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Kergin interpolation in Banach spaces

by

HENRIK PETERSSON (Växjö)

Abstract. We study the Kergin operator on the space $H_{\mathrm{Nb}}(E)$ of nuclearly entire functions of bounded type on a Banach space E. We show that the Kergin operator is a projector with interpolating properties and that it preserves homogeneous solutions to homogeneous differential operators. Further, we show that the Kergin operator is uniquely determined by these properties. We give error estimates for approximating a function by its Kergin polynomial and show in this way that for any given bounded sequence of interpolation points and any nuclearly entire function, the corresponding sequence of Kergin polynomials converges.

1. Introduction. Let $(p_0, \ldots, p_d)$ be a sequence of points in the complex plane. Let f be an entire function in one variable. Then there is a unique polynomial F of degree at most d such that

$$F^{(k)}(p_j) = f^{(k)}(p_j), \quad k = 0, \ldots, k_j - 1,\ j = 0, \ldots, d, \tag{1}$$

where k_j is the number of repetitions of the point p_j in the sequence $(p_0, \ldots, p_d)$. If all the points are different we obtain the Lagrange polynomial and at the other extreme case, when all are equal, F is the Taylor polynomial for f.

Consider now the n-variable case and a sequence of points $(p_0, \ldots, p_d)$ in $\mathbb{C}^n$. Then for every given entire function f there is a polynomial F, of degree at most d, that interpolates f analogously to (1), i.e. we replace derivatives of order k with directional derivatives of order k. In contrast to the one variable case, the polynomial F is, in general, not uniquely determined by this property.

More generally, let X be any (complex) vector space and let $(p_0, \ldots, p_d)$ be a sequence of points in X. Let f be an arbitrary Gateaux holomorphic function on X, i.e. f is an entire function on all finite-dimensional subspaces. We denote by $H_{\mathrm{G}}(X)$ the space of Gateaux holomorphic functions on X.

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We shall see (Theorem 1) that there is a polynomial F of degree at most d on X such that

$$(2) \quad D_x^k F(p_j) = D_x^k f(p_j), \quad x \in X, \quad k = 0, \dots, k_j - 1, \ j = 0, \dots, d,$$

where k_j is the number of repetitions of the point p_j in the sequence $(p_0, \dots, p_d)$. Here, and in what follows, D_h denotes the directional derivative along h. In fact, we show that the Kergin polynomial $F = K_p f$, with respect to the points $p = (p_0, \dots, p_d)$, satisfies (2) for any given Gateaux holomorphic function f. Thus, in the one variable case, the Kergin polynomial is nothing but the Lagrange resp. Taylor polynomial in the above-mentioned two extreme cases.

Even though the Kergin polynomial is not uniquely determined by its interpolating property (2), it is possible to characterize Kergin interpolation. In n variables, the following result is due to Filipsson and Calvi (see [3] or [7]): the Kergin operator, $f \mapsto K_p f$, on the space of entire functions is a projector that preserves homogeneous solutions to homogeneous partial differential operators. That is, if $P(D)f = 0$ for some homogeneous polynomial P, then $P(D)K_p f = 0$. Moreover, the Kergin projector is uniquely determined by its interpolating property and this property. An infinite-dimensional generalization of this result can be found in [11].

In this article we study the Kergin operator on the space of nuclearly entire functions of bounded type, $H_{\mathrm{Nb}}(E)$, on a Banach space E. An excellent exposition of the theory of $H_{\mathrm{Nb}}(E)$ can be found in Gupta's work [9] and we shall also refer to Dineen [4] and Dwyer [5]. We show that the Kergin operator is a continuous projector from $H_{\mathrm{Nb}}(E)$ onto the nuclear polynomials and that, in addition to its interpolating property, it preserves homogeneous solutions to homogeneous differential equations (Theorem 8). Further, we give error estimates for approximating a function by its Kergin polynomial (Theorem 11). Our result shows that for any given bounded sequence of interpolation points and any nuclearly entire function, the corresponding sequence of Kergin polynomials converges to the function under consideration. In the finite-dimensional theory, such convergence problems have been studied in [2], [10] and in [8] where Gelfond's classical theorem about convergence of Lagrange interpolants to entire functions in one variable can be found.

Some of the results in this article are taken from the author's doctoral thesis [11].

2. The Kergin operator. Let X be a complex vector space. For any Gateaux holomorphic function $f \in H_{\mathrm{G}}(X)$ and sequence $p = (p_0, \dots, p_d)$ of points in X let

$$(3) \qquad \int\limits_{[p]} f \equiv \int\limits_{S_d} f(p_0 + s_1(p_1 - p_0) + \ldots + s_d(p_d - p_0))\, ds,$$

where $S_d \equiv \{s = (s_k \geq 0) \in \mathbb{R}^d : \sum s_k \leq 1\}$ denotes the simplex in $\mathbb{R}^d$. If $p = (p_0)$, i.e. if p consists only of one point, let $\int_{[p]} f \equiv f(p_0)$. Note that $\int_{[p]} f$ does not depend on the ordering of the points p_j. Indeed, with an obvious change of variables in (3) we may permute any of the points with p_0 and thus any two points. Next, for a given sequence of points $p = (p_0, \ldots, p_d)$, the *Kergin operator* with respect to p is the operator $K_p : H_{\mathrm{G}}(X) \to H_{\mathrm{G}}(X)$ defined by

$$(4) \qquad K_p f(x) \equiv \int\limits_{[p_0]} f + \int\limits_{[p_0,p_1]} D_{x-p_0} f + \ldots + \int\limits_{[p_0,\ldots,p_d]} D_{x-p_{d-1}} \ldots D_{x-p_0} f.$$

Recall that $D_h f(x)$ denotes the directional derivative at x along h.

REMARK 1. Suppose $f \in H_{\mathrm{G}}(X)$ has the form $f = h \circ \pi_F$ for some entire function h in n variables. Here $F = (y_1, \ldots, y_n) \in X^* \times \ldots \times X^*$, where X^* denotes the algebraic dual, and π_F is the projector $\pi_F : X \to \mathbb{C}^n$ defined by $\pi_F(x) \equiv (\langle x, y_k \rangle)$. We say that f is an X^*-*cylinder function* of order n. If $p = (p_0, \ldots, p_d)$ then $\int_{[p]} f = \int_{[\pi_F(p)]} h$ where $\pi_F(p) \equiv (\pi_F(p_0), \ldots, \pi_F(p_d))$. In view of this it is easily checked that $K_p[h \circ \pi_F] = [K_{\pi_F(p)} h] \circ \pi_F$ and, in particular, $K_p f = [K_{\pi_y(p)} h] \circ \pi_y$ if $f = h \circ \pi_y$ is a cylinder function of order one.

The Kergin operator on function spaces with finite-dimensional domains is studied in [1, 6]. Many of its properties depend only on the properties of the functions under consideration restricted to finite-dimensional subspaces and are therefore easily extended to Gateaux holomorphic functions. If V is any finite-dimensional subspace containing the points p_j, then $[K_p f]_V = K_p f_V$ where $[\cdot]_V$ denotes the restriction of $[\cdot]$ to V. We denote by $\mathcal{P}_{\mathrm{A}}^d(X)$ the space of polynomials on X of degree at most d. That is, $\mathcal{P}_{\mathrm{A}}^d(X)$ is the space spanned by $\bigcup_{n \leq d} \mathcal{P}_{\mathrm{A}}({}^n X)$ in $H_{\mathrm{G}}(X)$, where $\mathcal{P}_{\mathrm{A}}({}^n X)$ denotes the n-homogeneous polynomials on X.

THEOREM 1. *Let X be any complex vector space and let $f \in H_{\mathrm{G}}(X)$. For any finite sequence $p = (p_0, \ldots, p_d)$ of points in X we have*:

(i) *$K_p f$ does not depend on the ordering of the points.*

(ii) *$K_p f$ interpolates f at the points of p in the sense of* (2).

(iii) *$K_p K_p f = K_p f$, and $K_p f = f$ if $f \in \mathcal{P}_{\mathrm{A}}^d(X)$, i.e. K_p is a projector in $H_{\mathrm{G}}(X)$ onto $\mathcal{P}_{\mathrm{A}}^d(X)$.*

Proof. We prove (i) by induction on d. When $d = 1$ we have $D_{x-p_0} f = D_{x-p_1} f + D_{p_1-p_0} f$ and the assertion follows easily. Assume now that it holds for $d = k$. Note that $K_{p^{k+1}} f(x) = K_{p^k} f(x) + \int_{[p^{k+1}]} D_{x-p_k} \ldots D_{x-p_0} f$ where

$p^j \equiv (p_0, \dots, p_j)$. By the assumption that (i) holds for $d = k$, we may permute any of the points $(p_0, \dots, p_k)$ in the first term $K_{p^k} f$ and this is also true for the second term on the right hand side. In view of this, it suffices to show that we can exchange the roles of p_{k+1} and, say, p_k. In the same way as above,

$$K_{p^{k+1}} f(x) = K_{p^{k-1}} f(x) + \int_{[p^k]} D_{x-p_{k-1}} \dots D_{x-p_0} f + \int_{[p^{k+1}]} D_{x-p_k} \dots D_{x-p_0} f.$$

The first term on the right hand side does not depend on p_k and p_{k+1}. Hence we are left with establishing that we may exchange the positions of p_k and p_{k+1} in the last two expressions. Since integrals like (3) do not depend on the ordering of the points, this follows easily as in the case $d = 1$. We leave the details to the reader.

Next we prove (ii). This holds for finite-dimensional spaces ([7, Theorem 5.7]). Let $x_1, \dots, x_k$ be arbitrary points in X where $k \le k_j - 1$, $j \le d$. Let V be any finite-dimensional subspace containing the point x and the points of p. By the finite-dimensional result,

$$\begin{aligned} D_x^k K_p f(p_j) &= [D_x^k K_p f]_V(p_j) = D_x^k [K_p f]_V(p_j) = D_x^k K_p f_V(p_j) \\ &= D_x^k f_V(p_j) = [D_x^k f]_V(p_j) = D_x^k K_p f(p_j). \end{aligned}$$

This proves (ii).

(iii) follows easily from the corresponding result in n variables ([7, Theorem 5.7]). Indeed, let $x \in X$ be arbitrary and let $V \subseteq X$ be any finite-dimensional vector space containing x and the points p_j. If f_V denotes the restriction to V then $K_p f_V = [K_p f]_V$ and

$$\begin{aligned} K_p K_p f(x) &= [K_p K_p f]_V(x) = K_p [K_p f]_V(x) = K_p K_p f_V(x) \\ &= K_p f_V(x) = [K_p f]_V(x) = K_p f(x). \end{aligned}$$

In the same way $K_p f = f$ if $f \in \mathcal{P}^d(X)$ since $f_V \in \mathcal{P}^d_{\mathrm{A}}(V)$. ∎

The interpolating property of the Kergin operator implies the following "difference formula".

Lemma 2. *Let X be a complex vector space and let $f \in H_{\mathrm{G}}(X)$. Then for any finite sequence $p = (p_0, \dots, p_d)$ of points in X,*

$$f(x) - K_p f(x) = \int_{[p_0, \dots, p_d, x]} D_{x-p_0} \dots D_{x-p_d} f.$$

Proof. By Theorem 1 we have $f(x) = K_{(p_0, \dots p_d, x)} f(x)$ and the lemma follows from the observation that

$$K_{(p_0, \dots p_d, x)} f(x) = K_{(p_0, \dots p_d)} f(x) + \int_{[p_0, \dots, p_d, x]} D_{x-p_0} \dots D_{x-p_d} f. \quad ∎$$

The next theorem shows that, in some sense, the Kergin operator approximates functions with geometric error (compare [7, Theorem 6.2] and [11, Theorem 9.3]).

THEOREM 3. *Let X be a vector space and let $(p_0, p_1, \ldots)$ be a sequence of points in an absolutely convex set $P \subseteq X$. Then for an arbitrary absolutely convex set $B \subseteq X$ and $f \in H_{\mathrm{G}}(X)$ we have, for any j and $s > 0$,*

$$\|f - K_{p^j} f\|_B \le e^{-1}\left(\frac{e}{s}\right)^{j+1} \|f\|_{3P+B+s(P+B)},$$

where $p^j = (p_0, \ldots, p_j)$ and $\|\cdot\|_A$ is the supremum over $A \subseteq X$.

Proof. We can assume that f is bounded on $[3P + B + s(P + B)]$. By Lemma 2,

$$(5)\qquad [f - K_{p^j} f](x) = \int\limits_{[p^j,x]} D_{x-p_0} \ldots D_{x-p_j} f = \int\limits_{S_{j+1}} D_{x-p_0} \ldots D_{x-p_j} f(\xi(s) + s_{j+1}x)\, ds$$

where

$$\xi(s) \equiv p_0 + s_1(p_1 - p_0) + \ldots + s_j(p_j - p_0) + s_{j+1}(-p_0) \in 3P, \quad s \in S_{j+1}.$$

Thus $\xi(s) + s_{j+1}x \in 3P + B \equiv U$ for all $s \in S_{j+1}$ and $x \in B$. By Cauchy's formula and the polarization formula (see for example [4, Proposition 1.8])

$$\begin{aligned}
|D_{x-p_0} \ldots D_{x-p_j} f(\xi(s) + s_{j+1}x)| &\le \sup_{h_k \in V,\, \xi \in U} |D_{h_0} \ldots D_{h_j} f(\xi)| \\
&\le \frac{(j+1)^{j+1}}{(j+1)!} \sup_{h \in V,\, \xi \in U} |D_h^{j+1} f(\xi)| \\
&\le \frac{(j+1)^{j+1}}{2\pi s^{j+1}} \sup_{h \in V,\, \xi \in U} \Big| \int\limits_0^{2\pi} f(\xi + she^{it}) e^{-i(j+1)t}\, dt \Big| \\
&\le \frac{(j+1)^{j+1}}{s^{j+1}} \|f\|_{U+sV}, \qquad s \in S_{j+1},\ x \in B,
\end{aligned}$$

where $V \equiv P + B$. Since $\int_{S_j} ds = 1/j!$ and $j^j \le e^{j-1} j!$, $j \in \mathbb{N}$, we have, for any $x \in B$,

$$|[f - K_{p^j} f](x)| \le \frac{(j+1)^{j+1}}{(j+1)!} \cdot \frac{1}{s^{j+1}} \|f\|_{U+sV} \le \frac{e^j}{s^{j+1}} \|f\|_{U+sV}$$

and hence the assertion. ∎

3. The Kergin operator on the nuclearly entire functions. In this section we study the Kergin operator on the space of nuclearly entire functions of bounded type, $H_{\mathrm{Nb}}(E)$, on a Banach space E. The objective is to prove that the Kergin operator on $H_{\mathrm{Nb}}(E)$, in addition to its interpolating property (Theorem 1), is a projector that preserves homogeneous solutions to homogeneous differential operators, and that it is uniquely determined by these properties. We end up by deriving an error formula, in terms of the seminorms on $H_{\mathrm{Nb}}(E)$, for approximating a function by its Kergin polynomial.

A *Banach pairing* is a pairing (E, F) where E and F are Banach spaces and where $F = E'$ or the other way round. Thus the norm topologies on E and F are the (strong) topologies $\beta(E, F)$ and $\beta(F, E)$ respectively. Let (E, F) be a fixed Banach pairing. We denote by $\mathcal{P}(^nE)$ the space of continuous n-homogeneous polynomials on E equipped with the usual norm topology, and by $\mathcal{P}_{\mathrm{N}}(^nE)$ the nuclear n-homogeneous polynomials. That is, $\mathcal{P}_{\mathrm{N}}(^nE)$ is the space of all polynomials P of the form

$$P(x) = \sum_{k=1}^{\infty} \lambda_k \langle \cdot, y_k \rangle^n, \qquad (|\lambda_k| \cdot \|y_k\|^n) \in \ell_1,$$

where λ_k are complex numbers and y_k are vectors in F, provided with the nuclear norm $\|\cdot\|_n$ (to simplify the notation, we do not specify F in the notation $\mathcal{P}_{\mathrm{N}}(^nE)$). The *finite type polynomials* on E, i.e. the (dense) subspace of $\mathcal{P}_{\mathrm{N}}(^nE)$ spanned by polynomials of the form $\langle \cdot, y \rangle^n$, $y \in F$, is denoted by $\mathcal{P}_{\mathrm{F}}(^nE)$. If F has the approximation property, then the map $\mathcal{F}_n$ defined by $\mathcal{F}_n\lambda(y) \equiv \langle \lambda, \langle \cdot, y \rangle^n \rangle$ defines a topological isomorphism between $\mathcal{P}'_N(^nE)$ and $\mathcal{P}(^nF)$ (when $(E, F) = (E, E')$ see Dineen [4, Proposition 2.10] or Gupta [9, Lemma 4, page 59], and when $(E, F) = (F', F)$ see Dwyer [5, Proposition I.1]). In this way we may put $\mathcal{P}_{\mathrm{N}}(^nE)$ and $\mathcal{P}(^nF)$ in duality by $\langle P, Q \rangle_n \equiv \mathcal{F}_n^{-1}Q(P)$ where $P \in \mathcal{P}_{\mathrm{N}}(^nE)$ and $Q \in \mathcal{P}(^nF)$.

If (E, F) is a Banach pairing, then $H_{\mathrm{Nb}}(E)$ is the space of all Gateaux holomorphic functions $f \in H_{\mathrm{G}}(E)$ such that $f_n \equiv D^n_{(\cdot)} f(0)/n! \in \mathcal{P}_{\mathrm{N}}(^nE)$, $n = 0, 1, \ldots,$ and

$$\|f\|_{N:r} \equiv \sum r^n \|f_n\|_n < \infty, \qquad r > 0,$$

equipped with the seminorms thus defined. In this way, the nuclearly entire functions of bounded type, $H_{\mathrm{Nb}}(E)$, are a Fréchet space. Note that $f = \sum f_n$, $f_n \equiv D^n_{(\cdot)} f(0)/n!$, with convergence in $H_{\mathrm{Nb}}(E)$, for every $f \in H_{\mathrm{Nb}}(E)$. The exponential type functions on F, $\mathrm{Exp}(F)$, are the space of functions $\varphi \in H_{\mathrm{G}}(F)$ such that $\varphi_n \equiv D^n_{(\cdot)}\varphi(0)/n! \in \mathcal{P}(^nF)$, $n = 0, 1, \ldots,$ and $\limsup\, [n! |||\varphi_n|||_n]^{1/n} < \infty$, or equivalently, such that $|\varphi(y)| \le Ce^{c\|y\|}$ for some $c, C \ge 0$. Here $|||\cdot|||_n$ denotes the norm on $\mathcal{P}(^nF)$. By the duality between $\mathcal{P}_{\mathrm{N}}(^nE)$ and $\mathcal{P}(^nF)$, when F has the approximation property, the Fourier–Borel transform $\mathcal{F}$, defined by $\mathcal{F}\lambda(y) \equiv \langle \lambda, e_y \rangle$, is an isomorphism between $H'_{\mathrm{Nb}}(E)$ and $\mathrm{Exp}(F)$ and we may put $H_{\mathrm{Nb}}(E)$ and $\mathrm{Exp}(F)$

in duality by

$$(6)\quad \langle f,\varphi\rangle \equiv \mathcal{F}^{-1}\varphi(f)=\sum_{n=0}^{\infty} n!\langle f_n,\varphi_n\rangle_n,\quad f\in H_{\mathrm{Nb}}(E),\ \varphi\in \operatorname{Exp}(F).$$

Multiplication $\psi\mapsto\varphi\psi$, $\varphi\in\operatorname{Exp}(F)$, is weakly continuous on $\operatorname{Exp}(F)$ for the duality between $H_{\mathrm{Nb}}(E)$ and $\operatorname{Exp}(F)$. Thus, for any given $\varphi\in\operatorname{Exp}(F)$, $\varphi(D)\equiv {}^t\varphi : H_{\mathrm{Nb}}(E)\to H_{\mathrm{Nb}}(E)$. Such differential operators are continuous. In this way one obtains the family of all continuous convolution operators on $H_{\mathrm{Nb}}(E)$ (a *convolution operator* is an operator that commutes with translations). In particular, homogeneous differential operators $P(D)$, $P\in\mathcal{P}({}^nF)$, $n=0,1,\ldots$, are homogeneous convolution operators.

REMARK 2. If $P=\langle x_1,\cdot\rangle\ldots\langle x_n,\cdot\rangle$, $x_i\in E$, then $P(D)=D_{x_1}\ldots D_{x_n}$. Hence, if $P=\sum_k\lambda_k\langle x_k,\cdot\rangle^n\in\mathcal{P}_{\mathrm{N}}({}^nF)$ $(\subseteq\mathcal{P}({}^nF))$, then $P(D)f=\sum_k\lambda_k D^n_{x_k}f$ with absolute convergence in $H_{\mathrm{Nb}}(E)$.

DEFINITION 1 ([3, 7]). Let (E,F) be a Banach pairing. A continuous linear projector $\Pi : H_{\mathrm{Nb}}(E)\to\mathcal{P}^d_{\mathrm{N}}(E)$ is *PDE-preserving* on $H_{\mathrm{Nb}}(E)$ if

$$P(D)f=0\ \Rightarrow\ P(D)\Pi f=0,\qquad \forall P\in\mathcal{P}({}^nF),\ n=0,1,\ldots,$$

i.e. if it preserves homogeneous solutions to homogeneous differential (convolution) operators.

LEMMA 4. *Let (E,F) be a Banach pairing and let $p=(p_0,\ldots,p_d)$ be points in E. The map $f\mapsto\int_{[p]}f$ defines a continuous linear functional on $H_{\mathrm{Nb}}(E)$.*

Proof. The (nuclear) topology on $H_{\mathrm{Nb}}(E)$ is finer than the topology of uniform convergence on bounded sets in E. Thus it suffices to prove the continuity for this latter topology. If $\|p_j\|\le r$, $j=0,\ldots,d$, we obtain

$$\Big|\int_{[p]} f\Big| = \Big|\int_{S_d} f\Big(p_0+\sum_{k=1}^{d}s_k(p_j-p_0)\Big)\,ds\Big| \le \sup_{\|x\|\le 3r}|f(x)|/d!.$$

This completes the proof. ∎

We denote by $\mathcal{P}^d_{\mathrm{N}}(E)$ the nuclear polynomials on E of degree at most d. That is, $\mathcal{P}^d_{\mathrm{N}}(E)$ is the space spanned by $\bigcup_{n\le d}\mathcal{P}_{\mathrm{N}}({}^nE)$ in $H_{\mathrm{Nb}}(E)$.

LEMMA 5. *Let (E,F) be a Banach pairing and let $p=(p_0,\ldots,p_d)$ be points in E. Then the Kergin operator $f\mapsto K_pf$ defines a continuous linear projector from $H_{\mathrm{Nb}}(E)$ onto $\mathcal{P}^d_{\mathrm{N}}(E)$.*

Proof. A map of the form $f\mapsto D_{h_0}\ldots D_{h_j}f$, $h_j\in E$, is continuous on $H_{\mathrm{Nb}}(E)$. In view of this, and by Theorem 1, it suffices to prove that $f\mapsto\int_{[p]}D^m_{(\cdot)}f$ defines a continuous map between $H_{\mathrm{Nb}}(E)$ and $\mathcal{P}_{\mathrm{N}}({}^mE)$.

Assume first that $f_n = \sum_k \lambda_k \langle \cdot, y_k\rangle^n \in \mathcal{P}_{\mathrm{N}}({}^nE)$. If $m > n$, then $D_x^m f_n = 0$ so assume that $m \le n$. Then

$$D_x^m f_n = \sum_k \lambda_k \frac{n!}{(n-m)!} \langle x, y_k\rangle^m \langle \cdot, y_k\rangle^{n-m}$$

with absolute convergence in $H_{\mathrm{Nb}}(E)$. Hence, by Lemma 4,

$$\int_{[p]} D_x^m f_n = \sum_k \Big[\lambda_k \frac{n!}{(n-m)!} \int_{[p]} \langle \cdot, y_k\rangle^{n-m}\Big] \langle x, y_k\rangle^m = \sum_k \mu_k \langle x, y_k\rangle^m.$$

Now if $\|p_j\| \le r$, $j = 0, \ldots, d$, then

$$\begin{aligned} |\mu_k| \cdot \|y_k\|^m &\le |\lambda_k| \frac{n!}{(n-m)!} \|y_k\|^m \Big| \int_{[p]} \langle \cdot, y_k\rangle^{n-m} \Big| \\ &\le |\lambda_k| \frac{n!}{(n-m)!} \cdot \frac{[3r]^{n-m}}{d!} \|y_k\|^n. \end{aligned}$$

Hence $f_n^m \equiv \int_{[p]} D_{(\cdot)}^m f_n \in \mathcal{P}_{\mathrm{N}}({}^mE)$ and

$$\|f_n^m\|_m \le \frac{n!}{(n-m)!} \cdot \frac{[3r]^{n-m}}{d!} \|f_n\|_n. \tag{7}$$

Now let $f = \sum f_n \in H_{\mathrm{Nb}}(E)$ where $f_n \in \mathcal{P}_{\mathrm{N}}({}^nE)$. By the continuity of $f \mapsto D_x^m f$ on $H_{\mathrm{Nb}}(E)$ we have $D_x^m f = \sum_{n\ge m} D_x^m f_n$. Hence, by Lemma 4, $\int_{[p]} D_x^m f = \sum_{n\ge m} f_n^m$ where $f_n^m(x) = \int_{[p]} D_x^m f_n$. By (7) this series converges absolutely in $\mathcal{P}_{\mathrm{N}}({}^mE)$ and

$$\begin{aligned} \sum_{n\ge m} \|f_n^m\|_m &\le \frac{[3r]^{-m}}{d!} \sum_{n\ge m} \frac{n![3r]^n}{(n-m)!} \|f_n\|_n \le \frac{m!}{d![3r]^m} \sum_{n\ge m} [6r]^n \|f_n\|_n \\ &\le \frac{m!}{d![3r]^m} \|f\|_{N:6r}. \end{aligned}$$

This also shows the continuity of $f \mapsto \int_{[p]} D_{(\cdot)}^m f$ and the proof is complete. ■

LEMMA 6. *Let* (E, F) *be a Banach pairing where* F *has the approximation property. Then* $\mathcal{P}_{\mathrm{F}}({}^nF)$ *is dense in* $\mathcal{P}({}^nF)$ *for the weak topology* $\sigma(\mathcal{P}({}^nF), \mathcal{P}_{\mathrm{N}}({}^nE))$ *and thus for the topology* $\sigma(\mathcal{P}({}^nF), H_{\mathrm{Nb}}(E))$ (*duality induced by the duality* (6) *between* $\mathrm{Exp}(F)$ *and* $H_{\mathrm{Nb}}(E)$).

Proof. Assume that $\mathcal{P}_{\mathrm{F}}({}^nF)$ is not dense in $\mathcal{P}({}^nF)$. Then there is a $0 \ne P \in \mathcal{P}_{\mathrm{N}}({}^nE)$ such that $\langle P, Q\rangle = 0$ for every $Q \in \mathcal{P}({}^nF)$. But with $Q = \langle x, \cdot\rangle^n$ this gives $0 = \langle P, Q\rangle = P(x)$ and hence $P = 0$ as x was arbitrary. This is a contradiction so $\mathcal{P}_{\mathrm{F}}({}^nF)$ is dense in $\mathcal{P}({}^nF)$. This also implies that $\mathcal{P}_{\mathrm{F}}({}^nF)$ is dense for $\sigma(\mathcal{P}({}^nF), H_{\mathrm{Nb}}(E))$ since by formula (6), $\langle f, Q\rangle = n!\langle f_n, Q\rangle_n$ for every $f \in H_{\mathrm{Nb}}(E)$ and $Q \in \mathcal{P}({}^nF) \subseteq \mathrm{Exp}(F)$. ■

LEMMA 7. *Let (E,F) be a Banach pairing, where F has the approximation property, and let $\Pi : H_{\mathrm{Nb}}(E) \to \mathcal{P}_{\mathrm{N}}^d(E)$ be a continuous projector. Assume that there are functionals $\lambda_j \in H'_{\mathrm{Nb}}(E)$, $j = 0,\ldots,d$, such that $\langle \lambda_j, 1\rangle = 1$ and*

$$\lambda_j D_x^j \Pi = \lambda_j D_x^j, \qquad x \in E,\ j = 0,\ldots,d. \tag{8}$$

Then Π is PDE-preserving.

REMARK 3. (8) is equivalent to $\lambda_j P(D)\Pi = \lambda_j P(D)$ for all $P \in \mathcal{P}_{\mathrm{F}}({}^jF)$, $j = 0,\ldots,d$. Since any derivative of the form $D_{x_1}\ldots D_{x_j}$ can be written as $P(D)$ for some $P \in \mathcal{P}_{\mathrm{F}}({}^jF)$ (see [9, Proposition 2]), (8) is also equivalent to $\lambda_j D_{x_1}\ldots D_{x_j}\Pi = \lambda_j D_{x_1}\ldots D_{x_j}$, $x_i \in E$, $j = 0,\ldots,d$.

Proof of Lemma 7. Assume that $P(D)f = 0$ where $P \in \mathcal{P}({}^nF)$ and $f \in H_{\mathrm{Nb}}(E)$. We must prove that $r \equiv P(D)\Pi f = 0$. If $d < n$ this is obvious so we can assume that $d \ge n$. We have $r \in \mathcal{P}_{\mathrm{N}}^{d-n}(E)$ and hence

$$r = r_0 + \ldots + r_{d-n}, \quad r_j \in \mathcal{P}_{\mathrm{N}}({}^jE).$$

We prove that $r_{d-n} = 0$. Let $x \in E$ be arbitrary and put $Q \equiv \langle x,\cdot\rangle^{d-n}/(d-n)! \in \mathcal{P}_{\mathrm{F}}({}^{d-n}F)$. Then $Q(D) = D_x^{d-n}/(d-n)!$ and $r_{d-n}(x) = Q(D)r_{d-n} = Q(D)r$ (= constant). By Lemma 6 there is a net $P_\alpha \in \mathcal{P}_F({}^{d-n}E)$ such that $P_\alpha \to P$ in $\sigma(\mathcal{P}({}^nF), H_{\mathrm{Nb}}(E))$. Now $QP_\alpha \in \mathcal{P}_{\mathrm{F}}({}^dF)$ and $Q(D)\circ P_\alpha(D) = QP_\alpha(D)$. Hence for $\varphi_d \equiv \mathcal{F}\lambda_d \in \mathrm{Exp}(F)$ we obtain, in view of Remark 3,

$$\begin{aligned} r_{d-n}(x) &= \lambda_d(Q(D)r) = \langle Q(D)P(D)\Pi f, \varphi_d\rangle = \langle \varphi_d(D)Q(D)\Pi f, P\rangle \\ &= \lim \langle \varphi_d(D)Q(D)\Pi f, P_\alpha\rangle = \lim \lambda_d(Q(D)P_\alpha(D)\Pi f) \\ &= \lim \lambda_d(Q(D)P_\alpha(D)f) = \lim\langle \varphi_d(D)Q(D)f, P_\alpha\rangle \\ &= \langle \varphi_d(D)Q(D)f, P\rangle = \lambda_d(Q(D)P(D)f) = 0. \end{aligned}$$

As x was arbitrary we conclude that $r_{d-n} = 0$. Repeating these arguments for $d-n-1, d-n-2, \ldots$ down to zero we obtain $r = 0$. ■

THEOREM 8. *Let (E,F) be a Banach pairing, where F has the approximation property, and let $p = (p_0,\ldots,p_d)$ be a sequence of points in E. Then the Kergin operator K_p on $H_{\mathrm{Nb}}(E)$ is PDE-preserving.*

Proof. By Lemma 4 the functionals λ_j defined by

$$\langle \lambda_j, f\rangle \equiv j! \int_{[p^j]} f, \qquad p^j = (p_0,\ldots,p_j),\ j = 0,\ldots,d,$$

are continuous. Moreover, $\langle \lambda_j, 1\rangle = 1$ for all j. Thus, by Theorem 7, we are done if we can prove that $\langle \lambda_j, D_x^j(f - K_p f)\rangle = 0$, that is,

$$\int_{[p^j]} D_x^j(f - K_p f) = 0$$

for all $x \in X$ and $j = 0, \ldots, d$. This is known to hold in the finite-dimensional case (see [7]). Thus if V is any finite-dimensional subspace containing the points in p and x we have

$$\int\limits_{[p^j]} D_x^j(f - K_p f) = \int\limits_{[p^j]} [D_x^j(f - K_p f)]_V = \int\limits_{[p^j]} D_x^j(f_V - [K_p f]_V)$$
$$= \int\limits_{[p^j]} D_x^j(f_V - K_p f_V) = 0$$

and hence the conclusion. ∎

The next theorem shows that the Kergin operator is uniquely determined by its interpolating and PDE-preserving properties on $H_{\mathrm{Nb}}(E)$. The n-dimensional version of the theorem was obtained by Filipsson [7]. An infinite-dimensional generalization can be found in [11] for entire functions in the ring of formal power series in an infinite number of variables.

Theorem 9. *Let (E, F) be a Banach pair where F has the approximation property. Let $\Pi : H_{\mathrm{Nb}}(E) \to \mathcal{P}_{\mathrm{N}}^d(E)$ be a continuous PDE-preserving projector that interpolates function values at the points $(p_0, \ldots, p_d)$ in E in the sense of* (2). *Then Π is the Kergin operator K_p on $H_{\mathrm{Nb}}(E)$.*

Proof. Linear combinations of functions of the form $h_y = h(\langle \cdot, y \rangle)$, where $y \in F$ and h is an entire function in one variable (F-*cylinder functions* of order one), are dense in $H_{\mathrm{Nb}}(E)$. Thus it suffices to prove that $K_p = \Pi$ on such functions. If $y = 0$, then h_y is constant and hence $K_p h_y = h_y = \Pi h_y$. Further, every F-cylinder function, with non-zero element in F, is the limit of cylinder functions h_y for which $\widetilde{p}_j \equiv \langle p_j, y \rangle \neq 0$ whenever $p_j \neq 0$. Hence it suffices to prove $\Pi f = K_p f$ for an arbitrary function of the form $f = h_y$. By Remark 1, $K_p h_y = q(\langle \cdot, y \rangle)$ where $q = K_{\widetilde{p}} h$ is the Kergin polynomial of h with respect to the points $\widetilde{p} = (\widetilde{p}_j)$. Thus, q is the unique polynomial of degree at most d that interpolates h at the points $\widetilde{p}$ in the sense of (1). Thus we must prove that $\widetilde{f} = q(\langle \cdot, y \rangle)$ where $\widetilde{f} \equiv \Pi f$.

We note that $D_{x_0} f = \langle x_0, y \rangle h'(\langle \cdot, y \rangle) = 0$ for every $x_0 \in [y]^\perp$ ($[\cdot] =$ linear hull). The assumption that Π is PDE-preserving implies that such derivatives also vanish for $\widetilde{f}$. Hence $\widetilde{f}$ is constant on each hyperplane $Y(z) \equiv \{x : \langle x, y \rangle = z\}$, $z \in \mathbb{C}$. Indeed, let $Y(z_0)$ be a fixed hyperplane and let $x_0 \in Y(z_0)$. We must show $\widetilde{f}(x') = \widetilde{f}(x_0)$ for $x' \in Y(z_0)$, that is, $\widetilde{f}_{x_0;x'-x_0}(1) = \widetilde{f}_{x_0;x'-x_0}(0)$ where $\widetilde{f}_{x;\xi}(z) \equiv \widetilde{f}(x + z\xi)$. Thus it suffices to prove that $\widetilde{f}_{x_0;x'-x_0}$ is constant on $[0, 1]$. But since $x' - x_0 \in [y]^\perp$ we have

$$\widetilde{f}'_{x_0;x'-x_0}(t) = D_{x'-x_0} \widetilde{f}(x_0 + t(x' - x_0)) = 0, \quad t \in [0, 1],$$

and $\widetilde{f}_{x_0;x'-x_0}$ is constant. We now define the function $\widetilde{q}$ by $\widetilde{q}(z) \equiv \widetilde{f}(Y(z))$, where $\widetilde{f}(Y(z))$ is the constant value of $\widetilde{f}$ on the hyperplane $Y(z)$. Hence

$\widetilde{f}(x) = \widetilde{q}(\langle x, y\rangle)$ for all x. Moreover, $\widetilde{q}(z) = \widetilde{f}(x_z)$ where x_z is any element in $Y(z)$. If $x_1 \in Y(1)$ is arbitrary then $x_z \equiv zx_1 \in Y(z)$ for any given z. Hence $\widetilde{q}(z) = \widetilde{f}(zx_1)$ and $\widetilde{q}$ is entire. Since $\widetilde{f} \in \mathcal{P}^d_{\mathrm{N}}(E)$ is a polynomial of degree at most d, $\widetilde{q}$ must be a polynomial of degree less than or equal to d. If $x_1 \in Y(1)$, then $\widetilde{q}(z) = \widetilde{f}(zx_1)$ and hence

$$\widetilde{q}^{(k)}(z) = D^k_{x_1}\widetilde{f}(zx_1). \tag{9}$$

Thus if $p_j \neq 0$, and hence $\widetilde{p}_j \neq 0$ by assumption, for $z = \widetilde{p}_j$ and $x_1 = p_j/\widetilde{p}_j \in Y(1)$ we obtain, for $k \le k_j - 1$,

$$\widetilde{q}^{(k)}(\widetilde{p}_j) = D^k_{x_1}\widetilde{f}(p_j) = D^k_{x_1} f(p_j) = \langle x_1, y\rangle^k h^{(k)}(\widetilde{p}_j) = h^{(k)}(\widetilde{p}_j).$$

On the other hand, if $p_j = 0$, and hence $\widetilde{p}_j = 0$, for $z = 0 = \widetilde{p}_j$ and arbitrary $x_1 \in Y(1)$ we obtain

$$\widetilde{q}^{(k)}(0) = \widetilde{q}^{(k)}(\widetilde{p}_j) = D^k_{x_1}\widetilde{f}(p_j) = D^k_{x_1} f(p_j) = \langle x_1, y\rangle^k h^{(k)}(\widetilde{p}_j) = h^{(k)}(0)$$

for $k \le k_j - 1$. Summing up, $\widetilde{q}$ is a polynomial of degree at most d and interpolates h at the points $\widetilde{p}_j$ in the sense of (1). Hence $\widetilde{q} = q$ and we are done. ∎

As a consequence we obtain the following.

COROLLARY 10. *Let (E, F) be a Banach pairing where F has the approximation property. Let $\Pi : H_{\mathrm{Nb}}(E) \to \mathcal{P}^d_{\mathrm{N}}(E)$ be a continuous PDE-preserving projector. Then Π cannot interpolate function values at more than $d+1$ points in E in the sense of* (2).

Proof. Assume that there is a PDE-preserving projector Π that interpolates function values at the points $p = (p_0, \ldots, p_{d+1})$ in the sense of (2). By Theorem 9, Π is the Kergin projector with respect to the first $d+1$ points $(p_0, \ldots, p_d)$. By Remark 1, the Kergin polynomial for every entire function is the same for the points $(\widetilde{p}_0, \ldots, \widetilde{p}_d)$ as for the points $(\widetilde{p}_0, \ldots, \widetilde{p}_{d+1})$, $\widetilde{p}_j \equiv \langle p_j, y\rangle$, for every $y \in F$. This is impossible and the proof is complete. ∎

We now derive an error formula for Kergin approximation. Let B_r denote the closed ball with radius r centered at the origin. By Theorem 3 with $P = B_r$ and $B = B_\varrho$,

$$\|f - K_{p^j} f\|_{B_\varrho} \le e^{-1}\left(\frac{e}{s}\right)^{j+1}\|f\|_{B_R}, \quad R = R(r, \varrho, s) \equiv 3r + \varrho + s(r + \varrho).$$

However, our objective is to give an error formula in terms of the seminorms on $H_{\mathrm{Nb}}(E)$. In the finite-dimensional theory, such error formulae have been studied in [2], [8] and [10].

THEOREM 11. *Let (E, F) be a Banach pairing and let $f \in H_{\mathrm{Nb}}(E)$. Then for any sequence $(p_0, p_1, \ldots)$ of points in the closed ball B_r, we have,* *for any j and $\varrho > 0$,*

(10) $$\|f - K_{p^j} f\|_{N:\varrho} \le \Big\| f - \sum_{n=0}^{j} f_n \Big\|_{N:[2(2r+\varrho)]} \Big(= \sum_{n=j+1}^{\infty} [2(2r+\varrho)]^n \|f_n\|_n \Big),$$

where $p^j = (p_0, \dots, p_j)$.

Proof. By Lemmas 2 and 4 we have

(11) $$[f - K_{p^j} f](x) = \int_{[p^j,x]} D_{x-p_0} \dots D_{x-p_j} f = \sum_{n=j+1}^{\infty} \int_{[p^j,x]} D_{x-p_0} \dots D_{x-p_j} f_n = \sum_{n=j+1}^{\infty} F_n(x)$$

for $f = \sum f_n$ in $H_{\mathrm{Nb}}(E)$. Let $f_n = \sum \lambda_k \langle \cdot, y_k \rangle^n$ be an arbitrary representation of $f_n \in \mathcal{P}_{\mathrm{N}}({}^n E)$. Then F_n is a finite sum of terms of the form

(12) $$\int_{[p^j,x]} D_x^m D_{q_0} \dots D_{q_{j-m}} f_n, \qquad m \le j+1,$$

where $q_i \in \{p_0, \dots, p_j\}$. We note that

(13) $$D_x^m D_{q_0} \dots D_{q_{j-m}} f_n = \sum_{k=1}^{\infty} n \dots (n-j) \lambda_k \langle x, y_k \rangle^m \langle q_0, y_k \rangle \dots \langle q_{j-m}, y_k \rangle \langle \cdot, y_k \rangle^{n-j-1}.$$

Now $F_n^m(x) = \int_{S_{j+1}} D_x^m D_{q_0} \dots D_{q_{j-m}} f_n(\xi(s) + s_{j+1} x)\, ds$ where

$$\xi(s) \equiv p_0 + s_1(p_1 - p_0) \dots s_j(p_j - p_0) + s_{j+1}(-p_0) \in 3B_r, \quad s \in S_{j+1}.$$

The binomial formula gives

$$\langle \xi(s) + s_{j+1} x, y_k \rangle^{n-j-1} = \sum_{i=0}^{n-j-1} \binom{n-j-1}{i} \langle x, y_k \rangle^i s_{j+1}^i \langle \xi(s), y_k \rangle^{n-j-1-i}.$$

Hence, the expression (12) for F_n^m can be rewritten as

$$\begin{aligned} F_n^m(x) &= \sum_{i=0}^{n-j-1} \sum_{k=1}^{\infty} n \dots (n-j) \binom{n-j-1}{i} \lambda_k \langle q_0, y_k \rangle \dots \langle q_{j-m}, y_k \rangle \xi_{nk}^{ij} \langle x, y_k \rangle^{m+i} \\ &= \sum_{i=0}^{n-j-1} \sum_{k=1}^{\infty} \mu_{nk}^{ij} \langle x, y_k \rangle^{m+i}, \qquad \xi_{nk}^{ij} \equiv \int_{S_{j+1}} s_{j+1}^i \langle \xi(s), y_k \rangle^{n-j-1-i}\, ds. \end{aligned}$$

Hence $F_n^{mi} \equiv \sum \mu_{nk}^{ij} \langle \cdot, y_k \rangle^{m+i} \in \mathcal{P}_{\mathrm{N}}({}^{m+i}E)$. In fact, since $\xi(s) \in 3B_r$ and

$\int_{S_j} ds = 1/j!$, we see that $|\xi_{nk}^{ij}| \le [3r\|y_k\|]^{n-j-1-i}/(j+1)!$ and

$$\begin{aligned}|\mu_{nk}^{ij}|\cdot\|y_k\|^{m+i} &\le \frac{n\ldots(n-j)}{(j+1)!}\binom{n-j-1}{i} r^{j-m+1}[3r]^{n-j-1-i}|\lambda_k|\cdot\|y_k\|^n\\ &= \binom{n}{j+1}\binom{n-j-1}{i} r^{j-m+1}[3r]^{n-j-1-i}|\lambda_k|\cdot\|y_k\|^n.\end{aligned}$$

This implies $F_n^{mi} \in \mathcal{P}_{\mathrm{N}}({}^{m+i}E)$ and

$$\|F_n^{mi}\|_{m+i} \le \binom{n}{j+1}\binom{n-j-1}{i} r^{j-m+1}[3r]^{n-j-1-i}\|f_n\|_n.$$

Hence

$$\begin{aligned}\|F_n^m\|_{N:\varrho} &= \sum_{i=0}^{n-j-1} \varrho^{m+i}\|F_n^{mi}\|_{m+i}\\ &\le \|f_n\|_n\binom{n}{j+1}\varrho^m r^{j-m+1}\sum_{i=0}^{n-j-1}\binom{n-j-1}{i}[3r]^{n-j-1-i}\varrho^i\\ &= \|f_n\|_n\binom{n}{j+1}\varrho^m r^{j-m+1}[3r+\varrho]^{n-j-1}.\end{aligned}$$

Now, F_n is a finite sum $\sum_{m=0}^{j+1} F_{nm}$ and each F_{nm} is a sum of $\binom{j+1}{m}$ terms of the form F_n^m. By the binomial formula,

$$\begin{aligned}\|F_n\|_{N:\varrho} &\le \|f_n\|_n\binom{n}{j+1}[3r+\varrho]^{n-j-1}\sum_{m=0}^{j+1}\binom{j+1}{m} r^{j-m+1}\varrho^m\\ &= \|f_n\|_n\binom{n}{j+1}[3r+\varrho]^{n-j-1}[r+\varrho]^{j+1} \le [4r+2\varrho]^n\|f_n\|_n.\end{aligned}$$

Substituting into (11) completes the proof. ∎

Estimate (10) shows that for any bounded sequence of interpolation points and $f \in H_{\mathrm{Nb}}(E)$, the corresponding sequence of Kergin polynomials converges to f. More generally, from the proof it follows that for an arbitrary sequence of interpolation points, the estimate (10) holds with r replaced by $r_j \equiv \max\{\|p_i\| : 0 \le i \le j\}$. This gives a sufficient condition on a function in order that its sequence of Kergin polynomials converges. This holds for $f \in H_{\mathrm{Nb}}(E)$ when the (increasing) sequence $\sum_{n=0}^{j} r_j^n \varrho^n \|f_n\|_n$ is bounded for every $\varrho > 0$ (compare with [7, Theorem 7.1]).

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School of Mathematics and Systems Engineering
SE-351 95 Växjö, Sweden
E-mail: Henrik.Petersson@msi.vxu.se

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