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α-times integrated semigroups: local and global

by

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Abstract. We investigate the relations between local α-times integrated semigroups and $(\alpha+1)$-times integrated Cauchy problems, and then the relations between global α-times integrated semigroups and regularized semigroups.

1. Introduction. Let X be a Banach space. A strongly continuous family $(S(t))_{0\le t<\tau}\subset \mathbf{B}(X)$ is called a *local α-times integrated semigroup* if

$$S(t+s)x=\frac{1}{\Gamma(\alpha)}\Big[\int_t^{t+s}(t+s-r)^{\alpha-1}S(r)x\,dr-\int_0^s(t+s-r)^{\alpha-1}S(r)x\,dr\Big]$$

for all $x\in X$ and $0\le s,t,s+t<\tau$. If $\tau=\infty$, we call $(S(t))_{t\ge 0}$ a *global α-times integrated semigroup*, or simply, α-times integrated semigroup. $S(\cdot)$ is said to be *nondegenerate* if $S(t)x=0$ for all $t\in[0,\tau)$ implies $x=0$. For a nondegenerate local α-times integrated semigroup $S(\cdot)$, we define its *generator*, A, by

$$x\in D(A) \text{ with } Ax=y \ \Leftrightarrow\ S(t)x-\frac{t^\alpha}{\Gamma(\alpha+1)}\,x=\int_0^t S(s)y\,ds,\ \forall t\in[0,\tau).$$

In the first section, we consider the relations between local α-times integrated semigroups and the $(\alpha+1)$-times integrated Cauchy problem

$$C_{\alpha+1}(\tau)\qquad \begin{cases} v\in C([0,\tau),D(A))\cap C^1([0,\tau),X),\\ v'(t)=Av(t)+\dfrac{t^\alpha}{\Gamma(\alpha+1)}\,x, \quad t\in[0,\tau),\\ v(0)=0,\end{cases}$$

where α is a positive number. This is motivated by the work by Arendt *et al.* in [AEK], where they investigate such relations for α an integer. By using an asymptotic formula for Kummer's function, we obtain analogous results for

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local fractional times integrated semigroups. Our results generalize those in [AEK], and enrich the existing local theory for semigroups (see [TO], [HH], [AEK], [Ga] and [LHZ]).

On the other hand, deLaubenfels proved in [dL1] that A generates an n-times integrated semigroup if and only if it is a generator of a $(\lambda - A)^{-n}$-regularized semigroup for any $\lambda \in \varrho(A)$. Using these relations, we can reduce some problems concerning integrated semigroups to problems for regularized semigroups. Also, the solutions of the abstract Cauchy problem

$$\text{(ACP)} \qquad u'(t) = Au(t) \quad (t \geq 0), \qquad u(0) = x_0,$$

can be expressed more directly by a regularized semigroup than by an integrated semigroup. However, in many cases, especially in the study of differential operators on function spaces such as $L^p(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$, we have a fractional times integrated semigroup instead of an integer times integrated one. So our second aim is to clarify the relations between fractional times integrated semigroups and regularized semigroups (Theorems 3.1, 3.2). Moreover, these relations will be applied to the study of the asymptotic behavior of the abstract Cauchy problem (ACP) for A being a generator of an α-times integrated semigroup.

2. $(\alpha+1)$-times integrated Cauchy problems and local α-times integrated semigroups. The Cauchy problem $C_{\alpha+1}(\tau)$ is called *well-posed* if it has a unique solution for every $x \in X$. In this section, we will extend the results of [AEK] to the case that α is not necessarily an integer. First, we characterize the well-posedness by the resolvent. For an operator A on a Banach space X, we write $D(A)$, $R(A)$ and $\varrho(A)$ for its domain, range and resolvent set, respectively. If $\lambda \in \varrho(A)$, we let $R(\lambda, A) = (\lambda - A)^{-1}$, and $\mathbf{B}(X)$ is the space of all bounded linear operators on X.

THEOREM 2.1. *Let $\alpha > 0$, $0 < \tau \leq \infty$. Suppose that $C_{\alpha+1}(\tau)$ is well-posed. Then for every $0 < a < \tau/\alpha$, there exist constants $b > 0$, $M \geq 0$ such that*

$$E(a,b) := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq b,\ |\operatorname{Im}\lambda| \leq e^{a\operatorname{Re}\lambda}\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leq M|\lambda|^{\alpha}, \quad \lambda \in E(a,b). \tag{2.1}$$

Proof. Let $x \in X$. Define $S(t)x = v'(t)$, where $v(t)$ is the unique solution of $C_{\alpha+1}(\tau)$ at x. One may prove that $S(t) \in \mathbf{B}(X)$ and

$$A\int_0^t S(s)x\,ds = S(t)x - \frac{t^{\alpha}}{\Gamma(\alpha+1)}x \quad \text{for all } x \in X$$

in a similar way to [AEK, Proposition 2.3]. Next, given $t \in [0, \tau)$, define the finite Laplace transform of $S(t)$ by

$$L_\lambda(t)x = \int_0^t e^{-\lambda s} S(s)x\, ds, \qquad \forall x \in X.$$

By a simple computation, we have, for every $x \in X$, $L_\lambda(t)x \in D(A)$ and

$$(\lambda - A)L_\lambda(t)x = e^{-\lambda t}(g_\lambda(t) - S(t)x), \qquad \forall \lambda \in \mathbb{C}, \tag{2.2}$$

where

$$g_\lambda(t) = \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)}\, ds.$$

When $\alpha > 1$, integration by parts yields

$$g_\lambda(t) = \frac{e^{\lambda t}}{\lambda^{[\alpha]}} \int_0^t e^{-\lambda s} \frac{s^{\alpha-[\alpha]-1}}{\Gamma(\alpha - [\alpha])}\, ds - q_\lambda(t), \tag{2.3}$$

where $[\alpha]$ is the maximal integer less than or equal to α, and

$$q_\lambda(t) = \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\lambda^2\Gamma(\alpha-1)} + \ldots + \frac{t^{\alpha-[\alpha]}}{\lambda^{[\alpha]}\Gamma(\alpha - [\alpha] + 1)}.$$

For $|\lambda| > 1$, we have

$$|q_\lambda(t)| \le q_1(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + \ldots + \frac{t^{\alpha-[\alpha]}}{\Gamma(\alpha - [\alpha] + 1)}. \tag{2.4}$$

If we define $q_\lambda(t) = q_1(t) \equiv 0$ for $0 < \alpha < 1$, then (2.3) and (2.4) are valid for all $\alpha > 0$.

Let $0 < t' < t < \tau$. Substituting $s = tu$ in the following, one obtains

$$\begin{aligned}\int_0^t e^{-\lambda s} \frac{s^{\alpha-[\alpha]-1}}{\Gamma(\alpha-[\alpha])}\, ds &= t^{\alpha-[\alpha]} \int_0^1 e^{-\lambda t u} \frac{u^{\alpha-[\alpha]-1}}{\Gamma(\alpha-[\alpha])}\, du \\ &= t^{\alpha-[\alpha]} K(1, \alpha - [\alpha] - 1, -\lambda t),\end{aligned}$$

where $K(1, \alpha - [\alpha] - 1, -\lambda t)$ is the Kummer function. By its asymptotic formula (cf. [Er, p. 278]) one has

$$\begin{aligned}K(1, \alpha - [\alpha] - 1, -\lambda t) &= (-\lambda t)^{-1} e^{-\lambda t}(1 + O(|\lambda t|^{-1})) \\ &\quad + \Gamma(\alpha - [\alpha]) e^{\pm i(\alpha - [\alpha])\pi} (-\lambda)^{[\alpha]-\alpha} (1 + O(|\lambda t|^{-1})) \\ &= \lambda^{-1} e^{-\lambda t} O(1) + \lambda^{[\alpha]-\alpha} O(1), \qquad |\lambda| \to \infty,\end{aligned}$$

where we choose $+$ (respectively $-$) if $-\pi/2 < \arg\lambda < 3\pi/2$ (respectively $-3\pi/2 < \arg\lambda < \pi/2$). Since $[\alpha] - \alpha < 1$, there are constants C_1 and C_2

such that when $|\lambda| \geq C_1$ we have

$$\int_0^t e^{-\lambda s} \frac{s^{\alpha-[\alpha]-1}}{\Gamma(\alpha-[\alpha])}\, ds \geq C_2|\lambda|^{[\alpha]-\alpha};$$

combining (2.3), (2.4) with this leads to

$$|g_\lambda(t)| \geq \frac{|e^{\lambda t}|}{|\lambda|^\alpha} - |q_1(t)| = \frac{e^{t\,\mathrm{Re}\,\lambda}}{|\lambda|^\alpha} - |q_1(t)| \tag{2.5}$$

for $|\lambda| > \max\{C_1, 1\}$. Let $0 < q < 1$ and choose $b' \geq \max\{C_1, 1\}$ large enough such that, when $\mathrm{Re}\,\lambda \geq b'$, we have

$$e^{(2t/\alpha)\,\mathrm{Re}\,\lambda} \exp\biggl(-\frac{2}{\alpha}\ln\biggl(\frac{\|S(t)\|}{q} + |q_1(t)|\biggr)\biggr) \geq e^{(2t'/\alpha)\,\mathrm{Re}\,\lambda} + (\mathrm{Re}\,\lambda)^2.$$

It follows that, for $a = t'/\alpha$ and $\lambda \in E(a, b')$,

$$|\lambda|^2 \leq e^{(2t/\alpha)\,\mathrm{Re}\,\lambda} \exp\biggl(-\frac{2}{\alpha}\ln\biggl(\frac{\|S(t)\|}{q} + |q_1(t)|\biggr)\biggr),$$

which implies that $e^{t\,\mathrm{Re}\,\lambda}/|\lambda|^\alpha \geq |q_1(t)| + \|S(t)\|/q$. Therefore for $\lambda \in E(a, b')$, we have $|g_\lambda(t)| \geq \|S(t)\|/q$. So the operator $g_\lambda(t) - S(t)$ is invertible. From (2.2), as in the proof of [AEK, Proposition 2.5], we can show that $E(a, b') \subset \varrho(A)$ and $R(\lambda, A) = L_\lambda(t)e^{\lambda t}(g_\lambda(t) - S(t))^{-1}$, so

$$\begin{aligned}\|R(\lambda, A)\| &\leq \|L_\lambda(t)\| \cdot \|e^{\lambda t}(g_\lambda(t) - S(t))^{-1}\| \\ &\leq M_1 e^{t\,\mathrm{Re}\,\lambda}(g_\lambda(t))^{-1}(1-q)^{-1}.\end{aligned}$$

Thus (2.1) follows easily from (2.5) for $0 < \alpha < 1$; and for $\alpha > 1$,

$$e^{t\,\mathrm{Re}\,\lambda}(g_\lambda(t))^{-1} \leq e^{t\,\mathrm{Re}\,\lambda}\biggl(\frac{e^{t\,\mathrm{Re}\,\lambda}}{|\lambda|^\alpha} - |q_\lambda(t)|\biggr)^{-1} = \biggl[\frac{1}{\lambda^\alpha}\biggl(1 - \frac{1}{e^{t\,\mathrm{Re}\,\lambda}}h_\lambda(t)\biggr)\biggr]^{-1},$$

where

$$h_\lambda(t) := \frac{1}{e^{t\,\mathrm{Re}\,\lambda}}\biggl(\frac{(|\lambda|t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(|\lambda|t)^{\alpha-2}}{\Gamma(\alpha-1)} + \ldots + \frac{(|\lambda|t)^{\alpha-[\alpha]}}{\Gamma(\alpha-[\alpha]+1)}\biggr).$$

For $\lambda \in E(a, b')$, we have $e^{t\,\mathrm{Re}\,\lambda} \geq C|\lambda|^\alpha$ for some positive constant C. So $h_\lambda(t) \to 0$ as $\mathrm{Re}\,\lambda \to \infty$. Choosing $b \geq b'$ large enough such that $h(\lambda) < 1/2$ when $\mathrm{Re}\,\lambda \geq b$, one gets (2.1). ■

The proofs of the following results are similar to [AEK, Theorem 2.2], with the exception that β here can be chosen to be a noninteger.

THEOREM 2.2. *Let $a > 0$, $b > 0$, $-1 < \alpha$. Suppose that $E(a, b) \subset \varrho(A)$ and* (2.1) *holds. Then for every $\beta > \alpha + 1$ and $\tau = a(\beta - \alpha - 1)$, $C_{\beta+1}(\tau)$ is well-posed.*

From the proof of Theorem 2.1, we know that if $C_{\alpha+1}(\tau)$ is well-posed, then the solution operators $(S(t))_{0\leq t<\tau}$ defined by $S(t)x = u'(t, x)$, where

$u(t,x)$ is the solution of $C_{\alpha+1}(\tau)$ at x, are bounded operators. And $S(\cdot)$ has all the properties in [AEK, Proposition 3.1] with k therein replaced by α. Moreover, the proof of [LHZ, Proposition 2.4] provides a way to show that $S(\cdot)$ is a local α-times integrated semigroup.

THEOREM 2.3. *Assume that $C_{\alpha+1}(\tau)$ is well-posed. Then the family $(S(t))_{0\le t<\tau}$ of its solution operators defined above is a local α-times integrated semigroup generated by A.*

We conclude this section with a rescaling result.

PROPOSITION 2.4. (a) *Let $0<\tau\le\infty$, $0<\alpha<\beta$. If $C_{\alpha+1}(\tau)$ is well-posed, then so is $C_{\beta+1}(\tau)$; and if $v(t)$ is the solution of $C_{\alpha+1}(\tau)$, then $j_{\beta-\alpha-1}*v(t):=\int_0^t j_{\beta-\alpha-1}(t-s)v(s)ds$ $(t\in[0,\tau))$ is the solution of $C_{\beta+1}(\tau)$, where $j_\beta(t):=t^\beta/\Gamma(\beta+1)$.*

(b) *If $C_{\alpha+1}(\tau)$ is well-posed and its solution is $v(t)$, then for each $r\in\mathbb{R}$, the problem $C^r_{\alpha+1}(\tau)$ corresponding to $A-r$ (obtained by replacing A by $A-r$ in $C_{\alpha+1}(\tau)$) is also well-posed and its solution is given by*

$$v^r(t)=e^{-rt}v(t)+\int_0^t\sum_{k=1}^{\infty}\frac{\langle\alpha+1\rangle_k r^k}{k!}\cdot\frac{s^{k-1}}{(k-1)!}\,e^{-r(t-s)}v(t-s)\,ds,$$

where $\langle\alpha+1\rangle_k=(\alpha+1)\alpha\ldots(\alpha-k+2)$.

Proof. (a) is easy to verify.

(b) Define

$$v^r(t)=e^{-rt}v(t)+\int_0^t\sum_{k=1}^{\infty}\frac{\langle\alpha+1\rangle_k r^k}{k!}\cdot\frac{s^{k-1}}{(k-1)!}\,e^{-r(t-s)}v(t-s)\,ds.$$

Differentiating this leads to

$$\begin{aligned}(v^r(t))'&=e^{-rt}\Big[Av(t)-rv(t)+\frac{t^\alpha}{\Gamma(\alpha+1)}\,x\Big]+\int_0^t\sum_{k=1}^{\infty}\frac{\langle\alpha+1\rangle_k r^k}{k!}\cdot\frac{s^{k-1}}{(k-1)!}\\&\qquad\cdot\Big[(A-r)v(t-s)+\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\,x\Big]e^{-r(t-s)}\,ds\\&=(A-r)\Big[e^{-rt}v(t)+\int_0^t\sum_{k=1}^{\infty}\frac{\langle\alpha+1\rangle_k r^k}{k!}\frac{s^{k-1}}{(k-1)!}\,e^{-r(t-s)}v(t-s)\,ds\Big]\\&\qquad+e^{-rt}\cdot\frac{t^\alpha}{\Gamma(\alpha+1)}\,x\\&\qquad+\int_0^t\sum_{k=1}^{\infty}\frac{\langle\alpha+1\rangle_k r^k}{k!}\cdot\frac{s^{k-1}}{(k-1)!}\,e^{-r(t-s)}\,\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\,x\,ds.\end{aligned}$$

Now we compute the Laplace transform of the last integral as follows:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\Big(\int_0^t \sum_{k=1}^\infty \frac{\langle \alpha+1\rangle_k r^k}{k!}\cdot\frac{s^{k-1}}{(k-1)!}\,e^{-r(t-s)}\,\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\,x\,ds\Big)dt \\
&= \sum_{k=1}^\infty \frac{\langle \alpha+1\rangle_k r^k}{k!}\Big(\int_0^\infty e^{-\lambda t}\,\frac{t^{k-1}}{(k-1)!}\,dt\Big)\Big(\int_0^\infty e^{-(\lambda+r)t}\,\frac{t^\alpha}{\Gamma(\alpha+1)}\,dt\Big)x \\
&= \sum_{k=1}^\infty \frac{\langle \alpha+1\rangle_k r^k}{k!}\cdot\frac{1}{\lambda^k}\cdot\frac{1}{(\lambda+r)^{\alpha+1}}\,x \\
&= \Big[\Big(1+\frac{r}{\lambda}\Big)^{\alpha+1}-1\Big]\frac{1}{(\lambda+r)^{\alpha+1}}\,x = \Big(\frac{1}{\lambda^{\alpha+1}}-\frac{1}{(\lambda+r)^{\alpha+1}}\Big)x \\
&= \int_0^\infty e^{-\lambda t}\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} - e^{-rt}\,\frac{t^\alpha}{\Gamma(\alpha+1)}\Big)x\,dt.\end{aligned}$$

By the uniqueness of Laplace transform, we have

$$e^{-rt}\,\frac{t^\alpha}{\Gamma(\alpha+1)}\,x + \int_0^t \sum_{k=1}^\infty \frac{\langle \alpha+1\rangle_k r^k}{k!}\cdot\frac{s^{k-1}}{(k-1)!}\,e^{-r(t-s)}\,\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}x\,ds = \frac{t^\alpha}{\Gamma(\alpha+1)}\,x;$$

so $v^r(t)$ is the solution of $C^r_{\alpha+1}(\tau)$. ■

3. The relations between α-times integrated semigroups and regularized semigroups. Let A be densely defined and generate an α-times integrated semigroup $(S^\alpha(t))_{t\ge 0}$ with $\|S^\alpha(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Then $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \varrho(A)$ and

$$R(\lambda, A) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S^\alpha(t)\,dt \quad \text{with} \quad \|R(\lambda,A)\| \le \frac{M|\lambda|^\alpha}{\operatorname{Re}\lambda - \omega}$$

for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > \omega$.

For $\sigma > 0$, set $A_{\omega+\sigma} = A - (\omega+\sigma)I$. Let $0 < a < \pi/2$, $0<\delta<\sigma$. Then $\Sigma_a \cup B_\delta \subseteq \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > -\sigma\} \subseteq \varrho(A_{\omega+\sigma})$, where

$$\Sigma_a := \{\lambda\in\mathbb{C} : |\arg\lambda| \le a\}\cup\{0\} \quad \text{and} \quad B_\delta := \{\lambda\in\mathbb{C} : |\lambda|\le\delta\},$$

and

$$\begin{aligned}\|R(\lambda, A_{\omega+\sigma})\| &= \|R(\lambda+\omega+\sigma, A)\| \le M\,\frac{|\lambda+\omega+\sigma|^\alpha}{\operatorname{Re}\lambda+\sigma} \\
&\le M\,\frac{(|\lambda|+\omega+\sigma)^\alpha}{|\lambda|\cos a+\sigma} \le M'(|\lambda|+1)^{\alpha-1}, \quad \forall \lambda\in\Sigma_a.\end{aligned}$$

Therefore, we can define the fractional powers of $-A_{\omega+\sigma}$. For every $\varepsilon > 0$, the operator defined by

$$(3.1)\qquad (-A_{\omega+\sigma})^{-(\alpha+\varepsilon)} = \frac{1}{2\pi i}\int\limits_{\Gamma}(-\lambda)^{-(\alpha+\varepsilon)}R(\lambda, A_{\omega+\sigma})x\,d\lambda$$

is bounded and injective, where Γ is a curve in Σ_a, and $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ is independent of σ (see [St2]). From [NS, Theorem 1.1] and its proof we know that, for every $x_0 \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$, the abstract Cauchy problem (ACP) has a unique mild solution $u(t)$ and for every $\sigma' > \sigma$ there exists a constant $M > 0$ such that $\|u(t)\| \le Me^{(\omega+\sigma')t}\|(-A_{\omega+\sigma})^{\alpha+\varepsilon}x_0\|$; the constant M may depend on σ' and ε but is independent of x_0. By [dL2, Theorem 5.17], we have the following

THEOREM 3.1. *Suppose that a densely defined operator A generates an α-times integrated semigroup $(S^\alpha(t))_{t\ge0}$ and $\|S^\alpha(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Then for every $\varepsilon > 0$ and $\sigma > 0$, $(-A_{\omega+\sigma})^{-(\alpha+\varepsilon)}$ defined by* (3.1) *is a bounded injective operator and A generates a $(-A_{\omega+\sigma})^{-(\alpha+\varepsilon)}$-regularized semigroup $(T(t))_{t\ge0}$. Moreover, for every $\sigma' > \sigma$, there exists a constant $M \ge 0$ such that $\|T(t)\| \le Me^{(\omega+\sigma')t}$.*

Now we turn to the converse problem. Suppose that

($A1$) A is a densely defined operator and there are constants $M, \omega \ge 0$, $\beta > 0$ and $0 < \varphi < \pi/2$ such that $\Sigma_\varphi + \omega \subseteq \varrho(A)$ and

$$\|R(\lambda, A)\| \le M(1+|\lambda|)^{\beta-1}, \quad \forall\lambda \in \Sigma_\varphi + \omega.$$

Then for every $\alpha > \beta$, $\sigma > 0$, $(-A_{\omega+\sigma})^{-\alpha}$, the fractional power of $-A_{\omega+\sigma}$, is well-defined and is a bounded injective operator on X.

If in addition,

($A2$) for some $\alpha > \beta$, $\sigma > 0$, A generates an exponentially bounded $(-A_{\omega+\sigma})^{-\alpha}$-regularized semigroup $(T(t))_{t\ge0}$,

then for every $x \in D((-A_{\omega+\sigma})^{\alpha+1})$, the abstract Cauchy problem (ACP) has a unique classical solution $u(t)$ given by $T(t)(-A_{\omega+\sigma})^\alpha x$ and $u'(t) = (\omega+\sigma)T(t)(-A_{\omega+\sigma})^\alpha x - T(t)(-A_{\omega+\sigma})^{\alpha+1}x$; both are exponentially bounded. So by [St2, Theorem 5.6], we have

THEOREM 3.2. *Suppose A satisfies* ($A1$) *and* ($A2$). *Then for every $\varepsilon > 0$, A generates an $(\alpha+\varepsilon)$-times integrated semigroup.*

REMARK 3.3. Since A is densely defined, a sufficient and necessary condition for A to satisfy ($A2$) is that for all $n \in \mathbb{N}$ and $\operatorname{Re}\lambda > \omega$, $D((-A_{\omega+\sigma})^\alpha) \subset R((\lambda - A)^n)$ and there exists a constant $M \in \mathbb{R}$ such that

$$\|(\operatorname{Re}\lambda - \omega)^n(\lambda - A)^{-n}(-A_{\omega+\sigma})^{-\alpha}x\| \le Mn!\|x\| \quad \text{for all } x \in X.$$

Under the conditions of Theorem 3.1 we know that if $\sigma > \sigma' > 0$, then $A_{\omega+\sigma}$ generates a uniformly bounded $(-A_{\omega+\sigma'})^{-(\alpha+\varepsilon)}$-regularized semigroup. Since $D((-A_{\omega+\sigma'})^{\alpha+\varepsilon})$ and $D((-A_{\omega+\sigma'})^{\alpha+\varepsilon+1})$ are independent of σ', the Cauchy problem

$$(\mathrm{ACP})_{\omega+\sigma} \qquad u'(t) = A_{\omega+\sigma}u(t) \quad (t \geq 0), \qquad u(0) = x,$$

has a bounded (respectively mild) solution for every $x \in D((-A_{\omega+\sigma})^{\alpha+\varepsilon+1})$ (respectively $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$). Therefore we can discuss some asymptotic properties of such solutions.

By [ZL, Theorem 2.1] and Theorem 3.1 we have

THEOREM 3.4. *Suppose that a densely defined operator A generates an α-times integrated semigroup $(S^\alpha(t))_{t\geq 0}$ on X and $\|S^\alpha(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Suppose also that for some $\sigma > 0$,* $\mathrm{span}\{x \in D(A) : Ax = (\omega+\sigma+ir)x$ *for some* $r \in \mathbb{R}\}$ *is dense in X. Then all solutions of $(\mathrm{ACP})_{\omega+\sigma}$ for initial values in $D((-A_{\omega+\sigma})^{\alpha+\varepsilon+1})$ are almost periodic.*

The following ergodic result follows from Theorem 3.1 and [LHC].

THEOREM 3.5. *Suppose that a densely defined operator A generates an α-times integrated semigroup $(S^\alpha(t))_{t\geq 0}$ on X and $\|S^\alpha(t)\| \leq Me^{\omega t}$ for every $t \geq 0$. Let $\varepsilon > 0$. Then the following statements are equivalent.*

(a) *all mild solutions of $(\mathrm{ACP})_{\omega+\sigma}$ for initial values in $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ are strongly Abel-ergodic*;

(b) *all mild solutions of $(\mathrm{ACP})_{\omega+\sigma}$ for initial values in $D((-A_{\omega+\sigma})^{\alpha+\varepsilon})$ are weakly Abel-ergodic*;

(c) *for every $x \in X$, $\{\lambda(\lambda - A_{\omega+\sigma})^{-1}(-A_{\omega+\sigma})^{-(\alpha+\varepsilon)}x : 0 < \lambda < 1\}$ is weakly sequentially compact*, *i.e. there exists a sequence $\{\lambda_n\}$, $\lambda_n \to 0$, and some $y \in X$ such that*

$$\text{weak-}\lim_{n\to\infty} \lambda_n(\lambda_n - A_{\omega+\sigma})^{-1}(-A_{\omega+\sigma})^{-(\alpha+\varepsilon)}x = y;$$

(d) $D((-A_{\omega+\sigma})^{\alpha+\varepsilon}) \subseteq \ker A_{\omega+\sigma} \oplus \overline{R(A_{\omega+\sigma})}$.

If a family $\{(S_n(t))_{t\geq 0}\}_{n=1}^{\infty}$ of α-times integrated semigroups satisfying $\|S_n(t)\| \leq Me^{\omega t}$ for some constants $M, \omega \geq 0$ converges to an α-times integrated semigroup $(S_0(t))_{t\geq 0}$, then so do the resolvents of their generators. Since the solution of the corresponding Cauchy problem can be obtained by an integral of the resolvents (see [NS]), we have the following approximation theorem.

THEOREM 3.6. *Suppose that for every $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, A_n generates an α-times integrated semigroup $(S_n(t))_{t\geq 0}$ on X and there exists a constant $\omega > 0$ such that $\|S_n(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and $n \in \mathbb{N}_0$. If $S_n(t)x \to S_0(t)x$ as $n \to \infty$ for all $x \in X$ and $t \geq 0$, then, for every $\varepsilon > 0$ and every $x \in D((-(A_0)_{\omega+\sigma})^{\alpha+\varepsilon+1})$, the solution of $u'(t) = A_0u(t), u(0) = x$ can be* *approximated by $u_n(t)$ which is the solution of $u_n'(t) = A_n u_n(t), u_n(0) = x_n$ and $x_n \to x$.*

Finally, we apply Theorem 3.1 to elliptic differential operators. From [Zh] we know that if a polynomial P is elliptic and $\omega := \sup_{\xi\in\mathbb{R}^n} \operatorname{Re} P(\xi) < \infty$, then for every $\alpha > n_p := n|1/2-1/p|$, $P(D)$ generates an α-times integrated semigroup; thus Theorem 3.1 yields

THEOREM 3.7. *Let P be an elliptic polynomial on $\mathbb{R}^n$. If*

$$\omega := \sup_{\xi\in\mathbb{R}^n} \operatorname{Re} P(\xi) < \infty,$$

then for every $\alpha > n_p := n|1/2 - 1/p|$ and $\omega' > \omega$, $P(D)$ generates an $(\omega' - P(D))^{-\alpha}$-regularized semigroup.

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