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Quotients of indecomposable Banach spaces of continuous functions

by

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Abstract. Assuming $\diamondsuit$, we construct a connected compact topological space K such that for every closed $L \subset K$ the Banach space $C(L)$ has few operators, in the sense that every operator on $C(L)$ is multiplication by a continuous function plus a weakly compact operator. In particular, $C(K)$ is indecomposable and has continuum many non-isomorphic indecomposable quotients, and K does not contain a homeomorphic copy of $\beta\mathbb{N}$.

Moreover, assuming CH we construct a connected compact K where $C(K)$ has few operators and K contains a homeomorphic copy of $\beta\mathbb{N}$.

1. Introduction. A Banach space X is called *indecomposable* if there are no infinite-dimensional subspaces Y and Z such that $X = Y \oplus Z$.

The first example of an indecomposable Banach space is due to Gowers and Maurey [GM]. Their space is hereditarily indecomposable, which means that all its subspaces are also indecomposable. Ferenczi [Fe] modified Gowers and Maurey's construction in order to obtain a quotient hereditarily indecomposable Banach space, which means that all its quotients are hereditarily indecomposable.

We may ask similar questions about Banach spaces of the form $C(K)$, i.e., the space of all real continuous functions on a compact topological space K normed by the supremum. The first indecomposable $C(K)$ was built by Koszmider [Ko1], using the concept of few operators.

Since $C(K)$ always contains an isomorphic copy of c_0, it cannot be hereditarily indecomposable. By a result of Lacey and Morris [LM], every $C(K)$ either contains a complemented copy of c_0 or has l_2 as quotient. So, $C(K)$ cannot be quotient indecomposable, i.e., it always has a decomposable quotient.

Although it is not possible to make the properties of indecomposability or having few operators hereditary to all quotients, one may ask if it is possible

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to make these properties hereditary to a large class of quotients. In this paper we answer a question posed at the end of [Ko1], obtaining, under axiom $\diamondsuit$, an indecomposable $C(K)$ such that for all closed $L \subset K$ the space $C(L)$ has few operators. Since a closed subspace of K induces a quotient of $C(K)$, we conclude that $C(K)$ has many quotients which also have few operators, and several of them are indecomposable (whenever L is connected).

This result is related to *Efimov's problem*, which asks whether a compact space K can have neither a non-trivial convergent sequence nor a homeomorphic copy of $\beta\mathbb{N}$. The problem was solved affirmatively under CH (see [Ta]) and remains open in ZFC. Our construction provides an alternative solution (under $\diamondsuit$, which implies CH) to Efimov's problem, because the space K cannot have a convergent sequence, since this would imply having c_0 as a complemented subspace, and cannot have a homeomorphic copy of $\beta\mathbb{N}$, since $C(\beta\mathbb{N}) \equiv l_\infty$, which has many operators.

All topological spaces appearing in this paper are Hausdorff.

2. Weak multipliers.

Indecomposable Banach spaces are closely related to the property of having few operators. In Gowers and Maurey's construction, every operator is multiplication by a real number plus a weakly compact operator. Assuming the continuum hypothesis (which was further eliminated by Plebanek [Pl]), Koszmider built an indecomposable $C(K)$ space such that every operator on it is multiplication by a continuous function plus a weakly compact operator. With no extra set-theoretical assumption, he showed that all operators on the space he constructed have a special property, similar to the above one, and which we will describe in this section.

DEFINITION 2.1 ([Ko1, 2.1]). An operator $T : C(K) \to C(K)$ is called a *weak multiplier* if for every bounded sequence $(e_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $C(K)$ (i.e., $e_n \cdot e_m = 0$ for $n \neq m$) and any sequence $(x_n)_{n\in\mathbb{N}} \subset K$ such that $e_n(x_n) = 0$ we have

$$\lim_{n\to\infty} T(e_n)(x_n) = 0.$$

LEMMA 2.2 ([Ko1, 2.3]). *Let $T : C(K) \to C(K)$ be a weak multiplier. Then T is onto if and only if T is an isomorphism onto its range.*

Recall that $Y \subset X$ is C^*-*embedded* in X if every bounded continuous function from Y into $\mathbb{R}$ extends to a bounded continuous function from X to $\mathbb{R}$.

LEMMA 2.3 ([Ko1, 2.8]). *Let K be a compact topological space such that for any disjoint open sets U_1 and U_2 we have either $\overline{U}_1 \cap \overline{U}_2 = \emptyset$ or $|\overline{U}_1 \cap \overline{U}_2| \geq 2$. Then for every $x \in K$ the space $K \setminus \{x\}$ is C^*-embedded in K.*

Theorem 2.4 ([Ko1, 2.7]). *The following are equivalent for a compact space K:*

(i) *All operators $T : C(K) \to C(K)$ are of the form $gI + S$ where $g \in C(K)$ and S is weakly compact.*
(ii) *All operators on $C(K)$ are weak multipliers and for every $x \in K$ the space $K \setminus \{x\}$ is C^*-embedded in K.*

We will use a definition which was first introduced in [Sc].

Definition 2.5. A compact space K is a *Koszmider space* if all operators on $C(K)$ have the form $gI + S$, where $g \in C(K)$ and S is weakly compact.

The following lemma is an adaptation of Lemma 2.5 of [Ko1].

Lemma 2.6. *If K is a connected Koszmider space then $C(K)$ is indecomposable.*

Proof. Let K be as in the hypothesis and suppose that there are closed subspaces X and Y of $C(K)$ such that $C(K) = X \oplus Y$. We will show that either X or Y is finite-dimensional.

Let P be the projection on $C(K)$ such that $\mathrm{Im}(P) = X$ and $\mathrm{Ker}(P) = Y$. By hypothesis there are $g \in C(K)$ and S weakly compact such that $P = gI + S$. Since $P^2 = P$ we have $g^2I + S^2 + gS + Sg = gI + S$. Hence $S' = (g^2 - g)I$ is weakly compact, and so strictly singular (see [Pe]).

Suppose $(g^2 - g)(x) \neq 0$ for some $x \in K$. Then there is an open neighborhood V of x such that $|(g^2 - g)(y)| > \varepsilon$ for some $\varepsilon > 0$ and every $y \in \overline{V}$. Let Z be the subspace of $C(K)$ consisting of all continuous functions whose supports are included in V. Since K is connected and therefore it has no isolated point, Z is infinite-dimensional. But $S'|_Z$ is an isomorphism onto its range, since $(g^2 - g)^{-1}$ is well defined and continuous on $\overline{V}$, and provides an inverse of S'. This contradicts S' being strictly singular.

Therefore $(g^2 - g)(x) = 0$ for every $x \in K$, so $g(x) \in \{0, 1\}$ for every $x \in K$. Since K is connected, $g \equiv 0$ or $g \equiv 1$. So we have either $P = S$ or $P = I + S$, which implies that either P or $I - P$ is weakly compact. In the first case, $P|_{\mathrm{Im}(P)}$ is an isomorphism onto its range and so $\mathrm{Im}(P))$ has finite dimension. In the second case, $(I - P)|_{\mathrm{Ker}(P)}$ is the identity and so $\mathrm{Ker}(P)$ is finite-dimensional, concluding the proof. ■

3. Extensions by continuous functions. The first construction of $C(K)$ having few operators, in the sense described below, was made in [Ko1] with K being the Stone space of a boolean algebra. The algebra was obtained by transfinite induction, adding the supremum of a pairwise disjoint sequence of elements of the algebra at each step.

Although this construction provides a Banach space $C(K)$ with few operators, it cannot provide an indecomposable Banach space, since the Stone space of a boolean algebra is always zero-dimensional, i.e., it has a basis of closed-open sets. If K can be written as the union of two infinite disjoint closed-open sets L_1 and L_2, then $C(K)$ is isomorphic to $C(L_1) \oplus C(L_2)$ and therefore it cannot be indecomposable.

To fix this problem and get an indecomposable $C(K)$, Koszmider created an analogous procedure to add suprema of continuum many pairwise disjoint functions defined on a connected compact space. In this section we describe Koszmider's main results and develop some new lemmas that will be useful in our constructions.

We now introduce some notation. Let K be a compact space and $(f_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$. We denote by $D((f_n)_{n\in\mathbb{N}})$ the set of all $x \in K$ such that there exists an open neighborhood U of x such that $U \cap \operatorname{supp}(f_n) = \emptyset$ for all but finitely many $n \in \mathbb{N}$. We denote its complement $K \setminus D((f_n)_{n\in\mathbb{N}})$ by $\Delta((f_n)_{n\in\mathbb{N}})$.

We recall that f is the supremum of $\{f_n : n \in \mathbb{N}\}$ in $C(K)$ if $f_n \le f$ for all $n \in \mathbb{N}$ and $f \le g$ whenever $g \in C(K)$ satisfies $f_n \le g$ for every $n \in \mathbb{N}$.

Lemma 3.1 ([Ko1, 4.1]). *Let K be a compact space and let $(f_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$. Then:*

(i) *$f \in C(K)$ is $\sup\{f_n : n \in \mathbb{N}\}$ in the lattice $C(K)$ if and only if*
$$\Big\{x \in K : \sum_{n\in\mathbb{N}} f_n(x) \neq f(x)\Big\}$$
is nowhere dense in K.

(ii) *$D((f_n)_{n\in\mathbb{N}})$ is an open dense set in K and $\sum_{n\in\mathbb{N}} f_n$ is continuous on $D((f_n)_{n\in\mathbb{N}})$.*

Definition 3.2 ([Ko1, 4.2]). Suppose that K is a compact space, $L \subset K \times [0,1]$ and $(f_n)_{n\in\mathbb{N}}$ is a pairwise disjoint sequence of continuous functions from K into $[0,1]$. We say that L is an *extension of K by $(f_n)_{n\in\mathbb{N}}$*, written $L = K((f_n)_{n\in\mathbb{N}})$, if L is the closure of the graph $\sum_{n\in\mathbb{N}} f_n | D((f_n)_{n\in\mathbb{N}})$.

The next lemma is an easy consequence of Lemma 3.1 and the above definition.

Lemma 3.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of continuous functions from a compact space K into $[0,1]$, and let L be the extension of K by $(f_n)_{n\in\mathbb{N}}$. Then*
$$\{x \in K : |\pi_{L,K}^{-1}(x)| > 1\} \subset \Delta((f_n)_{n\in\mathbb{N}}).$$

Lemma 3.4 ([Ko1, 4.3]). *Let $(f_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of continuous functions from a compact space K into $[0,1]$, and take $L = K((f_n)_{n\in\mathbb{N}})$. Let π be the projection of L to K. Then:*

(i) *If* $M \subset K$ *is nowhere dense in* K *then* $\pi^{-1}[M]$ *is nowhere dense in* L.

(ii) $\sup\{f_n \circ \pi : n \in \mathbb{N}\}$ *exists in* $C(L)$.

Part (i) of the above lemma guarantees that the supremum obtained at some step of the inductive construction is not spoiled at further steps. One may easily verify that the supremum in (ii) is the function which maps the pair $(x,t) \in L \subset K \times [0,1]$ into $t \in [0,1]$.

The main problem of working with extensions by continuous functions is the preservation of connectedness. We do not know yet if extensions, in general, preserve connectedness. Koszmider proved [Ko1] that this occurs when the extension contains the graph of $\sum_{n\in\mathbb{N}} f_n$. But for our purpose we need the following definition:

Definition 3.5. Let L be an extension of K. We say that L is a *complete extension* of K if for every $x \in K$, $\pi_{L,K}^{-1}[\{x\}]$ is either a singleton or $\{x\} \times [0,1]$.

Lemma 3.6. *If* K *is a compact connected space and* L *is a complete extension of* K, *then* L *is also connected.*

Proof. Let π be the standard projection from L to K. Suppose that L is not connected. Let U and V be non-empty open sets of L such that $U \cap V = \emptyset$ and $U \cup V = L$. Since U and V are closed, and therefore compact, $\pi[U]$ and $\pi[V]$ are closed in K. It is clear that $\pi[U] \cup \pi[V] = K$, because $U \cup V = L$ and $\pi[L] = K$. Hence, by connectedness of K, we conclude that $\pi[U] \cap \pi[V] \neq \emptyset$. Let x be in this intersection. Then we have $x \in K$, $\pi^{-1}[\{x\}] \cap U \neq \emptyset$ and $\pi^{-1}[\{x\}] \cap V \neq \emptyset$. But $\pi^{-1}[\{x\}] \subset U \cup V$ and $\pi^{-1}[\{x\}] \cap (U \cap V) = \emptyset$, proving that $\pi^{-1}[\{x\}]$ is not connected, contradicting the hypothesis that it is either a singleton or homeomorphic to $[0,1]$. ∎

We recall the definition of the inverse limit of topological spaces. Let $\prod_{\alpha<\kappa} X_\alpha$ be a product of topological spaces, where κ is a limit ordinal. Let Y_α be subspaces of $\prod_{\beta<\alpha} X_\beta$ such that $\pi_\beta[Y_\alpha] = Y_\beta$ whenever $\beta < \alpha$. We define the inverse limit of $(Y_\alpha)_{\alpha<\kappa}$ as

$$\varprojlim\, (Y_\alpha)_{\alpha<\kappa} = \Big\{(y_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} X_\alpha : \forall \alpha < \kappa \ ((y_\beta)_{\beta<\alpha} \in Y_\alpha)\Big\}.$$

Inverse limits preserve compactness and connectedness (see [Eng, 2.5.1]).

Lemma 3.7. *Suppose that* β *is an ordinal and let* $(K_\alpha)_{\alpha\le\beta}$ *be a sequence such that* $K_2 = [0,1]^2$, $K_\alpha \subset [0,1]^\alpha$ *is compact,* K_α *is the inverse limit of* $(K_\gamma)_{\gamma<\alpha}$ *if* α *is a limit ordinal, and* $K_{\alpha+1}$ *is a complete extension of* K_α *by a pairwise disjoint sequence of continuous functions from* K_α *into* $[0,1]$. *Then:*

(i) *If* $f, f_n \in C(K_\alpha)$ *for* $n \in \mathbb{N}$ *and* $\alpha \le \beta$ *are such that*
$$f = \sup\{f_n : n \in \mathbb{N}\},$$
then
$$f \circ \pi_{\beta,\alpha} = \sup\{f_n \circ \pi_{\beta,\alpha} : n \in \mathbb{N}\}.$$

(ii) *For every compact connected* $K' \subset K_2$, $\pi^{-1}_{K_\beta,K_2}[K']$ *is connected. In particular,* K_β *is connected.*

Proof. Item (i) follows from Lemma 3.1 and Lemma 3.4(i). Item (ii) is a consequence of Lemma 3.6 and the fact that inverse limits preserve connectedness. ∎

We now present three variations of Lemma 4.5 from [Ko1].

Lemma 3.8. *Let* K *be a metric compactum without isolated points. Suppose that* $(\varepsilon_n)_{n\in\mathbb{N}}$ *is a sequence of positive real numbers,* $(g_n)_{n\in\mathbb{N}}$ *is a pairwise disjoint sequence of continuous functions from* K *into* $[0,1]$, $(\mu_n)_{n\in\mathbb{N}}$ *is a sequence of regular measures on* K *and* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of points in* K *such that* $g_n(x_n) = 1$. *Then there exist continuous functions* $f_n : K \to [0,1]$ *such that:*

(i) *For every* $n \in \mathbb{N}$,
$$\operatorname{supp}(f_n) \subset \operatorname{supp}(g_n), \qquad f_n(x_n) = 1, \qquad \int |f_n - g_n| \, d|\mu_n| < \varepsilon_n.$$

(ii) *For every* $x \in K \setminus D((f_n)_{n\in\mathbb{N}})$ *we have* $\pi^{-1}[\{x\}] = \{x\} \times [0,1]$, *where* π *is the standard projection from* $K((f_n)_{n\in\mathbb{N}})$ *into* K.

Proof. Let d be a metric on K. For each $n \in \mathbb{N}$ we fix a finite cover $\{V_n^i : i \in I_n\}$ of K such that each V_n^i has diameter at most $1/n$ (i.e., $\sup\{d(x,y) : x, y \in V_n^i\} \le 1/n$). We take
$$I'_n = \{i \in I_n : \{j \in \mathbb{N} : V_n^i \cap \operatorname{supp}(g_j) \neq \emptyset\} \text{ is finite}\}.$$

We construct, by induction on n, pairwise disjoint finite sets $F_n \subset \mathbb{N}$ and functions $\{f_k : k \in F_n\}$ from K into $[0,1]$ satisfying:

1. For every $k \in F_n$, $f_k(x_k) = 1$.
2. For every $k \in F_n$, $\operatorname{supp}(f_k) \subset \operatorname{supp}(g_k)$.
3. For every $k \in F_n$, $\int |f_k - g_k| \, d|\mu_k| < \varepsilon_k$.
4. For all $m \le n$ and $i \in I'_n$ there exist $k \in F_n$ and $y \in V_n^i$ such that $f_k(y) = m/n$.

At the inductive step n, suppose that we already have F_j and $\{f_k : k \in F_j\}$ for every $j < n$. We will define F_n and $\{f_k : k \in F_n\}$.

For every $i \in I'_n$ and $m \le n$ we fix $k^n_{i,m} \in \mathbb{N} \setminus \bigcup_{j<n} F_j$ such that $\operatorname{supp}(g_{k^n_{i,m}}) \cap V_n^i \neq \emptyset$. We may assume that $k^n_{i,m} \neq k^n_{i',m'}$ whenever $i \neq i'$ or $m \neq m'$.

Define

$$U_{k^n_{i,m}} = \{x \in K : g_{k^n_{i,m}}(x) > 0\} \cap V^i_n.$$

Since K has no isolated points, $U_{k^n_{i,m}}$ is infinite. Hence there exists $y_{k^n_{i,m}} \in U_{k^n_{i,m}}$ such that $y_{k^n_{i,m}} \neq x_{k^n_{i,m}}$ and $|\mu_{k^n_{i,m}}|(\{y_{k^n_{i,m}}\}) < \varepsilon_{k^n_{i,m}}$. Using regularity of the measures we may choose an open neighborhood $V_{k^n_{i,m}} \subset U_{k^n_{i,m}} \setminus \{x_{k^n_{i,m}}\}$ of $y_{k^n_{i,m}}$ such that $|\mu_{k^n_{i,m}}|(V_{k^n_{i,m}}) < \varepsilon_{k^n_{i,m}}$.

Define $F_n = \{k^n_{i,m} : i \in I'_n,\, m \leq n\}$. By Urysohn's lemma, for each $i \in I'_n$ and $m \leq n$ we may define a continuous function $f_{k^n_{i,m}} : K \to [0,1]$ such that $f_{k^n_{i,m}}|_{K \setminus V_{k^n_{i,m}}} = g_{k^n_{i,m}}|_{K \setminus V_{k^n_{i,m}}}$ and $f_{k^n_{i,m}}(y_{k^n_{i,m}}) = m/n$. Properties 1 to 4 of the inductive construction clearly hold for F_n and $\{f_k : k \in F_n\}$.

At the end of the construction we have defined f_n for every $n \in \bigcup_{j\in\mathbb{N}} F_j$. For $n \in \mathbb{N} \setminus \bigcup_{j\in\mathbb{N}} F_j$ we take $f_n = g_n$. We will prove (i) and (ii).

Item (i) follows immediately from hypotheses 1 to 3 of the inductive step. Now we prove (ii). Suppose $x \in K \setminus D((f_n)_{n\in b})$. In particular $x \notin D((g_n)_{n\in b})$, since $\mathrm{supp}(f_n) \subset \mathrm{supp}(g_n)$. Let us show that $\pi^{-1}_{K((f_n)_{n\in b}),K}(x) = \{x\} \times [0,1]$. It is sufficient to prove that for every $t \in [0,1]$ and $n \in \mathbb{N}$ there exist $j \in \mathbb{N}$ and $y \in K$ such that $d(x,y) < 1/n$ and $|f_j(y) - t| < 1/n$.

Take $i \in I_n$ such that $x \in V^i_n$. Since $x \notin D((g_n)_{n\in b})$ we have $i \in I'_n$, because every open neighborhood of x intersects $\mathrm{supp}(g_n)$ for infinitely many n's. Fix $m \leq n$ such that $|t - m/n| < 1/n$. By item (iv) of the inductive hypothesis there exist $y \in V^i_n$ and $j \in \mathbb{N}$ such that $f_j(y) = m/n$. Since $\mathrm{diam}(V^i_n) = 1/n$ we have $d(x,y) < 1/n$, concluding the proof. ∎

Lemma 3.9. *Let K be a compact metric space without isolated points. Let $\{X_n : n \in \mathbb{N}\}$ and $\{Y_n : n \in \mathbb{N}\}$ be families of countable subsets of K such that $X_n \cap Y_n = \emptyset$ and $\overline{X}_n \cap \overline{Y}_n \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be a relatively discrete sequence in K which is disjoint from $\bigcup_{m\in\mathbb{N}}(X_m \cup Y_m)$. Then there exists a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from K into $[0,1]$ such that:*

(i) *$f_n(x_n) = 1$ for every $n \in \mathbb{N}$.*
(ii) *For every $x \in K \setminus D((f_n)_{n\in\mathbb{N}})$, $\pi^{-1}[\{x\}] = \{x\} \times [0,1]$, where π is the standard projection from $K((f_n)_{n\in\mathbb{N}})$ into K.*
(iii) *For every $n \in \mathbb{N}$, $\overline{X'_n} \cap \overline{Y'_n} \neq \emptyset$ in $K((f_n)_{n\in\mathbb{N}})$, where*

$$X'_n = \Big\{\Big(x, \sum_{n\in\mathbb{N}} f_n(x)\Big) : x \in X_n\Big\} \quad \textit{and} \quad Y'_n = \Big\{\Big(x, \sum_{n\in\mathbb{N}} f_n(x)\Big) : x \in Y_n\Big\}.$$

(iv) *$K((f_n)_{n\in\mathbb{N}})$ is a complete extension of K.*

Proof. Take a pairwise disjoint sequence $(g_n)_{n\in\mathbb{N}}$ of functions from K into $[0,1]$ such that $g_n(x_n) = 1$. We will modify the functions g_n in order to obtain (ii) and (iii).

By metrizability there are $\{q_n^m : n \in \mathbb{N}\} \subset X_m$ and $\{r_n^m : n \in \mathbb{N}\} \subset Y_m$ for every m, where $q_{n_1}^m \neq q_{n_2}^m$ and $r_{n_1}^m \neq r_{n_2}^m$ whenever $n_1 \neq n_2$, and there are $q^m \in K$ such that $q_n^m \xrightarrow{n} q^m$ and $r_n^m \xrightarrow{n} q^m$ for each m.

If $q^m \in D((g_n)_{n\in\mathbb{N}})$ and $\mathrm{supp}(f_n) \subset \mathrm{supp}(g_n)$ for every $n \in b$, item (iii) clearly holds for X'_m and Y'_m in $K((f_n)_{n\in b})$, since, in that case, $\sum_{n\in b} f_n$ is continuous on an open neighborhood of q^m. Therefore we may assume without loss of generality that $q^m \notin D((g_n)_{n\in b})$ for every $m \in \mathbb{N}$. So $g_n(q^m) = 0$ for all $n, m \in \mathbb{N}$. Hence, reducing $\mathrm{supp}(g_n)$ for every n, we may assume $q^m \notin \mathrm{supp}(g_n)$ for all n, m.

Let us build $(f_n)_{n\in\mathbb{N}}$ such that $f_n(x_n) = 1$, $\mathrm{supp}(f_n) \subset \mathrm{supp}(g_n)$, $f_n(q_i^m) = 0$ and $f_n(r_j^m) = 0$ for every $n, m \in \mathbb{N}$ and for all but finitely many $i \in \mathbb{N}$ and $j \in \mathbb{N}$.

Using induction on n, we build finite sets $F_n \subset \mathbb{N}$, continuous functions $\{f_i : i \in F_n\}$ from K into $[0, 1]$ and positive integers $\{k_{n,m} : m \leq n\}$ and $\{l_{n,m} : m \leq n\}$ such that:

1. If $j \leq n$ then $F_j \subset F_n$.
2. $k_{n,m} > k_{n',m'}$ and $l_{n,m} > l_{n',m'}$ for all m, m', n' such that $m \leq n$, $m' \leq n'$ and $n > n'$.
3. $h_i \leq g_i$ and $h_i(x_i) = 1$ for every $i \in F_n$.
4. $q_{k_{j,m}}^m, r_{l_{j,m}}^m \notin \mathrm{supp}(h_i)$ for all $i \in F_n$, $m \leq j \leq n$.
5. $q_{k_{j,m}}^m, r_{l_{j,m}}^m \notin \mathrm{supp}(h_i)$ for all $i \in \mathbb{N} \setminus F_n$, $m \leq j \leq n$.

For $n = 0$ we define $F_0 = \emptyset$ and $k_{0,0} = l_{0,0} = 0$. Suppose we have F_n, $\{h_i : i \in F_n\}$, $\{k_{n,m} : m \leq n\}$ and $\{l_{n,m} : m \leq n\}$. We will define F_{n+1}, $\{h_i : i \in F_{n+1}\}$, $\{k_{n+1,m} : m \leq n+1\}$ and $\{l_{n+1,m} : m \leq n+1\}$.

Since $\mathrm{supp}(h_i) \subset \mathrm{supp}(g_i)$ for every $i \in F_n$, and $q^m \notin \mathrm{supp}(g_i)$ for all $i \in \mathbb{N}$, for each $m \leq n+1$ we may fix integers k_m and l_m such that $h_i(q_{k_m}^m) = h_i(r_{l_m}^m) = 0$, $k_m > k_{j,m'}$ and $l_m > l_{j,m'}$ for every $m' \leq j \leq n$ and every $i \in F_n$. For $m \leq n+1$ take $i_m \in \mathbb{N} \setminus F_n$ such that $g_{i_m}(q_{k_m}^m) \neq 0$ and $j_m \in \mathbb{N} \setminus F_n$ such that $g_{j_m}(r_{l_m}^m) \neq 0$, when they exist. Otherwise, take any i_m or j_m in $\mathbb{N} \setminus F_n$. Note that we may have $i_m = i_{m'}$ for some $m \neq m'$.

Define $F_{n+1} = F_n \cup \{i_m : m \leq n+1\} \cup \{j_m : m \leq n+1\}$, $k_{n+1,m} = k_m$ and $l_{n+1,m} = l_m$. We have already defined h_i for $i \in F_n$. It remains to define h_{i_m} for every $m \leq n+1$ in such a way that $h_{i_m} \leq g_{i_m}$, $h_{i_m}(x_{i_m}) = 1$ and, for every $m' \leq n+1$, there exist open neighborhoods $U_{m'}$ and $V_{m'}$ of $q_{k_{m'}}^{m'}$ and $r_{k_{m'}}^{m'}$, respectively, satisfying $h_{i_m}|_{U_{m'}} = h_{i_m}|_{V_{m'}} = 0$. To find h_{i_m}, we use that $x_i \neq q_j^m$ for all i, j, m, and apply Urysohn's lemma. The functions h_{j_m} are defined analogously. Properties 1 to 5 will be clearly preserved at step $n+1$.

For $i \notin \bigcup_{n\in\mathbb{N}} F_n$ we define $h_i = g_i$.

Using Lemma 3.8 we build f_n with $\mathrm{supp}(f_n) \subset \mathrm{supp}(h_n)$, satisfying (ii) and preserving (i). Items 4 and 5 are clearly preserved for f_n in place of h_n. From (ii) it follows that $(q^m, 0) \in K((f_n)_{n\in\mathbb{N}}$, as we assumed $q^m \notin D((f_n)_{n\in\mathbb{N}})$. So, by 1 to 5 we conclude $(q^m, 0) \in \overline{X}'_m \cap \overline{Y}'_m$, proving (iii).

Item (iv) follows immediately from (ii) and (iii). ∎

LEMMA 3.10. *Let K be a compact metric space with no isolated points. Given*

(a) *a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from K into $[0,1]$;*
(b) *a sequence $(x_n)_{n\in\mathbb{N}}$ in K;*
(c) *an $\varepsilon > 0$;*
(d) *a bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of regular measures on K such that $|\int f_n\, d\mu_n| > \varepsilon$ for every $n \in \mathbb{N}$,*

there exist $\delta > 0$, $a \subset \mathbb{N}$ infinite, and continuous functions $f'_n : K \to [0,1]$ such that $\mathrm{supp}(f'_n) \subset \mathrm{supp}(f_n)$ and, for every $b \subset a$, we have:

(e) *$|\int f'_n\, d\mu_n| > \delta$ and $\sum\{\int f'_m\, d|\mu_n| : m \neq n,\ m \in a\} < \delta/3$ for every $n \in a$;*
(f) *$L = K((f'_n)_{n\in b})$ is a complete extension;*
(g) *$\Delta((f'_n)_{n\in b})$ is a singleton or disjoint from $\{x_n : n \in \mathbb{N}\}$.*

Proof. Given a pairwise disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of open sets of K, define $\Delta((A_n)_{n\in\mathbb{N}}) = \{x \in K :$ every open neighborhood of x intersects infinitely many A_n's$\}$.

Take $A_n = \{x \in K : f_n(x) > 0\}$ for every $n \in \mathbb{N}$. Clearly $\Delta((A_n)_{n\in\mathbb{N}}) = \Delta((f_n)_{n\in\mathbb{N}})$.

Note that if A_n's are closed open sets, $\Delta((A_n)_{n\in\mathbb{N}}) = \overline{\bigcup_{n\in\mathbb{N}} A_n} \setminus \bigcup_{n\in\mathbb{N}} A_n$, as stated in Lemma 7 of [Ko2].

The next case is an adaptation for the non-0-dimensional case.

CLAIM 1. *There exist $N_1 \subset \mathbb{N}$ infinite, $\delta > 0$, and open sets A'_n for $n \in N_1$, such that $A'_n \subset A_n$, $|\mu_n(A'_n)| > \delta$ and*

1. *$\Delta((A'_n)_{n\in N_1})$ is a singleton, and*
2. *$x_m \notin \Delta((A'_n)_{n\in N_1})$ for every $m \in N_1$.*

We prove the claim dividing it into two cases:

CASE 1: There exist $\delta' > 0$ and $x \in K$ such that, for every open neighborhood V of x and for every $m \in \mathbb{N}$ there exists $k > m$ such that $|\mu_k|(A_k \cap V) > \delta'$.

In this case, as K is metrizable, we take a decreasing open basis $(V_n)_{n\in\mathbb{N}}$ of neighborhoods of x. If $x = x_n$ for some $n \in \mathbb{N}$, in order to take care of part 2 of the claim, we define $N_0 = \mathbb{N} \setminus \{n\}$. Otherwise we take $N_0 = \mathbb{N}$. We will build, by induction, an infinite $N_1 \subset N_0$ and integers k_n with no repetitions such that $|\mu_n|(A_n \cap V_{k_n}) > \delta'$ for every $n \in N_1$. Take $A'_n = A_n \cap V_{k_n}$. Using the definition of variation of measures we may assume that $|\mu_n(A'_n)| > \delta'$ for every $n \in N_1$, replacing δ' by $\delta'/2$. Then we have $\Delta((A'_n)_{n\in\mathbb{N}}) = \{x\}$, since every open neighborhood of x contains all but finitely many A'_n's, by the fact that $(V_{k_n})_{n\in N_1}$ is an open basis at x. This proves the claim, where $\delta = \delta'$.

Case 2: Case 1 does not occur. For every $n \in \mathbb{N}$ and $\delta' > 0$ there exist $m(n, \delta') \in \mathbb{N}$ and an open neighborhood $V(n, \delta')$ of x_n such that

$$|\mu_k|(A_k \cap V(n, \delta')) < \delta'$$

for every $k > m(n, \delta')$. Replacing $V(n, \delta')$ by $V'(n, \delta')$ such that $x_n \in V'(n, \delta') \subset \overline{V'(n, \delta')} \subset V(n, \delta')$, we may asume that

$$|\mu_k|(A_k \cap \overline{V(n, \delta')}) < \delta' \tag{1}$$

for every $k > m(n, \delta')$.

We choose by induction a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of integers such that $k_n > m(j, \varepsilon/2^{j+2})$ for every $j < n$. Let

$$A'_{k_n} = A_{k_n} \setminus \bigcup\{\overline{V(j, \varepsilon/2^{j+2})} : j < n\}.$$

By (1) we have $|\mu_{k_n}|(A_{k_n} \cap \overline{V(j, \varepsilon/2^{j+2})}) < \varepsilon/2^{j+2}$ for $j < n$, and

$$|\mu_{k_n}|(A'_{k_n}) > \varepsilon/2. \tag{2}$$

Take $N_1 = \{k_n : n \in \mathbb{N}\}$ and $\delta = \varepsilon/2$. Since $V(n, \varepsilon/2^{n+2})$ is disjoint from A'_{k_i} for every $i > n$, we have $x_n \notin \Delta((A'_n)_{n\in N_1})$, proving the claim.

For every $n \in N_1$ we fix $\delta_n > \delta$ such that $|\mu_n(A'_n)| > \delta_n$. By regularity of μ_n there are closed $B_n \subset A'_n$ such that $|\mu_n(B_n)| > \delta_n$ and $|\mu_n|(B_n - A'_n) < \delta_n - \delta$. By Tietze's theorem there are continuous functions f'_n from K into $[0, 1]$ such that $f'_n|_{B_n} = 1$ and $f'_n|_{K\setminus A'_n} = 0$. So we have $|\int f'_n\, d\mu_n| > \delta$ and $\operatorname{supp}(f'_n) \subset \operatorname{supp}(f_n)$. Since $\Delta((f'_n)_{n\in N_1}) \subset \Delta((A'_n)_{n\in N_1})$ we obtain (g).

By Rosenthal's lemma (see [Di, p. 82]) and the fact that $\int f'_m\, d|\mu_n| < |\mu_n|(A_m)$, there is an infinite $N_2 \subset N_1$ satisfying the second part of (e). To get (f), we use Lemma 3.8 to modify f'_n's in order to obtain (f) preserving (b). Item (g) is preserved, since we did not increase $\operatorname{supp}(f'_n)$. ■

4. Axiom $\diamondsuit$. In this section we briefly discuss a set-theoretical axiom that will be used in our construction. As the continuum hypothesis (CH), axiom $\diamondsuit$ provides an enumeration of countable sets, but in a way stronger than the one given by CH.

Axiom $\diamondsuit$ is relatively consistent with ZFC, it holds in the constructible universe, and implies CH. See [Ku] or [Ve] for references.

Before stating axiom $\diamondsuit$ we need some definitions.

We say that a subset C of ω_1 is a *cub* if it is closed and unbounded, i.e., it is uncountable and for every increasing countable sequence in C, its supremum in ω_1 belongs to C. And we say that a subset S of ω_1 is *stationary* if it intersects every cub.

All stationary sets are uncountable (equivalently, unbounded in ω_1), since $\{\alpha < \omega_1 : \alpha > \beta\}$ is a cub, for a fixed $\beta < \omega_1$.

The next lemma is proved in [Ku, Chapter II, Lemma 6.8].

LEMMA 4.1. *A countable intersection of cubs is a cub itself. In particular, if S is stationary and C is a cub, then $S \cap C$ is stationary.*

Now we recall the axiom.

AXIOM $\diamondsuit$. There exists a sequence $(X_\alpha)_{\alpha\in\omega_1}$ such that $X_\alpha \subset \alpha$ for all $\alpha \in \omega_1$, and $\{\alpha \in \omega_1 : X \cap \alpha = X_\alpha\}$ is stationary for all $X \subset \omega_1$.

The sequence $(X_\alpha)_{\alpha\in\omega_1}$ is called a $\diamondsuit$-*sequence.*

The next lemma is an easy consequence of the definition, since we may identify functions on ω_1 into $\{0,1\}$ with subsets of ω_1.

LEMMA 4.2. *Axiom $\diamondsuit$ implies:*

(i) *If $(B_\alpha)_{\alpha<\omega_1}$ is a sequence of sets of cardinality ω_1, there exists a sequence $(x_\alpha)_{\alpha<\omega_1}$ such that $x_\alpha \in \prod_{\beta<\alpha} B_\beta$ for all $\alpha < \omega_1$, and $\{\alpha < \omega_1 : x|\alpha = x_\alpha\}$ is stationary for all $x \in \prod_{\alpha<\omega_1} B_\alpha$.*

(ii) *There exists a sequence $\{x_n(\alpha) : n \in \mathbb{N},\ \alpha < \omega_1\}$ such that $x_n(\alpha) \in [0,1]^\alpha$ for all $\alpha < \omega_1$, and $\{\alpha \in \omega_1 : \forall n \in \mathbb{N}\ (x_n|\alpha = x_n(\alpha))\}$ is stationary for all sequences $(x_n)_{n\in\mathbb{N}}$ in $[0,1]^{\omega_1}$.*

(iii) *There exists $(x_\alpha)_{\alpha<\omega_1}$ such that $x_\alpha \in [0,1]^{\alpha\times\alpha}$ for all $\alpha < \omega_1$, and $\{\alpha < \omega_1 : x|_{\alpha\times\alpha} = x_\alpha\}$ is stationary for all $x \in [0,1]^{\omega_1\times\omega_1}$.*

(iv) *There is a sequence $(A_\alpha)_{\alpha<\omega_1}$ of subsets of ω_1 such that $\{\alpha \in \omega_1 : \{z_\beta|\alpha : \beta < \alpha\} = A_\alpha\}$ is stationary whenever $(z_\beta)_{\beta\in\omega_1}$ is a sequence in $[0,1]^{\omega_1}$.*

The following lemma will be used in a topological application of $\diamondsuit$, in the construction of K.

LEMMA 4.3. *Let $Y \subset [0,1]^{\omega_1}$ and $(x_\alpha)_{\alpha<\omega_1}$ be dense sequences in Y. Then $\{\alpha < \omega_1 : (x_\beta|\alpha)_{\beta<\alpha}$ is dense in $\pi_\alpha[Y]\}$ is a cub in ω_1.*

Proof. First we show that $C = \{\alpha < \omega_1 : (x_\beta|\alpha)_{\beta<\alpha}$ is dense in $\pi_\alpha[Y]\}$ is closed in ω_1, i.e., for every countable increasing sequence in C its supremum belongs to C. Let $(\alpha_n)_{n\in\mathbb{N}}$ be an increasing sequence in C and let α be the supremum of $(\alpha_n)_{n\in\mathbb{N}}$ in ω_1. We will prove that $\alpha \in C$, which means that $(x_\beta|\alpha)_{\beta<\alpha}$ is dense in $\pi_\alpha[Y]$.

Let U be an elementary open set of $[0,1]^\alpha$ which intersects $\pi_\alpha[Y]$. Since U depends on a finite number of coordinates, there exists $n \in \mathbb{N}$ such that α_n includes all these coordinates. So we have $U = \pi_{\alpha_n}^{-1}[\pi_{\alpha_n}[U]]$ and $\pi_{\alpha_n}[U]$ is open in $[0,1]^{\alpha_n}$. Then, since $\alpha_n \in C$, there exists $\beta < \alpha$ such that $x_\beta|\alpha_n \in \pi_{\alpha_n}[U]$ and so $x_\beta|\alpha \in U$.

Now we will prove that C is unbounded in ω_1. Let α_0 be an ordinal in ω_1. By the continuity of projection, $(x_\beta|\alpha_0)_{\beta<\omega_1}$ is dense in $\pi_{\alpha_0}[Y]$. Since $\pi_{\alpha_0}[Y]$ has countable basis, choosing one $x_\beta|\alpha$ for each open set from the countable basis, we find $\alpha_1 < \omega_1$, which can be taken greater than α_0, such that $(x_\beta|\alpha_0)_{\beta<\alpha_1}$ is dense in $\pi_{\alpha_0}[Y]$. Iterating this process, we find an increasing sequence $(\alpha_n)_{n\in\omega_1}$ such that $(x_\beta|\alpha_n)_{\beta<\alpha_{n+1}}$ is dense in $\pi_{\alpha_n}[Y]$. Letting α be the supremum of $(\alpha_n)_{n\in\mathbb{N}}$ and repeating the argument above, we conclude that $(x_\beta|\alpha)_{\beta<\alpha}$ is dense in $\pi_\alpha[Y]$, proving that $\alpha \in C$. ∎

5. Construction of K. The construction presented in this section is an adaptation of the one in [Ko1]. The main improvement is that the separation obtained in item (g) of Theorem 5.2 is made for any sequence $(x_n)_{n\in\mathbb{N}}$ relatively discrete in K, and not only for sequences in a dense set previously fixed, as in [Ko1]. This allows us to transfer the property of $C(K)$ having few operators to every closed subspace of K. Nevertheless, we pay the price of assuming axiom $\diamondsuit$, to enumerate the sequences in K properly.

The construction will be by transfinite induction. The space K obtained in Theorem 5.2 below is the inverse limit of a sequence $(K_\alpha)_{\alpha<\omega_1}$, where $K_0 = [0,1]^2$ and K_α is the inverse limit of $(K_\beta)_{\beta<\alpha}$ when α is a limit ordinal. Theorem 5.1 will take care of the successor step of the induction. Theorem 5.2 will explain the construction itself. In that theorem we will use axiom $\diamondsuit$ to fix in advance some parameters used in the successor step. Finally, Theorem 5.3 proves that the properties announced in Theorem 5.2 imply that every closed subspace of K is a Koszmider space.

Before stating Theorem 5.1 we need to introduce some terminology. We say that a sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets *converges* to a point x if for every open neighborhood U of x we have $F_n \subset U$ for all but finitely many $n \in \mathbb{N}$.

THEOREM 5.1. *Let K be a compact metric space with no isolated points. Given*

- (a) *a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from K into $[0,1]$;*
- (b) *a relatively discrete sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of K such that $x_n \notin \operatorname{supp}(f_m)$ for all $n, m \in \mathbb{N}$;*
- (c) *a countable set $\mathcal{P}$ of pairs $((F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}})$ such that $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ are sequences of disjoint closed subsets of K which both converge to the same point in K;*
- (d) *an $\varepsilon > 0$;*
- (e) *a bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of regular measures on K such that $|\int f_n \, d\mu_n| > \varepsilon$ for every $n \in \mathbb{N}$,*

there exist $\delta > 0$, $b \subset \mathbb{N}$ infinite, and continuous functions $f'_n : K \to [0,1]$ such that $\operatorname{supp}(f'_n) \subset \operatorname{supp}(f_n)$ *and:*

(f) $|\int f'_n \, d\mu_n| > \delta$ *and* $\sum\{\int f'_m \, d|\mu_n| : m \neq n,\, m \in a\} < \delta/3$ *for every* $n \in a$;
(g) $L = K((f'_n)_{n\in b})$ *is a complete extension;*
(h) $(\pi^{-1}_{L,K}[x_n])_{n\in b}$ *and* $(\pi^{-1}_{L,K}[x_n])_{n\in \mathbb{N}\setminus b}$ *converge to the same point in* L;
(i) *for all* $((F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}}) \in \mathcal{P}$ *there exist infinite* $N', N'' \subset \mathbb{N}$ *and* $z \in L$ *such that* $(\pi^{-1}_{L,K}[F_n])_{n\in N'}$ *and* $(\pi^{-1}_{L,K}[G_n])_{n\in N''}$ *converge to* z.

Proof. By metrizability of K we may assume that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence. By Lemma 3.10 we get an infinite $a \subset \mathbb{N}$ and continuous functions $(f'_n)_{n\in a}$ satisfying (f) and (g) for every $b \subset a$. Moreover, letting Z be the set of all limits of $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$, for all $((F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}}) \in \mathcal{P}$, by Lemma 3.10(g) we may assume that $\Delta((f'_n)_{n\in b})$ is either a singleton or disjoint from $Z \cup \{x_n : n \in \mathbb{N}\} \cup \{\lim_{n\in\mathbb{N}} x_n\}$. We may also assume that $\mathbb{N}\setminus a$ is infinite.

It remains to prove (h) and (i). Thus, we have to prove that some sequences which cannot be separated in K still cannot be separated in L.

By Lemma 3.3, if L is an extension of K by $(f'_n)_{n\in b}$ for some $b \subset a$, then $|\pi^{-1}_{L,K}(x)| = 1$ for every $x \notin \Delta((f'_n)_{n\in a})$.

Now we separate our construction into two cases.

Case 1: $\Delta((f'_n)_{n\in a})$ is disjoint from $Z \cup \{x_n : n \in \mathbb{N}\} \cup \{\lim_{n\in\mathbb{N}} x_n\}$. In this case we take any infinite $b \subset a$ such that $a \setminus b$ is also infinite.

Let $((F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}}) \in \mathcal{P}$ and let z be the limit of both sequences. Let z' be the unique point such that $\pi_{L,K}(z') = z$. Since L is the graph of a continuous function when restricted to an open neighborhood of z, it follows that $\pi^{-1}[F_n]$ and $\pi^{-1}[G_n]$ converge to z' in L. This yields (h), and (i) is proved analogously.

Case 2: $\Delta((f'_n)_{n\in a})$ is a singleton. Let y be the single point in this set. Then $\mathrm{supp}(f'_n) \xrightarrow{n\in a} y$. In fact, otherwise there would exist an open neighborhood V of y and an infinite $c \subset a$ such that, for every $n \in c$, there is $y_n \in \mathrm{supp}(f'_n) \setminus V$. Letting y' be a limit point of $\{y_n : n \in c\}$ we would have $y' \in \Delta((f'_n)_{n\in a})$ and $y' \neq y$, contradicting that $\Delta((f'_n)_{n\in a})$ is a singleton.

To simplify the notation we assume that $((x_{k_n})_{n\in\mathbb{N}}, (x_{l_n})_{n\in\mathbb{N}}) \in \mathcal{P}$, where $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ are enumerations of a and $\mathbb{N}\setminus a$, respectively. This allows us to take care just of (h), with (i) being an immediate consequence. In particular, we assume that $\lim_{n\in\mathbb{N}} x_n \in Z$.

If $y \notin Z$ we proceed as in Case 1. For the elements of $\mathcal{P}$ whose limit does not belong to Z, we also proceed as in Case 1 to prove (h).

Let $((F_{n,m})_{n\in\mathbb{N}}, (G_{n,m})_{n\in\mathbb{N}})$ be an enumeration (for $m \in \mathbb{N}$) of all elements of $\mathcal{P}$ such that $\lim_{n\in\mathbb{N}} F_{n,m} = \lim_{n\in\mathbb{N}} G_{n,m} = y$. This enumeration can have repetitions, so we need not deal separately with the case of only finitely many elements in $\mathcal{P}$ under these conditions.

Going to a subsequence, we may assume that $y \notin \mathrm{supp}(f'_n)$, $y \notin F_{n,m}$ and $y \notin G_{n,m}$ for all n, m.

We need the following claim:

CLAIM 2. *There exist infinite subsets $b, c_m, d_m \subset \mathbb{N}$, for $m \in \mathbb{N}$, such that $F_{n,m} \cap \mathrm{supp}(f'_k) = \emptyset$ and $G_{n,m} \cap \mathrm{supp}(f'_k) = \emptyset$ for all $n \in c_m$, $m \in \mathbb{N}$ and $k \in b$.*

Let U_0 be any open neighborhood of y. Suppose we have defined U_n, $(k_j)_{j<n}$ and $(l_j)_{j<n}$. We choose k_n such that $k_n > k_j$ for every $j < n$, and $F_{k_n,m} \subset U_n$, for every $m \leq n$. Let $V_n \subset U_n$ be an open neighborhood of y disjoint from $F_{k_j,m}$ for all $j \leq n$ and $m \leq j$. Take l_n such that $l_n > l_j$ for every $j < n$, and $\mathrm{supp}(f'_{l_n}) \subset V_n$. Let U_{n+1} be an open neighborhood of y disjoint from $\mathrm{supp}(f'_{l_n})$.

Define $b = \{l_n : n \in \mathbb{N}\}$ and $c_m = \{k_n : n \geq m\}$. For any $m, j \in \mathbb{N}$ and $n \geq m$ we have $F_{k_n,m} \subset U_n \setminus V_n$ and $\mathrm{supp}(f'_{l_j}) \subset V_j \setminus U_{j+1}$. If $n \leq j$ then $F_{k_n,m} \cap V_n = \emptyset$ and $\mathrm{supp}(f'_{l_j}) \subset V_j \subset V_n$. If $n > j$ then $F_{k_n,m} \subset U_n \subset U_j$ and $\mathrm{supp}(f'_{l_j}) \cap U_j = \emptyset$. In both cases we have $F_{k_n,m} \cap \mathrm{supp}(f'_{l_j}) = \emptyset$. Proceeding analogously for $G_{n,n}$, we obtain d_m and prove the claim.

Take $L = K((f'_n)_{n\in b})$ and denote $\pi_{L,K}$ by π.

By the claim, we have $\lim_{n\in c_m} \pi^{-1}[F_{n,m}] = (y, 0)$ and $\lim_{n\in d_m} \pi^{-1}[G_{n,m}] = (y, 0)$ for every $m \in \mathbb{N}$.

Now it remains to prove item (i) for the pairs $((F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}}) \in \mathcal{P}$ which do not converge to y. Suppose that $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ both converge to $z \neq y$. As in Case 1, π^{-1} is a homeomorphism on an open neighborhood of z. Thus, $\pi^{-1}[F_n]$ and $\pi^{-1}[G_n]$ converge to $\pi^{-1}(z)$. ∎

THEOREM 5.2. *Assume $\diamondsuit$. There is a compact connected K such that:*

(i) *Given*

(a) *a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from K into $[0, 1]$;*

(b) *a relatively discrete sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of K such that $f_m(x_n) = 0$ for all $n, m \in \mathbb{N}$;*

(c) *an $\varepsilon > 0$;*

(d) *a bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of regular measures on K such that $|\int f_n \, d\mu_n| > \varepsilon$ for every $n \in \mathbb{N}$,*

there exist $\delta > 0$, infinite $b \subset a \subset \mathbb{N}$, and continuous functions f'_n with $\mathrm{supp}(f'_n) \subset \mathrm{supp}(f_n)$, such that:

(e) *$|\int f'_n \, d\mu_n| > \delta$ and $\sum\{\int f'_m \, d|\mu_n| : m \neq n,\, m \in a\} < \delta/3$ for every $n \in a$;*

(f) *$(f'_n)_{n\in b}$ has supremum in $C(K)$;*

(g) *$\overline{\{x_n : n \in b\}} \cap \overline{\{x_n : n \in a \setminus b\}} \neq \emptyset$.*

(ii) *If L is a closed subspace of K, and V_1 and V_2 are disjoint open sets in L such that $\overline{V}_1 \cap \overline{V}_2 \neq \emptyset$, then $\overline{V}_1 \cap \overline{V}_2$ has at least two elements.*

Proof. For every $\alpha \leq \omega_1$ let $\mathcal{B}_\alpha$ be the basis for $[0,1]^\alpha$ consisting of all open sets of the form $\prod_{\xi<\alpha} V_\xi$, where V_ξ is an interval with rational endpoints for all $\xi < \alpha$, and $V_\xi = [0,1]$ for all but finitely many $\xi < \alpha$.

By regularity of the measures and compactness of $[0,1]^\alpha$, for any given measure μ and any Borel set A, we can approximate $\mu(A)$ by finite unions of basic open sets. Therefore, every regular measure is uniquely determined by its value on a basis, which allows us to identify all regular measures on $[0,1]^\alpha$ with functions from $\mathcal{B}_\alpha$ into $\mathbb{R}$.

Let *Even*, *Odd* be respectively the sets of all even and all odd ordinals in ω_1. Recall that α is an *even ordinal* if it has the form $\beta + n$, where β is a limit ordinal and n a even integer. Otherwise we say that α is *odd*.

We know that if X is an uncountable subset of ω_1 then it is order-isomorphic to ω_1. So we will use the following terminology: a subset of X is a *cub in X* if it is a cub via this isomorphism between X and ω_1, and similarly for *stationary in X* and *$\diamondsuit$-sequence in X*.

This terminology will be used for *Even* and *Odd*.

Using $\diamondsuit$ and Lemmas 4.1 and 4.2 we fix enumerations $\{f_n(\alpha) : n \in \mathbb{N}\}$, $\varepsilon(\alpha)$, $\{\mu_n(\alpha) : n \in \mathbb{N}\}$, $\{x_n(\alpha) : n \in \mathbb{N}\}$, for $\alpha \in$ *Even*, such that

A.1. $\{f_n(\alpha) : n \in \mathbb{N}\}$ are continuous functions from $[0,1]^{\omega_1}$ into $[0,1]$;
A.2. $\varepsilon(\alpha) > 0$;
A.3. $(\mu_n(\alpha))_{n\in\mathbb{N}}$ is a bounded sequence of functions from $\mathcal{B}_\alpha$ into $\mathbb{R}$;
A.4. $(x_n(\alpha))_{n\in\mathbb{N}}$ is a sequence in $[0,1]^\alpha$;

and given $\beta < \omega_1$ and

B.1. a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from $[0,1]^{\omega_1}$ into $[0,1]$;
B.2. an $\varepsilon > 0$;
B.3. a bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{B}_{\omega_1}$ into $\mathbb{R}$ representing Radon measures;
B.4. a sequence $(x_n)_{n\in\mathbb{N}}$ relatively discrete in $[0,1]^{\omega_1}$,

there exists $\alpha > \beta$, with $\alpha \in$ *Even*, such that

C.1. $f_n(\alpha) = f_n$ for every n;
C.2. $\varepsilon(\alpha) = \varepsilon$;
C.3. $\mu_n(\alpha) = \mu_n | \mathcal{B}_\alpha$ for every n;
C.4. $x_n(\alpha) = x_n | \alpha$ for every n.

Using $\diamondsuit$ in *Odd* we fix sequences $(U_\alpha, V_\alpha, A_\alpha, B_\alpha)_{\alpha\in Odd}$, where

D.1. U_α and V_α are countable unions of elementary open sets of $[0,1]^{\omega_1}$ such that $U_\alpha \cap V_\alpha = \emptyset$ and $\overline{U}_\alpha \cap \overline{V}_\alpha \neq \emptyset$;
D.2. A_α and B_α are countable subsets of $[0,1]^\alpha$,

and given

E.1. countable unions U and V of elementary open sets of $[0,1]^{\omega_1}$ such that $U \cap V = \emptyset$ and $\overline{U} \cap \overline{V} \neq \emptyset$;

E.2. sequences $(x_\beta)_{\beta<\omega_1}$ and $(y_\beta)_{\beta<\omega_1}$ in $[0,1]^{\omega_1}$,

the set

$$\{\alpha \in Odd : U_\alpha = U,\, V_\alpha = V,\, \{x_\beta|\alpha : \beta \in Odd \cap \alpha\} = A_\alpha, \\ \{y_\beta|\alpha : \beta \in Odd \cap \alpha\} = B_\alpha\}$$

is stationary in Odd.

Let $\alpha \in Odd$. If $\overline{\pi_\alpha[U_\alpha] \cap A_\alpha} \cap \overline{\pi_\alpha[V_\alpha] \cap B_\alpha} \neq \emptyset$ we fix $(x_n(\alpha))_{\alpha\in\mathbb{N}}$ such that $x_n(\alpha) \xrightarrow{n\in\mathbb{N}} z$ for some $z \in \overline{\pi_\alpha[U_\alpha] \cap A_\alpha} \cap \overline{\pi_\alpha[V_\alpha] \cap B_\alpha}$, and

$$\{x_n(\alpha) : n \in 2\mathbb{N}\} \subset A_\alpha, \qquad \{x_n(\alpha) : n \in \mathbb{N} \setminus 2\mathbb{N}\} \subset B_\alpha.$$

If $\overline{\pi_\alpha[U_\alpha] \cap A_\alpha} \cap \overline{\pi_\alpha[V_\alpha] \cap B_\alpha} = \emptyset$ then we take any sequence $(x_n(\alpha))_{n\in\mathbb{N}}$ in $A_\alpha \cup B_\alpha$.

Now we will construct by induction compact spaces $(K_\alpha)_{\alpha<\omega_1}$ with $K_\alpha \subset [0,1]^\alpha$, sequences $P_\alpha = \{(L^\alpha_{(\beta,i)}, R^\alpha_{(\beta,i)}, z^\alpha_{(\beta,i)}) : (\beta,i) \in \alpha \times \{0,1\}\}$, where $L^\alpha_{(\beta,i)}, R^\alpha_{(\beta,i)} \subset \mathbb{N}$ are disjoint and $z^\alpha_{(\beta,i)} \in K_\alpha$, and closed sets $F^\beta_n(\alpha) \subset K_\alpha$ for $\beta \leq \alpha$.

Once K_α is defined, for every $\beta \leq \alpha$ we define

$$F^\beta_n(\alpha) = \pi^{-1}_{K_\alpha,K_\beta}[\{x_n(\beta)\}].$$

Suppose we have $(K_\gamma)_{\gamma<\alpha}$ and $(P_\gamma)_{\gamma<\alpha}$ satisfying the following inductive hypothesis, for every $\gamma < \alpha$:

F.1. for every $(\beta,i) \in \gamma \times \{0,1\}$,

$$\lim_{n\in L^\gamma_{(\beta,i)}} F^\beta_n(\gamma) = \lim_{n\in R^\gamma_{(\beta,i)}} F^\beta_n(\gamma) = z^\gamma_{(\beta,i)};$$

F.2. for every $\beta < \gamma' < \gamma$ and $i \in \{0,1\}$, we have $\pi_{\gamma'}[K_\gamma] = K_{\gamma'}$ and $z^\gamma_{(\beta,i)}|\gamma' = z^{\gamma'}_{(\beta,i)}$;

F.3. for every $\beta < \gamma' < \gamma$ and $i \in \{0,1\}$, $L^\gamma_{(\beta,i)} \setminus L^{\gamma'}_{(\beta,i)}$ and $R^\gamma_{(\beta,i)} \setminus R^{\gamma'}_{(\beta,i)}$ are finite.

If α is a limit ordinal, we define

G.1. K_α is the inverse limit $(K_\gamma)_{\gamma<\alpha}$;

G.2. $z^\alpha_{(\beta,i)} = \bigcup_{\beta<\gamma<\alpha} z^\gamma_\beta$ for all $\beta < \alpha$ and $i \in \{0,1\}$;

G.3. $L^\alpha_{(\beta,i)}$ is an infinite pseudointersection of $(L^\gamma_{(\beta,i)})_{\beta<\gamma<\alpha}$, that is, $L^\alpha_{(\beta,i)} \setminus L^\gamma_{(\beta,i)}$ is finite for every $\gamma < \alpha$ (the existence of such a pseudointersection is shown in [Do, Theorem 3.1]);

G.4. $R^\alpha_{(\beta,i)}$ is an infinite pseudointersection of $(R^\gamma_{(\beta,i)})_{\beta<\gamma<\alpha}$.

Now we handle the case of a successor ordinal. Suppose we have defined $(K_\gamma)_{\gamma\le\alpha}$ and $(P_\gamma)_{\gamma\le\alpha}$. We will define $K_{\alpha+1}$ and $P_{\alpha+1}$.

We say that a step $\alpha \in Even$ is *non-trivial* if:

H.1. $(x_n(\alpha))_{n\in\mathbb{N}}$ is a relatively discrete sequence of distinct points of K_α;
H.2. there exist continuous functions $g_n : [0,1]^\alpha \to [0,1]$ such that $f_n(\alpha) = g_n \circ \pi_\alpha$;
H.3. $(g_n|_{K_\alpha})_{n\in\mathbb{N}}$ as above is pairwise disjoint;
H.4. $x_n(\alpha) \notin \operatorname{supp}(g_m)$ for all $n, m \in \mathbb{N}$ and g_m as in **H.2**;
H.5. $|\int_{K_\alpha} g_n \, d\mu_n(\alpha)| > \varepsilon(\alpha)$ for every $n \in \mathbb{N}$.

We say that a step $\alpha \in Odd$ is *non-trivial* if:

I.1. $A_\alpha, B_\alpha \subset K_\alpha$;
I.2. $U_\alpha = \pi_\alpha^{-1}[\pi_\alpha[U_\alpha]]$ and $V_\alpha = \pi_\alpha^{-1}[\pi_\alpha[V_\alpha]]$;
I.3. $\overline{\pi_\alpha[U_\alpha] \cap A_\alpha} \cap \overline{\pi_\alpha[V_\alpha] \cap B_\alpha} \neq \emptyset$;

If step α is trivial, we take $K_{\alpha+1} = K_\alpha \times \{0\}$, $L^{\alpha+1}_{(\beta,i)} = L^\alpha_{(\beta,i)}$, $R^{\alpha+1}_{(\beta,i)} = R^\alpha_{(\beta,i)}$, $z^{\alpha+1}_{(\beta,i)} = z^\alpha_{(\beta,i)}{}^\frown 0$, $L^{\alpha+1}_{(\alpha,i)} = R^{\alpha+1}_{(\alpha,i)} = \emptyset$ and any $z^{\alpha+1}_{(\alpha,i)}$.

We assume now that we are in a non-trivial step and $\alpha \in Even$.

Let g_n be functions as in **H.2**. Define $h_n = g_n|_{K_\alpha}$.

By Theorem 5.1 there exist infinite $b \subset \mathbb{N}$, a $\delta > 0$ and continuous functions $h'_n : K_\alpha \to [0,1]$ for $n \in b$ such that

J.1. $\operatorname{supp}(h'_n) \subset \operatorname{supp}(h_n)$ for every $n \in b$;
J.2. $K_{\alpha+1} = K_\alpha((h'_n)_{n\in b})$ is a complete extension;
J.3. for every $n \in b$, $|\int h'_n \, d\mu_n(\alpha)| > \delta$ and $\sum\{\int h'_n d|\mu_n(\alpha)| : m \neq n, m \in b\} < \delta/3$;
J.4. for every $(\beta, i) \in \alpha \times \{0,1\}$ there exist infinite $L^{\alpha+1}_{(\beta,i)} \subset L^\alpha_{(\beta,i)}$ and $R^{\alpha+1}_{(\beta,i)} \subset R^\alpha_{(\beta,i)}$ and a point $z^{\alpha+1}_{(\beta,i)}$ such that $\lim_{n\in L^{\alpha+1}_{(\beta,i)}} \pi^{-1}[F^\beta_n(\alpha)] = \lim_{n\in R^{\alpha+1}_{(\beta,i)}} \pi^{-1}[F^\beta_n(\alpha)] = z^{\alpha+1}_{(\beta,i)}$, where π is the projection from $K_{\alpha+1}$ to K_α;
J.5. $(\pi^{-1}[x_n])_{n\in b}$ and $(\pi^{-1}[x_n])_{n\in\mathbb{N}\setminus b}$ converge to a point $z^{\alpha+1}_{(\alpha,0)} \in K_{\alpha+1}$.

We define $z^\alpha_{\alpha,0} = z^{\alpha+1}_{(\alpha,0)}|\alpha$, $z^\alpha_{(\alpha,1)} = z^\alpha_{(\alpha,0)}$, $z^{\alpha+1}_{(\alpha,1)} = z^{\alpha+1}_{(\alpha,0)}$ and $L^\alpha_{(\alpha,i)} = b$ and $R^\alpha_{(\alpha,i)} = \mathbb{N} \setminus b$ for $i \in \{0,1\}$. Finally, we take $F^\alpha_n(\alpha) = \{x_n\}$ and $F^\beta_n(\alpha+1) = \pi^{-1}[F^\beta_n(\alpha)]$ for all $n \in \mathbb{N}$ and $\beta \le \alpha$.

This concludes the construction of $K_{\alpha+1}$ when $\alpha \in Even$.

Let $\alpha \in Odd$ be a non-trivial step. Take $z = \lim_{n\to\infty} x_n(\alpha)$ in K_α. By Urysohn's lemma there is a pairwise disjoint sequence $(h_n)_{n\in 2\mathbb{N}}$ of continuous functions from K_α into $[0,1]$ such that $\Delta((h_n)_{n\in 2\mathbb{N}}) = \{z\}$ and, for every

$n \in 2\mathbb{N}$ and $m \in \mathbb{N}$,

$$h_n(x_m(\alpha)) = \begin{cases} 1 & \text{if } m = n \text{ or } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $K_{\alpha+1} = K_\alpha((h_n)_{n\in b})$, $L^{\alpha+1}_{(\alpha,1)} = b$, $L^{\alpha+1}_{(\alpha,0)} = 2\mathbb{N} \setminus b$, $R^{\alpha+1}_{(\alpha,1)} = \{n+1 : n \in L^{\alpha+1}_{(\alpha,1)}\}$, $R^{\alpha+1}_{(\alpha,0)} = \{n+1 : n \in L^{\alpha+1}_{(\alpha,0)}\}$, $z^{\alpha+1}_{(\alpha,0)} = z^\frown 0$ and $z^{\alpha+1}_{(\alpha,1)} = z^\frown 1$.

Note that $F^\alpha_n(\alpha+1) = \{x_n(\alpha)^\frown 1\}$ if $n \in L^{\alpha+1}_{(\alpha,1)} \cup R^{\alpha+1}_{(\alpha,1)}$, and $F^\alpha_n(\alpha+1) = \{x_n(\alpha)^\frown 0\}$ otherwise. Therefore $F^\alpha_n(\alpha+1) \to z^{\alpha+1}_{(\alpha,0)}$ for $n \in L^{\alpha+1}_{(\alpha,0)} \cup R^{\alpha+1}_{(\alpha,0)}$, and $F^\alpha_n(\alpha+1) \to z^{\alpha+1}_{(\alpha,1)}$ for $n \in L^{\alpha+1}_{(\alpha,1)} \cup R^{\alpha+1}_{(\alpha,1)}$.

The remaining construction of $P_{\alpha+1}$—namely, $L^{\alpha+1}_{(\beta,i)}$, $R^{\alpha+1}_{(\beta,i)}$ and $z^{\alpha+1}_{(\beta,i)}$ for $\beta < \alpha$—is made as in Case 2 at a non-trivial step $\alpha \in \mathit{Even}$.

Finally we define K to be the inverse limit of $(K_\alpha)_{\alpha<\omega_1}$. Let $(f_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$ and ε be as in the hypothesis of the theorem. Using Urysohn's lemma and regularity of μ_n to reduce the supports of f_n preserving condition (d) of the hypothesis, we may assume without loss of generality that $x_n \notin \operatorname{supp}(f_m)$ for all $n, m \in \mathbb{N}$.

By Tietze's theorem we extend f_n continuously to $\tilde{f}_n : [0,1]^{\omega_1} \to [0,1]$. By a theorem of Mibu (see [Mi]) there exist $\alpha < \omega_1$ and continuous functions $g_n : [0,1]^\alpha \to [0,1]$ such that $\tilde{f}_n = g_n \circ \pi$. Note that $f_n = g_n|_{K_\alpha} \circ \pi_\alpha$. As the existence of such functions still holds for some $\alpha' > \alpha$, since $\tilde{f}_n \circ \pi_{\alpha'} = g_n \circ \pi_\alpha \circ \pi_{\alpha'}$, we may choose a non-trivial step $\alpha \in \mathit{Even}$ such that:

K.1. $f_n(\alpha) = \tilde{f}_n$ for every $n \in \mathbb{N}$;
K.2. $x_n(\alpha) = x_n|\alpha$ for every $n \in \mathbb{N}$;
K.3. $\varepsilon(\alpha) = \varepsilon$;
K.4. $\mu_n(\alpha) = \mu_n|\mathcal{B}_\alpha$ for every $n \in \mathbb{N}$.

Let $b = L^{\alpha+1}_{(\alpha,0)}$ and $a = L^{\alpha+1}_{(\alpha,0)} \cup R^{\alpha+1}_{(\alpha,0)}$. Define $f'_n = h'_n \circ \pi$, whith h'_n as in **J.1** to **J.5**, regarding that $K_{\alpha+1} = K_\alpha((h'_n)_{n\in b})$. Fix $\delta > 0$ as in **J.3**.

By Lemma 3.4, $(h'_n \circ \pi)_{n\in b}$ has supremum in $C(K_{\alpha+1})$. By Lemma 3.7, $(f'_n)_{n\in b}$ has supremum in $C(K)$.

Note that $\int f'_n \, d\mu_m = \int h'_n \, d\mu_m(\alpha)$ for all n, m, since f'_n is determined by the coordinates below α. So we infer (e).

Connectedness of K follows from Lemma 3.7.

It remains to prove (g). Suppose that there exist open subsets U_1 and U_2 of K such that $x_n(\alpha) \in U_1$ for every $n \in b$, and $x_n(\alpha) \in U_2$ for every $n \in a \setminus b$. By compactness, we may assume that U_1 and U_2 are finite unions of elementary open sets. Hence there exists $\beta < \omega_1$, which may be taken greater than α, such that the separation occurs below β, i.e., $\overline{\{x_n|\beta : n \in b\}} \cap \overline{\{x_n|\beta : n \in a \setminus b\}} = \emptyset$ in K_β. Since $x_n|\alpha = x_n(\alpha)$, we have $x_n|\beta \in F^\alpha_n(\beta)$. By the construction, $L^\beta_{(\alpha,0)} \setminus L^{\alpha+1}_{(\alpha,0)}$ is finite. Since $L^{\alpha+1}_{(\alpha,0)} = b$

and $\lim_{n\in L^\beta_{(\alpha,0)}} F^\alpha_n(\beta) = z^\beta_{(\alpha,0)}$, we have $z^\beta_{(\alpha,0)} \in \overline{\{x_n|\beta : n \in b\}}$. We conclude analogously that $z^\beta_{(\alpha,0)} \in \overline{\{x_n|\beta : n \in a \setminus b\}}$, contradicting $\overline{\{x_n|\beta : n \in b\}} \cap \overline{\{x_n|\beta : n \in a \setminus b\}} = \emptyset$.

Now we prove part (ii) of the theorem. Let L be a closed subset of K and let V_1 and V_2 be disjoint open subsets of L such that $\overline{V}_1 \cap \overline{V}_2 \neq \emptyset$. Take open subsets U and V of $[0,1]^{\omega_1}$ such that $V_1 = U \cap L$ and $V_2 = V \cap L$. Since L is closed, $\overline{V}_1 \cap \overline{V}_2 = \overline{U} \cap \overline{V} \cap L$, because $\overline{U \cap L} = \overline{U} \cap L$.

By separability of $[0,1]^{\omega_1}$ (see [Eng, 2.3.16]), $[0,1]^{\omega_1}$ satisfies the countable chain condition, i.e., it does not contain an uncountable pairwise disjoint sequence of open sets. So, if we let $U' \subset U$ be the union of a maximal family of elementary open subsets of U, we have $\overline{U'} = \overline{U}$, and the same holds for V. Therefore we may assume that U and V are countable unions of elementary open sets.

Let $(y_\alpha)_{\alpha<\omega_1}$ and $(z_\alpha)_{\alpha<\omega_1}$ be dense sequences in V_1 and V_2, respectively. Take $\beta < \omega_1$ containing all coordinates which determine U and V, i.e., satisfying $\pi^{-1}[\pi_\beta[U]] = U$ and $\pi^{-1}[\pi_\beta[V]] = V$. By Lemma 4.3 there is $\alpha > \beta$ such that $\alpha \in Odd$, $U_\alpha = U$, $V_\alpha = V$, $(y_\beta|\alpha)_{\beta<\alpha}$ is dense in $\pi_\alpha[V_1]$, $(z_\beta|\alpha)_{\beta<\alpha}$ is dense in $\pi_\alpha[V_2]$, and

$$\{y_\beta|\alpha : \beta < \alpha\} = A_\alpha, \quad \{z_\beta|\alpha : \beta < \alpha\} = B_\alpha.$$

Let $x \in \overline{V}_1 \cap \overline{V}_2$. Since A_α and B_α are dense in $\pi_\alpha[V_1]$ and $\pi_\alpha[V_2]$, respectively, we have $x|\alpha \in \overline{\pi_\alpha[U_\alpha] \cap A_\alpha} \cap \overline{\pi_\alpha[V_\alpha] \cap B_\alpha}$. So $\alpha \in Odd$ is a non-trivial step. Therefore $x_n(\alpha) \xrightarrow{n\in\mathbb{N}} x|\alpha$ and $x_n(\alpha) \in \pi_\alpha[V_1]$ if n is even, while $x_n(\alpha) \in \pi_\alpha[V_2]$ if n is odd.

For every even n, take α_n such that $y_{\alpha_n}|\alpha = x_n(\alpha)$. For every odd n, take α_n such that $z_{\alpha_n}|\alpha = x_n(\alpha)$. As in the proof of item (g) of part (i), we have

$$\overline{\{y_{\alpha_n} : n \in L^{\alpha+1}_{(\alpha,0)}\}} \cap \overline{\{z_{\alpha_n} : n \in R^{\alpha+1}_{(\alpha,0)}\}} \neq \emptyset,$$
$$\overline{\{y_{\alpha_n} : n \in L^{\alpha+1}_{(\alpha,1)}\}} \cap \overline{\{z_{\alpha_n} : n \in R^{\alpha+1}_{(\alpha,1)}\}} \neq \emptyset.$$

Let $z_1 \in \overline{\{y_{\alpha_n} : n \in L^{\alpha+1}_{(\alpha,0)}\}} \cap \overline{\{z_{\alpha_n} : n \in R^{\alpha+1}_{(\alpha,0)}\}}$ and $z_2 \in \overline{\{y_{\alpha_n} : n \in L^{\alpha+1}_{(\alpha,1)}\}} \cap \overline{\{z_{\alpha_n} : n \in R^{\alpha+1}_{(\alpha,1)}\}}$. We have $z_1, z_2 \in \overline{V}_1 \cap \overline{V}_2$ and

$$z_1|(\alpha+1) = z^{\alpha+1}_{(\alpha,0)} \neq z^{\alpha+1}_{(\alpha,1)} = z_2|(\alpha+1).$$

So $|\overline{V}_1 \cap \overline{V}_2| \geq 2$, proving the theorem. ∎

THEOREM 5.3. *Assuming $\diamondsuit$ there exists a compact connected space K such that every closed $L \subset K$ is a Koszmider space. In particular, $C(L)$ is indecomposable whenever L is a closed connected subspace of K.*

Proof. Let K be the space of Theorem 5.2. Let L be a closed subspace of K and $T : C(L) \to C(L)$ be a bounded operator. We will prove that T is a weak multiplier. The following proof is an adaptation of Lemma 5.2 of [Ko1].

Suppose that T is not a weak multiplier, i.e., there exist a pairwise disjoint sequence $(e_n)_{n\in\mathbb{N}}$ in $C(L)$ with ranges included in $[-1,1]$, an $\varepsilon > 0$ and points $x_n \in L$ such that $e_n(x_n) = 0$ for every $n \in \mathbb{N}$, and $|T(e_n)(x_n)| > \varepsilon$ for infinitely many n's. Taking a subsequence, we assume that this holds for all n.

Since finite sums of e_n are uniformly bounded, if x_n were constant for infinitely many n's, T would not be bounded. Hence we may assume that $x_n \neq x_m$ whenever $n \neq m$.

We may assume without loss of generality that $e_m(x_n) = 0$ for all $n, m \in \mathbb{N}$. In fact, if there exists k_0 such that $e_{k_0}(x_n) \neq 0$ for n belonging to some infinite $N' \subset \mathbb{N}$, we pass to the subsequence indexed by $N' \setminus \{k_0\}$ and use disjointness of $(e_n)_n$ to show that $e_m(x_n) = 0$ for all n, m. Otherwise, we can easily construct by induction a subsequence with this property, namely, for each n we find $k_n > k_{n-1}$ such that $e_m(x_{k_n}) = 0$ for all $m \leq k_{n-1}$.

Taking $\max(e_n, 0) - \min(e_n, 0)$ instead of e_n, we may assume that e_n has its range included in $[0,1]$.

For every n, let $\mu_n = T^*(\delta_{x_n})$, i.e., μ_n is the measure given by the relation

$$T(f)(x_n) = \int f\, d\mu_n$$

for all $f \in C(L)$. We have $|\int e_n\, d\mu_n| > \varepsilon$ for every n. Note that $(\mu_n)_{n\in\mathbb{N}}$ is bounded. By Rosenthal's lemma (see [Di, p. 82]) we find some infinite $N' \subset \mathbb{N}$ such that

$$\sum\Big\{\Big|\int e_m\, d\mu_n\Big| : n \neq m,\ m \in N'\Big\} < \varepsilon/3.$$

Now we use Tietze's theorem and Urysohn's lemma to extend e_n continuously to K preserving disjointness and range in $[0,1]$.

Since $L \subset K$, we view μ_n as measures on K, i.e., $\mu_n(A) = \mu'_n(A \cap L)$ for every Borel $A \subset K$. So, for all $n, m \in \mathbb{N}$, we have

$$\int_K f_m\, d\mu_n = \int_L e_m\, d\mu_n.$$

Applying Theorem 5.2 for $(f_n)_{n\in N'}$, $(x_n)_{n\in N'}$, $(\mu_n)_{n\in N'}$ and ε, we find $b \subset a \subset N'$, $\delta > 0$ and functions $(f'_n)_{n\in a}$ as stated in the theorem.

We may assume that

$$\int \sup\{f'_m : m \in b\}\, d\mu_n = \int \sum_{m\in b} f'_m\, d\mu_n$$

for every $n \in \mathbb{N}$. In fact, take a family $(N_\xi)_{\xi<\omega_1}$ of infinite subsets of N' such that $N_\xi \cap N_\eta$ is finite for all $\xi \neq \eta$ (we may construct such a family by identifying $\mathbb{N}$ with $\mathbb{Q}$ and ω_1 with $\mathbb{R}$, and taking for N_ξ a rational sequence converging to ξ). For any ξ take $b_\xi \subset a_\xi \subset N_\xi$ as a and b of Theorem 5.2.

For every $\xi < \omega_1$ and $n \in b_\xi$ fix f_n^ξ as f_n' in the theorem, i.e., properties (e)–(g) of Theorem 5.2 hold for $f_n' = f_n^\xi$, $a = a_\xi$ and $b = b_\xi$.

CLAIM 3. *There exists $\xi < \omega_1$ such that*

$$\int \Big[\sup\{f_m^\xi : m \in b_\xi\} - \sum_{m\in b_\xi} f_m^\xi\Big]\, d\mu_n = 0$$

for every $n \in \mathbb{N}$.

For every $\xi < \omega_1$ and $c \subset b_\xi$ we define $f_c^\xi = \sup\{f_m^\xi : m \in c\} - \sum_{m\in c} f_m^\xi$ whenever the supremum exists. Note that, for any finite $F \subset b_\xi$, we have

$$\sup\{f_m^\xi : m \in b_\xi\} = \sup\{f_m^\xi : m \in b_\xi \setminus F\} + \sum_{m\in F} f_m^\xi$$

and therefore $f_{b_\xi\setminus F}^\xi = f_{b_\xi}^\xi$. In particular $f_{b_\xi}^\xi = f_{b_\xi\setminus b_{\xi'}}^\xi$ for all $\xi \neq \xi'$ in ω_1, once $b_\xi \cap b_{\xi'}$ is finite.

Let ξ and ξ' be different ordinals in ω_1. Take $g = \sup\{f_n^\xi : n \in b_\xi \setminus b_{\xi'}\}$ and $h = \sup\{f_n^{\xi'} : n \in b_{\xi'} \setminus b_\xi\}$. Since $\mathrm{supp}(f_n^\eta) \subset \mathrm{supp}(f_n)$ for every $n \in \mathbb{N}$ and $\eta < \omega_1$, we have $f_n^\xi \cdot f_m^{\xi'} = 0$ for all $n \neq m$. We will prove that $g \cdot h = 0$.

Suppose that there exists $x \in K$ such that $g(x) > 0$ and $h(x) > 0$. Then there exists an open neighborhood V of x such that the restrictions of g and h to V are both strictly positive. Hence there exist $y \in V$ and $n \in b_\xi \setminus b_{\xi'}$ such that $f_n^\xi(y) > 0$. Let φ be a continuous function from K into $[0,1]$ such that $\varphi(y) = 1$ and φ is null wherever f_n^ξ is null. Since $f_n^\xi \cdot f_m^{\xi'} = 0$ for every $m \in b_{\xi'} \setminus b_\xi$, we have $f_m^{\xi'} \le h \cdot \varphi < h$ for every $m \in b_{\xi'} \setminus b_\xi$, contradicting the definition of h.

Since $f_{b_\xi}^\xi = f_{b_\xi\setminus b_{\xi'}}^\xi \le g$ and $f_{b_{\xi'}}^{\xi'} = f_{b_{\xi'}\setminus b_\xi}^{\xi'} \le h$, it follows that $f_{b_\xi}^\xi \cdot f_{b_{\xi'}}^{\xi'} = 0$ for all $\xi \neq \xi'$. So there exists $\xi < \omega_1$ such that

$$\int f_{b_\xi}^\xi\, d\mu_n = 0$$

for all n, proving the claim.

Taking $f = \sup\{f_n' : n \in b\}$ and $n \in b$ we have

$$\begin{aligned}|T(f|_L)(x_n)| &= \Big|\int_K f\, d\mu_n\Big| = \Big|\int f_n'\, d\mu_n + \int \sum\{f_m' : m \neq n,\, m \in b\}\, d\mu_n\Big| \\ &\ge \delta - \delta/3 = 2\delta/3.\end{aligned}$$

On the other hand, if $n \in a \setminus b$ then

$$|T(f|_L)(x_n)| = \Big|\int_K \sum_{m\in b} f_m'\, d\mu_n\Big| \le \delta/3.$$

By continuity of $T(f|_L)$, we conclude that the closures of $\{x_n : n \in b\}$ and $\{x_n : n \in a \setminus b\}$ are disjoint, contradicting Theorem 5.2(g).

We have proved that all operators on $C(L)$ are weak multipliers, for all closed $L \subset K$. Connectedness of K is Theorem 5.2(ii). By Lemmas 2.3 and 2.6, and Theorem 2.4, we conclude that $C(L)$ is indecomposable whenever L is a connected closed subspace of K. ∎

Corollary 5.4. *The space $C(K)$ as above has at least continuum many non-isomorphic indecomposable quotients of the form $C(L)$.*

Proof. First we note that a Banach space $C(L)$ with few operators cannot be isomorphic to any of its proper quotients. Indeed, suppose that X is a proper quotient of $C(L)$. Let $T : C(L) \to X$ be a surjective and non-injective bounded linear transformation. Suppose that there exists an isomorphism $S : X \to C(L)$. Since $C(L)$ has few operators, $S \circ T$ is a weak multiplier and, by Lemma 2.2, $S \circ T$ is surjective iff it is an isomorphism onto its range. But S and T are both surjective, which implies that $S \circ T$ is surjective, and therefore it is an isomorphism on $C(K)$. This leads to a contradiction, since T is not injective.

Now, for every $r \in [0,1]$ take $K_r = \pi^{-1}_{K,[0,1]^2}([0,r]^2)$. By Lemma 3.7 we conclude that K_r is connected for every $r \in [0,1]$, and Theorem 5.3 states that $C(K_r)$ has few operators. If $r < s$ we have $K_r \subset K_s$, and so $C(K_r)$ is a proper quotient of $C(K_s)$. Therefore, these are non-isomorphic indecomposable quotients of $C(L)$. ∎

6. An indecomposable $C(K)$ where K contains a homeomorphic copy of $\beta\mathbb{N}$. Theorem 5.3 shows that, under $\diamondsuit$, there exists a compact K such that, for every $L \subset K$, $C(L)$ has few operators. In particular, K does not contain a homeomorphic copy of $\beta\mathbb{N}$. In this section, assuming CH, we construct a connected compact K containing $\beta\mathbb{N}$ such that $C(K)$ has few operators.

The next lemma is quite standard and we will omit the proof.

Lemma 6.1. *Let K be a compact space. The following statements are equivalent:*

(i) *K contains a subspace which is homeomorphic to $\beta\mathbb{N}$.*
(ii) *There exists a relatively discrete sequence $(x_n)_{n\in\mathbb{N}}$ in K such that, for every $a \subset \mathbb{N}$, $\overline{\{x_n : n \in a\}} \cap \overline{\{x_n : n \in \mathbb{N} \setminus a\}} = \emptyset$.*

Theorem 6.2. (CH) *There exists a compact connected K such that $C(K)$ is indecomposable and K contains a homeomorphic copy of $\beta\mathbb{N}$.*

Proof. As in [Ko1], we will construct by induction compact connected spaces $(K_\alpha)_{\alpha\le\omega_1}$, sequences $\{q_n|\alpha : n \in \mathbb{N}\} \subset K_\alpha$, and sets $b_\alpha \subset a_\alpha \subset \mathbb{N}$ for $\alpha < \omega_1$. We construct $(K_\alpha)_{\alpha\le\omega_1}$ so that $K_0 = [0,1]^2$ and $K_{\alpha+1}$ is an extension of K_α by a pairwise disjoint sequence of functions from K_α into $[0,1]$. If α is a limit ordinal, we define K_α as the inverse limit of $(K_\beta)_{\beta<\alpha}$. For every ordinal α we have $b_\beta \subset a_\beta \subset \mathbb{N}$ for $\beta<\alpha$, and the set $\{q_n|\alpha : n\in\mathbb{N}\}$ is dense in K_α and such that $\{q_n|\alpha : n \in a_\beta\}$ is relatively discrete and

$$\overline{\{q_n|\alpha : n\in b_\beta\}}\cap\overline{\{q_n|\alpha : n\in a_\beta\setminus b_\beta\}}\neq\emptyset$$

in K_α.

In $K_0=[0,1]^2$ let $\{x_n|0 : n\in\mathbb{N}\}$ be a relatively discrete sequence which is disjoint from $\{q_n|0 : n\in\mathbb{N}\}$. Let *Even* be the set of even ordinals in ω_1 and *Odd* the set of odd ordinals in ω_1. In every K_α we have constructed a relatively discrete sequence $\{x_n|\alpha : n\in\mathbb{N}\}$ of points that extend $x_n|0$. In the inductive construction, if $\alpha\in \mathit{Even}$ we proceed as in [Ko1] to get $K_{\alpha+1}$, identifying *Even* with ω_1. Let us fix an enumeration $\{N_\alpha : \alpha\in \mathit{Odd}\}$ of all subsets of $\mathbb{N}$. Let $\alpha\in\mathit{Odd}$. By CH, K_α has countable weight, and no isolated points since it is connected. So, we can apply Lemma 3.9 to K_α, obtaining pairwise disjoint functions f_n from K_α into $[0,1]$, for $n\in N_\alpha$, such that $f_n(x_m|\alpha)=1$ if $n=m$, and $f_n(x_m|\alpha)=0$ if $n\neq m$, and the extension of K_α by $(f_n)_{n\in N_\alpha}$ is strong (see [Ko1, Definition 4.2]). Define $K_{\alpha+1}$ to be that extension, and $x_n|(\alpha+1)=(x_n|\alpha,1)$ if $n\in N_\alpha$, and $x_n|(\alpha+1)=(x_n|\alpha,0)$ otherwise. We have

$$\overline{\{x_n|(\alpha+1) : n\in N_\alpha\}}\cap\overline{\{x_n|(\alpha+1) : n\in\mathbb{N}\setminus N_\alpha\}}=\emptyset$$

in $K_{\alpha+1}$. For $\alpha\in\mathit{Even}$, take for $x_n|(\alpha+1)$ any extension of $x_n(\alpha)$.

For all $\beta<\alpha<\omega_1$ we have

$$\overline{\{x_n|\alpha : n\in N_\beta\}}\cap\overline{\{x_n|\alpha : n\in\mathbb{N}\setminus N_\beta\}}=\emptyset$$

in K_α. Setting $x_n=\bigcup_{\alpha<\omega_1}x_n|\alpha$, for every $a\subset\mathbb{N}$ we have $\overline{\{x_n : n\in a\}}\cap\overline{\{x_n : n\in\mathbb{N}\setminus a\}}=\emptyset$ in $K=K_{\omega_1}$. From Lemma 6.1 we conclude that K contains a subspace homeomorphic to $\beta\mathbb{N}$. ∎

7. Final remarks. It remains open whether CH is necessary for Theorem 6.2 to hold. The necessity of axiom $\diamondsuit$ for 5.3 also remains open. We remark that a construction in ZFC of a compact K as in 5.3 would completely solve Efimov's problem, which has only consistent answers until now (assuming CH, which is weaker than $\diamondsuit$).

We say that a 0-dimensional compact space K has the *Subsequential Completeness Property* (SCP) if for any pairwise disjoint sequence of closed-open sets there is a subsequence which has supremum in the algebra of closed-open sets in K. By a result of Haydon (see [Ha, Proposition 1G]), this implies, under CH, that K contains a homeomorphic copy of $\beta\mathbb{N}$. It is easy to see that the 0-dimensional Koszmider space constructed in [Ko1] has the SCP. In the connected construction, an analogous property holds, with suprema of sequences of closed-open sets replaced by suprema of continuous functions. If the result of Haydon can be adapted to the connected case, we can prove that the construction of Koszmider, under CH, also satisfies the condition of Theorem 6.2.

Note that, in the construction of Theorem 5.2, it was necessary to change the functions $(f_n)_{n\in\mathbb{N}}$ before adding the supremum. This can be related to the result of Haydon, and one may investigate whether the connected version of SCP implies (under CH, perhaps) the presence of $\beta\mathbb{N}$ as a subspace.

Both Theorems 5.3 and 6.2 at first sight state topological properties of K which seem not to be preserved by isomorphisms of Banach spaces. Nevertheless, Schlakow proved ([Sc, Theorem 1.44]) that if $C(K)$ and $C(L)$ have few operators and are isomorphic, for perfect compact sets K and L, then K and L are homeomorphic.

Although we know that, at least for compact sets with no isolated points, those properties are preserved by isomorphisms, we do not know yet if the property stated in Theorem 5.3 is preserved by isomorphism.

Problem 7.1. *Suppose that $C(K_1)$ is isomorphic to $C(K_2)$ and that all closed $L \subset K_1$ are Koszmider spaces. Does this imply that every closed $L \subset K_2$ is a Koszmider space?*

The most important open problem that stems from this paper is the following question, which would generalize Theorem 5.3:

Problem 7.2. *Is there an indecomposable Banach space $C(K)$ whose quotients of the form $C(L)$ also have few operators?*

A weaker but still important version of the above problem is the following:

Problem 7.3. *Is there an indecomposable Banach space $C(K)$ which does not have l_∞ as quotient?*

The answers to Problems 7.2 and 7.3 are negative under Martin's axiom and the negation of continuum hypothesis, since it is proved in [HLO] that, under MA$+\neg$CH, non-reflexive Grothendieck spaces do have l_∞ as quotient.

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