Diameter-preserving maps on various classes of function spaces

by

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Abstract. Under some mild assumptions, non-linear diameter-preserving bijections between (vector-valued) function spaces are characterized with the help of a well-known theorem of Ulam and Mazur. A necessary and sufficient condition for the existence of a diameter-preserving bijection between function spaces in the complex scalar case is derived, and a complete description of such maps is given in several important cases.

Introduction. Several papers on diameter-preserving linear bijections of function spaces have appeared in recent years (see, for example, [GM], [GU], [S], [RR]). The present paper is a contribution to this circle of ideas, its principal motivation coming from an attempt to clarify and extend some of the results in [RR] in response to an interesting question raised there.

To give a summary in this introduction to our main results, let us first explain briefly the notations and terminology that we use (unexplained terms and notations will be found in [RR]). K, L, and S will generally denote compact convex sets in a Hausdorff locally convex topological vector space $X$ over $\mathbb{C}$ or $\mathbb{R}$, as the context will make clear. The (non-empty) set of extreme points of $K$ will be denoted by $\text{ext}(K)$, and $A(K)$ will stand for the space of complex-valued continuous affine functions on $K$ endowed with the sup-norm. Our general references for facts concerning compact convex sets are [A] and [AE].

Let $Q$ be a compact Hausdorff space, and $X$ a Banach space. Then the diameter of $f \in C(Q, X)$ is defined by $d(f) = \sup_{x,y \in Q} \|f(x) - f(y)\|$. If $A_1 \subseteq C(Q_1, X_1)$ and $A_2 \subseteq C(Q_2, X_2)$ are function spaces, i.e. sup-norm closed subspaces, containing the constant functions, and separating points, then a (not necessarily linear) map $T : A_1 \rightarrow A_2$ is called a diameter-preserving (d-preserving) bijection if $T$ is a 1-1 map of $A_1$ onto $A_2$, and $d(a) = d(Ta)$ for all $a \in A_1$.

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We begin, in Section 1, by discussing non-linear d-preserving bijections between (vector-valued) function spaces \( A_1 \) and \( A_2 \) of the kind described in the last paragraph and show, with the help of a well-known theorem of Mazur and Ulam (see [MU] and [B]), that under certain mild assumptions, \( T \) can be characterized as \( \overline{T} + \varphi \) where \( \overline{T} : A_1 \to A_2 \) is a linear d-preserving bijection, and \( \varphi : A_1 \to X_2 \) is a function, generally non-linear, with well-defined properties. We give several examples of such maps in Section 2.

In [RR, Theorem 1, p. 5], it was found that all linear d-preserving bijections between the spaces \( A(K) \) and \( A(L) \) are essentially induced by affine homeomorphisms between \( K \) and \( L \) when \( K, L \) are compact convex sets with all points of \( \text{ext}(K) \) and \( \text{ext}(L) \) split. It was asked there whether the same conclusion could be derived by assuming the “splittability” property only for \( \text{ext}(K) \). We exhibit in Section 3 a very simple two-dimensional counterexample to this question involving a simplex \( K \) and a hexagon \( S \) for which a linear d-preserving bijection exists between \( A_{\mathbb{R}}(K) \) and \( A_{\mathbb{R}}(S) \). [Here the subscript \( \mathbb{R} \) signifies real-valued functions.] The problem therefore naturally arises of characterizing compact convex sets \( K_1 \) and \( K_2 \) for which linear d-preserving bijections exist between \( A_{\mathbb{R}}(K_1) \) and \( A_{\mathbb{R}}(K_2) \). We consider the question more generally for function spaces \( A_i \subseteq C(Q_i) \), \( i = 1, 2 \), and show that the existence of an affine homeomorphism between certain compact convex sets in \( A_1 \) and \( A_2 \)—these sets being associated in a very natural manner with the state spaces of \( A_1 \) and \( A_2 \)—is necessary and sufficient for this purpose. (For example, for the spaces \( A_{\mathbb{R}}(K) \) and \( A_{\mathbb{R}}(S) \) as above, the condition is that \( K - K \) must be affinely homeomorphic to \( S - S \).) We pursue this further by describing in Theorem 9 (in Section 4) the precise form of these d-preserving bijections from \( A(K) \) to \( A(S) \) where all the points in \( \text{ext}(K) \) are split and \( S \) is any compact convex set when the aforementioned condition is satisfied.

The discussion in Section 3 leads naturally to the question of solvability in \( S \) (a compact convex set) of the equation \( S - S = K - K \) where \( K \) is a given compact convex set with all points of \( \text{ext}(K) \) split. The question in general appears too difficult, but we can exhibit, in \( \mathbb{R}^n \), the class of polytopes \( S \) for which \( S - S = K - K \) when \( K \) is a given simplex in \( \mathbb{R}^n \). This is the content of Section 5.

1. d-Preserving maps on function spaces. Assume that \( E \) is a normed linear space. Let \( C(X, E) \) be the space of all continuous \( E \)-valued functions on \( X \). For \( v \in E \), we also denote by \( v \) the constant function, \( v(x) = v \) for all \( x \in X \). Thus, vectors and constant functions are denoted in the same way; the meaning can be determined from the context. Also, we use \( E \) to denote the closed subspace of \( C(X, E) \) consisting of all the constant functions.
A subspace $A \subseteq C(X, E)$ is a (vector-valued) function space if $A$ is closed in the sup-norm, $E \subseteq A$, and $A$ separates points of $X$. For $a \in A$, let $[a] = a + E$ be the residue class of $a$ in the quotient space $A/E$. We remark that $A$ is linearly isomorphic to $(A/E) \oplus E$. For let $\psi : A \to (A/E) \oplus E$ be defined by $\psi(a) = [a] \oplus a(x_0)$ where $x_0 \in X$ is fixed. Then $\psi$ is 1-1 since if $a, b \in A$ with $\psi(a) = \psi(b)$, then $[a] = [b]$, so $a$ and $b$ differ by a constant, and also $a(x_0) = b(x_0)$. Thus, $a = b$. The linear map $\psi$ maps $A$ onto $(A/E) \oplus E$, since given $[a] \oplus v$, setting $c = a + (v - a(x_0))$, we have $\psi(c) = [a] \oplus v$.

The diameter of $a \in A$ is defined by $d(a) \equiv \sup_{x,y \in X} \|a(x) - a(y)\|$. Note that $d(a) = 0$ if, and only if, $a$ is a constant function. For $[a] \in A/E$, define $d[a] = d(a)$. Then $[a] \mapsto d[a]$ is a norm on $A/E$ which we call the $d$-norm.

**Proposition 1.** Assume that $A_1 \subseteq C(X_1, E_1)$ and $A_2 \subseteq C(X_2, E_2)$ are function spaces. Make the linear identifications $A_1 \cong (A_1/E_1) \oplus E_1$ and $A_2 \cong (A_2/E_2) \oplus E_2$ as above, using the maps $\psi_1$ and $\psi_2$, $\psi_k(a_k) = [a_k] \oplus a_k(x_k)$ where $x_k \in X_k$ are fixed points, $a_k \in A_k$ for $k = 1, 2$. Assume that $D : A_1/E_1 \to A_2/E_2$ and $J : E_1 \to E_2$ are linear bijections, and that $D$ is an isometry with respect to the $d$-norms. Define $D \oplus J : (A_1/E_1) \oplus E_1 \to (A_2/E_2) \oplus E_2$ by $(D \oplus J)([a] \oplus v) = D[a] \oplus J(v)$. Finally, define $\overline{D} : A_1 \to A_2$ by $\overline{D}(a) = \psi_2^{-1}(D \oplus J)\psi_1(a)$. Then $\overline{D}$ is a linear bijection which is $d$-preserving and has the property that the induced map $\tilde{D} : A_1/E_1 \to A_2/E_2$ defined by $\tilde{D}([a]) = [\overline{D}(a)]$ is $D$.

Proof. That $\overline{D}(a)$ is a linear bijection is clear, since $\psi_2^{-1}$, $D \oplus J$ and $\psi_1$ are all linear bijections. Let $a \in A_1$. Then by a straightforward computation,

$$\overline{D}(a) = b + -b(x_2) + J(a(x_1)) \quad \text{where} \quad [b] = D[a].$$

Thus, $d(\overline{D}(a)) = d(b) = d[b] = d[D[a]] = d[a] = d(a)$, so $\overline{D}$ is $d$-preserving. □

Remark. When $J : E_1 \to E_2$ is a bijection, but not linear, and $D \oplus J$ and $\overline{D}$ are defined as above, then $\overline{D}$ is still a bijection and a $d$-preserving function which is not linear.

Let $A_1$ and $A_2$ be function spaces as above, and assume that $T : A_1 \to A_2$ is a function (not necessarily linear). Now we prove a result which shows that with fairly weak assumptions on $T$ plus the assumption that $E_1$ and $E_2$ are linearly isomorphic, there exists a linear bijection $\overline{T} : A_1 \to A_2$ with $\overline{T}$ $d$-preserving. The key tool here is the Ulam–Mazur Theorem [MU, B] which we state for the convenience of the reader: Assume that $(E_1, \| \cdot \|_1)$ and $(E_2, \| \cdot \|_2)$ are normed linear spaces and that $D : E_1 \to E_2$ is a bijection with the properties: (a) $\|D(x) - D(y)\|_2 = \|x - y\|_1$ for all $x, y \in E_1$, (b) $D(0) = 0$, and in the case of complex scalars, (c) $D(ia) = iD(a)$ for all $a \in E_1$. Then $D$ is linear.
Theorem 2. Assume that $T : A_1 \to A_2$ is a function with the properties:

1. $T$ is a bijection;
2. $T$ has the property $d(T(a) - T(b)) = d(a - b)$ for all $a, b \in A_1$;
3. $T(0) = 0$;
4. [in the case of complex scalars] $T(ia) = iT(a)$ for all $a \in A_1$.

Also, assume that there exists $J : E_1 \to E_2$ such that $J$ is a linear bijection. Then $T(a) = \overline{T}(a) + \varphi(a)$ where $\overline{T} : A_1 \to A_2$ is a $d$-preserving linear bijection with $\overline{T}(v) = J(v)$, $v \in E_1$, and $\varphi : A_1 \to E_2$ is a function with the properties:

(i) $\varphi(0) = 0$;
(ii) [in the complex scalar case] for all $a \in A$, $\varphi(ia) = i\varphi(a)$;
(iii) the map $a \mapsto a + J^{-1}(\varphi(a))$ from $A_1$ to $A_1$ is 1-1;
(iv) given $a \in A_1$, the equation $J(x) + \varphi(a + x) = 0$ is solvable for $x \in E_1$.

Conversely, assume $\overline{T} : A_1 \to A_2$ is a $d$-preserving linear bijection, and define $J : E_1 \to E_2$ by $J(v) = \overline{T}(v)$, $v \in E_1$. Further, assume that $\varphi : A_1 \to E_2$ is a function with properties (i)–(iv) above. Then $T(a) = \overline{T}(a) + \varphi(a)$ is a (possibly non-linear) map from $A_1$ onto $A_2$ having the properties (1)–(4) listed in the theorem.

Proof. Assume that $T : A_1 \to A_2$ has properties (1)–(4). Define $\tilde{T} : A_1/E_1 \to A_2/E_2$ in the obvious way: $\tilde{T}([a]) = [Ta]$. From properties (1) and (2), $\tilde{T}$ is a bijection and has property (a) above (in the statement of the Ulam–Mazur Theorem). Also, $\tilde{T}$ inherits properties (3) and (4). Therefore the Ulam–Mazur Theorem applies, so $\tilde{T}$ is linear. Now using the construction in Proposition 1, define $\overline{T}(a) = \psi_2^{-1}(\tilde{T} \oplus J)\psi_1(a)$. By Proposition 1, $\overline{T}$ is a linear bijection which is $d$-preserving. For $a \in A_1$, $\overline{T}(a) = T(a) + c$ for some $c \in E_2$. Define $\varphi(a) = T(a) - \overline{T}(a) \in E_2$. Thus, for all $a \in A_1$, $T(a) = \overline{T}(a) + \varphi(a)$. It follows from the definition of $\overline{T}$ that $\overline{T}(a) = J(a)$ when $a \in E_1$. That $\varphi$ satisfies (i) and (ii) is clear. We verify that (iii) and (iv) hold:

(iii) The map $a \mapsto a + J^{-1}(\varphi(a))$ from $A_1$ to $A_1$ is 1-1: Suppose $a + J^{-1}(\varphi(a)) = b + J^{-1}(\varphi(b))$. Then $v = b - a \in E_1$, and $v = J^{-1}(\varphi(a) - \varphi(b))$. Thus, $\overline{T}(v) = J(v) = \varphi(a) - \varphi(b)$. Therefore, $\overline{T}(b) - \overline{T}(a) = \varphi(a) - \varphi(b)$, so $\overline{T}(a) = T(b)$. Then since $T$ is 1-1, $a = b$.

(iv) Given $a \in A_1$, the equation $J(x) + \varphi(a + x) = 0$ is solvable for $x \in E_1$; Choose $b$ such that $T(b) = \overline{T}(a)$ ($T$ is surjective). Then $0 = d(T(a) - T(b)) = d(a - b)$, so $b = a + v$ for some $v \in E_1$. Thus, $\overline{T}(a) = T(b) = T(a + v) = \overline{T}(a + v) + \varphi(a + v) = \overline{T}(a) + \overline{T}(v) + \varphi(a + v) = \overline{T}(a) + J(v) + \varphi(a + v)$. Therefore, $J(v) + \varphi(a + v) = 0$.

Now we do the converse. Assume that $\overline{T}$, $J$, and $\varphi$ are as stated in the last paragraph of the theorem. Also, define $T : A_1 \to A_1$ as above: $T(a) =$
That $T$ has properties (2), (3), and (4) follows immediately. We prove that $T$ is bijective.

$T$ is 1-1: Assume $T(a) = T(b)$, so

$$T(a) + \varphi(a) = T(b) + \varphi(b).$$

Then $T(a - b) = \varphi(b) - \varphi(a)$. Therefore, $a - b = x \in E_1$. Then $T(a) = T(b) + J(x)$ (since $J = T$ on $E_1$). Substituting this equality into (1), we have $T(b) + J(x) + \varphi(a) = T(b) + \varphi(b)$, so $J(x) = \varphi(b) - \varphi(a)$. Thus, $J(x) + \varphi(a) = \varphi(b) = \varphi(a - x)$. Then $a = b + x = b + J^{-1}(\varphi(b) - \varphi(a))$. Finally, this implies $a + J^{-1}(\varphi(a)) = b + J^{-1}(\varphi(b))$. Applying (iii), it follows that $a = b$.

$T$ maps onto $A_2$: Assume $c \in A_2$. We want to find $a \in A_1$ such that $T(a) + \varphi(a) = c$. There exists $b \in A_1$ such that $c = T(b)$. Now $T(a) + \varphi(a) = T(b)$ implies $T(b - a) = \varphi(a)$. Therefore, $b - a \in E_1$. Setting $x = b - a$, we have $T(b) = T(a) + J(x)$, so $T(a) + \varphi(a) = T(a) + J(x)$. Thus, $\varphi(a) = J(x)$. Then $\varphi(b - x) = J(x)$ and $J(-x) + \varphi(b - x) = 0$. Finally, applying (iv), we can solve this last equation for $x$. Therefore, $a = b - x$ will be a solution of $T(a) = T(a) + \varphi(a) = c$.

**Remark.** $T$, the linear part of $T$, has been characterized in [RR, Theorem 2 and Proposition 2] for some vector-valued function spaces.

The assumption in Theorem 2 of the existence of the linear bijection $J : E_1 \rightarrow E_2$ is not very restrictive. First, it is not assumed that $J$ is continuous, so the existence of $J$ is equivalent to $E_1$ and $E_2$ having (algebraic) bases of the same cardinality.

We use $\text{card}(S)$ to denote the cardinality of a set $S$, and set $c = \text{card}(\mathbb{R})$. Also, the dimension of a linear space $E$ is denoted by $\text{dim}(E)$. Here is a useful known result:

If $\text{dim}(E) \geq c$, then $\text{card}(E) = \text{dim}(E)$ [LT, Problem 2, p. 43].

Note that because of the hypotheses in Theorem 2, $T(E_1) = E_2$, and $T$ is 1-1. It follows that $\text{card}(E_1) = \text{card}(E_2)$. Thus, if $\text{dim}(E_1) \geq c$ and $\text{dim}(E_2) \geq c$, then

$$\text{dim}(E_1) = \text{card}(E_1) = \text{card}(E_2) = \text{dim}(E_2).$$

**Conclusion.** If $\text{dim}(E_1) \geq c$ and $\text{dim}(E_2) \geq c$, then there exists a linear bijection of $E_1$ onto $E_2$.

Our main concern in this paper being with function spaces whose members take values in Banach spaces, it is pertinent to point out that, as a consequence of the Baire Category Theorem, the algebraic dimension of a Banach space is either finite or uncountable ($\geq c$).

Now assume the dimensions of $E_1$ and $E_2$ are both finite, $E_1 \approx \mathbb{R}^m$ and $E_2 \approx \mathbb{R}^n$. Assume that $T$ and $T^{-1}$ are continuous, so that $T$ is a homeo-
morphism of $E_1 \approx \mathbb{R}^m$ onto $E_2 \approx \mathbb{R}^n$. Then the Invariance of Dimension Theorem [D, p. 359] implies that $m = n$.

**Conclusion.** If $E_1$ and $E_2$ are both finite-dimensional and $T$ is a homeomorphism, then there exists a linear bijection of $E_1$ onto $E_2$.

2. **Examples of non-linear d-preserving maps.** In this section we present several examples of non-linear d-preserving maps $T$ which satisfy the hypotheses (1)–(4) of Theorem 2. Of course, by that theorem, $T$ must be the sum of a linear d-preserving bijection and a non-linear part.

First we give an example when $E_1 = E_2 = \mathbb{C}$. Let $A$ be a function space, $A \subseteq C(X)$. Note the relations for a function $a \in A$: $a = \text{Re}(a) + i\text{Im}(a)$; $ia = i\text{Re}(a) - \text{Im}(a)$; $\text{Re}(ia) = -\text{Im}(a)$; $\text{Im}(ia) = \text{Re}(a)$. Now set

$$
\varphi(a) = \sup(\text{Re}(a)) + i\sup(\text{Im}(a)) - \sup(-\text{Re}(a)) - i\sup(-\text{Im}(a)).
$$

For $a \in A$, $\varphi(ia) = i\varphi(a)$. Proof:

$$
\varphi(ia) = \sup(\text{Re}(ia)) + i\sup(\text{Im}(ia)) - \sup(-\text{Re}(ia)) - i\sup(-\text{Im}(ia))
= \sup(-\text{Im}(a)) + i\sup(\text{Re}(a)) - \sup(\text{Im}(a)) - i\sup(-\text{Re}(a))
= i[\sup(\text{Re}(a)) - \sup(-\text{Re}(a)) + i\sup(\text{Im}(a)) - i\sup(-\text{Im}(a))]
= i\varphi(a).
$$

**Note.** For $b \in A$, $\mu \in \mathbb{C}$, $\varphi(b + \mu) = \varphi(b) + 2\mu$. Proof:

$$
\varphi(b + \mu) = [\sup(\text{Re}(b)) + \text{Re}(\mu)] + i[\sup(\text{Im}(b)) + \text{Im}(\mu)]
- [\sup(-\text{Re}(b)) - \text{Re}(\mu)] - i[\sup(-\text{Im}(b)) - \text{Im}(\mu)]
= \varphi(b) + 2\mu.
$$

Now define $T(a) = a + \varphi(a)$. Then $T$ maps onto $A$. Proof: Given $b \in A$, we want $a \in A$ such that $b = a + \varphi(a)$. Set $a = b + \mu$ where $\mu \in \mathbb{C}$. Then we want that $b = (b + \mu) + \varphi(b + \mu) = b + \mu + \varphi(b) + 2\mu$ (from Note). Solving, we see that $3\mu = -\varphi(b)$. Letting $\mu = -\frac{1}{3}\varphi(b)$, we find that $T(a) = b$.

$T$ is 1-1. Proof: Suppose that $a, b \in A$ with $a + \varphi(a) = b + \varphi(b)$, so $a - b = \varphi(b) - \varphi(a)$. Then $a = b + \mu$ where $\mu \in \mathbb{C}$. Therefore, $\mu = \varphi(b) - \varphi(a) = \varphi(a - \mu) - \varphi(a) = (\text{from Note}) \varphi(a) - 2\mu - \varphi(a) = -2\mu$, so $\mu = 0$.

Thus, $T(a) = a + \varphi(a)$ is non-linear, but satisfies the hypotheses of Theorem 2.

When the scalar field is $\mathbb{R}$, there is a much simpler example. Assume that $A$ is a function space of $\mathbb{R}$-valued functions. The reader can check that $T(a) = a + \sup(a)$ or $T(a) = a + \max[\sup(a), 0]$, $a \in A$, is a non-linear bijection of $A$ onto $A$ that satisfies the hypotheses of Theorem 2. The latter definition of $T$ shows that the function $\varphi(x)$ in the statement of Theorem 2 is not generally surjective.
Now we give an example in the vector-valued case. Let $Y$ be a compact Hausdorff space, and set $E_1 = E_2 = C(Y)$. We construct a function $J$ with the properties:

(i) $J$ is a bijection of $C(Y)$ onto $C(Y)$;
(ii) $J(0) = 0$ and $J(if) = iJ(f)$ for all $f \in C(Y)$;
(iii) $J$ is not linear.

For $f \in C(Y)$, define

$$J(f) = [\text{Re}(f)]^3 - [\text{Im}(f)]^3 - i[\text{Re}(if)]^3 + i[\text{Im}(if)]^3.$$  

[In the real scalar case, $J(f) = f^3$ will work.] (ii) follows from a straightforward computation, and (iii) is clear. We verify (i).

$J$ is 1-1. Proof: Suppose $f_k \in C(Y)$, $f_k = u_k + iv_k$, where $u_k$, $v_k$ are $\mathbb{R}$-valued functions in $C(Y)$, $k = 1, 2$. Note that $if_k = -v_k + iu_k$, $k = 1, 2$. Suppose that $J(f_1) = J(f_2)$. Then $v_1^3 - v_2^3 = u_3^3 - v_2^3$, and $u_1^3 + v_1^3 = u_2^3 + v_2^3$. Therefore, $2u_3^3 = 2u_2^3$, so $u_3 = u_2$. Then $v_2^3 = v_3^3$, so $v_1 = v_2$.

$J$ maps onto $C(Y)$. Proof: Assume that $g = u + iv \in C(Y)$, $u, v \in \mathbb{R}$-valued. Set

$$f = \left[\frac{u + v}{2}\right]^{1/3} + i\left[\frac{u - v}{2}\right]^{1/3}, \quad \text{so} \quad if = i\left[\frac{u + v}{2}\right]^{1/3} - i\left[\frac{u - v}{2}\right]^{1/3}.$$  

Then

$$J(f) = \left[\frac{u + v}{2}\right] - \left[\frac{v - u}{2}\right] + i\left[\frac{v - u}{2}\right] + i\left[\frac{u + v}{2}\right] = u + iv = g.$$  

Now let $A \subseteq C(X, C(Y))$ be a function space. Let $I$ denote the identity map on $A$. Use Proposition 1 to define $T = \psi^{-1}(I + J)\psi$ where $\psi$ is the map defined in the discussion just prior to Proposition 1. Then $T$ is a non-linear bijection with properties (1)–(4) in Theorem 2.

3. d-Preserving maps on complex-valued function spaces. The real-scalar version of Theorem 1 in [RR] is:

Let $K$ and $S$ be compact convex sets, both of which have the property that every extreme point is a split face. If $D : A(K) \to A(S)$ is a d-preserving linear bijection, then there exists an affine homeomorphism $\tau : S \to K$ and a functional $\alpha$ defined on $A(K)$ such that for all $a \in A(K)$,

$$D(a) = c(a \circ \tau) + \alpha(a), \quad \text{where} \ c = \pm 1, \ \text{and} \ \alpha(1) \neq -c.$$  

A question raised in [RR, Remark 1] is: Does the same result hold if the hypothesis, “every extreme point is a split face”, is assumed only for $K$? We now give an example which answers this question in the negative.

In $\mathbb{R}^2$, let $K = \text{co}\{(1, 0), (-1, 0), (0, 1)\}$. The three extreme points of $K$ are split faces in the sense defined in [A] because $K$ is a simplex, and for each extreme point $x$ of $K$, $\{x\}^c$ (i.e. the complementary set of $\{x\}$) is a face,
and every point \( p \in K \) can be written uniquely as \( p = \alpha x + (1 - \alpha)y \), where \( y \in \{x\} \) and \( 0 \leq \alpha \leq 1 \). [Strictly speaking, to use the definition given in [A], one should regard \( K \subseteq \{\varphi \in A(K)^* : \varphi(1) = 1\} \). The analysis could be done in \( A(K)^* \). But this seems an unnecessary and technical approach to the elementary example under consideration.]

Now let \( S = \frac{1}{2}(K - K) \). It is easily checked that \( S \) is a hexagon; in fact \( S = \co\{x_1, x_2, x_3, -x_1, -x_2, -x_3\} \) where \( x_1 = (1, 0) \), \( x_2 = (1/2, 1/2) \), and \( x_3 = (-1/2, 1/2) \).

Thus, there is no affine homeomorphism of \( S \) onto \( K \). Using the identity

\[
\frac{1}{2}(z - w) - \frac{1}{2}(x - y) = \left(\frac{z + y}{2}\right) - \left(\frac{w + x}{2}\right),
\]

it is easy to check that \( K - K = S - S \). Then it follows from Corollary 7 of this paper that there exists a \( d \)-preserving linear bijection of \( A(K) \) onto \( A(S) \).

Now we investigate function spaces \( A_i \subseteq C(Q_i), \ i = 1, 2 \), to determine necessary and sufficient conditions for the existence of a linear \( d \)-preserving \( T : A_1 \to A_2 \). First we give some relevant definitions and results which will be needed for this analysis.

Let \( S \) be a compact convex set which is symmetric (\( s \in S \Rightarrow -s \in S \)). When the scalar field is \( \mathbb{R} \), define \( A_0(S) = \{f \in A(S) : f(0) = 0\} \). When the scalar field is \( \mathbb{C} \), assume (\( s \in S \), \( \alpha \in \mathbb{C} \), \( |\alpha| = 1 \) \( \Rightarrow \alpha s \in S \). In this case define \( A_0(S) = \{f \in A(S) : f(is) = if(s) \text{ for all } s \in S\} \). Note that if \( f \in A_0(S) \), then \( f(i0) = if(0) \), so \( f(0) = 0 \).

**Proposition 3.** Let \( X \) be a Banach space. Assume that \( f \in A_0(X_1^*) \) (here \( X_1^* \) is the closed unit ball of the dual of \( X \), and the topology on \( X_1^* \) is the \( w^* \)-topology). Extend \( f \) to \( \tilde{f} \) by

\[
\tilde{f}(\varphi) = \|\varphi\|f(\varphi/\|\varphi\|), \quad \varphi \in X^*.
\]

Then \( \tilde{f} \) is a \( w^* \)-continuous linear functional on \( X^* \).

**Proof.** We do the complex scalar case, so it is assumed that \( f(i\varphi) = if(\varphi) \) for all \( \varphi \in X_1^* \). Also, note that \( f \) has the properties: \( f(-\varphi) = -f(\varphi) \); and \( (0 \leq \alpha \leq 1, \ \varphi \in X_1^* \Rightarrow \alpha f(\varphi) = f(\alpha \varphi) \). To be proved:

(a) \( \tilde{f}(\varphi + \psi) = \tilde{f}(\varphi) + \tilde{f}(\psi) \) for all \( \varphi, \psi \in X^* \);
(b) \( \tilde{f}(\alpha \varphi) = \alpha \tilde{f}(\varphi) \) for all \( \alpha \in \mathbb{C}, \ \varphi \in X^* \).

Assume that both (a) and (b) hold. Now \( \ker(\tilde{f}) \cap X_1^* = \{\text{the zero set of } f\} \), which is \( w^* \)-closed. Therefore by the Kreĭn–Shmul’yan Theorem (see [DS] or [LT]), \( \tilde{f} \) is \( w^* \)-continuous.
Now we prove (a). Let \( \varphi, \psi \in X^* \setminus \{0\} \). Since \( \|\varphi + \psi\|/\|\varphi\| + \|\psi\| \leq 1 \),
\[
\left( \frac{\|\varphi + \psi\|}{\|\varphi\| + \|\psi\|} \right) f \left( \frac{\varphi + \psi}{\|\varphi + \psi\|} \right) = f \left( \frac{\varphi + \psi}{\|\varphi\| + \|\psi\|} \right).
\]
Therefore
\[
\tilde{f}(\varphi + \psi) = \|\varphi + \psi\| f \left( \frac{\varphi + \psi}{\|\varphi + \psi\|} \right) = (\|\varphi\| + \|\psi\|) f \left( \frac{\varphi + \psi}{\|\varphi\| + \|\psi\|} \right)
\]
\[
= (\|\varphi\| + \|\psi\|) \left[ \frac{\varphi}{\|\varphi\| + \|\psi\|} f \left( \frac{\varphi}{\|\varphi\|} \right) + \frac{\psi}{\|\varphi\| + \|\psi\|} f \left( \frac{\psi}{\|\psi\|} \right) \right]
\]
\[
= \|\varphi\| f \left( \frac{\varphi}{\|\varphi\|} \right) + \|\psi\| f \left( \frac{\psi}{\|\psi\|} \right) = \tilde{f}(\varphi) + \tilde{f}(\psi).
\]

To prove (b), first note that
\[
\tilde{f}(-\varphi) = \|\varphi\| f \left( \frac{-\varphi}{\|\varphi\|} \right) = -\|\varphi\| f \left( \frac{\varphi}{\|\varphi\|} \right) = -\tilde{f}(\varphi).
\]

Now suppose \( \alpha \in \mathbb{R} \) and \( \alpha > 0 \). Then \( \tilde{f}(\alpha \varphi) = \|\alpha \varphi\| f(\alpha \varphi/\|\alpha \varphi\|) = \alpha \|\varphi\| f(\varphi/\|\varphi\|) = \alpha \tilde{f}(\varphi) \). The same equality for \( \alpha \in \mathbb{R} \) and \( \alpha < 0 \) follows from this and the fact that \( \tilde{f}(-\varphi) = -\tilde{f}(\varphi) \). Also, \( \tilde{f}(i \varphi) = \|i \varphi\| f(i \varphi/\|i \varphi\|) = \|\varphi\| f(i \varphi/\|i \varphi\|) = i \|\varphi\| f(\varphi/\|\varphi\|) = i \tilde{f}(\varphi) \).

Finally, assume that \( \alpha = \beta + i \delta \), \( \beta, \delta \in \mathbb{R} \). Then
\[
\tilde{f}(\alpha \varphi) = \tilde{f}(\beta \varphi + i \delta \varphi) = \tilde{f}(\beta \varphi) + \tilde{f}(i \delta \varphi) = \beta \tilde{f}(\varphi) + i \delta \tilde{f}(\varphi).
\]

Let \( A \subseteq C(Q) \) where \( Q \) is a compact Hausdorff space, be a function space equipped with the usual sup-norm \( \|a\|_{\infty} \). We work in the complex scalar case. For \( [a] \in A/\mathbb{C} \), define
\[
\|[a]\|_{\infty} = \inf \{ \|a + \lambda\|_{\infty} : \lambda \in \mathbb{C} \},
\]
the usual quotient norm.

**Note 4.** The d-norm on \( A/\mathbb{C} \) is equivalent to the quotient norm.

**Proof.** For \( a \in A \), fix \( x, y \in X \) such that \( d(a) = |a(x) - a(y)| \). Note that it is clear that \( d(a) \leq 2\|a\|_{\infty} \), so for all \( \lambda \in \mathbb{C} \), \( d(a) = d(a + \lambda) \leq 2\|a + \lambda\|_{\infty} \). It follows that \( d[a] = d(a) \leq 2\|[a]\|_{\infty} \). Also, \( \|[a]\|_{\infty} \leq \|a - a(y)\|_{\infty} = \sup_{z \in X} |a(z) - a(y)| = |a(x) - a(y)| = d(a) = d[a] \).

Note that by the Hahn–Banach Theorem, \( (A/\mathbb{C})^* \) is isometrically isomorphic to \( \{ \varphi \in A^* : \varphi(1) = 0 \} \). For \( a \in A \), \( \varphi \in (A/\mathbb{C})^* \), let \( \widehat{a}[\varphi] = \varphi([a]) \). Define \( \Gamma = \{ \alpha \in \mathbb{C} : \|\alpha\| = 1 \} \), and \( T = \{ \alpha(q - r) : \alpha \in \Gamma, q, r \in Q \} \subseteq (A/\mathbb{C})^* \) (here we identify \( Q \) as a subset of \( A^* \) via the evaluation map \( q \mapsto e_q \) where
$e_q \in A^*$ is defined by $e_q(a) = a(q)$, $a \in A$, $q \in Q$). Then

$$T^0 = \{[a] : \text{Re}(\alpha(a(q) - a(r))) \leq 1 \text{ for all } \alpha \in T, \text{ and all } q, r \in Q\}$$
$$= \{[a] : d[a] = d(a) \leq 1\}$$
$$= \{\text{the closed unit ball in } A/\mathbb{C} \text{ with respect to the d-norm}\}.$$  

$$T^{00} = \{\varphi \in (A/\mathbb{C})^* : \text{Re}(\widehat{\varphi}(\varphi)) \leq 1 \text{ for all } [a] \in T^0\}$$
$$= \{\text{the closed unit ball in } (A/\mathbb{C})^* \text{ with respect to the dual d-norm}\},$$
$$= \overline{\sigma}(T) \quad (\text{by the Bipolar Theorem [LT, Thm. 7.3, p. 162]}$$
$$= \overline{\sigma}(\Gamma(S_A - S_A)) = \overline{\alpha}(S_A - S_A)$$

where $S_A = \{\varphi \in A^* : \|\varphi\| = \varphi(1) = 1\}$ is the state space of $A$, and $\overline{\alpha}(S)$ is the absolute convex hull of a set $S (\subseteq A^*)$ where the closure is taken with respect to the $w^*$-topology in $A^*$.

We use this notation in the theorem below.

**Theorem 5.** The map $[a] \mapsto \widehat{\varphi}$ is a linear bijection of $A/\mathbb{C}$ onto $A_0(T^{00})$. Also, it is an isometry of $(A/\mathbb{C}, \text{d-norm})$ onto $(A_0(T^{00}), \|\|_\infty)$ where for $b \in A_0(T^{00})$,

$$\|b\|_\infty = \sup\{|b(\varphi)| : \varphi \in T^{00}\}.$$  

**Proof.** Since $T^{00} = \{\text{the closed unit ball in } (A/\mathbb{C})^* \text{ with respect to the dual d-norm}\}$, for $a \in A$, $d[a] = \sup\{|\varphi(a)| : \varphi \in T^{00}\} = \|\widehat{a}\|_\infty$. Thus, the map $[a] \mapsto \widehat{a}$ is an isometry. This map is clearly linear and 1-1. Now assume that $b \in A_0(T^{00})$. Proposition 3 applies with $X = (A/\mathbb{C}, \text{d-norm})$. Therefore $b$ has a $w^*$-continuous extension $\bar{b}$ in $(A\mathbb{C})^{**}$. It follows that there exists $[a] \in A/\mathbb{C}$ such that $\widehat{a} = b$. ■

Let $J$ and $K$ be convex circled subsets of a complex linear space. We say that an affine map $\tau : J \to K$ is a *complex affine map* if $\tau(ix) = i\tau(x)$ for all $x \in J$. [Note that the map $z \mapsto \overline{z}$ on the closed unit disk in the complex plane is affine, but not complex affine.]

Now let $A_k \subseteq C(Q_k)$, $k = 1, 2$, be function spaces on compact Hausdorff spaces $Q_1$ and $Q_2$. Let $D$ be a linear isometry of $(A_1/\mathbb{C}, \text{d-norm})$ onto $(A_2/\mathbb{C}, \text{d-norm})$. Letting $T_k = \{\alpha(q - r) : \alpha \in T, q, r \in Q_k\}$, $k = 1, 2$, we deduce by the discussion prior to Theorem 5 that $T^{00}_k$ is the closed unit ball in $(A_k\mathbb{C})^*$. Thus, $D^*(T^{00}_2) = T^{00}_1$, where $D^*$ is the adjoint of $D$. For $\varphi \in T^{00}_2$, define $\tau(\varphi) = D^*(\varphi)$. Then $\tau$ is a complex affine homeomorphism ($w^*$-topology) of $T^{00}_2$ onto $T^{00}_1$. Also, for $\varphi \in T^{00}_2$, $[a] \in A_1/\mathbb{C}$, $\overline{D[a]}(\varphi) = \overline{[a]}(D^*(\varphi)) = \overline{[a]}(\tau(\varphi))$, so $\overline{D[a]} = \overline{[a]} \circ \tau$, $[a] \in A_1/\mathbb{C}$.

We summarize this discussion in the following theorem.
Theorem 6. There exists a d-preserving linear bijection of $A_1$ onto $A_2$ if, and only if, there exists a complex affine homeomorphism of the set $\overline{acod}(S_{A_2} - S_{A_2})$ onto $\overline{acod}(S_{A_1} - S_{A_1})$.

Proof. First note that $T_k$ is compact as it is the continuous image of the compact set $T \times Q_k \times T$ under the map $(\alpha, q, r) \mapsto \alpha(q - r)$. Thus, $T_k^{00} = \overline{acod}(S_{A_k} - S_{A_k})$ is compact. Suppose that $\tau : \overline{acod}(S_{A_2} - S_{A_2}) \rightarrow \overline{acod}(S_{A_1} - S_{A_1})$ is a complex affine homeomorphism. Then for $a \in A_1$, $D[a] = [\tilde{a}] \circ \tau$ is a linear bijection of $A_0(T_1^{00})$ onto $A_0(T_2^{00})$ which is an isometry with respect to the sup-norm. By Theorem 5, this implies the existence of a linear bijection $\tilde{D}$ which is an isometry of $(A_1/\mathbb{C}, d$-norm) onto $(A_2/\mathbb{C}, d$-norm). Then by Proposition 1, $\tilde{D}$ lifts to a linear bijection of $A_1$ onto $A_2$ which is d-preserving.

Conversely, assume that $\tilde{D}$ is a linear bijection of $A_1$ onto $A_2$ which is d-preserving. Define $D : A_1/\mathbb{C} \rightarrow A_2/\mathbb{C}$, as usual, by $D[a] = [\tilde{D}(a)]$. Then $D$ is a linear bijection which is an isometry with respect to the d-norm. Then as argued in the discussion before the theorem, $D^*$ is a complex affine homeomorphism of $T_2^{00} = \overline{acod}(S_{A_2} - S_{A_2})$ onto $T_1^{00} = \overline{acod}(S_{A_1} - S_{A_1})$. ■

When $A_k$ is the space $A(K_k)$, i.e., the space of continuous affine functions on a compact convex set $K_k$ with the sup-norm, then $S_{A_k} = K_k$, $k = 1, 2$. In the real scalar case, we see that $T_k^{00} = \overline{cod}(K_k - K_k) = K_k - K_k$.

Also note that if $\tau : K_2 - K_2 \rightarrow K_1 - K_1$ is an affine homeomorphism, then $\tau$ carries a point of symmetry to a point of symmetry, and 0 is the only point of symmetry for both the above sets. Therefore, we must have $\tau(0) = 0$.

Using the same notation as in Theorem 6, we have the following corollary:

Corollary 7. In the case where the scalar field is $\mathbb{R}$, there exists a d-preserving linear bijection of $A(K_1)$ onto $A(K_2)$ if, and only if, there exists an affine homeomorphism of $K_2 - K_2$ onto $K_1 - K_1$.

The proof of Theorem 6 applies verbatim to Corollary 7, except that $T_k^{00} = K_k - K_k$, $k = 1, 2$, as we noted above.

Corollary 7 raises the natural question: When $K_1$ and $K_2$ are compact convex sets, under what conditions are $K_1 - K_1$ and $K_2 - K_2$ affinely homeomorphic? This question seems too difficult to answer in general, although in some cases conditions can be found. For example, the results in [RR] show that (in the real scalar case), when $K_1$ and $K_2$ both have the property that all their extreme points are split faces, then $K_1 - K_1$ and $K_2 - K_2$ are affinely homeomorphic if, and only if, $K_1$ and $K_2$ are affinely homeomorphic.

Here is an especially simple situation. Suppose that $K$ is a compact convex set which is symmetric. Then clearly $K + K = K - K$. Also, $K + K = 2K$, since for all $x, y \in K$, $x + y = 2\left(\frac{x + y}{2}\right)$. Thus, $K - K = 2K$. It follows that,
when both $K_1$ and $K_2$ are symmetric, then again $K_1 - K_1$ and $K_2 - K_2$ are affinely homeomorphic if, and only if, $K_1$ and $K_2$ are affinely homeomorphic.

We derive more information concerning this question in the last section.

4. A characterization of some linear $d$-preserving maps. Assume that $L$ and $S$ are compact convex sets, that $0 \in L$, $0 \in S$, and $\overline{\text{cmd}}(L - L) = \overline{\text{cmd}}(S - S)$ [in the case of real scalars, the assumption is $L - L = S - S$]. For $a \in A(L)$, the function $\hat{a} = [a]$ is in $A_0(\overline{\text{cmd}}(L - L))$. Now for $a \in A(L)$, define $a_S \in A(S)$ by $a_S = \hat{a}|_S + a(0)$ on $S$. Since the hypotheses are the same for $L$ and $S$, for $a \in A(S)$, we define $a_L$ in the same way. Note that $a_S \in A(S)$ and $a_L \in A(L)$. Also, $a_S(0) = a_L(0) = a(0)$. We use this notation in the next result.

**Proposition 8.** Assume that $L$ and $S$ are compact convex sets with the properties above.

1. For $a \in A(L)$, $\hat{a}_S = \hat{a}$; for $a \in A(S)$, $\hat{a}_L = \hat{a}$.

2. For $a \in A(L)$, $(a_S)_L = a$; for $a \in A(S)$, $(a_L)_S = a$.

3. For $a \in A(L)$, $d(a) = d(a_S)$; for $a \in A(S)$, $d(a) = d(a_L)$.

Also, if $\lambda \in A(L)$ is a constant function, then $\lambda_S = \lambda$ (and the same statement with $L$ and $S$ interchanged).

**Proof.** We do the proof in the complex scalar case.

First we prove (1) when $a \in A(L)$. It is enough to verify that $\hat{a}_S(\varphi) = \hat{a}(\varphi)$ for all $\varphi \in \overline{\text{cmd}}(S - S)$ of the form $\varphi = t(s_1 - s_2)$, $|t| = 1$, $s_1, s_2 \in S$, since these generate $\overline{\text{cmd}}(S - S)$. Assume that $\varphi = t(s_1 - s_2)$ as above, and $a \in A(L)$. Now $\frac{1}{2}(\varphi + ts_2) = \frac{1}{2}ts_1$, so $\frac{1}{2}\hat{a}(\varphi) + \frac{1}{2}\hat{a}(s_2) = \frac{1}{2}\hat{a}(s_1)$. Therefore, $\hat{a}_S(\varphi) = t(a_S(s_1) - a_S(s_2)) = t(\hat{a}(s_1) - \hat{a}(s_2)) = \hat{a}(\varphi)$.

Now we prove (2) for $a \in A(L)$. By definition $(a_S)_L = \hat{a}_S|_L + a_S(0)$, so by (1), $(a_S)_L = \hat{a}|_L + a(0)$. For $l \in L$, $l = l - 0$, so $\hat{a}(l) = a(l) - a(0)$. Thus, $a(l) = \hat{a}(l) + a(0)$. Then $(a_S)_L(l) = \hat{a}(l) + a(0) = a(l)$. This establishes (2).

Assume that $a \in A(L)$. By (1), $\hat{a} = \hat{a}_S$. Then $d(a) = \|\hat{a}\|_{\infty}$ and $d(a_S) = \|\hat{a}_S\|_{\infty}$ (Theorem 5), so (3) follows from these equalities.

We can now describe the general form of the $d$-preserving linear bijection raised in the question in [RR] that we mentioned at the beginning of Section 3.

**Theorem 9.** Let $D : A(K) \to A(S)$ be a $d$-preserving linear bijection where $K, S$ are compact convex sets with the former having the property that all the points of $\text{ext}(K)$ are split. [In particular, $K$ could be a Choquet simplex.] We assume, as we may by translation in $A(S)^*$, that $0 \in S$. Then there exist a compact convex set $L \subseteq \overline{\text{cmd}}(S - S)$ [$S - S$ in the real scalar case], affinely homeomorphic to $K$, such that $\overline{\text{cmd}}(S - S) = \overline{\text{cmd}}(L - L)$, $0 \in L$, ...
and an affine homeomorphism $\tau : L \to K$, and $\alpha \in A(L)'$ such that for all $a \in A(K)$,

$$D(a) = c(a \circ \tau) + \alpha(a), \quad \text{where } |c| = 1, \alpha(1) \neq -c.$$  

Proof. We do the proof in the complex scalar case. First we construct $L$. Define $\tilde{D} : A(K)/\mathbb{C} \to A(S)/\mathbb{C}$ in the usual way: $\tilde{D}[a] = [Da]$. Then as seen in the proof of Theorem 6, $\tilde{D}^* \text{ maps } \text{aco}(S - S) \text{ onto } \text{aco}(K - K)$. Fix $x_0 \in \text{ext}(K)$. Define $L = (\tilde{D}^*)^{-1}(K - \{x_0\})$. Then for $x \in L$, $x \mapsto \tilde{D}^*(x) + x_0 \in K$, and this map is an affine homeomorphism of $L$ onto $K$. Note $0 \in L$. Observe that $\text{ext}(L) - \text{ext}(L) = (\tilde{D}^*)^{-1}(\text{ext}(K) - \text{ext}(K))$, and consequently, $\text{aco}(L - L) = (\tilde{D}^*)^{-1}(\text{aco}(K - K)) = \text{aco}(S - S)$. Thus, $S \subseteq S - S \subseteq \text{aco}(S - S)$, $0 \in S$, and $L \subseteq L - L \subseteq \text{aco}(L - L)$, $0 \in L$. Therefore Proposition 8 applies.

Now for $a \in A(K), D(a) \in A(S)$ and $(D(a))_L \in A(L)$. By Proposition 8, $a \mapsto (D(a))_L$ is a d-preserving linear bijection of $A(K)$ onto $A(L)$. Applying [RR, Theorem 1], we have $(D(a))_L = c(a \circ \tau) + \alpha(a)$, where $|c| = 1, \alpha \in A(L)'$, $\tau : L \to K$ is an affine homeomorphism, and $\alpha(1) \neq -c$. Using Proposition 8 again, we have

$$D(a) = ((D(a))_L)_S = (c(a \circ \tau) + \alpha(a))_S = c(a \circ \tau)_S + \alpha(a).$$

5. A geometric problem involving $K - K$, $K$ a simplex. Let $K$ be a simplex. We assume that $K$ is embedded in $A(K)^*$ as the base of a cone $\tilde{K}$ which generates $A(K)^*$; see [P, p. 59]. It is important to keep in mind that distances between points will be computed in the dual norm on $A(K)^*$.

First, let $K$ be the simplex, $K = \text{co}\{x_1, x_2, x_3\}$. A simple observation using the linear independence of the vectors $x_1 - x_2$ and $x_2 - x_3$ is that

$$K - K = \{c(x_1 - x_2) + \beta(x_2 - x_3) : c, \beta \in \mathbb{R}, |c| + |\alpha - \beta| + |\beta| \leq 2\}.$$  

Also, representation of points in $K - K$ is unique. These remarks will be useful in what follows.

The problem is to find polytopes $S$ with the property that $K - K = S - S$.

We may assume by translating that $S \subseteq \tilde{K}$. Also, we assume that

$$S = \text{co}\{s_1, \tilde{s}_1, s_2, \tilde{s}_2, s_3, \tilde{s}_3\} \subseteq A(K)^*$$

is such that

$$s_1 - \tilde{s}_1 = x_1 - x_2 \quad (= y_1, \text{ say}), \quad s_2 - \tilde{s}_2 = x_2 - x_3 = y_2,$$

$$s_3 - \tilde{s}_3 = x_1 - x_3 = y_3,$$

where $\text{ext}(S) = \{s_1, \tilde{s}_1, s_2, \tilde{s}_2, s_3, \tilde{s}_3\}$. Therefore, $S - S = K - K$. Note that $S - S$ will have only the extreme points $\pm y_1, \pm y_2, \pm y_3$. 
By the decomposition property of vector lattices [P, p. 61], one sees easily that:

\[ s_1 = x_1 + ax, \quad s_2 = x_2 + by, \quad s_3 = x_1 + cz, \]

\[ \tilde{s}_1 = x_2 + ax, \quad \tilde{s}_2 = x_3 + by, \quad \tilde{s}_3 = x_3 + cz, \]

where \( a, b, c \geq 0 \), \( x, y, z \in K \).

We first consider the case where the vectors \( s_1 - \tilde{s}_1, s_2 - \tilde{s}_2 \), and \( s_3 - \tilde{s}_3 \) intersect in distinct points \( P, P', \) and \( P'' \); see Fig. 1.

Consider the point \( P \) as a typical case; refer to the quadrilateral \( \{s_1, s_2, \tilde{s}_1, \tilde{s}_2\} \) in Fig. 1. Here

\[ P = \alpha s_1 + (1 - \alpha)\tilde{s}_1 = \beta s_2 + (1 - \beta)\tilde{s}_2 \]

(it is assumed that \( 1/2 < \alpha < 1 \)).

Recall that \( \|s_1 - \tilde{s}_1\| = \|x_1 - x_2\| = 1 + 1 = 2 \) by the splitability of extreme points for a simplex. Thus, we have \( \alpha = \|P - \tilde{s}_1\|/\|s_1 - \tilde{s}_1\| = \frac{1}{2}\|P - \tilde{s}_1\| \). Now \( P = \alpha(s_1 - \tilde{s}_1) + \tilde{s}_1 = \beta(s_2 - \tilde{s}_2) + \tilde{s}_2 \). Also, \( \tilde{s}_1 - \tilde{s}_2 = \beta(x_2 - x_3) - \alpha(x_1 - x_2) = (\alpha + \beta)x_2 - \alpha x_1 - \beta x_3 \). Therefore, \( \|\tilde{s}_1 - \tilde{s}_2\| = \alpha + \beta + \alpha + \beta = 2(\alpha + \beta) \leq 2 \), so \( \alpha + \beta \leq 1 \). Also,

\[ s_1 - s_2 = (P - s_2) + (s_1 - P) \]

\[ = [\beta s_2 + (1 - \beta)\tilde{s}_2 - s_2] + [s_1 - \alpha s_1 - (1 - \alpha)\tilde{s}_1] \]

\[ = (1 - \beta)(\tilde{s}_2 - s_2) + (1 - \alpha)(s_1 - \tilde{s}_1) \]

\[ = (1 - \beta)(x_3 - x_2) + (1 - \alpha)(x_1 - x_2) \]

\[ = (1 - \alpha)x_1 - (2 - \alpha - \beta)x_2 + (1 - \beta)x_3. \]
This implies that $\|s_1 - s_2\| = 4 - 2(\alpha + \beta) \leq 2$, so $\alpha + \beta \geq 1$. Thus, $\alpha + \beta = 1$.

By referring to Fig. 1, we have similarly:

$$P' = \alpha' s_1 + (1 - \alpha') \tilde{s}_1$$
$$= \beta' \tilde{s}_3 + (1 - \beta') s_3, \quad \alpha' + \beta' = 1,$$
where $\alpha' = \frac{1}{2} \|P' - \tilde{s}_1\|,$

$$P'' = \alpha'' s_2 + (1 - \alpha'') \tilde{s}_2$$
$$= \beta'' \tilde{s}_3 + (1 - \beta'') s_3, \quad \alpha'' + \beta'' = 1,$$
where $\alpha'' = \frac{1}{2} \|P'' - \tilde{s}_2\|.$

To find the relations among $\alpha, \beta, \alpha', \beta', \alpha'', \beta''$, argue as follows:

$$P - s_3 = (P - P'') + (P'' - s_3) = (1 - \beta - \beta'') (\tilde{s}_2 - s_2) + \beta'' (\tilde{s}_3 - s_3)$$
$$= (\alpha - \beta'') (\tilde{s}_2 - s_2) + \beta'' (\tilde{s}_3 - s_3)$$
$$= (\beta'' - \alpha) (x_2 - x_3) - \beta'' (x_1 - x_3)$$
$$= -\beta'' (x_1 - x_2) - \alpha (x_2 - x_3).$$

Also,

$$P - s_3 = (P - P') + (P' - s_3) = (1 - \alpha' - \beta) (s_1 - \tilde{s}_1) + \beta' (\tilde{s}_3 - s_3)$$
$$= (\beta' - \beta) (x_1 - x_2) - \beta' (x_1 - x_3) = -\beta (x_1 - x_2) - \beta' (x_2 - x_3).$$

It follows by the uniqueness of representation of points in $K - K$ that $\beta = \beta''$ and $\alpha = \beta'$. Then $1 = \alpha + \beta = \beta' + \beta''$, so $\alpha'' = \beta'$ and $\alpha' = \beta''$. Therefore, $\alpha = \beta' = \alpha''$ and $\beta = \alpha' = \beta''$.

Note that now the various distances can be computed. For example,

$$s_2 - \tilde{s}_1 = (s_2 - P) + (P - \tilde{s}_1) = \alpha (s_2 - \tilde{s}_2) + \alpha (s_1 - \tilde{s}_1) = \alpha (s_3 - \tilde{s}_3).$$

Therefore, $\|s_2 - \tilde{s}_1\| = 2 \alpha$. This also shows that $s_2 - \tilde{s}_1$ is not extreme in $S - S$. The same kind of argument shows that $\{s_i - \tilde{s}_i\}_{i=1,2,3}$ are the only extreme points in $S - S$.

Returning to equation (§), we can explicitly write down the form of $s_1, \tilde{s}_1, s_2, \tilde{s}_2, s_3, \tilde{s}_3$ as follows (from Fig. 1):

$$P = \alpha s_1 + \beta \tilde{s}_1 = \alpha (x_1 + ax) + \beta (x_2 + ax) = \alpha x_1 + \beta x_2 + ax$$
$$= \alpha \tilde{s}_2 + \beta s_2 = \alpha (x_3 + by) + \beta (x_2 + by) = \alpha x_3 + \beta x_2 + by.$$

This implies $\alpha (x_1 - x_3) = by - ax$, so $b = a$ and $by = \alpha (x_1 - x_3) + ax$.

Similarly, $c = a$ and $cz = \alpha (x_2 - x_3) + ax$. Thus,

$$s_1 = x_1 + ax, \quad \tilde{s}_1 = x_2 + ax,$$

$$s_2 = x_2 + \alpha (x_1 - x_3) + ax, \quad \tilde{s}_2 = x_3 + \alpha (x_1 - x_3) + ax,$$

$$s_3 = x_1 + \alpha (x_2 - x_3) + ax, \quad \tilde{s}_3 = x_3 + \alpha (x_2 - x_3) + ax.$$

REMARKS. (1) The above is the general solution when $\alpha > 1/2$. When $\alpha \to 1/2$, we check that $P_0 = \frac{1}{2} (s_1 + \tilde{s}_1) = \frac{1}{2} (s_2 + \tilde{s}_2) = \frac{1}{2} (s_3 + \tilde{s}_3) = \frac{1}{2} (x_1 + x_2) + ax$, and we get the symmetric solution $S = \frac{1}{2} (K - K) + P_0$. 

(2) When \( \alpha \to 1 \), we get
\[
\begin{align*}
{s_1} &= x_1 + ax, \quad {s_2} = x_1 + x_2 - x_3 + ax, \quad {s_3} = x_1 + x_2 - x_3 + ax, \\
\tilde{s}_1 &= x_2 + ax, \quad \tilde{s}_2 = x_1 + ax, \quad \tilde{s}_3 = x_2 + ax.
\end{align*}
\]
Hence, \( s_1 = \tilde{s}_2, \tilde{s}_1 = \tilde{s}_3 \) and \( s_2 = s_3 \), and we get the simplex \( \text{co}\{x_1, x_1 + x_2 - x_3, x_2\} \). Translating by \( x_3 \), we have the simplex \( \text{co}\{x_1 + x_3, x_1 + x_2, x_2 + x_3\} \), and the latter is obtained by translating \( \text{co}\{-x_1, -x_2, -x_3\} \) by \( x_1 + x_2 + x_3 \).

(3) “Uniqueness” of solutions. To show that there cannot be points in \( S \) other than those specified, take \( s \) as a typical point outside \( \text{co}\{s_1, s_1, s_2, \tilde{s}_2, s_3, \tilde{s}_3\} \) of the form
\[
s = \tilde{s}_1 + t_0(\tilde{s}_1 - s_1) + t(\tilde{s}_2 - \tilde{s}_1), \quad t_0 > 0, \ 0 < t < 1.
\]
Then from (\%) we obtain \( s = x_2 + t_0(x_2 - x_1) + t\alpha(x_1 - x_3) + ax \) and \( \tilde{s}_2 = x_3 + \alpha(x_1 - x_3) + ax \). Hence,
\[
\begin{align*}
s - \tilde{s}_2 &= x_2 - x_3 + \alpha(t - 1)(x_1 - x_3) - t_0(x_1 - x_2) \\
&= -[\alpha(1 - t) + t_0](x_1 - x_2) + [1 - \alpha(1 - t)](x_2 - x_3) \\
&= a(x_1 - x_2) + b(x_2 - x_3).
\end{align*}
\]
Then \( |a| + |b| + |a - b| = \alpha(1 - t) + t_0 + 1 - \alpha(1 - t) + |1 + t_0| = 2(1 + t_0) > 2 \).
It follows that \( s - \tilde{s}_2 \notin K - K \).

The problem in \( \mathbb{R}^3 \). Consider the simplex \( K = \text{co}\{x_1, x_2, x_3, x_4\} \) in \( \mathbb{R}^3 \). Let \( S = \{s_1, \tilde{s}_1, s_2, \tilde{s}_2, \ldots, s_6, \tilde{s}_6\} \). Again, we want to find conditions under which \( S - S = K - K \). For this purpose we use the equations in (\%). By (\%), for \( \text{co}\{x_1, x_2, x_3\} \), with \( 1/2 < \alpha_1 < 1 \), we have:
\[
\begin{align*}
&{s_1} = x_1 + ax, \quad \tilde{s}_1 = x_2 + ax, \\
&(A) \quad {s_2} = x_2 + \alpha_1(x_1 - x_3) + ax, \quad \tilde{s}_2 = x_3 + \alpha_1(x_1 - x_3) + ax, \\
&{s_3} = x_1 + \alpha_1(x_2 - x_3) + ax, \quad \tilde{s}_3 = x_3 + \alpha_1(x_2 - x_3) + ax.
\end{align*}
\]
Similarly, for \( \text{co}\{x_1, x_2, x_4\} \), with \( 1/2 < \alpha_1 < 1, a_1 \tilde{x} \), we have:
\[
\begin{align*}
&{s_1} = x_1 + a_1 \tilde{x}, \quad \tilde{s}_1 = x_2 + a_1 \tilde{x}, \\
&(B) \quad {s_5} = x_2 + a_2(x_1 - x_4) + a_1 \tilde{x}, \quad \tilde{s}_5 = x_4 + a_2(x_1 - x_4) + a_1 \tilde{x}, \\
&{s_6} = x_1 + a_2(x_2 - x_4) + a_1 \tilde{x}, \quad \tilde{s}_6 = x_4 + a_2(x_2 - x_4) + a_1 \tilde{x}.
\end{align*}
\]
Note that from the first two equations in the systems (A) and (B), it is clear that \( ax = a_1 \tilde{x} \). For \( \text{co}\{x_1, x_3, x_4\} \), with \( \frac{1}{2} < \alpha_3 < 1, a_2 \tilde{x} \), we have:
\[
\begin{align*}
&{s_3} = x_1 + \alpha_3(x_4 - x_1) + a_2 \tilde{x}, \quad \tilde{s}_3 = x_3 + \alpha_3(x_4 - x_1) + a_2 \tilde{x}, \\
&(C) \quad {s_4} = x_4 + a_2 \tilde{x}, \quad \tilde{s}_4 = x_3 + a_2 \tilde{x}, \\
&{s_6} = x_1 + \alpha_3(x_3 - x_1) + a_2 \tilde{x}, \quad \tilde{s}_6 = x_4 + \alpha_3(x_3 - x_1) + a_2 \tilde{x}.
\end{align*}
\]
Equating \(s_6\) from (B) and (C), we have
\[
s_6 = x_1 + \alpha_2(x_2 - x_4) + a_1\overline{x} = s_6 = x_1 + \alpha_3(x_3 - x_1) + a_2\overline{x},
\]
and this implies \(a = a_2\) and \(a_2\overline{x} = \alpha_2(x_2 - x_4) + \alpha_3(x_1 - x_3) + ax\). Then using the fifth equation in (C), we obtain
\[
s_3 = x_1 + \alpha(x_4 - x_1) + \alpha_2(x_2 - x_4) + \alpha_3(x_1 - x_3) + ax = x_1 + \alpha_1(x_2 - x_3) + ax
\]
(by the fifth equation in (A)). It follows that
\[
\alpha_3(x_4 - x_1) + \alpha_2(x_2 - x_4) + \alpha_3(x_1 - x_3) = \alpha_1(x_2 - x_3),
\]
or
\[
\alpha_3(x_4 - x_1) - \alpha_2(x_2 - x_4) + \alpha_3(x_1 - x_4 + x_4 - x_3) = \alpha_1(x_2 - x_4 + x_4 - x_3),
\]
or
\[
(\alpha_1 - \alpha_2)(x_2 - x_4) + (\alpha_1 - \alpha_3)(x_4 - x_3) = 0.
\]

By the linear independence of the vectors \(\{x_1 - x_4, x_2 - x_4, x_3 - x_4\}\), we must have \(\alpha_1 - \alpha_2 = 0 = \alpha_1 - \alpha_3\), which implies \(\alpha_1 = \alpha_2 = \alpha_3\) (= \(\alpha\), say), and the solution can now be written
\[
s_1 = x_1 + ax,
\]
\[
s_2 = x_2 + \alpha(x_1 - x_3) + ax,
\]
\[
s_3 = x_1 + \alpha(x_2 - x_3) + ax,
\]
\[
s_4 = x_4 + a(x_1 + x_2 - x_3 - x_4) + ax,
\]
\[
s_5 = x_2 + \alpha(x_1 - x_4) + ax,
\]
\[
s_6 = x_1 + \alpha(x_2 - x_4) + ax,
\]
\[
\tilde{s}_1 = x_2 + ax,
\]
\[
\tilde{s}_2 = x_3 + \alpha(x_1 - x_3) + ax,
\]
\[
\tilde{s}_3 = x_3 + \alpha(x_2 - x_3) + ax,
\]
\[
\tilde{s}_4 = x_3 + a(x_1 + x_2 - x_3 - x_4) + ax,
\]
\[
\tilde{s}_5 = x_4 + \alpha(x_1 - x_4) + ax,
\]
\[
\tilde{s}_6 = x_4 + \alpha(x_2 - x_4) + ax.
\]
The form of this solution simplifies if one translates by \(\alpha(x_3 + x_4)\):
\[
s_1 = x_1 + \alpha(x_3 + x_4) + ax,
\]
\[
s_2 = x_2 + \alpha(x_1 + x_4) + ax,
\]
\[
s_3 = x_1 + \alpha(x_2 + x_4) + ax,
\]
\[
s_4 = x_4 + a(x_1 + x_2) + ax,
\]
\[
s_5 = x_2 + \alpha(x_1 + x_3) + ax,
\]
\[
s_6 = x_1 + \alpha(x_2 + x_3) + ax,
\]
\[
\tilde{s}_1 = x_2 + \alpha(x_3 + x_4) + ax,
\]
\[
\tilde{s}_2 = x_3 + \alpha(x_1 + x_4) + ax,
\]
\[
\tilde{s}_3 = x_3 + \alpha(x_2 + x_4) + ax,
\]
\[
\tilde{s}_4 = x_3 + a(x_1 + x_2) + ax,
\]
\[
\tilde{s}_5 = x_4 + \alpha(x_1 + x_3) + ax,
\]
\[
\tilde{s}_6 = x_4 + \alpha(x_2 + x_3) + ax.
\]

As before, one checks easily that \(s_i - s_j, \tilde{s}_i - \tilde{s}_j, \text{ and } \tilde{s}_i - \tilde{s}_j\) are not extreme for \(i \neq j\).

**Remarks.** (1) The points \(\{s_i - \tilde{s}_i\}_{i=1}^6\) are the extreme points of \(S - S\). It is a curious fact that the points in this set are equidistant from \(x_1 + x_2 + x_3 + x_4\) [in the \(A(K)^*\) metric].

(2) It is more or less clear that the set \(S\) is non-symmetric, but here is a formal proof of this fact: If \(P_0\) were the centre of symmetry of \(S\), then for each extreme point (say, \(s_1\)), there exists \(t_1 \in S\) such that \((s_1 + t_1)/2 = P_0\).
This implies $s_1 - P_0 = P_0 - t_1$. Similarly, $\tilde{s}_1 - P_0 = P_0 - t_2$ for some $t_2 \in S$. Then we must have $s_1 - \tilde{s}_1 = t_2 - t_1$, and so $s_1 = t_2$ and $\tilde{s}_1 = t_1$ (by the uniqueness of the expression of an extreme point in $S - S$). Thus, $P_0 = (s_1 + \tilde{s}_1)/2 = \frac{1}{2}(x_1 + x_2) + ax$. But $(s_2 + \tilde{s}_2)/2$ is something different, which is a contradiction (unless $\alpha = 1/2$).

(3) $\alpha \to 1/2$ gives the symmetric solution as before with centre of symmetry $\frac{1}{2}(x_1 + x_2)$.

(4) $\alpha \to 1$ gives a translate of $K$ as a solution.

**Uniqueness.** Since the solution for the simplex $K = co\{x_1, x_2, x_3, x_4\}$ in $\mathbb{R}^3$ was obtained by solving for each face, the solution should therefore be unique, modulo translation and the ordering of the points $s_1, \tilde{s}_1, s_2, \tilde{s}_2, \ldots, s_6, \tilde{s}_6$.

It is now apparent what the solution for $K = co\{x_1, x_2, x_3, x_4, x_5\}$ in $\mathbb{R}^4$ will be like:

$$
\{x_1, x_2\} \rightarrow s_1 = x_1 + \alpha(x_3 + x_4 + x_5) + ax,
\tilde{s}_1 = x_2 + \alpha(x_3 + x_4 + x_5) + ax,
\{x_1, x_3\} \rightarrow s_2 = x_1 + \alpha(x_1 + x_4 + x_5) + ax,
\tilde{s}_2 = x_3 + \alpha(x_1 + x_4 + x_5) + ax,
\{x_1, x_4\} \rightarrow
\{x_1, x_5\} \rightarrow
\vdots
\{x_4, x_5\} \rightarrow s_{10} = x_4 + \alpha(x_1 + x_2 + x_3) + ax,
\tilde{s}_{10} = x_5 + \alpha(x_1 + x_2 + x_3) + ax.
$$

It is now clear that a solution $S$ for a simplex $K = co\{x_1, x_2, \ldots, x_{n+1}\}$ in $\mathbb{R}^n$ can be written down by inspection.

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