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Noncommutative weak Orlicz spaces and martingale inequalities

by

Turdebek N. Bekjan (Urumqi), Zeqian Chen (Wuhan), Peide Liu (Wuhan) and Yong Jiao (Changsha)

Abstract. This paper is devoted to the study of noncommutative weak Orlicz spaces and martingale inequalities. The Marcinkiewicz interpolation theorem is extended to include noncommutative weak Orlicz spaces as interpolation classes. As an application, we prove the weak type Φ-moment Burkholder–Gundy inequality for noncommutative martingales through establishing a weak type Φ-moment noncommutative Khinchin inequality for Rademacher random variables.

1. Introduction. Recently, the first two named authors [BC] proved a Φ-moment Burkholder–Gundy inequality for noncommutative martingales, i.e., a noncommutative analogue of the following inequality [BDG]: Let Φ be an Orlicz function with $1 < p_\Phi \le q_\Phi < \infty$. If $f = (f_n)_{n\ge1}$ is an L_Φ-bounded martingale, then

$$(1.1)\qquad \int_\Omega \Phi\Big[\Big(\sum_{n=1}^{\infty} |df_n|^2\Big)^{1/2}\Big]\, dP \approx \sup_{n\ge1} \int_\Omega \Phi(|f_n|)\, dP,$$

where $df = (df_n)_{n\ge1}$ is the martingale difference of f and " $\approx$ " depends only on Φ. Notice that for convex powers $\Phi(t) = t^p$, (1.1) is the well-known Burkholder–Gundy inequality (see [BG]). In their remarkable paper [PX1], Pisier and Xu proved a noncommutative analogue of the Burkholder–Gundy inequality, which triggered a systematic research of noncommutative martingale inequalities. We refer to a recent book by Xu [Xu2] for an up-to-date exposition of the theory of noncommutative martingales. Evidently, the noncommutative Φ-moment Burkholder–Gundy inequality implies inequalites for L_Φ norms, which were already known as particular cases of more general ones established by the first named author in [B06].

In this paper, we continue this line of investigation. We will introduce noncommutative weak Orlicz spaces and prove the associated martingale

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inequalities. In particular, we will prove that noncommutative weak Orlicz spaces can be renormed as Banach spaces under a mild condition on Φ, and a weak type version of the Φ-moment inequalities for noncommutative martingles obtained recently by the first two authors [BC]. To the best of our knowledge, this kind of weak type Φ-moment inequality is new even in the commutative setting.

In [LHW], the authors prove the Burkholder–Gundy inequality for weak Orlicz spaces, using the arguments of stopping times and good-λ inequalities developed by Burkholder et al. [Burk]. However, the concepts of stopping times and good-λ inequalities are, up to now, not well defined in the general noncommutative setting (there are some works on this topic, see [AC] and references therein). Instead, interpolation and noncommutative Khinchin inequalities play crucial roles in the proof of the noncommutative Burkholder–Gundy inequality mentioned above. Thus, in order to prove the weak type Φ-moment Burkholder–Gundy inequality in the noncommutative setting, we need to prove the associated Khinchin type inequality. There is an extensive literature on various generalizations of the noncommutative Khinchin inequality in L_p setting (see [LP86, LPP, P09] and the references therein). Unfortunately, our weak type Φ-moment Khinchin inequality cannot be obtained directly from the ones established previously. To derive it, we adapt natural and classical techniques of [LP86, LP92, LPP, LPX, MS]. This is the key point of this paper.

The paper is organized as follows. In Section 2 we present some preliminaries and notation related to noncommutative weak L_p and Orlicz spaces. Noncommutative weak Orlicz spaces are presented in Section 3. In Section 4 we establish a Marcinkiewicz-type interpolation theorem for noncommutative weak Orlicz spaces and prove that noncommutative weak Orlicz spaces can be renormed as Banach spaces when Φ satisfies a mild condition. Finally, in Section 5, we prove the weak type Φ-moment Burkholder–Gundy inequality for noncommutative martingales through establishing a weak type Φ-moment noncommutative Khinchin inequality for Rademacher random variables. The proof follows mainly the arguments in [BC].

In what follows, C always denotes a constant, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y, by $X \lesssim Y$ we mean that there exists a constant $C > 0$ such that $X \leq CY$, and by $X \approx Y$ that $X \lesssim Y$ and $Y \lesssim X$.

2. Preliminaries

2.1. Noncommutative weak L_p spaces. We use standard notation and notions from the theory of noncommutative L_p spaces. Our main references are [PX2] and [Xu2] (see [PX2] for more historical references). Let

$\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace τ. For $0 < p < \infty$ let $L_p(\mathcal{M})$ denote the noncommutative L_p space with respect to $(\mathcal{M}, \tau)$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm. Also, let $L_0(\mathcal{M})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$.

For $x \in L_0(\mathcal{M})$ we define

$$\lambda_s(x) = \tau(e_s^\perp(|x|)) \ (s > 0) \quad \text{and} \quad \mu_t(x) = \inf\{s > 0 : \lambda_s(x) \le t\} \ (t > 0),$$

where $e_s^\perp(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞). The function $s \mapsto \lambda_s(x)$ is called the *distribution function* of x, and $\mu_t(x)$ the *generalized singular number* of x. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $s \mapsto \lambda_s(x)$ and $t \mapsto \mu_t(x)$, respectively. It is easy to check that both are decreasing and continuous from the right on $(0, \infty)$. For further information we refer the reader to [FK].

For $0 < p < \infty$, we have the Kolmogorov inequality

$$\lambda_s(x) \le \|x\|_p^p / s^p, \quad \forall s > 0, \tag{2.1}$$

for any $x \in L_p(\mathcal{M})$. If x, y in $L_0(\mathcal{M})$, then

$$\lambda_{2s}(x + y) \le \lambda_{s/2}(x) + \lambda_{s/2}(y), \quad \forall s > 0. \tag{2.2}$$

We will frequently use these two inequalities in what follows.

For $0 < p < \infty$, the *noncommutative weak L_p space $L_p^w(\mathcal{M})$* is defined as the space of all measurable operators x such that

$$\|x\|_{L_p^w} := \sup_{t>0} t^{1/p} \mu_t(x) < \infty.$$

Equipped with $\|\cdot\|_{L_p^w}$, $L_p^w(\mathcal{M})$ is a quasi-Banach space. However, for $p > 1$, $L_p^w(\mathcal{M})$ can be renormed as a Banach space by

$$x \mapsto \sup_{t>0} t^{-1+1/p} \int_0^t \mu_s(x)\, ds.$$

On the other hand, the quasi-norm admits the following useful description:

$$\|x\|_{L_p^w} = \inf\{c > 0 : t(\mu_t(x)/c)^p \le 1, \ \forall t > 0\}. \tag{2.3}$$

Also, we have a description in terms of the distribution function:

$$\|x\|_{L_p^w} = \sup_{s>0} s \lambda_s(x)^{1/p}. \tag{2.4}$$

Recall that noncommutative weak L_p spaces can be defined through noncommutative Lorenz spaces; for details see Dodds et al. [DDP] and Xu [Xu1].

2.2. Noncommutative Orlicz spaces. Recall that noncommutative Orlicz spaces were defined by Kunze [Kun] in an algebraic way (see also [ARZ] for more general cases), and by Dodds et al. [DDP] and Xu [Xu1] employing Banach space theory. The second approach, based on the concept of Banach function spaces, among other properties readily indicates similarities with the classical origins. We will take the second approach.

Let Φ be an *Orlicz function* on $[0,\infty)$, i.e., a continuous increasing and convex function satisfying $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. Recall that Φ is said to satisfy the $\triangle_2$-*condition* if there is a constant C such that $\Phi(2t) \le C\Phi(t)$ for all $t > 0$. In this case, we write $\Phi \in \triangle_2$. It is easy to check that $\Phi \in \triangle_2$ if and only if for any $a > 0$ there is a constant $C_a > 0$ such that $\Phi(at) \le C_a\Phi(t)$ for all $t > 0$.

We will work with some standard indices associated to Orlicz functions. Let Φ be an Orlicz function. Since Φ is convex, $\Phi'(t)$ is defined for each $t > 0$ except for a countable set of points at which we take $\Phi'(t)$ to be the derivative from the right. We define

$$a_\Phi = \inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t>0}\frac{t\Phi'(t)}{\Phi(t)}.$$

Note the following properties:

(1) $1 \le a_\Phi \le b_\Phi \le \infty$.

(2) The following characterizations of a_Φ and b_Φ hold:

$$a_\Phi = \sup\{p > 0 : t^{-p}\Phi(t) \text{ is nondecreasing for all } t > 0\},$$
$$b_\Phi = \inf\{q > 0 : t^{-q}\Phi(t) \text{ is nonincreasing for all } t > 0\}.$$

(3) $\Phi \in \triangle_2$ if and only if $b_\Phi < \infty$.

See [M85, M89] for more information on Orlicz functions and Orlicz spaces.

For an Orlicz function Φ, the noncommutative Orlicz space $L_\Phi(\mathcal{M})$ is defined as the space of all measurable operators x with respect to $(\mathcal{M},\tau)$ such that

$$\tau(\Phi(|x|/c)) < \infty$$

for some $c > 0$. The space $L_\Phi(\mathcal{M})$, equipped with the norm

$$\|x\|_\Phi = \inf\{c > 0 : \tau(\Phi(|x|/c)) < 1\},$$

is a Banach space. If $\Phi(t) = t^p$ with $1 \le p < \infty$ then $L_\Phi(\mathcal{M}) = L_p(\mathcal{M})$. Noncommutative Orlicz spaces are symmetric spaces of measurable operators as defined in [DDP, Xu1].

3. Noncommutative weak Orlicz spaces. Unless otherwise specified, we always denote by Φ an Orlicz function. Motivated by (2.3), we give the following definition:

Definition 3.1. For an Orlicz function Φ, define

$$L_\Phi^w(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \exists c > 0 \text{ such that } \sup_{t>0} t\Phi(\mu_t(x)/c) < \infty\},$$

equipped with

$$\|x\|_{L^w_\Phi} = \inf\{c > 0 : t\Phi(\mu_t(x)/c) \le 1, \forall t > 0\}.$$

$L^w_\Phi(\mathcal{M})$ is called a *noncommutative weak Orlicz space.*

REMARK 3.2.

(1) It is easy to check that

$$\|x\|_{L^w_\Phi} = \inf\left\{c > 0 : \frac{1}{\Phi^{-1}(1/t)}\mu_t(x)/c \le 1,\ \forall t > 0\right\}.$$

(2) For $0 < p < \infty$, if $\Phi(t) = t^p$ then $L^w_\Phi(\mathcal{M})$ is the noncommutative weak L_p space with the norm (2.3).

(3) Note that $L_\Phi(\mathcal{M})$ has the following description:

$$L_\Phi(\mathcal{M}) = \left\{x \in L_0(\mathcal{M}) : \exists\, c > 0,\ \int_0^\infty \left[t\Phi\left(\frac{\mu_t(x)}{c}\right)\right]\frac{dt}{t} < \infty\right\}$$

with the norm

$$\|x\|_{L_\Phi} = \inf\left\{c > 0 : \int_0^\infty \left[t\Phi\left(\frac{\mu_t(x)}{c}\right)\right]\frac{dt}{t} \le 1\right\}.$$

This shows that $L^w_\Phi(\mathcal{M})$ has a close connection with $L_\Phi(\mathcal{M})$.

We have the following useful characterization of $L^w_\Phi(\mathcal{M})$.

PROPOSITION 3.1. *Let Φ be an Orlicz function. For any $c > 0$ we have*

$$\sup_{t>0} t\Phi(\mu_t(x)/c) = \sup_{s>0} \lambda_s(x)\Phi(s/c), \qquad \forall x \in L_0(\mathcal{M}). \tag{3.1}$$

Consequently,

$$L^w_\Phi(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \exists c > 0 \text{ such that } \sup_{s>0} \lambda_s(x)\Phi(s/c) < \infty\},$$

and

$$\|x\|_{L^w_\Phi} = \inf\{c > 0 : \lambda_s(x)\Phi(s/c) \le 1, \forall s > 0\}.$$

Proof. Since $\lambda_s(x) = \lambda_{\mu(x)}(s)$, where $\lambda_{\mu(x)}$ is the distribution function of the function $t \mapsto \mu_t(x)$ with respect to the Lebesgue measure on $[0,\infty)$, it suffices to prove that

$$\sup_{t>0} t\Phi(f^*(t)/c) = \sup_{s>0} \lambda_f(s)\Phi(s/c) \tag{3.2}$$

for any nonnegative measurable function f on $(0,\infty)$, where λ_f is the distribution function of f with respect to the Lebesgue measure on $[0,\infty)$ and f^* is the rearrangement function of f defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \le t\}.$$

To this end, we consider a simple function $f = \sum_k a_k \chi_{A_k}$, where $a_k > 0$ and A_k are measurable subsets of $[0,\infty)$ such that $|A_k| < \infty$ and $A_k \cap A_j = \emptyset$

whenever $k \neq j$. An immediate computation yields (3.2) for such a function. Since each nonnegative measurable function can be approximated almost everywhere from below by a sequence of nonnegative simple functions, a standard argument proves (3.2) for any nonnegative measurable function. ■

We collect some basic properties of noncommutative weak Orlicz spaces.

Proposition 3.2. *Let Φ be an Orlicz function.*

(1) *If* $\|x\|_{L_\Phi^w} > 0$ *then*

$$\sup_{t>0} t\Phi(\mu_t(x)/\|x\|_{L_\Phi^w}) \leq 1 \quad \textit{and} \quad \sup_{s>0} \lambda_s(x)\Phi(s/\|x\|_{L_\Phi^w}) \leq 1.$$

(2) $\|\cdot\|_{L_\Phi^w}$ *is a quasi-norm on* $L_\Phi^w(\mathcal{M})$. *In particular,*

$$\|x+y\|_{L_\Phi^w} \leq 2(\|x\|_{L_\Phi^w} + \|y\|_{L_\Phi^w}), \quad \forall x, y \in L_\Phi^w(\mathcal{M}). \tag{3.3}$$

(3) *If* $\|x\|_{L_\Phi^w} \leq 1$, *then*

$$\sup_{t>0} t\Phi(\mu_t(x)) \leq \|x\|_{L_\Phi^w} \quad \textit{and} \quad \sup_{s>0} \lambda_s(x)\Phi(s) \leq \|x\|_{L_\Phi^w}.$$

(4) $\|x\|_{L_\Phi^w} \leq \|x\|_{L_\Phi}$ *for any* $x \in L_\Phi(\mathcal{M})$. *Consequently,* $L_\Phi(\mathcal{M}) \subset L_\Phi^w(\mathcal{M})$.

Proof. (1) By the definition of $\|x\|_{L_\Phi^w}$, there is a sequence $(c_k) \subset \mathbb{R}^+$ such that $c_k \downarrow \|x\|_{L_\Phi^w}$ and $t\Phi(\mu_t(x)/c_k) \leq 1$ for all $t > 0$. Since Φ is continuous, taking $k \to \infty$ we obtain the first inequality. The second inequality follows from (3.1) and the first one.

(2) If $\|x\|_{L_\Phi^w} = 0$, then there is a sequence $(c_k) \subset \mathbb{R}^+$ such that $c_k \downarrow 0$ and $t\Phi(\mu_t(x)/c_k) \leq 1$ for all $t > 0$. Since $\Phi(t) \to \infty$ as $t \to \infty$, it follows that $\mu_t(x) = 0$ for all $t > 0$, which implies $x = 0$ because $\lim_{t\to 0^+} \mu_t(x) = \|x\|$.

It is clear that $\|\alpha x\|_{L_\Phi^w} = |\alpha|\, \|x\|_{L_\Phi^w}$. To prove the generalized triangle inequality, let $x, y \in L_\Phi^w(\mathcal{M})$, with $\|x\|_{L_\Phi^w} = a$, $\|y\|_{L_\Phi^w} = b$ where $a, b > 0$. By (1), we have

$$\begin{aligned} t\Phi\Big(\frac{\mu_t(x+y)}{2(a+b)}\Big) &\leq t\Phi\Big(\frac{\mu_{t/2}(x)+\mu_{t/2}(y)}{2(a+b)}\Big) \leq \frac{t}{2}\Phi\Big(\frac{\mu_{t/2}(x)}{a+b}\Big) + \frac{t}{2}\Phi\Big(\frac{\mu_{t/2}(y)}{a+b}\Big) \\ &\leq \frac{a}{a+b}\,\frac{t}{2}\Phi\Big(\frac{\mu_{t/2}(x)}{a}\Big) + \frac{b}{a+b}\,\frac{t}{2}\Phi\Big(\frac{\mu_{t/2}(y)}{b}\Big) \leq 1. \end{aligned}$$

Hence, $\|x+y\|_{L_\Phi^w} \leq 2(a+b) = 2(\|x\|_{L_\Phi^w} + \|y\|_{L_\Phi^w})$.

(3) If $\|x\|_{L_\Phi^w} = 0$, by (2) the first inequality holds. Suppose $\|x\|_{L_\Phi^w} = a \leq 1$ and $a \neq 0$. By (1) we have $t\Phi(\mu_t(x)/a) \leq 1$ for all $t > 0$. From the convexity of Φ and the fact that $\Phi(0) = 0$, we have $\Phi(at) \leq a\Phi(t)$ for all $t > 0$, which implies that

$$\frac{t}{a}\Phi(\mu_t(x)) \leq t\Phi(\mu_t(x)/a) \leq 1, \quad \forall t > 0.$$

This gives the first inequality. The second follows from (3.1) and the first.

(4) Let $x \in L_\Phi(\mathcal{M})$, $x \neq 0$. Then, for any $t > 0$,

$$t\Phi\left(\frac{\mu_t(x)}{\|x\|_{L_\Phi}}\right) \le \int_0^t \Phi\left(\frac{\mu_s(x)}{\|x\|_{L_\Phi}}\right) ds \le \int_0^\infty \Phi\left(\frac{\mu_s(x)}{\|x\|_{L_\Phi}}\right) ds \le 1.$$

Hence, $\|x\|_{L_\Phi^w} \le \|x\|_{L_\Phi}$ and $L_\Phi(\mathcal{M}) \subset L_\Phi^w(\mathcal{M})$. ■

Recall that for measurable operators x_n, x with respect to $(\mathcal{M}, \tau)$, x_n converges to x in measure if and only if $\lim_n \mu_t(x_n - x) = 0$ for all $t > 0$.

Proposition 3.3. *Let Φ be an Orlicz function.*

(1) *If $\|x_n - x\|_{L_\Phi^w} \to 0$, then $x_n \to x$ in measure.*
(2) *$L_\Phi^w(\mathcal{M})$ is a quasi-Banach space.*

Proof. (1) Suppose $\|x_n - x\|_{L_\Phi^w} \to 0$. Then there is a sequence (c_n) of positive numbers with $\lim_n c_n = 0$ such that

$$t\Phi\left(\frac{\mu_t(x_n - x)}{c_n}\right) \le 1, \qquad \forall t > 0,$$

for all n. Since $\Phi(t) \to \infty$ as $t \to \infty$, it follows that $\lim_n \mu_t(x_n - x) = 0$ for any $t > 0$. Hence, $x_n \to x$ in measure.

(2) By Proposition 3.2(2), it suffices to prove that $L_\Phi^w(\mathcal{M})$ is complete. Suppose $x_n \in L_\Phi^w(\mathcal{M})$ with $\lim_{m,n\to\infty} \|x_n - x_m\|_{L_\Phi^w} = 0$. Then for any $1 > \varepsilon > 0$ there is an n_0 such that $\|x_n - x_m\|_{L_\Phi^w} < \varepsilon$ for all $n, m \ge n_0$. Since $L_0(\mathcal{M})$ is complete in the topology of convergence in measure, by (1) there exists $x \in L_0(\mathcal{M})$ such that

$$\lim_{n\to\infty} \mu_t(x_n - x) = 0, \qquad \forall t > 0.$$

Clearly,

$$x_n - x_m \to x_n - x \qquad \text{in measure}$$

as $m \to \infty$. By Proposition 3.2(3), for any $n \ge n_0$ we have

$$t\Phi\left(\frac{\mu_t(x_n - x)}{\varepsilon}\right) \le \lim_{m\to\infty} t\Phi\left(\frac{\mu_t(x_n - x_m)}{\varepsilon}\right) \le \liminf_{m\to\infty} \left\|\frac{x_n - x_m}{\varepsilon}\right\|_{L_\Phi^w} \le 1$$

for any $t > 0$. This yields $\|x_n - x\|_{L_\Phi^w} < \varepsilon$ and so $\lim_{n\to\infty} \|x_n - x\|_{L_\Phi^w} = 0$. Also, by (3.3) we obtain $x \in L_\Phi^w(\mathcal{M})$. Hence, $L_\Phi^w(\mathcal{M})$ is complete. ■

Remark 3.3. Clearly, $L_\Phi^w(\mathcal{M})$ is rearrangement invariant. Then, by Proposition 3.3(2), $L_\Phi^w(\mathcal{M})$ is a symmetric quasi-Banach space of measurable operators as defined in [Xu1].

The following two examples illustrate noncommutative weak Orlicz spaces.

Example 3.4. Let $\Phi(t) = t^a \ln(1 + t^b)$ with $a > 1$ and $b > 0$. It is easy to check that Φ is an Orlicz function and $p_\Phi = a$ and $q_\Phi = a + b$. Thus, L_Φ^w cannot coincide with any L_p^w.

EXAMPLE 3.5. Let $\Phi(t) = t^p(1 + c\sin(p\ln t))$ with $p > 1/(1-2c)$ and $0 < c < 1/2$. Then Φ is an Orlicz function and $p_\Phi = q_\Phi = p$. It is clear that Φ is equivalent to t^p and hence $L^w_\Phi = L^w_p$.

Let $a = (a_n)$ be a finite sequence in $L^w_\Phi(\mathcal{M})$. We define

$$\|a\|_{L^w_\Phi(\mathcal{M},\ell^2_C)} = \Big\|\Big(\sum_n |a_n|^2\Big)^{1/2}\Big\|_{L^w_\Phi} \quad \text{and} \quad \|a\|_{L^w_\Phi(\mathcal{M},\ell^2_R)} = \Big\|\Big(\sum_n |a^*_n|^2\Big)^{1/2}\Big\|_{L^w_\Phi}.$$

PROPOSITION 3.4. *$\|\cdot\|_{L^w_\Phi(\mathcal{M},\ell^2_C)}$ and $\|\cdot\|_{L^w_\Phi(\mathcal{M},\ell^2_R)}$ are two quasi-norms on the family of all finite sequences in $L^w_\Phi(\mathcal{M})$.*

Proof. To see this, let us consider the von Neumann algebra tensor product $\mathcal{M} \bar{\otimes} \mathcal{B}(\ell^2)$ with the product trace $\tau \bar{\otimes} \mathrm{tr}$, where $\mathcal{B}(\ell^2)$ is the algebra of all bounded operators on ℓ^2 with the usual trace tr. $\tau \otimes \mathrm{tr}$ is a semifinite normal faithful trace. The associated noncommutative weak Orlicz space is denoted by $L^w_\Phi(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$. Now, any finite sequence $a = (a_n)_{n\geq 0}$ in $L^w_\Phi(\mathcal{M})$ can be regarded as an element in $L^w_\Phi(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell^2))$ via the map

$$a \mapsto T(a) = \begin{pmatrix} a_0 & 0 & \dots \\ a_1 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

that is, the matrix of $T(a)$ has all entries zero except those in the first column which are the a_n's. Such a matrix is called a *column matrix*, and the closure in $L^w_\Phi(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell^2))$ of all column matrices is called the *column subspace* of $L^w_\Phi(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell^2))$. Since

$$\|a\|_{L^w_\Phi(\mathcal{M},\ell^2_C)} = \big\||T(a)|\big\|_{L^w_\Phi(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))} = \|T(a)\|_{L^w_\Phi(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))},$$

$\|\cdot\|_{L^w_\Phi(\mathcal{M},\ell^2_C)}$ defines a quasi-norm on the family of all finite sequences in $L^w_\Phi(\mathcal{M})$. Similarly, $\|\cdot\|_{L^w_\Phi(\mathcal{M},\ell^2_R)}$ defines a quasi-norm on the family of all finite sequences in $L^w_\Phi(\mathcal{M})$. ∎

We define $L^w_\Phi(\mathcal{M},\ell^2_C)$ (resp. $L^w_\Phi(\mathcal{M},\ell^2_R)$) to be the space of all sequences in $L^w_\Phi(\mathcal{M})$ under the norm $\|\cdot\|_{L^w_\Phi(\mathcal{M},\ell^2_C)}$ (resp. $\|\cdot\|_{L^w_\Phi(\mathcal{M},\ell^2_R)}$). Evidently, both are quasi-Banach spaces, but we will see in Sect. 4 that they can be renormed as Banach spaces provided Φ satisfies a mild condition.

4. Interpolation. The main result of this section is a Marcinkiewicz type interpolation theorem for noncommutative weak Orlicz spaces. We first introduce the following definition.

DEFINITION 4.1. Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a von Neumann algebra with a normal semifinite faithful trace τ (resp. ν). A map $T : L_0(\mathcal{M}) \to L_0(\mathcal{N})$ is said to be *quasilinear* if

(i) $|T(\alpha x)| \le |\alpha|\,|Tx|$ for all $x \in L_0(\mathcal{M})$ and $\alpha \in \mathbb{C}$;
(ii) there is a constant $K > 0$ such that for any operators $x, y \in L_0(\mathcal{M})$, there exist partial isometries $u, v \in \mathcal{N}$ such that

$$|T(x+y)| \le K(u^*|Tx|u + v^*|Ty|v).$$

In addition, if $K = 1$ we call T a *sublinear operator*.

This definition of sublinear operators in the noncommutative setting is due to Q. Xu and first appeared in Ying Hu's thesis [Hu07] (see also [Hu09]). Recall that for any $x, y \in L_0(\mathcal{N})$ there exist partial isometries $u, v \in \mathcal{N}$ such that

$$|x+y| \le u^*|x|u + v^*|y|v \tag{4.1}$$

(see [AAP]) and so every linear operator is sublinear. We recall that a quasilinear operator $T : L_0(\mathcal{M}) \to L_0(\mathcal{N})$ is of *weak type* (p,q) with $0 < p \le q \le \infty$ if

$$\|Tx\|_{L_q^w} \le C\|x\|_{L_p}, \quad \forall x \in L_p(\mathcal{M}).$$

The classical Marcinkiewicz interpolation theorem has been extended to include Orlicz spaces as interpolation classes by A. Zygmund, A. P. Calderón, S. Koizumi, I. B. Simonenko, W. Riordan, H. P. Heinig and A. Torchinsky (for references see [M89]). The following result is a noncommutative analogue of the Marcinkiewicz type interpolation theorem for weak Orlicz spaces.

THEOREM 4.2. *Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a von Neumann algebra with a normal semifinite faithful trace τ (resp. ν). Suppose $0 < p_0 < p_1 \le \infty$. Let $T : L_0(\mathcal{M}) \to L_0(\mathcal{N})$ be a quasilinear operator which is of weak type (p_i, p_i) for $i = 0, 1$ if $p_1 < \infty$, and of weak type (p_0, p_0) and strong type (p_1, p_1) if $p_1 = \infty$. If Φ is an Orlicz function with $p_0 < a_\Phi \le b_\Phi < p_1$, then there exists a constant $C > 0$ such that*

$$\sup_{t>0} t\Phi[\mu_t(Tx)] \le C \sup_{t>0} t\Phi[\mu_t(x)] \tag{4.2}$$

for all $x \in L_\Phi^w(\mathcal{M})$. Consequently,

$$\|Tx\|_{L_\Phi^w(\mathcal{N})} \lesssim \|x\|_{L_\Phi^w(\mathcal{M})}, \quad \forall x \in L_\Phi^w(\mathcal{M}). \tag{4.3}$$

Proof. We choose $\theta_1, \theta_2, r_0, r_1$ such that

$$p_0 < r_0 < a_\Phi \le b_\Phi < r_1 < p_1$$

and

$$0 < \theta_1, \theta_2 < 1, \quad \frac{1}{r_k} = \frac{1-\theta_k}{p_0} + \frac{\theta_k}{p_1}, \quad k = 0, 1.$$

Then, by real interpolation of noncommutative L_p spaces (cf. Corollary 1.6.11 of [Xu2]), we have

$$(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_{\theta_k,q} = L_{r_k,q}(\mathcal{M}), \quad k = 0, 1,$$

with equivalent quasi-norms. Since T is simultaneously of weak type (p_i, p_i) for $i = 0$ and $i = 1$, we obtain

$$\|Tx\|_{L^w_{r_0}} \le A_0 \|x\|_{L^w_{r_0}}, \quad \forall x \in L^w_{p_0}(\mathcal{M}), \tag{4.4}$$

$$\|Tx\|_{L^w_{r_1}} \le A_1 \|x\|_{L^w_{r_1}}, \quad \forall x \in L^w_{p_1}(\mathcal{M}), \tag{4.5}$$

where A_0, A_1 are constants which depend only on p_0, p_1, r_0, r_1 and the weak type (p_i, p_i) norms of T for $i = 0$ and $i = 1$.

Now, take $x \in L^w_\Phi(\mathcal{M})$. For any $\alpha > 0$ let $x = x_0^\alpha + x_1^\alpha$, where $x_0^\alpha = xe_{(\alpha,\infty)}(|x|)$. Since $t^{-r_0}\Phi(t)$ is an increasing function in $(0, \infty)$, by Proposition 3.2(1) and (4.4) we have

$$\begin{aligned}\lambda_\alpha(Tx_0^\alpha) &\le \alpha^{-r_0}\|Tx_0^\alpha\|^{r_0}_{L^w_{r_0}} \le \alpha^{-r_0}A_0^{r_0}\|x_0^\alpha\|^{r_0}_{L^w_{r_0}} = \alpha^{-r_0}A_0^{r_0}\sup_{t>0} t^{r_0}\lambda_t(x_0^\alpha)\\ &\le A_0^{r_0}\sup_{t>\alpha}\left(\frac{t}{\alpha}\right)^{r_0}\lambda_t(x) \le A_0^{r_0}\sup_{t>\alpha}\frac{\Phi(t)}{\Phi(\alpha)}\lambda_t(x) \le \frac{A_0^{r_0}}{\Phi(\alpha)}\sup_{t>0}\Phi(t)\lambda_t(x).\end{aligned}$$

Also, since $t^{-r_1}\Phi(t)$ is a decreasing function in $(0, \infty)$, by Proposition 3.2(1) and (4.5) we obtain similarly

$$\lambda_\alpha(Tx_1^\alpha) \le \frac{A_1^{r_1}}{\Phi(\alpha)}\sup_{t>0}\Phi(t)\lambda_t(x).$$

On the other hand, by the quasilinearity of T and the basic properties of the distribution function $\lambda(|x|)$, such as $\lambda(a^*a) = \lambda(aa^*)$ and $\lambda_{\alpha+\beta}(x+y) \le \lambda_\alpha(x) + \lambda_\beta(y)$ for any $x, y \ge 0$, we have

$$\begin{aligned}\lambda_{2K\alpha}(Tx) &\le \nu(E_{(2K\alpha,\infty)}[K(u^*|Tx_0^\alpha|u + v^*|Tx_1^\alpha|v)])\\ &\le \lambda_\alpha(u^*|Tx_0^\alpha|u) + \lambda_\alpha(v^*|Tx_1^\alpha|v)\\ &\le \lambda_\alpha(|Tx_0^\alpha|) + \lambda_\alpha(|Tx_1^\alpha|),\end{aligned} \tag{4.6}$$

where the first and third inequalities use the fact that $0 \le a \le b$ implies $E_{(\alpha,\infty)}(a)$ is equivalent to a subprojection of $E_{(\alpha,\infty)}(b)$ (see e.g. [FK]). By (4.6) we have

$$\begin{aligned}\lambda_{2K\alpha}(Tx) &\le \frac{A_0^{r_0}}{\Phi(\alpha)}\sup_{t>0}\Phi(t)\lambda_t(x) + \frac{A_1^{r_1}}{\Phi(\alpha)}\sup_{t>0}\Phi(t)\lambda_t(x)\\ &\le \frac{C}{\Phi(2K\alpha)}\sup_{t>0}\Phi(t)\lambda_t(x).\end{aligned}$$

By Proposition 3.1 we obtain the desired inequality (4.2). ∎

Remark 4.3. We set

$$L_p(\mathcal{N})_{\mathrm{Her}} = \{x \in L_p(\mathcal{N}) : x^* = x\}.$$

If T is simultaneously of weak types $L_{p_i}(\mathcal{M})_{\mathrm{Her}} \to L_{p_i}(\mathcal{N})_{\mathrm{Her}}$ for $i = 0$ and $i = 1$, then the conclusion of Theorem 4.2 holds for any hermitian operator $x \in L_\Phi(\mathcal{M})$. The proof is the same as above and is omitted.

Corollary 4.4. *Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a von Neumann algebra with a normal semifinite faithful trace τ (resp. ν). Suppose $0 < p_0 < p_1 \le \infty$. Let $T : L_0(\mathcal{M}) \mapsto L_0(\mathcal{N})$ be a quasilinear operator of strong type (p_i, p_i) for $i = 0, 1$, i.e.,*

$$\|Tx\|_{L_{p_0}} \lesssim \|x\|_{L_{p_0}}, \quad \forall x \in L_{p_0}(\mathcal{M}),$$
$$\|Tx\|_{L_{p_1}} \lesssim \|x\|_{L_{p_1}}, \quad \forall x \in L_{p_1}(\mathcal{M}).$$

Let Φ be an Orlicz function with $p_0 < a_\Phi \le b_\Phi < p_1$. Then the conclusion of Theorem 4.2 holds.

Proof. If T is of strong type (p, p), by the Kolmogorov inequality (2.1) we immediately conclude that T is of weak type (p, p). An appeal to Theorem 4.2 yields the result. ■

Corollary 4.5. *Let Φ be an Orlicz function with $1 < a_\Phi \le b_\Phi < \infty$. Then*

$$\|x\|_{L_\Phi^w} \approx \inf\left\{c > 0 : t\Phi\left(\frac{1}{t}\int_0^t \mu_s(x)\, ds/c\right) \le 1,\ \forall t > 0\right\}. \tag{4.7}$$

Consequently, $L_\Phi^w(\mathcal{M})$ can be renormed as a Banach space.

Proof. Since $\mu_t(x)$ is decreasing in $t \in (0, \infty)$, we immediately get

$$\|x\|_{L_\Phi^w} \le \inf\left\{c > 0 : t\Phi\left(\frac{1}{t}\int_0^t \mu_s(x)\, ds/c\right) \le 1, \forall t > 0\right\}.$$

Conversely, let $1 < p \le \infty$. Define $S : f(t) \mapsto \frac{1}{t}\int_0^t |f(s)|\, ds$ for $f \in L_p(0, \infty)$. Then by the classical Hardy–Littlewood inequality there exists a constant $A_p > 0$ such that

$$\|Sf\|_p \le C_p\|f\|_p, \quad \forall f \in L_p(0, \infty).$$

Consequently,

$$\|Tx\|_p \le A_p\|x\|_p, \quad \forall x \in L_p(\mathcal{M}),$$

where

$$Tx := \frac{1}{t}\int_0^t \mu_s(x)\, ds, \quad x \in L_0(\mathcal{M}).$$

Since T is sublinear, by Corollary 4.4 we obtain the reverse inequality and hence (4.7) holds. ■

The corollary above implies that $L_\Phi^w(0, \infty)$ is a symmetric function space, so we can consider the associated Boyd indices p_Φ^w and q_Φ^w.

Corollary 4.6. *Let Φ be an Orlicz function with $1 < a_\Phi \le b_\Phi < \infty$. Let p_Φ^w and q_Φ^w be respectively the lower and upper Boyd indices of $L_\Phi^w(\mathcal{M})$. Then*

$$a_\Phi \le p_\Phi^w \le q_\Phi^w \le b_\Phi. \tag{4.8}$$

Proof. Let $1 \le p < a_\Phi \le b_\Phi < q < \infty$. Suppose T is a linear operator defined on $L_{p,1}[0,\infty)+L_{q,1}[0,\infty)$, which is simultaneously of weak type (p,p) and weak type (q,q) in the sense of [LT]. Take p_0, q_0 such that $p < p_0 < a_\Phi \le b_\Phi < q_0 < q$. Then by Theorem 2.b.11 in [LT], T is simultaneously of strong type (p_0,p_0) and strong type (q_0,q_0). Using Corollary 4.4, we see that T maps $L_\Phi^w(\mathcal{M})$ into itself. Then, by Theorem 2.b.13 in [LT] we conclude that $p < p_\Phi^w \le q_\Phi^w < q$. This completes the proof. ∎

5. Martingale inequalities. In this section, we will prove the weak type Φ-moment versions of martingale transformations, Stein inequalities, Khinchin inequalities for Rademacher random variables, and Burkholder–Gundy martingale inequalities in the noncommutative setting. We mainly follow the arguments in [BC] using Theorem 4.3 and Corollary 4.4.

In the following, unless otherwise specified, we always denote by $\mathcal{M}$ a finite von Neumann algebra with a normalized normal faithful trace τ. Let $(\mathcal{M}_n)_{n\ge0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\bigcup_{n\ge0}\mathcal{M}_n$ generates $\mathcal{M}$ (in the w^*-topology). Then $(\mathcal{M}_n)_{n\ge0}$ is called a *filtration* of $\mathcal{M}$. The restriction of τ to $\mathcal{M}_n$ is still denoted by τ. Let $\mathcal{E}_n = \mathcal{E}(\cdot|\mathcal{M}_n)$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$.

Moreover, we let Φ be an Orlicz function with $1 < a_\Phi \le b_\Phi < \infty$. In this case, since $L_\Phi^w(\mathcal{M}) \subset L_1(\mathcal{M})$, the conditional expectation $\mathcal{E}_n$ extends to $L_\Phi^w(\mathcal{M})$.

A *noncommutative L_Φ^w-martingale* with respect to $(\mathcal{M}_n)_{n\ge0}$ is a sequence $x = (x_n)_{n\ge0}$ such that $x_n \in L_\Phi^w(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any $n \ge 0$. Let $\|x\|_{L_\Phi^w} = \sup_{n\ge0}\|x_n\|_{L_\Phi^w}$. If $\|x\|_{L_\Phi^w} < \infty$, then x is said to be a *bounded L_Φ^w-martingale.*

For convenience, we denote the weak type Φ-moment of x by

$$\|x\|_{\Phi_w(\mathcal{M})} := \sup_{t>0} t\Phi(\mu_t(x)), \quad x \in L_0(\mathcal{M}).$$

We write $\|x\|_{\Phi_w} = \|x\|_{\Phi_w(\mathcal{M})}$ for short when no confusion occurs.

Let $\alpha = (\alpha_n) \subset \mathbb{C}$ be a sequence. Recall that a map T_α on the family of martingale difference sequences defined by $T_\alpha(dx) = (\alpha_n dx_n)$ is called the *martingale transform* of symbol α. It is clear that $(\alpha_n dx_n)$ is indeed a martingale difference sequence. The corresponding martingale is $T_\alpha(x) = \sum_n \alpha_n dx_n$.

THEOREM 5.1. *Let $\alpha = (\alpha_n) \subset \mathbb{C}$ be a bounded sequence and T_α the associated martingale transform. Let Φ be an Orlicz function such that $1 < a_\Phi \le b_\Phi < \infty$. Then, for all bounded L_Φ^w-martingales $x = (x_n)$,*

$$\|T_\alpha x\|_{\Phi_w} \lesssim \|x\|_{\Phi_w}, \tag{5.1}$$

where $\lesssim$ *depends only on* Φ *and* $\sup_n |\alpha_n|$. *Consequently,*

$$\|x\|_{\Phi_w} \approx \Big\| \sum \varepsilon_n dx_n \Big\|_{\Phi_w}, \qquad \forall \varepsilon_n = \pm 1, \tag{5.2}$$

for any bounded L_Φ^w*-martingale* $x = (x_n)$, *where* " $\approx$ " *depends only on* Φ.

Proof. By the L_p-boundedness of martingale transforms (see [PX1]) and Corollary 4.4, we immediately deduce (5.1) and so (5.2). ∎

As in [PX1], consider the mapping T defined in $L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell^2))$ by

$$T\begin{pmatrix} a_{11} & \dots & a_{1n} & \dots \\ a_{21} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1(a_{11}) & 0 & 0 & \dots \\ \mathcal{E}_2(a_{21}) & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{E}_n(a_{n1}) & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is proved in [PX1] that T is bounded on $L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$ for any $1 < p < \infty$. Then, by Corollary 4.4 we have

THEOREM 5.2. *Let* Φ *be an Orlicz function with* $1 < a_\Phi \le b_\Phi < \infty$. *Then*

$$\Big\| \Big(\sum_n |\mathcal{E}_n(a_n)|^2 \Big)^{1/2} \Big\|_{\Phi_w} \lesssim \Big\| \Big(\sum_n |a_n|^2 \Big)^{1/2} \Big\|_{\Phi_w} \tag{5.3}$$

for any finite sequence (a_n) *in* $L_\Phi^w(\mathcal{M})$. *Similarly,*

$$\Big\| \Big(\sum_n |\mathcal{E}_n(a_n^*)|^2 \Big)^{1/2} \Big\|_{\Phi_w} \lesssim \Big\| \Big(\sum_n |a_n|^2 \Big)^{1/2} \Big\|_{\Phi_w} \tag{5.4}$$

for any finite sequence (a_n) *in* $L_\Phi^w(\mathcal{M})$.

The following is the weak type Φ-moment version of noncommutative Khinchin inequalities for Rademacher sequences.

THEOREM 5.3. *Let* Φ *be an Orlicz function and* (ε_k) *a Rademacher sequence on a probability space* (Ω, P).

(1) *If* $1 < a_\Phi \le b_\Phi < 2$, *then for any finite sequence* (x_k) *in* $L_\Phi^w(\mathcal{M})$,

$$\begin{aligned} &\Big\| \sum_k \varepsilon_k x_k \Big\|_{\Phi_w(L_\infty(\Omega)\bar{\otimes}\mathcal{M})} \\ &\qquad \approx \inf \Big\{ \Big\| \Big(\sum_k |y_k|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})} + \Big\| \Big(\sum_k |z_k^*|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})} \Big\} \end{aligned} \tag{5.5}$$

where the infimum runs over all decompositions $x_k = y_k + z_k$ *with* $y_k, z_k \in L_\Phi^w(\mathcal{M})$ *and* " $\approx$ " *depends only on* Φ.

(2) *If* $2 < a_\Phi \le b_\Phi < \infty$, *then for any finite sequence* (x_k) *in* $L_\Phi^w(\mathcal{M})$,

$$\Big\| \sum_k \varepsilon_k x_k \Big\|_{\Phi_w(L_\infty(\Omega)\bar{\otimes}\mathcal{M})} \gtrsim \Big\| \Big(\sum_k |x_k|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})} + \Big\| \Big(\sum_k |x_k^*|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})} \tag{5.6}$$

where " $\gtrsim$ *" depends only on* Φ.

Proof. (1) By the argument in [BC], we only need to prove the lower estimate of (5.5). By the analogous argument in [LPP], we are reduced to showing that for any finite sequence (x_k) in $L_\Phi^w(\mathcal{M})$,

$$\inf \Big\{ \Big\| \Big(\sum_{k=0}^n |y_k|^2 \Big)^{1/2} \Big\|_{\Phi_w} + \Big\| \Big(\sum_{k=0}^n |z_k^*|^2 \Big)^{1/2} \Big\|_{\Phi_w} \Big\} \lesssim \Big\| \sum_{k=0}^n x_k z^{3^k} \Big\|_{\Phi_w(L_\infty(\mathbb{T})\bar{\otimes}\mathcal{M})}, \tag{5.7}$$

where the infimum is taken over all decompositions $x_k = y_k + z_k$ with y_k and z_k in $L_\Phi^w(\mathcal{M})$.

To this end, we consider $\mathcal{N} = L_\infty(\mathbb{T}) \bar{\otimes} \mathcal{M}$ equipped with the tensor product trace $\nu = \int \otimes \tau$ and $\mathcal{A} = \mathcal{H}_\infty(\mathbb{T}) \bar{\otimes} \mathcal{M}$. Then $\mathcal{A}$ is a finite maximal subdiagonal algebra of $\mathcal{N}$ with respect to $\mathcal{E} = \int \otimes I_{\mathcal{M}} : \mathcal{N} \to \mathcal{M}$ (see e.g. [PX2]). Since $L_1(\mathcal{N}) = L_1(\mathbb{T}, L_1(\mathcal{M}))$ we can define the Fourier coefficients for any $f \in L_1(\mathcal{N})$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z)\bar{z}^n \, dm(z), \qquad \forall n \in \mathbb{Z},$$

where dm is the normalized Lebesgue measure on $\mathbb{T}$. It is easy to check that

$$\mathcal{A} = \{ f \in \mathcal{N} : \hat{f}(n) = 0, \, \forall n < 0 \}.$$

For any $n \in \mathbb{Z}$ we define the linear mapping F_n such that $F_n(f) = \hat{f}(n)$ for any $L_1(\mathcal{N})$. Then F_n is a contraction both from $L_1(\mathcal{N})$ into $L_1(\mathcal{M})$ and from $\mathcal{N}$ into $\mathcal{M}$. Hence, for an Orlicz function Φ with $1 < a_\Phi \le b_\Phi < \infty$, by Corollary 4.4 we have

$$\| \hat{f}(n) \|_{\Phi_w} \lesssim \| f \|_{\Phi_w}, \qquad \forall f \in L_\Phi^w(\mathcal{N}), \tag{5.8}$$

for any $n \in \mathbb{Z}$.

Lemma 5.4. *Let* Φ *be an Orlicz function with* $1 < a_\Phi \le b_\Phi < \infty$. *For any finite sequence* (f_k) *in* $L_\Phi^w(\mathcal{N})$ *and any* $n \in \mathbb{Z}$, *we have*

$$\Big\| \Big(\sum_k |\hat{f}_k(n)|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})} \lesssim \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{N})}.$$

Proof. Let $1 \le k \le K$. Applying (5.8) on $M_K(\mathcal{M})$ instead of $\mathcal{M}$ with

$$f = \sum_{k=1}^{K} E_{k1} \otimes f_k = \begin{pmatrix} f_1 & 0 & \dots & 0 \\ f_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ f_K & 0 & \dots & 0 \end{pmatrix}_{K\times K}$$

yields the required inequality. ∎

Since Φ is an Orlicz function with $1 < a_\Phi \le b_\Phi < \infty$, by Corollary 4.6 we have $L_\Phi^w(\mathcal{N}) \subset L_1(\mathcal{N})$. We define

$$\mathcal{H}_\Phi^w(\mathcal{A}) = \{f \in L_\Phi^w(\mathcal{N}) : \hat{f}(n) = 0,\ \forall n < 0\},$$

equipped with the quasinorm $\|\cdot\|_{L_\Phi^w(\mathcal{N})}$. In this case,

$$\mathcal{H}^1(\mathcal{A}) \cap L_\Phi^w(\mathcal{N}) = \mathcal{H}_\Phi^w(\mathcal{A}). \tag{5.9}$$

Lemma 5.5. *Let Φ be an Orlicz function with $1 < a_\Phi \le b_\Phi < \infty$. Let $\Phi^{(2)}(t) = \Phi(t^2)$. Then, for any $f \in \mathcal{H}_\Phi^w(\mathcal{A})$ and $\varepsilon > 0$, there exist $g, h \in \mathcal{H}_{\Phi^{(2)}}^w(\mathcal{A})$ such that $f = gh$ with*

$$\max\{\||g|^2\|_{\Phi_w(\mathcal{N})},\ \||h|^2\|_{\Phi_w(\mathcal{N})}\} \lesssim \||f|\|_{\Phi_w(\mathcal{N})} + \varepsilon.$$

Proof. We can prove this lemma by slightly modifying the proof of Lemma 4.1 in [BC]; we omit the details. ∎

Lemma 5.6. *Let Φ be an Orlicz function with $2 < a_\Phi \le b_\Phi < \infty$. Let $\{I_n = (3^n/2, 3^n] : n \in \mathbb{N}\}$ and $\triangle_n$ the Fourier multiplier by the indicator function χ_{I_n}, i.e.*

$$\triangle_n(f)(z) = \sum_{k\in I_n} \hat{f}(k) z^k$$

for any trigonometric polynomial f with coefficients in $L_\Phi^w(\mathcal{M})$. Then

$$\Big\|\Big(\sum_n \triangle_n(f)^* \triangle_n(f)\Big)^{1/2}\Big\|_{\Phi_w(\mathcal{N})} \lesssim \|f\|_{\Phi_w(\mathcal{N})}$$

for any $f \in \mathcal{H}_\Phi^w(\mathcal{N})$.

Proof. The proof is similar to the one of Lemma 4.2 in [BC] by using Corollary 4.4; the details are omitted. ∎

Now, we are ready to prove (5.7). Indeed, the proof can be obtained by using Lemmas 5.4–5.6 and is similar to the one of Theorem 4.1 in [BC]. We omit the details.

(2) In order to prove the inequality (5.6), using the argument for the proof of Lemma 4.3(2) in [BC] we have

$$\Big\| \sum_k \varepsilon_k x_k \Big\|_{\Phi_w(L_\infty(\Omega)\bar{\otimes}\mathcal{M})} \gtrsim \Big\| \Big(\sum_k |x_k|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})} + \Big\| \Big(\sum_k |x_k^*|^2 \Big)^{1/2} \Big\|_{\Phi_w(\mathcal{M})}. \tag{5.10}$$

This completes the proof. ∎

Remark 5.7.

(1) Note that the Khinchin inequality is valid for the L_1 norm in both commutative and noncommutative settings (cf. [LPP]). We could conjecture that the right condition in Theorem 5.3(1) should be $b_\Phi < 2$ without the additional restriction $1 < a_\Phi$. However, our argument seems to be inefficient in this case. We need new ideas to approach it.

(2) Evidently, the weak type Φ-moment Khinchin inequalities in Theorem 5.3 imply those for L_Φ^w norms, which, by Corollary 4.5, can be considered as particular cases of more general ones in [LP92] and in [LPX, MS].

(3) In the previous version of the paper, we claimed that the converse to the inequality (5.6) held true. Unfortunately, there was a gap in the proof as pointed out by the referee. At the time of this writing, this question remains open.

Now, we are in a position to state and prove the weak type Φ-moment version of noncommutative Burkholder–Gundy martingale inequalities.

Theorem 5.8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized normal faithful trace τ and $(\mathcal{M}_n)_{n\geq 0}$ an increasing filtration of subalgebras of $\mathcal{M}$. Let Φ be an Orlicz function and $x = (x_n)_{n\geq 0}$ a noncommutative L_Φ^w-martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$.*

(1) *If $1 < a_\Phi \leq b_\Phi < 2$, then*

$$\|x\|_{\Phi_w} \approx \inf \Big\{ \Big\| \Big(\sum_{n=0}^{\infty} |dy_n|^2 \Big)^{1/2} \Big\|_{\Phi_w} + \Big\| \Big(\sum_{n=0}^{\infty} |dz_n^*|^2 \Big)^{1/2} \Big\|_{\Phi_w} \Big\}, \tag{5.11}$$

where the infimum is taken over all decompositions $x_n = y_n + z_n$ with (dy_n) in $L_\Phi^w(\mathcal{M}, \ell_C^2)$ and $\{dz_n\}$ in $L_\Phi^w(\mathcal{M}, \ell_R^2)$ and " $\approx$ " depends only on Φ.

(2) *If $2 < a_\Phi \leq b_\Phi < \infty$, then*

$$\|x\|_{\Phi_w} \gtrsim \Big\| \Big(\sum_{n=0}^{\infty} |dx_n|^2 \Big)^{1/2} \Big\|_{\Phi_w} + \Big\| \Big(\sum_{n=0}^{\infty} |dx_n^*|^2 \Big)^{1/2} \Big\|_{\Phi_w}, \tag{5.12}$$

where " $\gtrsim$ " depends only on Φ.

Proof. The proof is similar to the one of Theorem 5.1 in [BC] through the use of Theorem 5.3; the details are omitted. ∎

REMARK 5.9.

(1) All inequalities in Theorems 5.3 and 5.8 are left open for $1 < a_\Phi \leq 2 \leq b_\Phi < \infty$, except for the case $a_\Phi = b_\Phi = 2$ in which $\Phi(t) = ct^2$ and the corresponding inequalities hold. At the time of this writing, we do not see how to formulate a meaningful statement for this case. However, our argument works in the commutative case for all cases $1 < a_\Phi \leq b_\Phi < \infty$.

(2) We expect that the converse to the inequality (5.12) holds true. This would be the case if one could prove the converse to (5.6).

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Turdebek N. Bekjan
College of Mathematics
and Systems Science
Xinjiang University
Urumqi 830046, China
E-mail: bek@xju.edu.cn

Zeqian Chen
Wuhan Institute of Physics and Mathematics
Chinese Academy of Sciences
West District 30
Xiao-Hong-Shan, Wuhan 430071, China
E-mail: zqchen@wipm.ac.cn

Peide Liu
School of Mathematics and Statistics
Wuhan University
Wuhan 430072, China
E-mail: pdliu@whu.edu.cn

Yong Jiao
Institute of Probability and Statistics
Central South University
Changsha 410075, China
E-mail: jiaoyong@mail.csu.edu.cn

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