i+j\neq 0}\Big\langle \int\limits_{[0,1]^2\setminus\{\widetilde{1}\}}(1-x_1^{i_1+j_1}x_2^{i_2+j_2})\,\frac{d(F_1+F_2)(x_1,x_2)}{2-x_1-x_2}\,f_i,f_j\Big\rangle.$$

Now, $\|f_0\|^2+\sum_{i+j\neq 0}\langle f_i,f_j\rangle=\|\sum_{i\geq 0}f_i\|^2=0$. Further,

$$\sum_{i+j\neq 0}(i_1+j_1)\langle F_1(\{\widetilde{1}\})f_i,f_j\rangle=\sum_{i,j\geq 0}(i_1+j_1)\langle F_1(\{\widetilde{1}\})f_i,f_j\rangle$$
$$=\sum_{i,j\geq 0}i_1\langle F_1(\{\widetilde{1}\})f_i,f_j\rangle+\sum_{i,j\geq 0}j_1\langle F_1(\{\widetilde{1}\})f_i,f_j\rangle,$$

and the last sum is easily seen to be zero in view of $\sum_{j\geq 0}f_j=\sum_{i\geq 0}f_i=0$. Similarly, one has

$$\sum_{i+j\neq 0}(i_2+j_2)\langle F_2(\{\widetilde{1}\})f_i,f_j\rangle=0.$$

We now verify that

$$\sum_{i+j\neq 0}\Big\langle \int\limits_{[0,1]^2\setminus\{\widetilde{1}\}}(1-x_1^{i_1+j_1}x_2^{i_2+j_2})\,\frac{d(F_1+F_2)(x_1,x_2)}{2-x_1-x_2}\,f_i,f_j\Big\rangle\leq 0.$$

For that purpose we set, for any Borel subset σ of $[0,1]^2$ and any positive integer $k\geq 1$,

$$G_k(\sigma)=\int\limits_{\sigma}\frac{d(F_1+F_2)(x_1,x_2)}{2-x_1-x_2+1/k}.$$

It should be noted that each G_k is a semi-spectral measure on $[0,1]^2$; to be specific, each G_k is a $\mathcal{B}(\mathcal{H})$-valued positive function defined on the Borel subsets of $[0,1]^2$ which is countably additive in the weak operator topology. Elementary measure-theoretic considerations convince one that it would suffice to verify that, for every k,

$$I_k\equiv\sum_{i+j\neq 0}\Big\langle \int\limits_{[0,1]^2\setminus\{\widetilde{1}\}}(1-x_1^{i_1+j_1}x_2^{i_2+j_2})\,dG_k(x_1,x_2)f_i,f_j\Big\rangle\leq 0.$$

By the Naimark Dilation Theorem (refer to Section 4 of [Ml], for example) there exist, for every k, a Hilbert space $\mathcal{K}_k$, a $\mathcal{B}(\mathcal{K}_k)$-valued (normalized) spectral measure E_k on $[0,1]^2$, and a bounded linear map R_k from $\mathcal{H}$ to $\mathcal{K}_k$ such that $G_k(\sigma)=R_k^*E_k(\sigma)R_k$ for every Borel subset σ of $[0,1]^2$. If we now set $A_{ki}=\int_{[0,1]^2}x_i\,dE_k(x_1,x_2)$ $(i=1,2)$ (and note that the value of I_k does not change by adding the point $\widetilde{1}$ to the domain of integration in its defining expression), then we have

$$\begin{aligned} I_k &= \sum_{i+j\neq 0} \langle R_k f_i, R_k f_j\rangle - \sum_{i+j\neq 0} \langle A_{k1}^{i_1} A_{k2}^{i_2} R_k f_i, A_{k1}^{j_1} A_{k2}^{j_2} R_k f_j\rangle \\ &= -\|R_k f_0\|^2 - \sum_{i+j\neq 0} \langle A_{k1}^{i_1} A_{k2}^{i_2} R_k f_i, A_{k1}^{j_1} A_{k2}^{j_2} R_k f_j\rangle \\ &= -\Big\|\sum_{i\geq 0} A_{k1}^{i_1} A_{k2}^{i_2} R_k f_i\Big\|^2 \leq 0. \ \blacksquare \end{aligned}$$

The dilation technique used in the proof of Theorem 2 has already appeared in the context of completely hyperexpansive operators in the work of Jabłoński ([Ja]). It is interesting to note that, in the very special case of a power-bounded completely hyperexpansive operator T, the conditions (2) in Theorem 1 above stand verified by Proposition 5.4 in [Ja]; indeed, as is shown there, for such an operator T there exists a positive operator Q such that

$$\sum_{i,j\geq 0} \langle T^{i+j} f_i, T^{i+j} f_j\rangle \leq \Big\|Q^{1/2} \sum_{i\geq 0} f_i\Big\|^2$$

for any finite set of vectors f_i in $\mathcal{H}$.

In the terminology employed by Stochel in [St], the conditions (2) of Theorem 2 can be described as follows: *An m-tuple of commuting operators in $\mathcal{B}(\mathcal{H})$ is completely hyperexpansive if and only if the Embry map $n \mapsto T^{*n}T^n$ is conditionally negative definite on* $\mathbb{N}^m$. As was shown by Stochel, the "conditional positive definiteness" of the Embry map is necessary and sufficient for the subnormality of a tuple of (commuting) contractions (refer to statement (iii) in Theorem 4.1 of [St]). Similarly, the conditions (3) of Theorem 2 are the complete hyperexpansivity analog of the Lambert conditions for subnormality (refer to [L], [Lu], [St]) and are the counterpart of statement (iv) in Theorem 4.1 of [St] pertaining to a subnormal tuple of contractions. The results of Theorem 2 thus highlight a general theme (as expounded in the previous works of the author and others) that completely hyperexpansive operator tuples are the antithesis of subnormal tuples of contractions.

Acknowledgments. The author is thankful to the referee for suggesting some changes making the original treatment more explicit and complete.

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Received March 5, 2002
Revised version July 22, 2002 (4898)