Characterising weakly almost periodic functionals on the measure algebra

by

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Abstract. Let G be a locally compact group, and consider the weakly almost periodic functionals on $M(G)$, the measure algebra of G, denoted by $\mathrm{WAP}(M(G))$. This is a C*-subalgebra of the commutative C*-algebra $M(G)^*$, and so has character space, say K_{WAP}. In this paper, we investigate properties of K_{WAP}. We present a short proof that K_{WAP} can naturally be turned into a semigroup whose product is separately continuous; at the Banach algebra level, this product is simply the natural one induced by the Arens products. This is in complete agreement with the classical situation when G is discrete. A study of how K_{WAP} is related to G is made, and it is shown that K_{WAP} is related to the weakly almost periodic compactification of the discretisation of G. Similar results are shown for the space of almost periodic functionals on $M(G)$.

1. Introduction. In [7], we developed a theory of corepresentations on reflexive Banach spaces, and used this to show, in particular, that the space of weakly almost periodic functionals on the measure algebra $M(G)$ forms a C*-subalgebra of $M(G)^* = C_0(G)^{**}$. We write $\mathrm{WAP}(M(G))$ for this space, so we see that $\mathrm{WAP}(M(G)) = C(K_{\mathrm{WAP}})$ for some compact Hausdorff space K_{WAP}. By analogy with the discrete case, when $M(G) = \ell^1(G)$ and when $\mathrm{WAP}(M(G))$ can be identified with $\mathrm{WAP}(G)$, we would expect K_{WAP} to become a semigroup in a natural way, perhaps with continuity properties, and perhaps with some sort of universal property related to G. For more information on weakly almost periodic functionals, see [13] or [1]; a recent study of the measure algebra is [6].

In this paper, we show that K_{WAP} does indeed carry a natural semigroup product which is separately continuous. By "natural", we mean that the product is directly induced by the product on G: at the level of Banach algebras, this is simply the Arens product. We show that, formally, the passage from G to K_{WAP} is a functor between natural categories. We might

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hope, as in the discrete case, to be able to give a satisfactory description of K_{WAP} in terms of G, but without reference to specific algebras of functions. We show some general properties of K_{WAP}, but at present we fail to find such a description of K_{WAP} purely in terms of G.

As well as weakly almost periodic functionals, one can consider almost periodic functionals. We do this in the first section, which the readers may skip on a first reading, if they are interested mainly in the weakly almost periodic case. Here the functional analytic tools required are simpler, but this easier setting allows us to develop some methods without undue worry about technicalities. We also make links with some classical notions, in particular, Taylor's Structure Semigroup for $M(G)$.

In the next section, we present a short proof that the character space of the C*-algebra of weakly almost periodic functionals becomes a semigroup whose product is separately continuous. The key idea is to use a suitable space of separately continuous functions. This was communicated to us by the anonymous referee of a previous version of this paper. The motivation of the construction in [7] was to build a theory which might be applicable to the non-commutative case; recently Runde has shown in [14] that this is rather unlikely to work. Similarly, the methods of the present paper, being essentially the study of function spaces, also seem unlikely to generalise directly to the non-commutative case.

So, we have $\mathrm{WAP}(M(G)) = C(K_{\mathrm{WAP}})$ where K_{WAP} becomes a semigroup whose product is separately continuous, a *semitopological* compact semigroup. In the final section we study K_{WAP} as a semigroup, much in the theme of Section 3. Our aim here is to give some sort of "functorial" description of K_{WAP} in terms of G; this is still a work in progress. Of course, to fully understand the semigroup K_{WAP} is rather hard, even for the classical case of, say, $G = \mathbb{Z}$.

2. Commutative Hopf von Neumann algebras. We now quickly recall some notions and results from [7]. A commutative Hopf von Neumann algebra is a commutative von Neumann algebra $L^\infty(X)$ equipped with a normal *-homomorphism $\Delta : L^\infty(X) \to L^\infty(X \times X)$ which is coassociative in the sense that $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$. In classical situations Δ is induced by some map $X \times X \to X$, but it is important for our applications that we work with more generality. The preadjoint of Δ, denoted by Δ_*, induces a map $L^1(X) \widehat{\otimes} L^1(X) = L^1(X \times X) \to L^1(X)$ which is associative, turning $L^1(X)$ into a Banach algebra. Here $\widehat{\otimes}$ denotes the projective tensor product. For the basics on tensor products, see [3, Appendix A.3], [12, Section 1.10] or [17, Section 2, Chapter IV].

Our main reason for considering such objects is because, for suitable X and Δ, we have $M(G) = L^1(X)$. Let us quickly recall how to see this. Define

$\Phi : C_0(G) \to C(G \times G)$ by

$$\Phi(f)(s,t) = f(st) \quad (f \in C_0(G),\ s,t \in G).$$

Here $C(G \times G)$ is the space of bounded continuous functions on $G \times G$. We can identify $C(G \times G)$ as a subspace of the dual of $M(G) \widehat{\otimes} M(G)$ by integration,

$$\langle F, \mu \otimes \lambda \rangle = \int_{G \times G} F(s,t)\, d\mu(s)\, d\lambda(t) \quad (F \in C(G \times G),\ s,t \in G).$$

Then, as $C_0(G)^{**} = M(G)^*$ is a commutative von Neumann algebra, there exists a measure space X with $C_0(G)^{**} = L^\infty(X)$, and so $M(G) = L^1(X)$ as Banach spaces. Thus we regard Φ as a $*$-homomorphism $C_0(G) \to L^\infty(X \times X)$. There exists a unique normal coassociative $*$-homomorphism $\Delta : L^\infty(X) \to L^\infty(X \times X)$ such that $\Delta \kappa_{C_0(G)}(f) = \Phi(f)$ for $f \in C_0(G)$. Here, for a Banach space E, $\kappa_E : E \to E^{**}$ is the canonical map from E to its bidual. A check shows that the preadjoint Δ_* induces the usual convolution product on $M(G)$. See [7, Section 2.1] for further details.

As in [7], it is convenient to work with the abstraction of commutative Hopf von Neumann algebras, using $M(G)$ as our main example.

Let $\mathcal{A}$ be a Banach algebra. We turn $\mathcal{A}^*$ into an $\mathcal{A}$-bimodule in the usual way:

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \quad (a, b \in \mathcal{A},\ \mu \in \mathcal{A}^*).$$

We define $\mu \in \mathcal{A}^*$ to be *weakly almost periodic* if the map

$$R_\mu : \mathcal{A} \to \mathcal{A}^*, \quad a \mapsto a \cdot \mu \quad (a \in \mathcal{A}),$$

is weakly compact. We write $\mu \in \mathrm{WAP}(\mathcal{A})$. Similarly, if R_μ is compact, then μ is *almost periodic*, written $\mu \in \mathrm{AP}(\mathcal{A})$. It is easy to see that $\mathrm{AP}(\mathcal{A})$ and $\mathrm{WAP}(\mathcal{A})$ are closed submodules of $\mathcal{A}^*$. Here we used actions on the left, but we get the same concepts if we instead look at the map $L_\mu(a) = \mu \cdot a$ for $a \in \mathcal{A}$.

We shall use the Arens products in a few places, so we define them here. We define contractive bilinear maps $\mathcal{A}^{**} \times \mathcal{A}^*, \mathcal{A}^* \times \mathcal{A}^{**} \to \mathcal{A}^*$ by

$$\langle \Phi \cdot \mu, a \rangle = \langle \Phi, \mu \cdot a \rangle, \quad \langle \mu \cdot \Phi, a \rangle = \langle \Phi, a \cdot \mu \rangle \quad (a \in \mathcal{A},\ \mu \in \mathcal{A}^*,\ \Phi \in \mathcal{A}^{**}).$$

Then we define contractive bilinear maps $\square, \diamond : \mathcal{A}^{**} \times \mathcal{A}^{**} \to \mathcal{A}^{**}$ by

$$\langle \Phi \square \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle, \quad \langle \Phi \diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle \quad (\mu \in \mathcal{A}^*,\ \Phi, \Psi \in \mathcal{A}^{**}).$$

These are actually associative algebra products such that $a \cdot \Phi = \kappa_\mathcal{A}(a) \square \Phi = \kappa_\mathcal{A}(a) \diamond \Phi$ for $a \in \mathcal{A}, \Phi \in \mathcal{A}^{**}$, and similarly for $\Phi \cdot a$. See [12, Section 1.4] or [3, Theorem 2.6.15] for further details.

3. Almost periodic case. In this section we shall investigate further properties of $\mathrm{AP}(L^1(X))$ for a commutative Hopf von Neumann algebra

$(L^\infty(X), \Delta)$. This case is easier than the weakly almost periodic case, and will allow us to build some general theory without added complication. By [7, Theorem 1], we know that $\mathrm{AP}(L^1(X))$ is a C*-subalgebra of $L^\infty(X)$, and so $\mathrm{AP}(L^1(X)) = C(K_{\mathrm{AP}})$ for some compact Hausdorff space K_{AP}.

In the following proof, we write $\check{\otimes}$ for the *injective tensor product*, which for commutative C*-algebras agrees with the *minimal* or *spacial tensor product*; see, for example, [17, Section 4, Chapter IV].

THEOREM 3.1. *Let $L^\infty(X)$ be a commutative Hopf von Neumann algebra, and let $\mathrm{AP}(L^1(X)) = C(K_{\mathrm{AP}})$. Then Δ restricts to a map $C(K_{\mathrm{AP}}) \to C(K_{\mathrm{AP}} \times K_{\mathrm{AP}})$, and hence naturally induces a jointly continuous semigroup product on K_{AP}.*

Proof. As in the proof of [7, Theorem 1], we know that $F \in \mathrm{AP}(L^1(X))$ if and only if $\Delta(F) \in L^\infty(X) \check{\otimes} L^\infty(X)$. That $\mathrm{AP}(L^1(X))$ is an $L^1(X)$-submodule of $L^\infty(X)$ is equivalent to

$$(a \otimes \mathrm{id})\Delta(F), (\mathrm{id} \otimes a)\Delta(F) \in \mathrm{AP}(L^1(X)) \quad (a \in L^1(X),\ F \in \mathrm{AP}(L^1(X))).$$

As $L^\infty(X)$ is commutative, this is equivalent to $\Delta(F) \in \mathrm{AP}(L^1(X)) \check{\otimes} \mathrm{AP}(L^1(X))$ for $F \in \mathrm{AP}(L^1(X))$. Thus Δ restricts to give a *-homomorphism $C(K_{\mathrm{AP}}) \to C(K_{\mathrm{AP}}) \check{\otimes} C(K_{\mathrm{AP}}) = C(K_{\mathrm{AP}} \times K_{\mathrm{AP}})$. Hence there is a continuous homomorphism $K_{\mathrm{AP}} \times K_{\mathrm{AP}} \to K_{\mathrm{AP}}$, which we shall write as $(s, t) \mapsto st$, such that

$$\Delta(f)(s, t) = f(st) \quad (f \in C(K_{\mathrm{AP}}),\ s, t \in K_{\mathrm{AP}}).$$

As Δ is coassociative, it easily follows that this product on K_{AP} is associative, as required. ∎

It is almost immediate that K_{AP} can be characterised, rather abstractly, as follows. Let S be a compact semigroup, and let $\Delta_S : C(S) \to C(S \times S)$ be the canonical coproduct given by $\Delta_S(f)(s, t) = f(st)$ for $f \in C(S)$ and $s, t \in S$. Then an operator $\theta : C(S) \to L^\infty(X)$ *intertwines* the coproducts if $(\theta \otimes \theta)\Delta_S = \Delta\theta$. This is equivalent to $\theta^* : L^1(X) \to M(S)$ being a Banach algebra homomorphism. If θ is also a *-homomorphism, then we write $\theta \in \mathrm{Mor}(S, L^\infty(X))$. The following is now immediate.

PROPOSITION 3.2. *Let S be a compact semigroup, and let $\theta \in \mathrm{Mor}(S, L^\infty(X))$. Then the image of θ is contained in $\mathrm{AP}(L^1(X))$. Furthermore, $\mathrm{AP}(L^1(X))$ is the union of the images of all such θ.*

Let $\mathbb{G}_1 = (L^\infty(X_1), \Delta_1)$ and $\mathbb{G}_2 = (L^\infty(X_2), \Delta_2)$ be commutative Hopf von Neumann algebras. A *morphism* between $\mathbb{G}_1$ and $\mathbb{G}_2$ is a normal unital *-homomorphism $T : L^\infty(X_2) \to L^\infty(X_1)$ which intertwines the coproducts, that is, $(T \otimes T) \circ \Delta_2 = \Delta_1 \circ T$. Again, this is equivalent to the preadjoint $T_* : L^1(X_1) \to L^1(X_2)$ being a homomorphism of Banach algebras.

LEMMA 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then T^* maps $\mathrm{AP}(\mathcal{B}^*)$ to $\mathrm{AP}(\mathcal{A}^*)$, and maps $\mathrm{WAP}(\mathcal{B}^*)$ to $\mathrm{WAP}(\mathcal{A}^*)$.*

Proof. This is folklore, and follows by observing that for $\mu \in \mathcal{B}^*$, we have $R_{T^*(\mu)} = T^* \circ R_\mu \circ T$. ∎

Given commutative Hopf von Neumann algebras $\mathbb{G}_1$ and $\mathbb{G}_2$, let $\mathrm{AP}(L^\infty(X_i)) = C(K_{\mathrm{AP}}^{(i)})$ for $i = 1, 2$. Given a morphism T from $\mathbb{G}_1$ to $\mathbb{G}_2$, the lemma shows that $T(\mathrm{AP}(L^\infty(X_2))) \subseteq \mathrm{AP}(L^\infty(X_1))$, and so as T is a $*$-homomorphism, we get a continuous map $T_{\mathrm{AP}} : K_{\mathrm{AP}}^{(1)} \to K_{\mathrm{AP}}^{(2)}$. As T intertwines the coproducts, it follows that T_{AP} is a semigroup homomorphism.

PROPOSITION 3.4. *The assignment of K_{AP} to $(L^\infty(X), \Delta)$, and of T_{AP} to T, is a functor between the category of commutative Hopf von Neumann algebras and the category of compact topological semigroups with continuous homomorphisms.*

Proof. The only thing to check is that maps compose correctly; but this is an easy, if tedious, verification. ∎

We now specialise to the case when $L^1(X) = M(G)$ for a locally compact group G. Let G and H be locally compact groups, and let $\theta : G \to H$ be a continuous group homomorphism. As θ need not be proper, we only get an induced map $\theta_* : C_0(H) \to C(G)$ given by

$$\theta_*(f)(s) = f(\theta(s)) \quad (f \in C_0(H),\ s \in G).$$

However, we embed $C(G)$ into $M(G)^*$ as in Section 2, which gives a $*$-homomorphism $T_0 : C_0(H) \to M(G)^*$ which satisfies

$$\langle T_0(f), \mu \rangle = \int_G f(\theta(s))\, d\mu(s) \quad (f \in C_0(H),\ \mu \in M(G)).$$

We can extend T_0 by weak*-continuity to get a normal $*$-homomorphism $T : M(H)^* \to M(G)^*$, which is easily checked to be unital. For $f \in C_0(H)$ and $\mu, \lambda \in M(G)$, we have

$$
\begin{aligned}
\langle T_*(\mu) T_*(\lambda), f \rangle &= \int_{H \times H} f(gh)\, dT_*(\mu)(g)\, dT_*(\lambda)(h) \\
&= \int_H \int_G f(\theta(s)h)\, d\mu(s)\, dT_*(\lambda)(h) \\
&= \int_G \int_G f(\theta(s)\theta(t)\, d\lambda(t)\, d\mu(s) \\
&= \int_{G \times G} f(\theta(st))\, d\mu(s)\, d\lambda(t) = \langle T_*(\mu\lambda), f \rangle.
\end{aligned}
$$

Hence T_* is a Banach algebra homomorphism.

Let us take a diversion briefly, and think about the converse. That is, let G and H be locally compact groups, and let $T : M(H)^* \to M(G)^*$ be a normal unital $*$-homomorphism such that $T_* : M(G) \to M(H)$ is a homomorphism. Then $T_0 = T\kappa_{C_0(H)} : C_0(H) \to M(G)^*$ is a $*$-homomorphism. Notice that if T were given by some θ as in the previous paragraph, then T_0 would map into $C(G) \subseteq M(G)^*$. For each $g \in G$, $\delta_g \in M(G)$ is a normal character on $M(G)^*$, and so $C_0(H) \to \mathbb{C}$, $f \mapsto \langle T_0(f), \delta_g \rangle$, is a character (which cannot be zero, as T is unital). We thus get a map $\theta : G \to H$ such that $T_*(\delta_g) = \delta_{\theta(g)}$ for $g \in G$. It follows that θ is a homomorphism; and if T_0 takes values in $C(G)$, it follows that θ is continuous. We have thus shown that T is associated to a continuous group homomorphism $G \to H$ if and only if $T(C_0(H)) \subseteq C(G)$.

The Borel measurable functions form a "large" subalgebra of $M(G)^*$, but we gain nothing by considering Borel measurable homomorphisms $G \to H$, as these are automatically continuous, [16].

The following is now easily proved.

PROPOSITION 3.5. *The assignment of K_{AP} to $(M(G)^*, \Delta)$, and of T_{AP} to θ, defines a functor between the category of locally compact groups with continuous homomorphisms and compact topological semigroups with continuous homomorphisms.*

From now on, fix a locally compact group G and a compact topological semigroup K_{AP} with $C(K_{\mathrm{AP}}) = \mathrm{AP}(M(G))$. The above proposition, in the abstract, tells us that K_{AP} depends only upon G. In the remainder of this section, we study some properties of K_{AP}, with the eventual aim (not realised yet) of describing K_{AP} "directly" using G. For example, if G is discrete, then K_{AP} is nothing but the usual almost periodic compactification of G, that is, *the* group compactification of G.

Let G_d be the group G with the discrete topology. For each $s \in G$, the point mass measure $\delta_s \in M(G)$ induces a normal character on $L^\infty(X)$, and hence by restriction a character on $\mathrm{AP}(M(G)) = C(K_{\mathrm{AP}})$. Hence we get a map $\theta_0 : G_d \to K_{\mathrm{AP}}$.

PROPOSITION 3.6. *The map $\theta_0 : G_d \to K_{\mathrm{AP}}$ is a semigroup homomorphism sending the unit of G_d to the unit of K_{AP}.*

Proof. Let $f \in C(K_{\mathrm{AP}}) = \mathrm{AP}(M(G))$ so that for $s, t \in G$,

$$f(\theta_0(st)) = \langle f, \delta_{st} \rangle = \langle f, \Delta_*(\delta_s \otimes \delta_t) \rangle = \langle \Delta(f), \delta_s \otimes \delta_t \rangle = f(\theta_0(s)\theta_0(t)).$$

This is enough to show that $\theta_0(st) = \theta_0(s)\theta_0(t)$, as required. Finally, let $e \in G_d$ be the unit. Then $\delta_e \in M(G)$ is the unit of the Banach algebra $M(G)$, and so the image of δ_e in $\mathrm{AP}(M(G))^*$ is a unit. It follows that $\theta_0(e)$ is a unit for K_{AP}. ∎

Following [1, Section 4.1], for example, let $\mathrm{AP}(G_d) \subseteq \ell^\infty(G)$ be the space of almost periodic functions on G_d. Then $\mathrm{AP}(G_d)$ is a commutative C*-subalgebra of $\ell^\infty(G)$ with character space $(G_d)^{\mathrm{AP}}$, the *almost periodic compactification* of G_d. As G_d is a group, this agrees with the *strongly almost periodic* compactification, so that $(G_d)^{\mathrm{AP}}$ is a group. This follows easily by extending the inverse from G_d, and using that the product in $(G_d)^{\mathrm{AP}}$ is jointly continuous. See [1, Corollary 4.1.12] for further details, for example.

As K_{AP} is a topological semigroup, by the universal property of the almost periodic compactification, there exists a continuous semigroup homomorphism $\theta : (G_d)^{\mathrm{AP}} \to K_{\mathrm{AP}}$ making the following diagram commute:

$$
\begin{array}{ccc}
G_d & \longrightarrow & (G_d)^{\mathrm{AP}} \\
& \theta_0 \searrow & \downarrow \theta \\
& & K_{\mathrm{AP}}
\end{array}
$$

We regard $\mathrm{AP}(G_d) = C((G_d)^{\mathrm{AP}})$ as a subalgebra of $l^\infty(G) = C(G_d)$. Recall (see [3, Section 3.3] for further details) that $M(G) = M_c(G) \oplus_1 \ell^1(G)$, where $\ell^1(G)$ is identified with the discrete measures in $M(G)$, and $M_c(G)$ is the space of continuous measures. Then $M_c(G)$ is an ideal in $M(G)$, and so the projection $P : M(G) \to \ell^1(G)$ is an algebra homomorphism.

LEMMA 3.7. $P^* : \ell^\infty(G) \to M(G)^*$ *is an algebra homomorphism which maps* $\mathrm{AP}(G_d)$ *into* $\mathrm{AP}(M(G))$.

Proof. Let $\mu \in M(G)$ and $P(\mu) = a = \sum_{s\in G} a_s \delta_s \in \ell^1(G)$. The product on $M(G)^*$ is simply the Arens product on $C_0(G)^{**}$. For $f, g \in C_0(G)$, we have $\langle \mu \cdot f, g \rangle = \int_G f(s)g(s)\,d\mu(s)$ and so $\mu \cdot f = f\mu$, the pointwise product. It is hence easy to see that

$$
P(\mu \cdot f) = P(f\mu) = \sum_{s \in G} f(s) a_s = P(\mu) \cdot f.
$$

For $\Phi = (\Phi_s)_{s\in G} \in \ell^\infty(G)$, we see that

$$
\langle P^*(\Phi) \cdot \mu, f \rangle = \langle \Phi, P(\mu \cdot f) \rangle = \sum_{s\in G} \Phi_s f(s) a_s = \Big\langle \sum_s \Phi_s a_s \delta_s, f \Big\rangle.
$$

Thus for $\Psi = (\Psi_s)_{s\in G} \in \ell^\infty(G)$,

$$
\langle P^*(\Psi) P^*(\Phi), \mu \rangle = \Big\langle P^*(\Psi), \sum_s \Phi_s a_s \delta_s \Big\rangle = \sum_{s\in G} \Psi_s \Phi_s a_s = \langle \Psi\Phi, P(\mu) \rangle
$$
$$
= \langle P^*(\Psi\Phi), \mu \rangle,
$$

showing that P^* is a homomorphism, as required.

As P is a Banach algebra homomorphism $M(G) \to \ell^1(G)$, by Lemma 3.3, we find that P^* maps $\mathrm{AP}(\ell^1(G)) = \mathrm{AP}(G_d)$ into $\mathrm{AP}(M(G))$, as claimed. ■

As P is an algebra homomorphism, dualising, we see that

$$\Delta \circ P^* = (P^* \otimes P^*) \circ \Phi_d,$$

where $\Phi_d : \ell^\infty(G) \to \ell^\infty(G \times G)$ is the coproduct for G_d. As $P^* : \mathrm{AP}(G_d) \to \mathrm{AP}(M(G)) = C(K_{\mathrm{AP}})$ is a homomorphism, we get a continuous map $\theta_1 : K_{\mathrm{AP}} \to (G_d)^{\mathrm{AP}}$. As P^* intertwines the coproducts, it follows that θ_1 is a semigroup homomorphism.

LEMMA 3.8. *Consider the continuous semigroup homomorphisms $\theta :$ $(G_d)^{\mathrm{AP}} \to K_{\mathrm{AP}}$ and $\theta_1 : K_{\mathrm{AP}} \to (G_d)^{\mathrm{AP}}$. Then $\theta_1 \circ \theta$ is the identity on $(G_d)^{\mathrm{AP}}$ and so θ is a homeomorphism onto its range.*

Proof. For $s \in G$ and $F \in \mathrm{AP}(G_d)$, we calculate that

$$F(\theta_1 \theta_0(s)) = \langle \delta_{\theta_0(s)}, P^*(F) \rangle = \langle P^*(F), \delta_s \rangle = \langle F, \delta_s \rangle = F(s).$$

Hence $\theta_1 \circ \theta_0 : G \to (G_d)^{\mathrm{AP}}$ is the canonical inclusion. By continuity, it follows that $\theta_1 \circ \theta$ is the identity on $(G_d)^{\mathrm{AP}}$, and so θ must be a homeomorphism onto its range. ∎

We now prove a simple fact about semigroups: this is surely a folklore result.

LEMMA 3.9. *Let K be a semigroup, let H be a group, let $\theta : H \to K$ and $\psi : K \to H$ be semigroup homomorphisms with $\psi\theta$ the identity on H and $\theta(e_H)$ a unit for K. Let K_0 be the kernel of ψ, so $K_0 = \psi^{-1}(\{e_H\})$. Then $K = H \ltimes K_0$ as semigroups.*

Furthermore, if K is a topological semigroup and H is a topological group, and θ and ψ are continuous, then $K = H \ltimes K_0$ as topological semigroups.

Proof. Let H act on K_0 by

$$s \cdot k = \theta(s) k \theta(s^{-1}) \quad (s \in H, \, k \in K_0).$$

As $\psi(\theta(s) k \theta(s^{-1})) = s\psi(k)s^{-1} = s e_H s^{-1} = e_H$, it follows that $s \cdot k \in K_0$ as claimed. Then $H \ltimes K_0$ is the set $H \times K_0$ with the semigroup product

$$(s, k)(t, l) = (st, k(s \cdot l)) \quad (s, t \in H, \, k, l \in K_0).$$

We define a map $\phi : H \ltimes K_0 \to K$ by $\phi(s, k) = k\theta(s)$. Then

$$\phi((s, k)(t, l)) = k\theta(s)l\theta(s^{-1})\theta(st) = k\theta(s)l\theta(t) = \phi(s, k)\phi(t, l),$$

so ϕ is a semigroup homomorphism. If $\phi(s, k) = \phi(t, l)$ then $k\theta(s) = l\theta(t)$, and so $s = \psi(k\theta(s)) = \psi(l\theta(t)) = t$ and $k = k\theta(e_H) = l\theta(ts^{-1}) = l\theta(e_H) = l$. Hence ϕ is injective. A calculation shows that for $k \in K$, $k\theta(\psi(k)^{-1}) \in K_0$ and $\phi(\psi(k), k\theta(\psi(k)^{-1})) = k$, so ϕ is a bijection, as required.

When K and H are topological and θ and ψ are continuous, then K_0 is a closed subsemigroup of K. The action of H on K_0 is continuous (by joint continuity) and ϕ is continuous, as required. ∎

In our situation, we immediately see the following.

COROLLARY 3.10. *Form the maps* $\theta : (G_d)^{\mathrm{AP}} \to K_{\mathrm{AP}}$ *and* $\theta_1 : K_{\mathrm{AP}} \to (G_d)^{\mathrm{AP}}$ *as above. Let* K_0 *be the kernel of* θ_1. *Then* $K_{\mathrm{AP}} = (G_d)^{\mathrm{AP}} \ltimes K_0$.

By importing some results of [5] relating to derivations, we can show that K_0 is not trivial.

PROPOSITION 3.11. *For a non-discrete group* G, *the semigroup* K_0 *is non-trivial.*

Proof. We have the augmentation character (see [3, Definition 3.3.29])

$$\varphi : \ell^1(G) \to \mathbb{C}, \qquad \sum_{s \in G} a_s \delta_s \mapsto \sum_{s \in G} a_s,$$

which, as $M(G) = M_c(G) \oplus \ell^1(G)$, has an extension $\tilde{\varphi} : M(G) \to \mathbb{C}$ given by

$$\tilde{\varphi}(a \oplus \mu) = \varphi(a) \quad (a \in \ell^1(G), \, \mu \in M_c(G)).$$

It is shown in [5, Theorem 3.2] that if G is non-discrete, then there is a non-zero continuous point derivation at $\tilde{\varphi}$. That is, there exists a non-zero $\Phi \in M(G)^*$ with

$$\langle \Phi, \mu\lambda \rangle = \tilde{\varphi}(\mu)\langle \Phi, \lambda \rangle + \langle \Phi, \mu \rangle \tilde{\varphi}(\lambda) \quad (\mu, \lambda \in M(G)).$$

Indeed, the proof proceeds as follows. There exists a non-zero, translation invariant $\Phi \in M(G)^*$ such that $\langle \Phi, \mu \rangle = 0$ for $\mu \in \ell^1(G)$ or $\mu \in M_c(G)^2$. That Φ is a point derivation follows by a calculation.

It follows immediately that

$$\Delta(\Phi) = \Phi \otimes \tilde{\varphi} + \tilde{\varphi} \otimes \Phi,$$

so that $\Phi \in \mathrm{AP}(M(G))$. Suppose towards a contradiction that $\Phi = P^*(\Psi)$ for some $\Psi \in \mathrm{AP}(G_d)$. Then

$$\langle \Phi, a \oplus \mu \rangle = \langle \Psi, a \rangle = \langle \Phi, a \oplus 0 \rangle = 0 \quad (a \in \ell^1(G), \, \mu \in M_c(G)),$$

giving a contradiction. Hence $P^*(\mathrm{AP}(G_d)) \subsetneq \mathrm{AP}(M(G))$, and so K_{AP} is strictly larger than $(G_d)^{\mathrm{AP}}$; equivalently, K_0 is non-trivial. ∎

Suppose that G is abelian, so that K_{AP} is also abelian. By [11, Theorem 2.8], as $(C(K_{\mathrm{AP}}), \Delta)$ is a *quantum semigroup*, we know that $C(K_{\mathrm{AP}})$ admits a "Haar state", that is, there exists $\mu \in C(K_{\mathrm{AP}})^* = M(K_{\mathrm{AP}})$ with

$$(\mu \otimes \mathrm{id})\Delta(F) = (\mathrm{id} \otimes \mu)\Delta(F) = \langle \mu, F \rangle 1 \quad (F \in C(K_{\mathrm{AP}})).$$

For $t \in K$, by applying δ_t, we see that

$$\int_K F(st) \, d\mu(s) = \int_K F(ts) \, d\mu(s) = \int_K F(s) \, d\mu(s) \quad (F \in C(K_{\mathrm{AP}})).$$

Let λ be the image of μ under θ_1, so that

$$\langle \lambda, f \rangle = \int_K f(\theta_1(s)) \, d\lambda(s) \quad (f \in C((G_d)^{\mathrm{AP}}) = \mathrm{AP}(G_d)).$$

A simple calculation shows that λ is the Haar measure on $(G_d)^{\mathrm{AP}}$.

As shown after [11, Theorem 2.8], it is not true that $C(L)$ always carries an invariant probability measure, for a compact semigroup L. It would be interesting to know if $C(K_{\mathrm{AP}}) = \mathrm{AP}(M(G))$ always carries an invariant probability measure.

3.1. Structure semigroup. Let $\mathrm{P}(M(G))$ be the closure of the collection of $F \in M(G)^*$ such that $\Delta(F)$ is a (finite-rank) tensor in $M(G)^* \otimes M(G)^*$. This is easily seen to be a C*-subalgebra of $M(G)^*$, and an $M(G)$-submodule of $M(G)^*$. Repeating the argument of Theorem 3.1 shows that $\mathrm{P}(M(G)) = C(K_{\mathrm{P}})$ for some topological semigroup K_{P}.

Taylor introduced the *structure semigroup* of G in [18] (we remark that this is usually only considered for abelian groups G). We shall follow the presentation of [6] instead, and define $\Phi = \Phi_{M(G)}$ to be the character space of $M(G)$. In our language, $F \in \Phi_{M(G)} \subseteq M(G)^*$ if and only if $\Delta(F) = F \otimes F$. Let X_G be the closed linear span of Φ in $M(G)^*$. Then $X_G \subseteq \mathrm{P}(M(G))$, and again it can be shown that X_G is a C*-subalgebra of $M(G)^*$, and an $M(G)$-submodule of $M(G)^*$. Then the structure semigroup of G, written $S(G)$, is the spectrum of X_G, which is again a topological semigroup.

It is asked in [6] (in the abelian case) whether $S(G) = K_{\mathrm{AP}}$. We can split this into two questions. Firstly, if G is abelian, does it follow that $S(G) = K_{\mathrm{P}}$, or equivalently, that $X_G = \mathrm{P}(M(G))$? This is true for a discrete group G, essentially because of the Peter–Weyl theorem, and Fourier analysis, applied to the compact abelian group G^{AP}.

Secondly, for a general G, do we have $K_{\mathrm{P}} = K_{\mathrm{AP}}$, or equivalently, $\mathrm{P}(M(G)) = \mathrm{AP}(M(G))$? For this question, consider $F \in \mathrm{AP}(M(G))$. Then by definition, $\Delta(F) : L^1(X) \to L^\infty(X)$ is compact. As $L^\infty(X)$ has the approximation property, it follows that there is a sequence (T_n) of finite-rank maps $L^1(X) \to L^\infty(X)$ such that $T_n \to \Delta(F)$. Then $K_{\mathrm{P}} = K_{\mathrm{AP}}$ if and only if we can always choose the T_n to be of the form $\Delta(F_n)$ (so that $\Delta(F_n)$ is finite-rank, that is, $F_n \in P(M(G))$). Again, in the discrete case, this follows from the Peter–Weyl theorem.

3.2. The antipode. Let $(L^\infty(X), \Delta)$ be a commutative Hopf von Neumann algebra. We shall call a normal *-homomorphism $R : L^\infty(X) \to L^\infty(X)$ an *antipode* if $R^2 = \mathrm{id}$ and $\Delta R = (R \otimes R)\chi\Delta$, where $\chi : L^\infty(X \times X) \to L^\infty(X \times X)$ is the swap map, $\chi(F)(s,t) = F(t,s)$, for $F \in L^\infty(X \times X)$ and $s, t \in X$.

For example, consider $(C_0(G), \Phi)$ for a locally compact group G. Then we define $r : C_0(G) \to C_0(G)$ by $r(f)(s) = f(s^{-1})$ for $f \in C_0(G)$ and $s \in G$. Then r is an antipode, if we extend the definition to C*-algebras in the obvious way. Let $(L^\infty(X), \Delta)$ be induced by $(C_0(G), \Phi)$ as before, so that $L^1(X) = M(G)$. Define $R_* : L^1(X) \to L^1(X)$ to be the map r^*, and let $R = R_*^*$. Then R is a normal *-homomorphism, and $R^2 = \mathrm{id}$. For

$a, b \in M(G)$ and $f \in C_0(G)$, we see that, as Δ_* induces the usual convolution product on $M(G)$,

$$\langle r^* \Delta_*(a \otimes b), f \rangle = \langle a \otimes b, \Delta r(f) \rangle = \langle b \otimes a, (r \otimes r)\Delta(f) \rangle$$
$$= \langle \Delta_* \chi(r^* \otimes r^*)(a \otimes b), f \rangle.$$

Hence $R_* \Delta_* = \Delta_* \chi(R_* \otimes R_*)$. So, for $F \in L^\infty(X)$ and $a, b \in L^1(X) = M(G)$, we see that

$$\langle \Delta R(F), a \otimes b \rangle = \langle F, R_* \Delta_*(a \otimes b) \rangle = \langle F, \Delta_* \chi(R_* \otimes R_*)(a \otimes b) \rangle$$
$$= \langle (R \otimes R)\chi\Delta(F), a \otimes b \rangle.$$

Hence R is an antipode on $(L^\infty(X), \Delta)$.

LEMMA 3.12. *Let $(L^\infty(X), \Delta)$ be a commutative Hopf von Neumann algebra, equipped with an anitpode R. Then R restricts to give $*$-homomorphisms on $\mathrm{AP}(L^1(X))$ and $\mathrm{WAP}(L^1(X))$.*

Proof. We know that $F \in \mathrm{AP}(L^1(X))$ if and only if $\Delta(F) \in L^\infty(X) \check{\otimes} L^\infty(X)$. Hence, for $F \in \mathrm{AP}(L^1(X))$, we see that $\Delta R(F) = (R \otimes R)\chi\Delta(F) \in L^\infty(X) \check{\otimes} L^\infty(X)$, and so $R(F) \in \mathrm{AP}(L^1(X))$, as required.

Now suppose that $F \in \mathrm{WAP}(L^1(X))$, so $\Delta(F) : L^1(X) \to L^\infty(X)$ is weakly compact. Then, for $a, b \in L^1(X)$,

$$\langle \Delta R(F)(a), b \rangle = \langle (R \otimes R)\chi\Delta(F), a \otimes b \rangle = \langle \Delta(F), R_*(b) \otimes R_*(a) \rangle$$
$$= \langle R\Delta(F)^* \kappa_{L^1(X)} R_*(a), b \rangle.$$

Thus $\Delta R(F) = R\Delta(F)^* \kappa_{L^1(X)} R_*$, which is weakly compact if $\Delta(F)$ is, as required. ∎

Hence R induces an *involution* on K_{AP}, written $s \mapsto s'$. This means that $(st)' = t's'$ for $s, t \in K$, and $R(F)(s) = F(s')$ for $F \in \mathrm{AP}(M(G)) = C(K_{\mathrm{AP}})$ and $s \in K_{\mathrm{AP}}$. There is no reason to expect this to be an inverse map on K_{AP}, but we do have the following.

PROPOSITION 3.13. *Consider the map $\theta : (G_d)^{\mathrm{AP}} \to K_{\mathrm{AP}}$ as above, and recall that $(G_d)^{\mathrm{AP}}$ is a (compact) group. Then $\theta(s^{-1}) = \theta(s)'$ for $s \in (G_d)^{\mathrm{AP}}$.*

Proof. Recall that, because of joint continuity, the inverse in $(G_d)^{\mathrm{AP}}$ has the following property. Let $s \in (G_d)^{\mathrm{AP}}$, so we can find a net (s_α) in G_d which converges to s. By possibly moving to a subnet, we have $s^{-1} = \lim_\alpha s_\alpha^{-1}$. Now let $F \in \mathrm{AP}(M(G))$, so that

$$F(\theta(s)') = R(F)(\theta(s)) = \lim_\alpha R(F)(\theta_0(s_\alpha)) = \lim_\alpha \langle R(F), \delta_{s_\alpha} \rangle$$
$$= \lim_\alpha \langle F, r^*(\delta_{s_\alpha}) \rangle = \lim_\alpha \langle F, \delta_{s_\alpha^{-1}} \rangle = \lim_\alpha F(\theta_0(s_\alpha^{-1})) = F(\theta(s^{-1})),$$

as required. ∎

We have hence demonstrated various properties of the compact semi-group K_{AP}. These do not, however, appear to be enough to characterise K_{AP} directly, just in terms of G.

4. Weakly almost periodic functionals. For a commutative Hopf von Neumann algebra $(L^\infty(X), \Delta)$, we know that $\mathrm{WAP}(L^1(X))$ is a unital commutative C*-algebra, say $C(K_{\mathrm{WAP}})$. In this section, we shall show that K_{WAP} is a compact *semitopological semigroup*, that is, a semigroup whose product is separately continuous. This is in complete agreement with what happens for $L^1(G)$ (see [19] and [1, Section 4.2]).

4.1. Embedding spaces of separately continuous functions. Let $L^\infty(X)$ be a commutative von Neumann algebra, and let $\Delta : L^\infty(X) \to L^\infty(X \times X)$ be a coassociative normal *-homomorphism, turning $L^1(X)$ into a Banach algebra. We can find a compact, Hausdorff, hyperstonian space K such that $L^\infty(X) = C(K)$ (see, for example, [17, Section 1, Chapter III]). Notice, however, that $L^\infty(X \times X)$ is, in general, much larger than $C(K \times K)$.

Let $SC(K \times K)$ be the space of functions $K \times K \to \mathbb{C}$ which are separately continuous. Obviously $SC(K \times K)$ is a C*-algebra. For $f \in SC(K \times K)$ and $\mu \in M(K)$, define functions $(\mu \otimes \iota)f, (\iota \otimes \mu)f : K \to \mathbb{C}$ by

$$(\mu \otimes \iota)f(k) = \int_K f(l, k)\, d\mu(l), \quad (\iota \otimes \mu)f(k) = \int_K f(k, l)\, d\mu(l) \quad (k \in K).$$

It is shown in [15, Lemma 2.2] (using a result of Grothendieck) that actually $(\mu \otimes \iota)f$ and $(\iota \otimes \mu)f$ are in $C(K)$.

Then [15, Lemma 2.4] shows that

$$\langle (\mu \otimes \iota)f, \lambda \rangle = \langle (\iota \otimes \lambda)f, \mu \rangle \quad (f \in SC(K \times K),\, \mu, \lambda \in M(K)).$$

We write $\langle \mu \otimes \lambda, f \rangle$ for this. Furthermore, [15, Lemma 2.4] shows that the map $M(K) \times M(K) \to \mathbb{C}, (\mu, \lambda) \mapsto \langle \mu \otimes \lambda, f \rangle$, is separately weak*-continuous in each variable. These results rely upon [9], which shows that each $f \in SC(K \times K)$ is μ-measurable for any $\mu \in M(K \times K)$.

For $a \in L^1(X)$, we have $\kappa_{L^1(X)}(a) \in L^\infty(X)^* = C(K)^*$ and hence induces a measure, say $\mu_a \in M(K)$. Define a map $\theta_{sc} : SC(K \times K) \to L^\infty(X \times X) = (L^1(X) \widehat{\otimes} L^1(X))^*$ by

$$\langle \theta_{sc}(f), a \otimes b \rangle = \langle \mu_a \otimes \mu_b, f \rangle \quad (f \in SC(K \times K),\, a, b \in L^1(X)).$$

PROPOSITION 4.1. *The map θ_{sc} is an isometric *-homomorphism.*

Proof. Clearly θ_{sc} is a contraction. For $k \in K$, let $\delta_k \in M(K) = L^1(X)^{**}$ be the point-mass at k, so that δ_k is the weak*-limit of norm-one elements of $L^1(X)$, say $a_\alpha^{(k)} \to \delta_k$. By separate weak*-continuity, for $f \in SC(K \times K)$

and $k, l \in K$,

$$f(k, l) = \lim_\alpha \lim_\beta \langle \mu_{a_\alpha^{(k)}} \otimes \mu_{a_\alpha^{(l)}}, f \rangle = \lim_\alpha \lim_\beta \langle \theta_{sc}(f), a_\alpha^{(k)} \otimes a_\alpha^{(l)} \rangle.$$

By taking the supremum over all k and l, this shows that θ_{sc} is an isometry.

To show that θ_{sc} is a $*$-homomorphism, we argue as follows. Let $f \in SC(K \times K)$ and $x, y \in L^\infty(X) = C(K)$, and set $g = x \otimes y$. Such g are linearly dense in $C(K \times K)$ and hence separate the points of $M(K) \widehat{\otimes} M(K)$. We also regard g as a member of $L^\infty(X \times X)$. Let $a, b \in L^1(X)$ and consider $\theta_{sc}(f)(a \otimes b)$, defined as usual by

$$\langle F, \theta_{sc}(f)(a \otimes b) \rangle = \langle F\theta_{sc}(f), a \otimes b \rangle \qquad (F \in L^\infty(X \times X)).$$

Then $\theta_{sc}(f)(a \otimes b) \in L^1(X) \widehat{\otimes} L^1(X)$, so we can find sequences $(c_n), (d_n)$ in $L^1(X)$ with $\theta_{sc}(f)(a \otimes b) = \sum_n c_n \otimes d_n$ and $\sum_n \|c_n\| \|d_n\| < \infty$. Then

$$\sum_n \langle x, c_n \rangle \langle y, d_n \rangle = \langle g, \theta_{sc}(f)(a \otimes b) \rangle = \langle \theta_{sc}(f), (a \otimes b)g \rangle$$

$$= \langle \theta_{sc}(f), xa \otimes yb \rangle = \langle \mu_{xa} \otimes \mu_{yb}, f \rangle.$$

However, it is easy to see that $\mu_{xa} = x\mu_a$, so

$$\sum_n \langle \mu_{c_n}, x \rangle \langle \mu_{d_n}, y \rangle = \int_{K \times K} x(k) y(l) f(k, l) \, d\mu_a(k) \, d\mu_b(l).$$

As $x, y \in C(K)$ were arbitrary, we conclude that

$$\sum_n \mu_{c_n} \otimes \mu_{d_n} = f(\mu_a \otimes \mu_b)$$

as measures on $K \times K$. Then, for $h \in SC(K \times K)$,

$$\langle \theta_{sc}(h)\theta_{sc}(f), a \otimes b \rangle = \langle \theta_{sc}(h), \theta_{sc}(f)(a \otimes b) \rangle = \sum_n \langle \theta_{sc}(h), c_n \otimes d_n \rangle$$

$$= \sum_n \langle \mu_{c_n} \otimes \mu_{d_n}, h \rangle = \langle f\mu_a \otimes \mu_b, h \rangle = \langle \mu_a \otimes \mu_b, hf \rangle$$

$$= \langle \theta_{sc}(hf), a \otimes b \rangle.$$

As $a, b \in L^1(X)$ were arbitrary, this shows that θ_{sc} is a homomorphism. A similar argument establishes that θ_{sc} is a $*$-homomorphism. ∎

We henceforth identify $SC(K \times K)$ with a $*$-subalgebra of $L^\infty(X \times X)$.

4.2. Application to WAP functionals. For a Banach algebra $\mathcal{A}$ we write WAP$(\mathcal{A})$ for the space of weakly almost periodic functionals on $\mathcal{A}$, which is a closed $\mathcal{A}$-submodule of $\mathcal{A}^*$. As shown in [10, Lemma 1.4] (for commutative algebras; the proof readily extends to the general case, cf. [8, Proposition 2.4]) the Arens products drop to a well-defined product on WAP$(\mathcal{A})^*$ which is separately weak*-continuous; that is, WAP$(\mathcal{A})^*$ is a dual Banach algebra.

Proposition 4.2. *Let $(L^\infty(X), \Delta)$ be a commutative Hopf von Neumann algebra, and let $L^\infty(X) = C(K)$ as before. For $F \in L^\infty(X)$, define $f : K \times K \to \mathbb{C}$ by*

$$f(k, l) = \langle \delta_k \,\square\, \delta_l, F \rangle \qquad (k, l \in K).$$

If $F \in \mathrm{WAP}(L^1(X))$ then $f \in SC(K \times K)$ and $\theta_{sc}(f) = \Delta(F)$. Conversely, if $\Delta(F) = \theta_{sc}(g)$ for some $g \in SC(K \times K)$, then $F \in \mathrm{WAP}(L^1(X))$, and $f = g$.

Proof. Suppose that $F \in \mathrm{WAP}(L^1(X))$. As $K \to L^\infty(X)^* = M(K)$, $k \mapsto \delta_k$ is continuous for the weak*-topology, and the product on $\mathrm{WAP}(L^1(X))^*$ is separately weak*-continuous, it follows that $f \in SC(K \times K)$. We claim that $\theta_{sc}(f) = \Delta(F)$. Indeed, let $a \in L^1(X)$, and observe that for $x \in L^\infty(X) = C(K)$,

$$\langle x, a \rangle = \langle \mu_a, x \rangle = \int_K \langle \delta_k, x \rangle \, d\mu_a(k).$$

Thus, for $a, b \in L^1(X)$, and using that $\mathrm{WAP}(\mathcal{A})$ is a $\mathrm{WAP}(\mathcal{A})^*$-bimodule,

$$
\begin{aligned}
\langle \theta_{sc}(f), a \otimes b \rangle &= \int_{K \times K} \langle \delta_k \,\square\, \delta_l, F \rangle \, d\mu_a(k) \, d\mu_b(l) \\
&= \int_K \int_K \langle \delta_k, \delta_l \cdot F \rangle \, d\mu_a(k) \, d\mu_b(l) \\
&= \int_K \langle \delta_l \cdot F, a \rangle \, d\mu_b(l) = \int_K \langle \delta_l, F \cdot a \rangle \, d\mu_b(l) \\
&= \langle F \cdot a, b \rangle = \langle \Delta(F), a \otimes b \rangle,
\end{aligned}
$$

as required.

Conversely, if $F \in L^\infty(X)$ with $\Delta(F) = \theta_{sc}(g)$ for some $g \in SC(K \times K)$, then for $a, b \in L^1(X)$,

$$\langle (a \otimes \iota)\Delta(F), b \rangle = \langle \theta_{sc}(g), a \otimes b \rangle = \langle \mu_a \otimes \mu_b, g \rangle = \langle \mu_b, (\mu_a \otimes \iota)g \rangle,$$

so that $(a \otimes \iota)\Delta(F) = (\mu_a \otimes \iota)g \in C(K) = L^\infty(X)$. Thus, for $\mu \in L^\infty(X)^* = M(K)$,

$$\langle \mu, (a \otimes \iota)\Delta(F) \rangle = \langle \mu, (\mu_a \otimes \iota)g \rangle = \langle \mu_a, (\iota \otimes \mu)g \rangle.$$

Now let (a_α) be a bounded net in $L^1(X)$. By moving to a subnet, we may suppose that $\mu_{a_\alpha} \to \lambda \in M(K)$ weak*. For $\mu \in M(K)$, we have

$$\lim_\alpha \langle \mu, (a_\alpha \otimes \iota)\Delta(F) \rangle = \lim_\alpha \langle \mu_{a_\alpha}, (\iota \otimes \mu)g \rangle = \langle \lambda, (\iota \otimes \mu)g \rangle = \langle \mu, (\lambda \otimes \iota)g \rangle,$$

so we see that $(a_\alpha \otimes \iota)\Delta(F) \to (\lambda \otimes \iota)g$ weakly. Thus $F \in \mathrm{WAP}(L^1(X))$. As θ_{sc} is injective, it follows that $f = g$. ∎

It is worth making a link with Theorem 3.1. Firstly, note that a simple check shows that θ_{sc} extends the natural embedding of $C(K \times K) = L^\infty(X) \,\check{\otimes}\, L^\infty(X)$ into $L^\infty(X \times X)$. If $F \in \mathrm{AP}(L^1(X))$, then as the product on

$AP(L^1(X))^*$ is jointly continuous, it follows that f will also be jointly continuous, so that $f \in C(K \times K) \subseteq SC(K \times K)$. Conversely, if $\Delta(F) = \theta_{sc}(g)$ for some $g \in C(K \times K)$, then $f = g$ and by the same weak*-approximation argument as used above, it follows that F is almost periodic.

The following is now immediate!

THEOREM 4.3. *Let $(L^\infty(X), \Delta)$ be a commutative Hopf von Neumann algebra. Then* $\mathrm{WAP}(L^1(X))$ *is a* C*-*algebra.*

Proof. The previous proposition shows that $F \in \mathrm{WAP}(L^1(X))$ if and only if $\Delta(F) \in \theta_{sc}(SC(K \times K))$. As Δ is a *-homomorphism, and $\theta_{sc}(SC(K \times K))$ is a C*-subalgebra of $L^\infty(X)$, it follows that $\mathrm{WAP}(L^1(X))$ is a C*-algebra. ∎

However, we can now easily prove more about the structure of $\mathrm{WAP}(L^1(X))$.

THEOREM 4.4. *Let $(L^\infty(X), \Delta)$ be a commutative Hopf von Neumann algebra, and let K_{WAP} be the character space of* $\mathrm{WAP}(L^1(X))$. *The map Δ, which restricts to a map* $\mathrm{WAP}(L^1(X)) \to \theta_{sc}(SC(K \times K))$, *induces a *-homomorphism*

$$\Delta_{\mathrm{WAP}} : C(K_{\mathrm{WAP}}) \to SC(K_{\mathrm{WAP}} \times K_{\mathrm{WAP}}).$$

The adjoint $\Delta_{\mathrm{WAP}}^* : M(K_{\mathrm{WAP}}) \widehat{\otimes} M(K_{\mathrm{WAP}}) \to M(K_{\mathrm{WAP}})$ is just the product on $\mathrm{WAP}(L^1(X))^*$. *Furthermore, Δ_{WAP} induces a separately continuous (that is, semitopological) semigroup product on K_{WAP}. At the level of Banach algebras, this product "is" the Arens product.*

Proof. Let $F \in \mathrm{WAP}(L^1(X))$, and let $f \in SC(K \times K)$ with $\theta_{sc}(f) = \Delta(F)$. Then, for $k, l \in K$,

$$f(k, l) = \langle \delta_k \,\square\, \delta_l, F \rangle = \langle \delta_k, (\iota \otimes \delta_l)f \rangle = \langle \delta_l, (\delta_k \otimes \iota)f \rangle.$$

So, with reference to the proof above, $(\delta_k \otimes \iota)f = F \cdot \delta_k \in \mathrm{WAP}(L^1(X))$ and $(\iota \otimes \delta_k)f = \delta_k \cdot F \in \mathrm{WAP}(L^1(X))$.

Hence $(\delta_k \otimes \iota)f, (\iota \otimes \delta_k)f$ are members of $C(K_{\mathrm{WAP}})$ for each $k \in K$. The inclusion $\mathrm{WAP}(L^1(X)) = C(K_{\mathrm{WAP}}) \to C(K)$ induces a continuous surjection $j : K \to K_{\mathrm{WAP}}$. We claim that we can define $f_0 \in SC(K_{\mathrm{WAP}} \times K_{\mathrm{WAP}})$ by

$$f_0(j(k), j(l)) = f(k, l) \quad (k, l \in K).$$

Indeed, this is well-defined, for if $j(k) = j(k')$ and $j(l) = j(l')$ then

$$f(k, l) = \langle \delta_l, (\delta_k \otimes \iota)f \rangle = \langle \delta_{l'}, (\delta_k \otimes \iota)f \rangle = f(k, l') = \langle \delta_k, (\iota \otimes \delta_{l'})f \rangle$$
$$= \langle \delta_{k'}, (\iota \otimes \delta_{l'})f \rangle = f(k', l').$$

That f_0 is separately continuous is immediate, as the same is true of f, and j is a closed map.

Denote f_0 by $\Delta_{\mathrm{WAP}}(F)$, so that Δ_{WAP} is a linear map $C(K_{\mathrm{WAP}}) \to SC(K_{\mathrm{WAP}} \times K_{\mathrm{WAP}})$. The map $f \mapsto f_0$ is clearly a *-homomorphism, and as

$f = \theta_{sc}^{-1} \Delta(F)$, it follows that Δ_{WAP} is also a $*$-homomorphism. So we have

$$\Delta_{\text{WAP}} : C(K_{\text{WAP}}) \to SC(K_{\text{WAP}} \times K_{\text{WAP}}),$$

a $*$-homomorphism. The adjoint Δ_{WAP}^* induces a map $M(K_{\text{WAP}}) \widehat{\otimes} M(K_{\text{WAP}}) \to M(K_{\text{WAP}})$, and this is simply the Arens product on $\text{WAP}(L^1(X))^* = M(K_{\text{WAP}})$; in this sense, we could say that Δ_{WAP} is coassociative. In particular, for $k, l \in K_{\text{WAP}}$, we see that $\delta_k \square \delta_l = \Delta_{\text{WAP}}^*(\delta_k \otimes \delta_l)$ is a character on $C(K_{\text{WAP}})$, and hence is identified with a point in K_{WAP}. So K_{WAP} carries a product, and it is easy to see that this is associative. As the product on $M(K_{\text{WAP}})$ is separately continuous, the semigroup product is separately continuous. ∎

The previous result is, to the author, still surprising, for the following reason. The fact that $L^\infty(X) = C(K)$ seems, naively, to be of little use, as the coproduct Δ is absolutely not (in general) associated with any product on K (indeed, [6, Section 8] shows that the (first) Arens product never induces a product on K, unless G is discrete). Hence, one might expect not to get far working with function spaces; nevertheless, this is exactly the approach which has worked above.

We now explore the weakly almost periodic version of Proposition 3.2. Let S be a compact, semitopological semigroup, and let $\Delta_S : C(S) \to SC(S \times S)$ be the canonical coproduct, given by $\Delta_S(f)(s,t) = f(st)$ for $f \in C(S)$ and $s, t \in S$. Now let $\theta : C(S) \to L^\infty(X) = C(K)$ be a unital $*$-homomorphism, so we have an induced continuous map $\theta_* : K \to S$. Then $\theta \otimes \theta : SC(S \times S) \to SC(K \times K)$ is defined to be the map $(\theta \otimes \theta)f(k,l) = f(\theta_*(k), \theta_*(l))$ for $f \in SC(S \times S)$ and $k, l \in K$. We can now say that θ *intertwines* the coproducts if $\Delta\theta = \theta_{sc}(\theta \otimes \theta)\Delta_S$. Again, this is equivalent to $\theta^* : L^1(X) \to M(S)$ being a Banach algebra homomorphism, and we write $\theta \in \text{Mor}(S, L^\infty(X))$ in this case.

PROPOSITION 4.5. *Let S be a compact semitopological semigroup, and let $\theta \in \text{Mor}(S, L^\infty(X))$. Then the image of θ is contained in $\text{WAP}(L^1(X))$. Furthermore, $\text{WAP}(L^1(X))$ is the union of the images of all such θ. In particular, K_{WAP} is the largest quotient of K which is a semitopological semigroup with the product induced by Δ.*

Proof. By Proposition 4.2, and the definition of $\theta \otimes \theta$, it is immediate that θ maps into $\text{WAP}(L^1(X))$. Taking $S = K_{\text{WAP}}$ and θ to be the inclusion, we see that $\text{WAP}(L^1(X))$ arises as the image of θ. ∎

Let $\mathbb{G}_1 = (L^\infty(X_1), \Delta_1)$ and $\mathbb{G}_2 = (L^\infty(X_2), \Delta_2)$ be commutative Hopf von Neumann algebras, and let $T : \mathbb{G}_1 \to \mathbb{G}_2$ be a morphism. For $i = 1, 2$ let $\text{WAP}(L^1(X_i)) = C(K_{\text{WAP}}^{(i)})$, so that $K_{\text{WAP}}^{(i)}$ is a compact semitopological semigroup. By Lemma 3.3, T maps $\text{WAP}(L^1(X_2)) = C(K_{\text{WAP}}^{(2)})$ to

$\mathrm{WAP}(L^1(X_1)) = C(K_{\mathrm{WAP}}^{(1)})$ and is a $*$-homomorphism, and so induces a continuous map $T_{\mathrm{WAP}} : K_{\mathrm{WAP}}^{(1)} \to K_{\mathrm{WAP}}^{(2)}$.

PROPOSITION 4.6. *The assignment of K_{WAP} to $(L^\infty(X), \Delta)$, and of T_{WAP} to T, defines a functor between the category of commutative Hopf von Neumann algebras and the category of compact semitopological semigroups with continuous homomorphisms.*

Proof. We first show that T_{WAP} is indeed a homomorphism. With reference to the proof of Theorem 4.4, for $s, t \in K_{\mathrm{WAP}}^{(1)}$, we have $\delta_{st} = \delta_s \,\square\, \delta_t$. As $T^* = T_*^{**}$, it is easy to see that $T^* : \mathrm{WAP}(L^1(X_1))^* \to \mathrm{WAP}(L^1(X_2))^*$ is a homomorphism, so that $\delta_{T_{\mathrm{WAP}}(st)} = T^*(\delta_{st}) = T^*(\delta_s \square \delta_t) = T^*(\delta_s) \square T^*(\delta_t) = \delta_{T_{\mathrm{WAP}}(s) T_{\mathrm{WAP}}(t)}$, which shows that T_{WAP} is a homomorphism.

It is now easy, though tedious, to check that we have defined a functor. ∎

5. For the measure algebra. Let G be a locally compact group, and consider $M(G) = L^1(X)$ as the predual of a commutative Hopf von Neumann algebra. By applying the results of the previous sections, we see that $\mathrm{WAP}(M(G)) = C(K_{\mathrm{WAP}})$ for some compact Hausdorff space K_{WAP}, and that K_{WAP} becomes a semitopological semigroup in a canonical fashion. The following now follows in exactly the same way as Proposition 3.5 (where, again, given a continuous group homomorphism θ, we define the associated morphism T and thus get T_{WAP} as above).

PROPOSITION 5.1. *The assignment of K_{WAP} to G, and of T_{WAP} to θ, is a functor between the category of locally compact spaces with continuous homomorphisms and compact semitopological semigroups with continuous homomorphisms.*

As $M(G)$ is a dual Banach algebra with predual $C_0(G)$, we deduce that $C_0(G) \subseteq \mathrm{WAP}(M(G))$; see [8, Section 2] and references therein. Clearly $1 \in \mathrm{WAP}(M(G))$. So the inclusion $\iota : C_0(G) \to C(K_{\mathrm{WAP}})$ factors through

$$C_0(G) \to C(G_\infty) \xrightarrow{\iota^\infty} C(K_{\mathrm{WAP}}),$$

where G_∞ is the one-point compactification of G (if G is already compact, we define $G_\infty = G$). We can turn G_∞ into a semigroup by letting the added point ∞ be a *semigroup zero*, so $s\infty = \infty s = \infty$ for $s \in G$. Then G_∞ is semitopological, for if $s_\alpha \to \infty$, then by definition, for each compact set $K \subseteq G$, there exists α_0 with $s_\alpha \notin K$ for $\alpha \geq \alpha_0$. So for $t \in G$, as $s_\alpha t \in K$ if and only if $s_\alpha \in Kt^{-1}$, and Kt^{-1} is compact, it follows that $s_\alpha t \to \infty$. Similarly $ts_\alpha \to \infty$.

Thus we have an induced continuous map $\iota_*^\infty : K_{\mathrm{WAP}} \to G_\infty$, which has dense and closed range, and hence must be surjective.

PROPOSITION 5.2. *The map $\iota_*^\infty : K_{\mathrm{WAP}} \to G_\infty$ is a homomorphism.*

Proof. Let $\kappa : C(G_\infty) \to M(G)^*$ be the canonical map, and let K be the compact space such that $M(G)^* = C(K)$. Hence $\kappa : C(G_\infty) \to C(K)$ is an injective $*$-homomorphism, and so there exists a continuous surjection $\phi : K \to G_\infty$. Notice then that we have the commutative diagrams

$$
\begin{array}{ccc}
C(G_\infty) \xrightarrow{\iota^\infty} C(K_{\mathrm{WAP}}) & \quad & G_\infty \xleftarrow{\iota_*^\infty} K_{\mathrm{WAP}} \\
\searrow{\kappa} \quad \downarrow & & \nwarrow{\phi} \quad \uparrow{j} \\
C(K) & & K
\end{array}
$$

Here $j : K \to K_{\mathrm{WAP}}$ is as in (the proof of) Theorem 4.4 above.

Let $f \in C(G_\infty)$, and define $\alpha : K \times K \to \mathbb{C}$ by

$$\alpha(k, l) = f(\phi(k)\phi(l)) \quad (k, l \in K).$$

Thus $\alpha \in SC(K \times K)$ as G_∞ is semitopological. For $l \in K$, let $f_l \in C(G_\infty)$ be defined by $f_l(s) = f(s\phi(l))$ for $s \in G_\infty$. For $a, b \in M(G)$, we see that

$$
\begin{aligned}
\langle \theta_{sc}(\alpha), a \otimes b \rangle = \langle \mu_a \otimes \mu_b, \alpha \rangle &= \int_K \int_K f(\phi(k)\phi(l)) \, d\mu_a(k) \, d\mu_b(l) \\
&= \int_K \langle \kappa(f_l, a) \, d\mu_b(l) = \int_G \int_K f_l(s) \, d\mu_b(l) \, da(s) \\
&= \int_G \int_K f(s\phi(l)) \, d\mu_b(l) \, da(s) = \int_G \int_G f(st) \, db(t) \, da(s),
\end{aligned}
$$

where the final equality comes from repeating the argument. Thus

$$\langle \theta_{sc}(\alpha), a \otimes b \rangle = \langle \kappa(f), ab \rangle = \langle \Delta(\kappa(f)), a \otimes b \rangle.$$

We conclude that $\theta_{sc}(\alpha) = \Delta(\kappa(f))$.

Now observe that $\Delta_{\mathrm{WAP}}(\iota_\infty(f))$ is the map $K_{\mathrm{WAP}} \times K_{\mathrm{WAP}} \to \mathbb{C}$ given by

$$(j(k), j(l)) \mapsto \theta_{sc}^{-1}\Delta(\kappa(f))(k, l) = \alpha(k, l) = f(\phi(k)\phi(l)) \quad (k, l \in K).$$

Let $s, t \in K_{\mathrm{WAP}}$, and pick $k, l \in K$ with $j(k) = s$ and $j(l) = t$. Then, as $\iota_*^\infty j = \phi$, we have

$$
\begin{aligned}
f(\iota_*^\infty(s)\iota_*^\infty(t)) = f(\phi(k)\phi(l)) &= \Delta_{\mathrm{WAP}}(\iota_\infty(f))(s, t) \\
&= \iota_\infty(f)(st) = f(\iota_*^\infty(st)).
\end{aligned}
$$

As $f \in C(G_\infty)$ was arbitrary, we conclude that $\iota_*^\infty(s)\iota_*^\infty(t) = \iota_*^\infty(st)$. Thus ι_*^∞ is a homomorphism, as required. ∎

Let $K_0 = (\iota_*^\infty)^{-1}(\{\infty\})$ a closed subset of K_{WAP}. As ι_*^∞ is a homomorphism, it follows that K_0 is an ideal in K_{WAP} and that $K_{\mathrm{WAP}} \setminus K_0$ is a locally compact subsemigroup of K_{WAP}.

Obviously each $s \in G$ induces a normal character δ_s on $M(G)^*$, and hence by restriction, a character on $\mathrm{WAP}(M(G))$. So we have a (possibly discontinuous) map $\theta : G \to K_{\mathrm{WAP}}$, which we shall henceforth consider as

a map $\theta : G_d \to K_{\mathrm{WAP}}$. Let $s, t \in G$ and $F \in \mathrm{WAP}(M(G))$, so that

$$F(\theta(s)\theta(t)) = \langle \delta_s \,\square\, \delta_t, F \rangle = \langle F, \delta_s \delta_t \rangle = \langle F, \delta_{st} \rangle = F(\theta(st)),$$

so we see that θ is a homomorphism.

Denote the unit of G by e_G. As δ_{e_G} is the unit of $M(G)$, it follows that δ_{e_G} is also the unit of $\mathrm{WAP}(M(G))^*$, and so $\theta(e_G)$ is the unit of K.

PROPOSITION 5.3. *Restrict ι_*^∞ to a homomorphism $K_{\mathrm{WAP}} \backslash K_0 \to G$. Let K_1 be the kernel of this homomorphism, so that K_1 is a closed subsemigroup of $K_{\mathrm{WAP}} \backslash K_0$. Then $\iota_*^\infty \circ \theta$ is the identity on G_d and θ maps into $K_{\mathrm{WAP}} \backslash K_0$. In particular, $K_{\mathrm{WAP}} \backslash K_0 = G_d \ltimes K_1$.*

Proof. For $t \in G$ and $f \in C_0(G)$, clearly $\langle \delta_{\theta(t)}, \iota(f) \rangle = f(t)$, showing that θ takes values in $K \backslash K_0$, and that $\iota_*^\infty(\theta(t)) = t$, as required. We now appeal to Lemma 3.9. ∎

By the universal property for WAP, as K_{WAP} is compact and semitopological, we have a factorisation

$$
\begin{array}{ccc}
G_d & \xrightarrow{\;\;\theta\;\;} & K_{\mathrm{WAP}} \\
\big\downarrow & \nearrow & \\
(G_d)^{\mathrm{WAP}} & \theta^{\mathrm{WAP}} &
\end{array}
$$

Recall that θ^{WAP} must satisfy the following condition: for $s \in (G_d)^{\mathrm{WAP}}$, if $(s_\alpha) \subseteq G_d$ is a net with $s_\alpha \to s$ in $(G_d)^{\mathrm{WAP}}$, then $\theta(s_\alpha) \to \theta^{\mathrm{WAP}}(s)$ in K_{WAP}.

We regard $\mathrm{WAP}(G_d) = C((G_d)^{\mathrm{WAP}})$ as a subalgebra of $\ell^\infty(G) = C(G_d)$. As discussed before Lemma 3.7, we consider the projection $P : M(G) \to \ell^1(G)$, which is an algebra homomorphism. The following has an almost identical proof to that of Lemma 3.7.

LEMMA 5.4. *$P^* : \ell^\infty(G) \to M(G)^*$ is an algebra homomorphism which maps $\mathrm{WAP}(G_d)$ into $\mathrm{WAP}(M(G))$.*

Again, we have $\Delta \circ P^* = (P^* \otimes P^*) \circ \Phi_d$, where $\Phi_d : \ell^\infty(G) \to \ell^\infty(G \times G)$ is the coproduct for G_d. We hence get a continuous semigroup homomorphism $\theta_1 : K \to (G_d)^{\mathrm{WAP}}$.

LEMMA 5.5. *Consider the continuous semigroup homomorphisms $\theta^{\mathrm{WAP}} : (G_d)^{\mathrm{WAP}} \to K_{\mathrm{WAP}}$ and $\theta_1 : K_{\mathrm{WAP}} \to (G_d)^{\mathrm{WAP}}$. Then $\theta_1 \circ \theta^{\mathrm{WAP}}$ is the identity on $(G_d)^{\mathrm{WAP}}$ and so θ^{WAP} is a homeomorphism onto its range.*

Proof. For $s \in G$ and $F \in \mathrm{WAP}(G_d)$, we calculate that

$$F(\theta_1 \theta(s)) = \langle \delta_{\theta(s)}, P^*(F) \rangle = \langle P^*(F), \delta_s \rangle = \langle F, \delta_s \rangle = F(s).$$

Hence $\theta_1 \circ \theta : G \to (G_d)^{\mathrm{WAP}}$ is the canonical inclusion. By continuity, it follows that $\theta_1 \circ \theta^{\mathrm{WAP}}$ is the identity on $(G_d)^{\mathrm{WAP}}$, and so θ^{WAP} must be a homeomorphism onto its range. ∎

LEMMA 5.6. *The following are equivalent:*
(1) *G is compact;*
(2) *K_0 is empty;*
(3) *θ^{WAP} maps into $K_{\mathrm{WAP}} \setminus K_0$.*

Proof. As K_0 is the inverse image of $\{\infty\}$ under ι_*^∞, it is immediate that if G is compact, then K_0 is empty. So (1) implies (2), and clearly (2) implies (3).

Suppose that G is not compact. Then we can find some net $(s_\alpha) \subseteq G$ which eventually leaves every compact subset of G. But moving to a subnet if necessary, we may suppose that (s_α) converges in $(G_d)^{\mathrm{WAP}}$, to s say. Notice that in K_{WAP}, we have $\theta^{\mathrm{WAP}}(s) = \lim_\alpha \theta^{\mathrm{WAP}}(s_\alpha) = \lim_\alpha \theta(s_\alpha)$. As $\iota_\infty^* : K_{\mathrm{WAP}} \to G_\infty$ is continuous, it follows that
$$\iota_\infty^* \theta^{\mathrm{WAP}}(s) = \lim_\alpha \iota_\infty^* \theta(s_\alpha) = \lim_\alpha s_\alpha = \infty.$$
Hence $\theta^{\mathrm{WAP}}(s) \in K_0$, and so we have shown that (3) implies (1). ∎

As $(G_d)^{\mathrm{WAP}}$ is not a group, we cannot apply Lemma 3.9. However, in [2], it is shown that unless G is finite, $\mathrm{WAP}(G_d)/c_0(G_d)$ contains a copy of ℓ^∞. In particular, $\mathrm{WAP}(G_d)$ is "large", and so also K_{WAP} is "large" in this sense. The following shows, again informally, that $K_{\mathrm{WAP}} \setminus K_0$ is also "large".

PROPOSITION 5.7. *Let G be non-discrete. For any compact, non-discrete subset $A \subseteq G$, the image of A in $(G_d)^{\mathrm{WAP}}$ is not closed. However, the image of the closure of A, under θ^{WAP}, is contained in $K_{\mathrm{WAP}} \setminus K_0$.*

Proof. The inclusion $G_d \to G^{\mathrm{WAP}}$ is continuous, so by the universal property, we get a continuous map $\phi : (G_d)^{\mathrm{WAP}} \to G^{\mathrm{WAP}}$ which has dense range. As $(G_d)^{\mathrm{WAP}}$ is compact, it follows that ϕ is surjective. We can see ϕ in a more concrete way. By [1, Section 4.2], $\mathrm{WAP}(G) = C(G) \cap \mathrm{WAP}(G_d)$. By considering both $\mathrm{WAP}(G)$ and $\mathrm{WAP}(G_d)$ as subalgebras of $\ell^\infty(G)$, we see that the inclusion map $\mathrm{WAP}(G) \to \mathrm{WAP}(G_d)$ is a $*$-homomorphism, and so induces a continuous map $\phi : (G_d)^{\mathrm{WAP}} \to G^{\mathrm{WAP}}$.

As also $C_0(G) \subseteq \mathrm{WAP}(G)$, the above argument (cf. Proposition 5.2) also shows the existence of a continuous homomorphism $\psi : (G_d)^{\mathrm{WAP}} \to G_\infty$ such that $\psi(s) = s$ for each $s \in G$.

Suppose that $A \subseteq G$ is compact and that the image of A in $(G_d)^{\mathrm{WAP}}$, say denoted by A_0, is closed. We can hence consider the restriction $\psi|_{A_0} : A_0 \to G_\infty$. Then $\psi|_{A_0}(s) = s$ for each $s \in A$, and so $\psi|_{A_0} : A_0 \to A$ is a continuous bijection between compact sets, and is hence a homeomorphism.

We then claim that for each $f \in \mathrm{WAP}(G_d)$, there exists $F \in C_0(G)$ such that, if $C_0(G)$ is considered as a subspace of $\ell^\infty(G)$, then $f(s) = F(s)$ for each $s \in A$. By the Tietze extension theorem, we can simply let F be an extension of the map $f \circ \psi|_{A_0}^{-1} : A \to \mathbb{C}$. Thus f is continuous on A.

However, $c_0(G) \subseteq \mathrm{WAP}(G_d)$, so we have shown that the restriction of any $c_0(G)$ function to A is continuous. This implies that A must be discrete, as required.

Finally, let $A \subseteq G$ be compact, let (s_α) be a net in A, and suppose that $s_\alpha \to s$ in $(G_d)^{\mathrm{WAP}}$. This means that $f(s_\alpha) \to f(s)$ for each $f \in \mathrm{WAP}(G_d)$, hence for all $f \in C_0(G) \subseteq \mathrm{WAP}(G) \subseteq \mathrm{WAP}(G_d)$. So (s_α) must converge in G, and hence in A, say to $t \neq \infty$. Then, as in the previous lemma, $\iota_\infty^* \theta^{\mathrm{WAP}}(s) = t$, so that $\theta^{\mathrm{WAP}}(s) \notin K_0$, as required. ∎

Exactly the same proof as used in Proposition 3.11 shows that $K_{\mathrm{WAP}} \neq (G_d)^{\mathrm{WAP}}$ when G is non-discrete. We finish by mentioning that, suitably modified, the results of Section 3.2 apply to the WAP case, although this seems to give little insight, given, again, that $(G_d)^{\mathrm{WAP}}$ is not a group. Similarly, it seems natural to ask about invariant measure on K_{WAP}, but we have made no progress in this direction.

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