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Ergodicity of $\mathbb{Z}^2$ extensions of irrational rotations

by

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Abstract. Let $\mathbf{T} = [0,1)$ be the additive group of real numbers modulo 1, $\alpha \in \mathbf{T}$ be an irrational number and $t \in \mathbf{T}$. We study ergodicity of skew product extensions $T\colon \mathbf{T}\times\mathbb{Z}^2 \to \mathbf{T}\times\mathbb{Z}^2$, $T(x,s_1,s_2) = (x+\alpha, s_1+2\chi_{[0,1/2)}(x)-1, s_2+2\chi_{[0,1/2)}(x+t)-1)$.

1. Introduction. The study of irrational rotations of the unit circle has led to various questions in number theory and ergodic theory. Let $\mathbf{T} = [0,1)$ be the additive group of real numbers modulo 1. Fix an irrational $\alpha \in \mathbf{T}$ and let $t \in \mathbf{T}$ satisfy the condition that neither t nor $t + 1/2$ is a multiple of α mod 1. Define a map $f\colon \mathbf{T} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1/2, \\ -1 & \text{for } 1/2 \le x < 1 \end{cases} \tag{1.1}$$

and an irrational rotation T_0 of $\mathbf{T}$ by

$$T_0 x = x + \alpha \bmod 1.$$

Set $\mathbf{X} = \mathbf{T} \times \mathbb{Z}^2$ and define $T\colon \mathbf{X} \to \mathbf{X}$ by

$$T(x,s_1,s_2) = (x+\alpha, s_1+f(x), s_2+f(x+t)). \tag{1.2}$$

It is a skew product extension of irrational rotations on the circle by $\mathbb{Z}^2$ determined by f and t. We study ergodicity of T on $\mathbf{X}$ relative to Haar measure, continuing a theme started by Schmidt [S1], [S2] and Veech [V]. It is known that ergodicity of skew product extensions of an irrational rotation arises from irregularity of distribution of $\mathbb{Z}\alpha$. For the case of cylinder flows, Oren [O] gave a complete solution to the problem of ergodicity of the map $F\colon \mathbf{T} \times E \to \mathbf{T} \times E$ defined by

$$F(x,s) = (x+\alpha, s+\chi_{[0,\beta)}(x)-\beta), \tag{1.3}$$

where $\beta \in \mathbf{T}$ and E is the closed subgroup of $\mathbb{R}$ generated by 1 and β. Earlier, special cases were settled by Schmidt for $\beta = 1/2$, $\alpha = (\sqrt{5}-1)/4$

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in [S2] and for $\beta = 1/2$, α irrational in [A], [C2], [S1]. Although ergodicity of cylinder flows (that is, (1.3)) is thoroughly understood, the situation of $\mathbb{Z}^2$ extensions of irrational rotations appears to be more complicated. [L] treated ergodicity of (1.2) for a proper subset of the set of α's with bounded partial quotients and pointed out its numerous applications, e.g. to the study of joinings of some Rokhlin cocycles. This paper extends the results of [L].

Note that by (1.2), we have

$$T^n(x, s_1, s_2) = (x + n\alpha, s_1 + a_n(x), s_2 + a_n(x + t)), \qquad \forall n \in \mathbb{Z}, \tag{1.4}$$

where

$$a_n(x) = \begin{cases} \sum_{i=0}^{n-1} f(x + i\alpha) = 2\sum_{i=0}^{n-1} \chi_{[0,1/2)}(x + i\alpha) - n, & n \geq 1, \\ 0, & n = 0, \\ -a_{-n}(T_0^{-n}x), & n \leq -1. \end{cases} \tag{1.5}$$

$a_n(x)$ satisfies the additive cocycle equation

$$a_n(T_0^m x) - a_{n+m}(x) + a_m(x) = 0, \qquad \forall m, n \in \mathbb{Z}, \forall x \in \mathbf{T}. \tag{1.6}$$

Also note that $a_n(x + t) \equiv a_n(x) \bmod 2$. The parity of $a_n(x)$ is always the same as that of n from (1.5). Hence T cannot be ergodic on the entire space $\mathbf{X}$. We set $G = \{(s_1, s_2) \in \mathbb{Z}^2 \mid s_1 \equiv s_2 \bmod 2\}$. Then G is cocompact in $\mathbb{Z}^2$.

Following [S1, Definition 2.1] we give

Definition 1.1. $(a, t)\colon \mathbb{Z} \times \mathbf{T} \to \mathbb{Z}^2$ defined by

$$(a, t)(n, x) = (a_n(x), a_n(x + t)) \tag{1.7}$$

is called a *cocycle* for T.

[S1] showed that ergodicity of T, or equivalently, ergodicity of the cocycle (a, t), is determined by the set $\overline{\mathbb{E}^2(a, t)}$ of essential values of (a, t). Put $\overline{\mathbb{Z}^2} = \mathbb{Z}^2 \cup \{\infty\}$, the one-point compactification of $\mathbb{Z}^2$. We recall the definition of essential values.

Definition 1.2. Let μ be Lebesgue measure on $\mathbf{T}$. An element $(k_1, k_2) \in \overline{\mathbb{Z}^2}$ is called an *essential value* of (a, t) if, for every neighbourhood $N(k_1, k_2)$ of (k_1, k_2) in $\overline{\mathbb{Z}^2}$, and for every measurable set $A \subset \mathbf{T}$ with $\mu(A) > 0$, we have

$$\mu\Big(\bigcup_{n\in\mathbb{Z}} (A \cap T_0^{-n}A \cap \{x \mid (a, t)(n, x) \in N(k_1, k_2)\})\Big) > 0.$$

We denote the set of essential values of (a, t) by $\overline{\mathbb{E}^2(a, t)}$. Set $\mathbb{E}^2(a, t) = \overline{\mathbb{E}^2(a, t)} \cap \mathbb{Z}^2$.

From [S1] we derive the following properties:

- $\mathbb{E}^2(a,t)$ is a closed subgroup of $\mathbb{Z}^2$ under addition. $(k_1,k_2)\in\mathbb{E}^2(a,t)$ only if $k_1\equiv k_2 \bmod 2$.
- (a,t) is a *coboundary* (that is, $(a,t)(n,x)=c(T_0^n x)-c(x)$ for a measurable map $c\colon \mathbf{T}\to\mathbb{Z}^2$) if and only if $\overline{\mathbb{E}^2(a,t)}=\{(0,0)\}$.

We say that two cocycles $(a,t),(b,t)\colon \mathbb{Z}\times\mathbf{T}\to\mathbb{Z}^2$ are *cohomologous* if $(a,t)-(b,t)$ is a coboundary. In this case, $\overline{\mathbb{E}^2(a,t)}=\overline{\mathbb{E}^2(b,t)}$. Given a cocycle $(a,t)\colon\mathbb{Z}\times\mathbf{T}\to\mathbb{Z}^2$, let $(a,t)^*\colon\mathbb{Z}\times\mathbf{T}\to\mathbb{Z}^2/\mathbb{E}^2(a,t)$ be the corresponding quotient cocycle. We have the following important result from [S1, Lemma 3.10]:

Lemma 1.3. $\mathbb{E}^2(a,t)^*=\{(0,0)\}$.

We say that the cocycle (a,t) is *regular* if $\overline{\mathbb{E}^2(a,t)^*}=\{(0,0)\}$. Otherwise (a,t) is called *nonregular* and in this case $\overline{\mathbb{E}^2(a,t)^*}=\{(0,0),\infty\}$. According to [L], if (a,t) is regular, then (a,t) is cohomologous to a cocycle $(b,t)\colon \mathbb{Z}\times\mathbf{T}\to\mathbb{E}^2(a,t)$ and the latter is ergodic as a cocycle with values in the closed subgroup $\mathbb{E}^2(a,t)$ (see also [S1]). In particular, if $\mathbb{E}^2(a,t)$ is cocompact in $\mathbb{Z}^2$ then (a,t) is regular.

The main results of this paper are the following theorems:

Main Theorem 1.4. *For arbitrary irrational $\alpha\in\mathbf{T}$, $\mathbb{E}^2(a,t)$ of the cocycle (a,t) defined in* (1.7) *is $G=\{(s_1,s_2)\in\mathbb{Z}^2 \mid s_1\equiv s_2 \bmod 2\}$ for almost all $t\in\mathbf{T}$. In particular, (a,t) is regular for almost all $t\in\mathbf{T}$.*

Main Theorem 1.5. *If α is badly approximable, then the group $\mathbb{E}^2(a,t)$ is G if and only if $t\notin\mathbb{Z}\alpha$ and $t\notin\mathbb{Z}\alpha+1/2$.*

Theorem 1.5 extends some of the results of [L]. Our methods, however, are based on those developed in [S1] and [O].

It is not hard to see that in Theorems 1.4 and 1.5, in order for the group of essential values to be equal to G, we must exclude $t\in\mathbb{Z}\alpha$ and $t\in\mathbb{Z}\alpha+1/2$. Note that for each nonnegative integer m, $|a_n(x+m\alpha)-a_n(x)|$ is bounded by $2m$ because for all $n>m$,

$$\begin{aligned}|a_n(x+m\alpha)-a_n(x)| &= \Big|\sum_{i=0}^{m-1} f(x+n\alpha+i\alpha)-\sum_{i=0}^{m-1} f(x+i\alpha)\Big| \\ &\le \sum_{i=0}^{m-1}|f(x+n\alpha+i\alpha)|+\sum_{i=0}^{m-1}|f(x+i\alpha)|\le 2m.\end{aligned}$$

From (1.1) we also have $f(x+1/2)=-f(x)$ and therefore

$$a_n(x+1/2)=-a_n(x),\qquad \forall x\in\mathbf{T},\ \forall n.$$

Hence $|a_n(x+1/2+m\alpha)+a_n(x)|$ is bounded from above by $2m$.

2. Period approximating sequences, partial convergents and other preliminaries. For $x \in \mathbb{R}$ we denote the closest integer to x by $[x]$, and set $\langle x \rangle = x - [x]$ and $\|x\| = |x - [x]|$. Throughout, n is assumed to be a nonnegative integer.

According to (1.5), $a_n(x)$ is locally constant except for jumps of $+2$ at $0, -\alpha, -2\alpha, \ldots, -(n-1)\alpha$ and jumps of -2 at $1/2, 1/2 - \alpha, 1/2 - 2\alpha, \ldots, 1/2 - (n-1)\alpha$. Also, $a_n(x+t)$ is locally constant except for jumps of $+2$ at $-t, -t-\alpha, -t-2\alpha, \ldots, -t-(n-1)\alpha$ and jumps of -2 at $1/2 - t$, $1/2 - t - \alpha, \ldots, 1/2 - t - (n-1)\alpha$.

If we set

$$S_n(x) = \sum_{i=0}^{n-1} \chi_{[0,1/2)}(x + i\alpha) = \#\{0 \le i \le n-1 \mid x + i\alpha \in [0, 1/2)\},$$

then from (1.5),

$$a_n(x) = 2S_n(x) - n.$$

The concept of essential values corresponds to that of periods in [O]. We have the following definition:

Definition 2.1. For fixed $(k_1, k_2) \in \mathbb{Z}^2$, a *period approximating sequence* is a sequence $\{(n_l, A_l)\}_{l=1}^{\infty}$ where

(1) $A_l \subset \mathbf{T}$, each A_l is measurable;
(2) a_{n_l} is constant on both A_l and $A_l + t$, that is, $a_{n_l}(A_l) = k_1$ and $a_{n_l}(A_l + t) = k_2$ for all l;
(3) $\inf_l \mu(A_l) > 0$;
(4) $\|n_l \alpha\| \to 0$.

The next lemma shows that a period approximating sequence defines an element in $\mathbb{E}^2(a, t)$.

Lemma 2.2. *For fixed $(k_1, k_2) \in \mathbb{Z}^2$, if there exists a period approximating sequence $\{(n_l, A_l)\}_{l=1}^{\infty}$ such that $a_{n_l}(A_l) = k_1$ and $a_{n_l}(A_l + t) = k_2$ for all l, then $(k_1, k_2) \in \mathbb{E}^2(a, t)$.*

Proof. Given the period approximating sequence $\{(n_l, A_l)\}_{l=1}^{\infty}$, for arbitrary $A \subset \mathbf{T}$ with $\mu(A) > 0$, because $\mu(A \cap T_0^{-n_l} A) \to \mu(A)$, there exists a subsequence $\{p_l\}_{l=1}^{\infty} \subset \{n_l\}_{l=1}^{\infty}$ such that the set

$$A_0 = \bigcap_{l=1}^{\infty} (A \cap T_0^{-p_l} A) \tag{2.1}$$

has positive measure. Without loss of generality, we assume that $\{p_l\}_{l=1}^{\infty}$ is the same as $\{n_l\}_{l=1}^{\infty}$.

Set $B = \limsup_{l\to\infty} A_l = \bigcap_{l=1}^{\infty} \bigcup_{i=l}^{\infty} A_i$; $\mu(B) > 0$ because $\inf_l \mu(A_l) > 0$.

There exist $m \in \mathbb{Z}$ and $A' \subset A_0$ such that $\mu(A') > 0$ and $T_0^m A' \subset B$ because the action T_0 is ergodic. Hence

$$\mu(B \cap T_0^m A') = \mu(T_0^m A') = \mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}(A_i \cap T_0^m A')\Big)$$
$$= \mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}(T_0^{-m} A_i \cap A')\Big) > 0.$$

Hence there exists some fixed positive number ϵ such that for each l, we have $\mu(\bigcup_{i=l}^{\infty}(T_0^{-m} A_i \cap A')) > \epsilon$. In other words, for each l, there exists a measurable set $A'_{p_l} \subset A'$ with $\mu(A'_{p_l}) > \epsilon$ and for all $x \in A'_{p_l}$ we have

$$a_{p_{l'}}(T_0^m x) = k_1, \quad a_{p_{l'}}(T_0^m x + t) = k_2, \quad \text{for some } l' \geq l.$$

From the cocycle identity

$$a_{p_{l'}}(x) + a_m(T_0^{p_{l'}} x) = a_{m+p_{l'}}(x) = a_m(x) + a_{p_{l'}}(T_0^m x),$$

we derive

$$(2.2) \qquad |a_{p_{l'}}(T_0^m x) - a_{p_{l'}}(x)| = |a_m(T_0^{p_{l'}} x) - a_m(x)|$$
$$= \Big|\sum_{i=0}^{m-1} f(x + i\alpha + p_{l'}\alpha) - \sum_{i=0}^{m-1} f(x + i\alpha)\Big|.$$

From $T_0^m(x+t) = T_0^m(x) + t$, we further derive

$$|a_{p_{l'}}(T_0^m x + t) - a_{p_{l'}}(x+t)| = \Big|\sum_{i=0}^{m-1} f(x + i\alpha + p_{l'}\alpha + t) - \sum_{i=0}^{m-1} f(x + i\alpha + t)\Big|.$$

Noting $\lim_{l'\to\infty}\|p_{l'}\alpha\| = 0$ as well as the fact that m is fixed and depends on A_0 only, we derive from (2.2) that $a_{p_{l'}}(T_0^m x) - a_{p_{l'}}(x) \to 0$ for almost all x. The set A' is also fixed and depends on A_0 only. Therefore there exist some $p_{l'}$ and $A'' \subset A' \subset A_0$ with $\mu(A'') > 0$ such that

$$a_{p_{l'}}(x) = a_{p_{l'}}(T_0^m x) = k_1, \quad a_{p_{l'}}(x+t) = a_{p_{l'}}(T_0^m x + t) = k_2, \quad \forall x \in A''.$$

We have $T_0^{-p_{l'}} A' \subset A$ by (2.1). Hence

$$\mu(A \cap T_0^{-p_{l'}} A \cap \{x \mid a_{p_{l'}}(x) = k_1\} \cap \{x \mid a_{p_{l'}}(x+t) = k_2\}) > 0,$$

and so $(k_1, k_2) \in \mathbb{E}^2(a, t)$. ■

We recall the Denjoy–Koksma inequality [O, Lemma 2], which plays a fundamental role in the proof.

Lemma 2.3 (Denjoy–Koksma). *If $p, q \in \mathbb{N}$ satisfy*

$$\Big|\alpha - \frac{p}{q}\Big| < \frac{1}{q^2} \quad \textit{and} \quad (p,q) = 1,$$

then $|a_q(x)| < 4$ for all $x \in \mathbf{T}$, where $a_q(x)$ is defined in (1.5).

It follows from the proof of the above lemma that every interval of the form $[i/q,(i+1)/q)$ contains exactly one of the points $j\alpha$ for $0 \le i, j \le q-1$. In other words, the points $j\alpha$ $(0 \le j \le q-1)$ are uniformly distributed on the unit circle.

We denote by $[a_0; a_1, a_2, \ldots,]$ the continued fraction of α and call the a_i the partial quotients of α. Denote by $\frac{p_k}{q_k}$ the kth partial convergent of α where $k \ge 0$. It is known from [K] that

$$\frac{p_k}{q_k} = [a_0; a_1, a_2, \ldots, a_k],$$

$$\|q_k\alpha\| < \frac{1}{q_{k+1}} < \frac{1}{q_k}, \tag{2.3}$$

$$\min_{q_k \le q < q_{k+1}} \|q\alpha\| = \|q_k\alpha\| > \frac{1}{q_k + q_{k+1}} > \frac{1}{2q_{k+1}}, \tag{2.4}$$

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k. \tag{2.5}$$

Set

$$D(\alpha) = \{q_k \mid p_k/q_k \text{ is a partial convergent of } \alpha\},$$

$$q^+ = \min\{q' \in D(\alpha) \mid q' > q\}, \qquad \forall q \in D(\alpha).$$

Adapting arguments of [S1, pp. 229–230] we are able to prove the following lemma which constitutes the first step in the entire proof:

Lemma 2.4.

$$\mathbb{E}^2(a,t) \cap \{(1,3),(1,-3),(1,1),(1,-1),(3,1),(3,-1),(3,3),(3,-3)\} \ne \emptyset.$$

Proof. From (2.5) we derive that there are infinitely many odd $q \in D(\alpha)$. For such q, from the Denjoy–Koksma inequality and (1.5) we derive that $a_q(x)$ can only be ± 3 or ± 1. Consequently, there exists a period approximating sequence $\{(q_l, A_l)\}_{l=1}^{\infty}$ which defines $(k_1,k_2) \in \mathbb{E}^2(a,t)$ and

$$\pm(k_1,k_2) \in \{(1,3),(1,-3),(1,1),(1,-1),(3,1),(3,-1),(3,3),(3,-3)\}.$$

The proof is completed by noting that $\mathbb{E}^2(a,t)$ is a subgroup of $\mathbb{Z}^2$. ∎

A major difficulty in proving Theorem 1.4 is therefore to show that $\mathbb{E}^2(a,t)$ is not isomorphic to $\mathbb{Z}$ because we aim to show that $\mathbb{E}^2(a,t)$ is G for almost all t. We will use period approximating sequences. From the properties of continued fractions we derive the following lemma:

Lemma 2.5. *For any nonzero $q \in D(\alpha)$, we have*

$$\min\{\|1/2 - j\alpha\| \mid |j| < q\} \ge \frac{1}{24q}.$$

Proof. We always have

$$\|1/2 - j\alpha\| \geq \frac{\|2(1/2 - j\alpha)\|}{2} = \frac{\|2j\alpha\|}{2}.$$

We consider five cases separately under the assumption that $0 < |j| < q$.

CASE 1: $q^+ \geq 3q$. Then since $\big||2j| - q\big| < q$ for $0 < |j| < q$, we have $\|(|2j| - q)\alpha\| > 1/2q$ from (2.4) and

$$\|2j\alpha\| = \|(|2j|-q)\alpha+q\alpha\| \geq \|(|2j|-q)\alpha\|-\|q\alpha\| > \frac{1}{2q}-\frac{1}{q^+} > \frac{1}{2q}-\frac{1}{3q} = \frac{1}{6q}.$$

Here we also used the inequality $\|q\alpha\| < 1/q^+$ from (2.3).

CASE 2: $q^+ < 3q$ and $q^{++} < 3q$. Then since $|2j| < 2q \leq q^{++}$, from (2.4) we have

$$\|2j\alpha\| \geq \|q^+\alpha\| \geq \frac{1}{2q^{++}} > \frac{1}{6q}.$$

CASE 3: $q^+ < 3q$, $q^{++} \geq 3q$ and $\big|q^+ - |2j|\big| < q$. Then $\|(|2j|-q^+)\alpha\| > \frac{1}{2q}$ from (2.4) and

$$\begin{aligned}\|2j\alpha\| &= \|(|2j| - q^+)\alpha + q^+\alpha\| \geq \|(|2j| - q^+)\alpha\| - \|q^+\alpha\| \\ &> \frac{1}{2q} - \frac{1}{q^{++}} > \frac{1}{2q} - \frac{1}{3q} = \frac{1}{6q}.\end{aligned}$$

CASE 4: $q^+ < 3q$, $q^{++} \geq 3q$, $\big|q^+ - |2j|\big| \geq q$ and $|2j| \leq q$. Then from (2.4) we get

$$\|2j\alpha\| \geq \|q\alpha\| > \frac{1}{2q^+} \geq \frac{1}{6q}.$$

CASE 5: $q^+ < 3q$, $q^{++} \geq 3q$, $\big|q^+ - |2j|\big| \geq q$ and $|2j| > q$. Then

$$\begin{gathered} q^+ - |4j| < 3q - 2q = q, \quad 2q - q^+ > 2q - 3q = -q; \\ |2j| \leq q^+ - q \;\Rightarrow\; q^+ - |4j| \geq q^+ - 2(q^+ - q) = 2q - q^+ > -q; \end{gathered}$$

hence $\big|q^+ - |4j|\big| < q$ and from (2.4),

$$\|4j\alpha\| = \|(q^+ - |4j|)\alpha - q^+\alpha\| \geq \|(q^+ - |4j|)\alpha\| - \|q^+\alpha\| > \frac{1}{2q} - \frac{1}{q^{++}} \geq \frac{1}{6q};$$

and $\|2j\alpha\| \geq \|4j\alpha\|/2$. The inequality is established. ∎

3. Proof of main theorems. Following [O] we set, for each $q \in D(\alpha)$,

$$\begin{aligned}\epsilon(q) &= q\min\{\|-t - j\alpha\| \mid |j| < q\}, \\ \theta(q) &= q\min\{\|1/2 - t - j\alpha\| \mid |j| < q\}.\end{aligned} \tag{3.1}$$

We immediately derive that $\epsilon(q) < 1$ and $\theta(q) < 1$ from the proof of the Denjoy–Koksma inequality.

Proposition 3.1. *If*

$$\limsup_{\substack{q\in D(\alpha)\\ q\to\infty}} \min\{\epsilon(q),\theta(q)\}>0, \tag{3.2}$$

then $\mathbb{E}^2(a,t)=\{(k_1,k_2)\in\mathbb{Z}^2 \mid k_1\equiv k_2 \bmod 2\}=G$.

Proof. Let $\{q_n\}_{n=1}^\infty\subset D(\alpha)$ be such that $\min\{\epsilon(q_n),\theta(q_n)\}>\delta>0$ for all n.

Recall $a_{q_n}(x)$ as set in (1.5) is locally constant except for jumps of $+2$ at $0,-\alpha,-2\alpha,\dots,-(q_n-1)\alpha$ and jumps of -2 at $1/2,1/2-\alpha,1/2-2\alpha,\dots,1/2-(q_n-1)\alpha$; and $a_{q_n}(x+t)$ is locally constant except for jumps of $+2$ at $-t,-t-\alpha,-t-2\alpha,\dots,-t-(q_n-1)\alpha$ and jumps of -2 at $1/2-t,1/2-t-\alpha,\dots,1/2-t-(q_n-1)\alpha$.

For fixed n, let $I_1,\dots,I_{4q_n}$ denote the intervals of constancy of both $a_{q_n}(x)$ and $a_{q_n}(x+t)$ in cyclic order. Since $a_{q_n}(\cdot)$ takes on at most four values by Lemma 2.3, there exists a union of intervals, A_n, such that $a_{q_n}(x)$ and $a_{q_n}(x+t)$ are constant on A_n and $\mu(A_n)\ge 1/16$. Let A_n' be the union of intervals contiguous on the right to those of A_n. Note that the distance between any discontinuity points of $a_{q_n}(x)$ and $a_{q_n}(x+t)$ is given by $\|(i-j)\alpha\|$ or $\|1/2+(i-j)\alpha\|$ or $\|-t+(i-j)\alpha\|$ or $\|1/2-t+(i-j)\alpha\|$ for $0\le i,j\le q_n-1$. From (2.4), Lemma 2.5 and (3.1), we see that $\min\{1/24q_n,\epsilon(q_n)/q_n,\theta(q_n)/q_n\}$ is a lower bound for the lengths $|I_i|$, $i=1,\dots,4q_n$. Since every interval of length $2/q_n$ must contain a $+2$ jump point by the discussion following Lemma 2.3, we have $|I_i|<2/q_n$. Therefore

$$\frac{|I_i|}{|I_j|}>\frac12\min\left\{\frac1{24},\epsilon(q_n),\theta(q_n)\right\},\qquad 1\le i,j\le 4q_n.$$

By setting $\epsilon=\min\{1/24,\delta\}$, we thus have $\mu(A_n')\ge\frac12\epsilon\mu(A_n)\ge\frac1{32}\epsilon$ for all n. Next, $(a,t)(q_n,x)=(a_{q_n}(x),a_{q_n}(x+t))$ can take on A_n' only the values $(a_{q_n}(A_n)\pm2,a_{q_n}(A_n+t))$ or $(a_{q_n}(A_n),a_{q_n}(A_n+t)\pm2)$ since each interval of A_n' is contiguous on the right to one of A_n. Thus, we can find $A_n''\subset A_n'$ such that $a_{q_n}(x)$ and $a_{q_n}(x+t)$ are both constant on A_n'', $\mu(A_n'')\ge\frac1{128}\epsilon$ and

$$\begin{aligned}(a_{q_n}(A_n''),a_{q_n}(A_n''+t))&=(a_{q_n}(A_n)\pm2,a_{q_n}(A_n+t))\\ &\quad\text{or } (a_{q_n}(A_n),a_{q_n}(A_n+t)\pm2).\end{aligned}\tag{3.3}$$

First we assume that $a_{q_n}(A_n)=1$ and $a_{q_n}(A_n+t)=3$ and consequently $(1,3)$ lies in $\mathbb{E}^2(a,t)$. We need to prove both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$. From (3.3) and the Denjoy–Koksma inequality, we derive that there exists a period approximating sequence $\{(q_n',A_n'')\}_{n=1}^\infty$ which defines either $(1+2,3)=(3,3)$ or $(1-2,3)=(-1,3)$ or $(1,3-2)=(1,1)$. We treat the three cases separately.

Case 1: Suppose that apart from $(1,3)\in\mathbb{E}^2(a,t)$, we also have $(3,3)\in\mathbb{E}^2(a,t)$. Then $(\pm2,0)$ lies in $\mathbb{E}^2(a,t)$ because $\mathbb{E}^2(a,t)$ is a subgroup of $\mathbb{Z}^2$.

Moreover, from our assumption, there exists a period approximating sequence $\{(q_n, A_n)\}_{n=1}^{\infty}$ which defines $(1,3) \in \mathbb{E}^2(a,t)$. That is, we have

(1) $A_n \subset \mathbf{T}$;
(2) a_{q_n} is constant on both A_n and $A_n + t$,
and $a_{q_n}(A_n) = 1, a_{q_n}(A_n + t) = 3$, for all n;
(3) $\inf_n \mu(A_n) > 0$;
(4) $\|q_n \alpha\| \to 0$.

Therefore there exists a period approximating sequence $\{(q'_n, B'_n)\}_{n=1}^{\infty}$ which defines $(k,1) \in \mathbb{E}^2(a,t)$ for some $k \in \{\pm 1, \pm 3\}$. That is, we have

(a) $\{q'_n\}$ is a subsequence of $\{q_n\}$, $B'_n + t \subset A'_n$, $\mu(B'_n) \geq \frac{1}{4}\mu(A'_n)$;
(b) $a_{q'_n}$ is constant on both B'_n and $B'_n + t$,
and $a_{q'_n}(B'_n) = k$, $a_{q'_n}(B'_n + t) = a_{q'_n}(A'_n) = 1$, for all n;
(c) $\inf_n \mu(B'_n) > 0$;
(d) $\|q'_n \alpha\| \to 0$.

Because $\mathbb{E}^2(a,t)$ is a subgroup of $\mathbb{Z}^2$ under addition, we have

$$(1,3) \in \mathbb{E}^2(a,t) \text{ and } (2,0) \in \mathbb{E}^2(a,t) \;\Rightarrow\; (k,3) \in \mathbb{E}^2(a,t);$$
$$(k,1) \in \mathbb{E}^2(a,t) \text{ and } (k,3) \in \mathbb{E}^2(a,t) \;\Rightarrow\; (0,2) \in \mathbb{E}^2(a,t).$$

Consequently, both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$.

Case 2: Suppose $(-1,3)$ and $(1,3)$ both lie in $\mathbb{E}^2(a,t)$. Then so does $(\pm 2, 0)$ because $\mathbb{E}^2(a,t)$ is a subgroup of $\mathbb{Z}^2$.

Moreover, there exists a period approximating sequence $\{(q_n, A_n)\}_{n=1}^{\infty}$ which defines $(1,3) \in \mathbb{E}^2(a,t)$, so (1)–(4) hold again.

Therefore there exists a period approximating sequence $\{(q'_n, B'_n)\}_{n=1}^{\infty}$ which defines $(k,1) \in \mathbb{E}^2(a,t)$ for some $k \in \{\pm 1, \pm 3\}$, so (a)–(d) hold.

From the above arguments we derive that

$$(1,3) \in \mathbb{E}^2(a,t) \text{ and } (2,0) \in \mathbb{E}^2(a,t) \;\Rightarrow\; (k,3) \in \mathbb{E}^2(a,t);$$
$$(k,1) \in \mathbb{E}^2(a,t) \text{ and } (k,3) \in \mathbb{E}^2(a,t) \;\Rightarrow\; (0,2) \in \mathbb{E}^2(a,t).$$

Consequently, both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$.

Case 3: Suppose $(1,1)$ and $(1,3)$ both lie in $\mathbb{E}^2(a,t)$. Then $(0,2)$ lies in $\mathbb{E}^2(a,t)$. Moreover, $(2,2)$ also lies in $\mathbb{E}^2(a,t)$ and therefore $(2,0)$ lies in $\mathbb{E}^2(a,t)$.

In all three cases we have shown both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$. Along with the assumption that $(1,3)$ lies in $\mathbb{E}^2(a,t)$, we derive that $\mathbb{E}^2(a,t) = G$ as desired. Other possibilities when infinitely many q's appearing in (3.2) are odd can be proved analogously.

Next, we assume that only finitely many q's appearing in (3.2) are odd.

We first assume that there exists a period approximating sequence $\{(q_n, A_n)\}_{n=1}^{\infty}$ which defines $(2,0) \in \mathbb{E}^2(a,t)$, so (1)–(4) hold with (2) replaced by

(2′) a_{q_n} is constant on both A_n and $A_n + t$, and $a_{q_n}(A_n) = 2, a_{q_n}(A_n + t) = 0$, for all n.

Therefore from the Denjoy–Koksma inequality, there exists a period approximating sequence $\{(q_n', B_n')\}_{n=1}^{\infty}$ which defines $(k,2) \in \mathbb{E}^2(a,t)$ for some $k \in \{\pm 2, 0\}$, so (a)–(d) hold with (b) replaced by

(b′) $a_{q_n'}$ is constant on both B_n' and $B_n' + t$, and $a_{q_n'}(B_n') = k$, $a_{q_n'}(B_n' + t) = a_{q_n'}(A_n') = 2$, for all n.

From the above argument we derive that

$$(k,2) \in \mathbb{E}^2(a,t) \text{ and } (2,0) \in \mathbb{E}^2(a,t) \;\Rightarrow\; (0,2) \in \mathbb{E}^2(a,t).$$

Consequently, both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$. Along with Lemma 2.4, we have $\mathbb{E}^2(a,t) = G$.

If there exists a period approximating sequence $\{(q_n, A_n)\}_{n=1}^{\infty}$ which defines $(0,0) \in \mathbb{E}^2(a,t)$, then from the Denjoy–Koksma inequality and (3.3) we can assume that there exists a period approximating sequence $\{(q_n', A_n')\}_{n=1}^{\infty}$ which defines $(\pm 2, 0)$ or $(0, \pm 2) \in \mathbb{E}^2(a,t)$. The rest of the argument is similar to what appeared above.

Other possibilities when only finitely many q's appearing in (3.2) are odd can be handled analogously. ∎

Proposition 3.2. *For every α the set of t satisfying* (3.2) *has full Lebesgue measure.*

Proof. For any positive δ and any $q \in D(\alpha)$, the size of the set of t with

$$\min\{\|-t - j\alpha\|, \|1/2 - t - j\alpha\| \mid |j| < q\} < \delta/q$$

is bounded from above by $\mathrm{const} \cdot \delta$. And the set of t not satisfying (3.2) has zero measure because δ can be arbitrarily small. ∎

Therefore for almost all $t \in \mathbf{T}$, we have $\mathbb{E}^2(a,t) = G$ and Theorem 1.4 is established. Next we prove Theorem 1.5. Note that α is badly approximable if and only if its partial quotients are bounded.

Proposition 3.3. *If α is badly approximable and*

$$\lim_{\substack{q \in D(\alpha) \\ q \to \infty}} \min\{\epsilon(q), \theta(q)\} = 0, \tag{3.4}$$

then $t \in \mathbb{Z}\alpha$ or $t \in \mathbb{Z}\alpha + 1/2$.

Proof. For each $q \in D(\alpha)$, let $|i_q|, |j_q| < q$ be such that

$$\epsilon(q) = q\|-t - i_q\alpha\|, \quad \theta(q) = q\|1/2 - t - j_q\alpha\|.$$

Then by assumption we have

$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} \min\{q\|-t-i_q\alpha\|, q\|1/2-t-j_q\alpha\|\}=0.$$

Because α is badly approximable, q^+/q and q^{++}/q have a uniform upper bound and

$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} \min\{q^{++}\|-t-i_q\alpha\|, q^{++}\|1/2-t-j_q\alpha\|\}=0.$$

Also for arbitrary n_1 and n_2 we have the following inequalities:

$$\|n_1\alpha-n_2\alpha\|\le\|-t-n_1\alpha\|+\|-t-n_2\alpha\|, \tag{3.5}$$

$$\|1/2+n_1\alpha-n_2\alpha\|\le\|1/2-t-n_1\alpha\|+\|-t-n_2\alpha\|. \tag{3.6}$$

If $q^{++}\|-t-i_{q^+}\alpha\|<1/100$ and $q^{++}\|1/2-t-j_q\alpha\|<1/100$, then by (3.6),

$$q^{++}\|1/2+i_{q^+}\alpha-j_q\alpha\|<\frac{1}{50}.$$

Because

$$|i_{q^+}-j_q|\le|i_{q^+}|+|j_q|<q^++q\le q^{++},$$

this contradicts Lemma 2.5, which asserts that $q^{++}\|1/2+i_{q^+}\alpha-j_q\alpha\| \ge 1/24$. Hence

$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q\|-t-i_q\alpha\|=0 \quad\text{or}\quad \lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q\|1/2-t-j_q\alpha\|=0. \tag{3.7}$$

Suppose the first limit is zero. Then by (3.5),

$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q^{++}\|i_{q^+}\alpha-i_q\alpha\|=0.$$

From (2.4) we derive that for q large enough $i_{q^+}=i_q$, that is, i_q is constant. Hence $t\in\mathbb{Z}\alpha$.

Suppose the second limit in (3.7) is zero. Then

$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q^{++}\|j_{q^+}\alpha-j_q\alpha\|=0.$$

From (2.4) we derive that for q large enough $j_{q^+}=j_q$, that is, j_q is constant. Hence $t\in\mathbb{Z}\alpha+1/2$. ■

When α is not badly approximable, Merrill [M] showed that if t belongs to an uncountable set of zero measure containing numbers well approximable by multiples of α, the cocycle $v=\chi_{[0,t)}-\chi_{[1/2,1/2+t)}$ is a coboundary. This implies $\mathbb{E}^2(a,t)$ cannot be cocompact in $\mathbb{Z}^2$ for such t. More importantly, when α is not badly approximable, [C1] showed that under certain circumstances, there exist cocycles similar to (1.7) that are not regular.

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