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On a binary relation between normal operators

by

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Abstract. The main goal of this paper is to clarify the antisymmetric nature of a binary relation $\ll$ which is defined for normal operators A and B by: $A \ll B$ if there exists an operator T such that $E_A(\varDelta) \leq T^* E_B(\varDelta) T$ for all Borel subset $\varDelta$ of the complex plane $\mathbb{C}$, where E_A and E_B are spectral measures of A and B, respectively (the operators A and B are allowed to act in different complex Hilbert spaces). It is proved that if $A \ll B$ and $B \ll A$, then A and B are unitarily equivalent, which shows that the relation $\ll$ is a partial order modulo unitary equivalence.

1. Introduction. Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. The set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\boldsymbol{B}(\mathcal{H}, \mathcal{K})$. We shall abbreviate $\boldsymbol{B}(\mathcal{H}, \mathcal{H})$ to $\boldsymbol{B}(\mathcal{H})$ and write I for the identity operator on $\mathcal{H}$. The kernel, the range and the adjoint of $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ are denoted by $\mathcal{N}(A)$, $\mathcal{R}(A)$ and A^*, respectively. As usual, $|A| := (A^*A)^{1/2}$ for $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$. We say that two operators $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are *unitarily equivalent* (in symbols $A \cong B$) if there exists a unitary operator $U \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $A = U^*BU$. Given an operator $A \in \boldsymbol{B}(\mathcal{H})$ with closed range, we write $\operatorname{rank} A$ for the orthogonal dimension of $\mathcal{R}(A)$ and call it the *rank* of A. For two vectors $u, v \in \mathcal{H}$, we define the finite rank operator $u \otimes v \in \boldsymbol{B}(\mathcal{H})$ by $(u \otimes v)(h) = \langle h, v\rangle u$ for $h \in \mathcal{H}$. The spectrum of $A \in \boldsymbol{B}(\mathcal{H})$ is denoted by $\sigma(A)$. Given two selfadjoint operators $A, B \in \boldsymbol{B}(\mathcal{H})$, we write $A \leq B$ (or $B \geq A$) if $\langle Ah, h\rangle \leq \langle Bh, h\rangle$ for all $h \in \mathcal{H}$. The spectral measure of a normal operator $A \in \boldsymbol{B}(\mathcal{H})$ is denoted by E_A. Recall that if $A \in \boldsymbol{B}(\mathcal{H})$ is a normal operator, then

$$\sigma(A) \text{ is equal to the closed support of } E_A. \tag{1.1}$$

For this and related facts concerning normal operators we refer the reader to the monographs [2] and [17].

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Given an operator $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ and two normal operators $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$, we write $A \ll_T B$ if

$$E_A(\Delta) \le T^* E_B(\Delta) T \quad \text{for all } \Delta \in \mathfrak{B}(\mathbb{C}), \tag{1.2}$$

where $\mathfrak{B}(\mathbb{C})$ stands for the σ-algebra of all Borel subsets of the complex plane $\mathbb{C}$. We shall abbreviate (1.2) to $E_A \le T^* E_B T$ (the same convention will be applied to other relations involving operator functions of Δ). We write $A \ll B$ if $A \ll_T B$ for some $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$. It is clear that the relation $\ll$ is reflexive and transitive. It is also obvious that if two normal operators $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are unitarily equivalent, then $A \ll B$ and $B \ll A$. Moreover, the relation $\ll$ is compatible with $\cong$ (i.e., $A \ll B$, $A \cong A'$ and $B \cong B'$ imply $A' \ll B'$). Theorem 5.2, which is the main result of this paper, states that if $A \ll B$ and $B \ll A$, then $A \cong B$. This means that the relation $\ll$ is a partial order modulo unitary equivalence.

It is worth mentioning that if $A \ll B$ and $B \ll A$, then the spectral measures E_A and E_B are mutually absolutely continuous, and so their spectral types coincide (cf. [2] for the terminology). Since, in general, the equality of spectral types does not imply the equality of multiplicity functions, one cannot expect normal operators with equal spectral types to be unitarily equivalent ([1]) (e.g. two normal operators with the same pure point spectra but of different multiplicity are not unitarily equivalent). However, according to the well-known result [3, Theorem 11.4], two star-cyclic normal operators are unitarily equivalent if and only if their spectral types coincide. In view of the above discussion, it is clear that this result implies a particular case of Theorem 5.2 for star-cyclic normal operators. Proofs of these two theorems are completely different.

The concept of the relation $\ll$ is somewhat linked to the binary relation $\le_u$, where $A \le_u B$ means that $A, B \in \boldsymbol{B}(\mathcal{H})$ are selfadjoint and $A \le U^* B U$ for some unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ (cf. [9, 11]). Note that if $B \ge 0$, then $A \ll_U B$ implies $A \le U^* B U$ (cf. Proposition 2.1), but the converse is not true, as is immediately seen by taking $A = 0$ and $B = U = I$. Though selfadjoint trace class operators A, B which satisfy $A \le_u B$ and $B \le_u A$ are necessarily unitarily equivalent, we can construct a unitary operator U and selfadjoint operators A, B such that $B \le A \le U^* B U$ and $A \ncong B$. It is shown in [11, Theorem 5] that if $B \le A \le U^* B U$, where A and B are selfadjoint operators with null spectra with respect to the Lebesgue measure on the real line $\mathbb{R}$, and U is a unitary operator whose spectrum does not fill up the whole unit circle, then $B = A = U^* B U$.

The *spectral order* $\preccurlyeq$, another concept which is linked to the relation $\ll$, is defined as follows: given two selfadjoint operators $A, B \in \boldsymbol{B}(\mathcal{H})$, we

([1]) Recall that the spectral type and the multiplicity function form a complete set of unitary invariants for normal operators (cf. [2, Theorem 7.4.3]).

write $A \preccurlyeq B$ if

$$E_B((-\infty,\ \lambda]) \leq E_A((-\infty,\ \lambda]), \qquad \lambda \in \mathbb{R}.$$

For commuting pairs, the spectral order $\preccurlyeq$ agrees with the usual one $\leq$. In general, these two notions do not coincide (see [7, 12, 8, 13]). Note that $B \ll_I A$ implies $A \preccurlyeq B$, but the converse is not true because, in general, $A \preccurlyeq B$ does not imply $A \cong B$, while $B \ll_I A$ does (see Corollary 3.4).

Some basic facts concerning the relation $\ll$ are included in Proposition 2.1. The case when the spectral measure E_A is a perturbation of the positive operator valued measure T^*E_BT by the Dirac measure at 0 with coefficient $I - T^*T$ is discussed in Theorem 2.3. Theorem 3.1 asserts that $A \ll B$ if and only if B decomposes into an orthogonal sum of two operators one of which is unitarily equivalent to A, or equivalently if and only if there exists an operator $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ with dense range which intertwines B and A. In turn, Theorem 4.2 states that $A \ll B$ if and only if $S^*E_AS \leq E_B$ for some operator $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ with dense range. The concepts of the paper are illustrated by examples in Section 6. In the Appendix we discuss conditions (iii) and (iv′) of Theorem 3.1.

2. Basic properties of $\ll$. We begin by formulating some preparatory facts concerning the relation $\ll$.

PROPOSITION 2.1. *Suppose that $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators and $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$. If $A \ll_T B$, then*

- (i) $0 \leq T^*E_B(\Delta)T - E_A(\Delta) \leq T^*T - I$ *for* $\Delta \in \mathfrak{B}(\mathbb{C})$,
- (ii) $\sigma(A) \subseteq \sigma(B)$,
- (iii) $\operatorname{rank} E_A(\Delta) \leq \operatorname{rank} E_B(\Delta)$ *for all* $\Delta \in \mathfrak{B}(\mathbb{C})$,
- (iv) $\phi(A) \leq T^*\phi(B)T$ *for any bounded nonnegative Borel function* ϕ *on* $\sigma(B)$,
- (v) $\phi(A) \ll_T \phi(B)$ *for any bounded complex Borel function* ϕ *on* $\sigma(B)$,
- (vi) $|A|^n \leq T^*|B|^nT$ *for all integers* $n \geq 0$,
- (vii) $A^n \leq T^*B^nT$ *for all integers* $n \geq 0$ *provided that* $B \geq 0$.

Proof. (i) Substitute $\mathbb{C} \setminus \Delta$ into (1.2) in place of Δ.

(ii) By (1.1), we have

$$E_A(\mathbb{C} \setminus \sigma(B)) \leq T^*E_B(\mathbb{C} \setminus \sigma(B))T = 0,$$

which implies that $E_A(\mathbb{C} \setminus \sigma(B)) = 0$. Hence $\mathbb{C} \setminus \sigma(B) \subseteq \mathbb{C} \setminus \sigma(A)$.

(iii) By assumption, $E_A(\Delta) \leq (T^*E_B(\Delta))(T^*E_B(\Delta))^*$. This combined with [4, Theorem 1] yields $\mathcal{R}(E_A(\Delta)) \subseteq T^*\mathcal{R}(E_B(\Delta))$. Since (bounded and linear) operators with dense range do not increase the orthogonal dimension, we have

$$\operatorname{rank} E_A(\Delta) \leq \dim \overline{T^*\mathcal{R}(E_B(\Delta))} \leq \dim \mathcal{R}(E_B(\Delta)) = \operatorname{rank} E_B(\Delta).$$

(iv) By (ii), we have

$$\begin{aligned}\langle\phi(A)h,h\rangle &= \int\limits_{\sigma(B)}\phi(z)\langle E_A(dz)h,h\rangle\\ &\le \int\limits_{\sigma(B)}\phi(z)\langle E_B(dz)Th,Th\rangle = \langle T^*\phi(B)Th,h\rangle, \qquad h\in\mathcal{H}.\end{aligned}$$

(v) Employ (ii) and the fact that for a normal operator $N\in\boldsymbol{B}(\mathcal{H})$ and a bounded complex Borel function ϕ on $\sigma(N)$,

$$E_{\phi(N)}(\Delta)=E_N(\phi^{-1}(\Delta)), \qquad \Delta\in\mathfrak{B}(\mathbb{C}).$$

Conditions (vi) and (vii) follow from (ii) and (iv). ∎

It follows from Proposition 2.1(i) that $A\ll_T B$ implies

$$E_A(\Delta)\ge T^*E_B(\Delta)T+I-T^*T, \qquad \Delta\in\mathfrak{B}(\mathbb{C}).$$

Replacing Δ by $\mathbb{C}\setminus\Delta$, we see that the converse is true as well.

In view of Proposition 2.1(i), the inequality $I\le T^*T$ is necessary for the relation $A\ll_T B$ to hold. Hence $A\ll_T B$ and $\|T\|\le 1$ imply that T is an isometry. In this particular situation the inequality (iv) in Proposition 2.1 turns into equality.

Corollary 2.2. *Suppose that $A\in\boldsymbol{B}(\mathcal{H})$ and $B\in\boldsymbol{B}(\mathcal{K})$ are normal operators and $T\in\boldsymbol{B}(\mathcal{H},\mathcal{K})$ is a contraction. If $A\ll_T B$, then $\phi(A)=T^*\phi(B)T$ for any bounded complex Borel function ϕ on $\sigma(B)$.*

Proof. Since T is an isometry, we infer from Proposition 2.1(i) that

$$E_A(\Delta)=T^*E_B(\Delta)T, \qquad \Delta\in\mathfrak{B}(\mathbb{C}). \tag{2.1}$$

Arguing as in the proof of Proposition 2.1(iv) completes the proof. ∎

By Corollary 2.2, if $A\ll_T B$ and T is an isometry, then

$$A^mA^{*n}=T^*B^mB^{*n}T, \qquad m,n\ge 0,\ m+n\ge 1. \tag{2.2}$$

We now discuss in more detail the relationship between (2.2) and $A\ll_T B$. In what follows, χ_Δ stands for the characteristic function of a subset Δ of $\mathbb{C}$.

Theorem 2.3. *Let $A\in\boldsymbol{B}(\mathcal{H})$ and $B\in\boldsymbol{B}(\mathcal{K})$ be normal operators and $T\in\boldsymbol{B}(\mathcal{H},\mathcal{K})$. Then (2.2) holds if and only if*

$$E_A(\Delta)=T^*E_B(\Delta)T+\chi_\Delta(0)(I-T^*T), \qquad \Delta\in\mathfrak{B}(\mathbb{C}). \tag{2.3}$$

Moreover, if (2.2) holds, then

(i) *T is an isometry if and only if $E_A=T^*E_BT$,*
(ii) *$I\le T^*T$ if and only if $E_A\le T^*E_BT$,*
(iii) *$T^*T\le I$ if and only if $T^*E_BT\le E_A$,*
(iv) *if $T^*T\ne I$, then $0\in\sigma(A)\cup\sigma(B)$,*
(v) *if $0\in\sigma(B)$, then $\sigma(A)\subseteq\sigma(B)$,*
(vi) *if $0\notin\sigma(B)$, then T is a contraction.*

Proof. Set $\Omega = \sigma(A) \cup \sigma(B) \cup \{0\}$.

We first prove that (2.2) implies (2.3) and assertions (i) to (vi). Assume that (2.2) holds. Take a continuous function $f\colon \Omega \to \mathbb{C}$. By the Stone–Weierstrass theorem, there exists a sequence $\{p_k\}_{k=1}^{\infty}$ of complex polynomials in two complex variables such that $\lim_{k\to\infty} \sup_{z\in\Omega} |f(z) - p_k(z,\bar{z})| = 0$. It follows from (2.2) that

$$p_k(A, A^*) = T^* p_k(B, B^*) T + p_k(0,0)(I - T^*T), \qquad k \geq 1. \tag{2.4}$$

After passage to the limit in (2.4) as $k \to \infty$ (which is possible due to the uniform continuity of the Stone–von Neumann operator calculus), we obtain

$$f(A) = T^* f(B) T + f(0)(I - T^*T), \qquad f\colon \Omega \to \mathbb{C} \text{ continuous.} \tag{2.5}$$

Fix $\Delta \in \mathfrak{B}(\mathbb{C})$ and $h \in \mathcal{H}$. Since the measures $\mu(\cdot) := \langle E_A(\cdot)h, h\rangle$ and $\nu(\cdot) := \langle E_B(\cdot)Th, Th\rangle$ are regular (cf. [15, Theorem 2.18]), there exist an ascending sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets in $\mathbb{C}$ and a descending sequence $\{G_n\}_{n=1}^{\infty}$ of open sets in $\mathbb{C}$ such that $K_n \subseteq \Delta \subseteq G_n$, $\mu(G_n \setminus K_n) < 1/n$ and $\nu(G_n \setminus K_n) < 1/n$. There is no loss of generality in assuming that $0 \in K_n$ for all $n \geq 1$ provided that $0 \in \Delta$, and $0 \notin G_n$ for all $n \geq 1$ provided that $0 \notin \Delta$. Set $K_\infty = \bigcup_{n=1}^{\infty} K_n$ and $G_\infty = \bigcap_{n=1}^{\infty} G_n$. Then $K_\infty, G_\infty \in \mathfrak{B}(\mathbb{C})$, $K_\infty \subseteq \Delta \subseteq G_\infty$, $\mu(G_\infty \setminus K_\infty) = 0$ and $\nu(G_\infty \setminus K_\infty) = 0$. For every $n \geq 1$ there exists a continuous function $f_n\colon \mathbb{C} \to [0,1]$ such that $f_n(K_n) = \{1\}$ and $f_n(\mathbb{C} \setminus G_n) = \{0\}$. It is now clear that $\lim_{n\to\infty} f_n(z) = \chi_\Delta(z)$ for all $z \in \mathbb{C} \setminus (G_\infty \setminus K_\infty)$. Hence $\lim_{n\to\infty} f_n(0) = \chi_\Delta(0)$ and $\lim_{n\to\infty} f_n(z) = \chi_\Delta(z)$ for almost every $z \in \mathbb{C}$ with respect to μ and ν. By Lebesgue's dominated convergence theorem, we have

$$\|f_n(A)h - E_A(\Delta)h\|^2 = \int_{\mathbb{C}} |f_n - \chi_\Delta|^2 \, d\mu \to 0 \quad \text{as } n \to \infty.$$

Similar reasoning gives $\lim_{n\to\infty} T^* f_n(B) Th = T^* E_B(\Delta) Th$. Hence, by (2.5), we have $E_A(\Delta)h = T^* E_B(\Delta) Th + \chi_\Delta(0)(I - T^*T)h$, which yields (2.3).

(i) Necessity follows from (2.3), while sufficiency from (2.1) with $\Delta = \mathbb{C}$.

(ii) Necessity follows from (2.3), while sufficiency from Proposition 2.1(i).

(iii) Argue as in the proof of (ii).

(iv) Suppose contrary to our claim that $0 \notin \sigma(A) \cup \sigma(B)$. Then there exists an open set Δ in $\mathbb{C}$ such that $0 \in \Delta$ and $\Delta \cap (\sigma(A) \cup \sigma(B)) = \emptyset$. Hence, by (1.1) and (2.3), we have $T^*T = I$, which is a contradiction.

(v) This can be deduced from (2.3) by substituting $\Delta = \mathbb{C} \setminus \sigma(B)$.

(vi) If $0 \notin \sigma(B)$, then there exists an open set Δ in $\mathbb{C}$ such that $0 \in \Delta$ and $\Delta \cap \sigma(B) = \emptyset$. This and (2.3) imply that $I - T^*T = E_A(\Delta) \geq 0$.

To complete the proof it is therefore enough to show that (2.3) implies (2.2). It follows from (2.3) that

$$\phi(A) = T^* \phi(B) T + \phi(0)(I - T^*T) \tag{2.6}$$

for any simple Borel function $\phi\colon \mathbb{C} \to \mathbb{C}$. Applying the standard approximation procedure, we see that (2.6) holds for any bounded complex Borel function ϕ on Ω. In particular, substituting $\phi(z) = z^m \bar{z}^n$ ($z \in \Omega$) into (2.6), we get (2.2). ∎

Corollary 2.4. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be selfadjoint operators and $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$. Then $A^k = T^* B^k T$ for all $k \geq 1$ if and only if*

$$(2.7)\quad E_A(\Delta) = T^* E_B(\Delta) T + \chi_\Delta(0)(I - T^*T) \quad \textit{for all Borel sets } \Delta \subseteq \mathbb{R}.$$

Moreover, if (2.7) *holds, then the assertions* (i) *to* (vi) *of Theorem* 2.3 *hold.*

Combining Corollaries 2.2 and 2.4, we see that if $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are selfadjoint operators and $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ is an isometry, then $E_A \leq T^* E_B T$ if and only if $E_A = T^* E_B T$, or equivalently if and only if $A^k = T^* B^k T$ for all $k \geq 1$.

For examples illustrating Theorem 2.3 we refer the reader to Section 6 (see Examples 6.1 and 6.2). It is shown there that (2.3) may hold with $I - T^*T \neq 0$.

3. Characterizations of $\ll$

Theorem 3.1. *Suppose that $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators. Then the following conditions are equivalent:*

(i) *there exists $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $A \ll_T B$,*
(ii) *there exist $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ and a surjective contraction $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ such that $VB = AV$ and $VT = I$,*
(ii′) *there exists an operator $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ with dense range such that $VB = AV$,*
(iii) *there exists an isometry $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $A = W^* B W$ and $\mathcal{R}(W)$ reduces B,*
(iv) *there exists an isometry $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $E_A = W^* E_B W$ and $\mathcal{R}(W)$ reduces B,*
(iv′) *there exists an isometry $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $E_A = W^* E_B W$.*

Moreover, if T is as in (i), *then the same T can be chosen in* (ii), *and vice versa. This is also true for W appearing in* (iii) *and* (iv).

Remark 3.2. By the Putnam–Fuglede theorem (cf. [14]), if $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators and $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$, then $VB = AV$ if and only if $VE_B = E_A V$. Arguing as in the proof of the implication (iii)⇒(iv), we see that (iii) is equivalent to the existence of a closed linear subspace $\mathcal{N}$ of $\mathcal{K}$ reducing B such that $A \cong B|_{\mathcal{N}}$. For a discussion concerning the assumption that $\mathcal{R}(W)$ reduces B, which appears in (iii) and (iv), we refer the reader to the Appendix. It is clear that if $E_A = W^* E_B W$, then W

is an isometry. It follows from Proposition 7.3 in the Appendix that if W is as in (iv$'$), then $\mathcal{R}(W)$ reduces B.

Proof of Theorem 3.1. (i)$\Rightarrow$(ii). Let $\varDelta_1, \ldots, \varDelta_n \in \mathfrak{B}(\mathbb{C})$ and $h_1, \ldots, h_n \in \mathcal{H}$ be fixed finite systems. Then there exist finite systems $\varDelta'_1, \ldots, \varDelta'_m \in \mathfrak{B}(\mathbb{C})$ and $J_1, \ldots, J_n \subseteq \{1, \ldots, m\}$ such that $\varDelta'_k \cap \varDelta'_l = \emptyset$ for all $k \neq l$, and $\varDelta_i = \bigcup_{j \in J_i} \varDelta'_j$ for all $i \in \{1, \ldots, n\}$. Set $h'_j = \sum_{i=1}^n \chi_{J_i}(j) h_i$ for $j \in \{1, \ldots, m\}$. Then we have

$$\begin{aligned}
\text{(3.1)} \qquad \Big\| \sum_{i=1}^n E_A(\varDelta_i) h_i \Big\|^2 &= \Big\| \sum_{i=1}^n \sum_{j \in J_i} E_A(\varDelta'_j) h_i \Big\|^2 \\
&= \Big\| \sum_{i=1}^n \sum_{j=1}^m \chi_{J_i}(j) E_A(\varDelta'_j) h_i \Big\|^2 \\
&= \Big\| \sum_{j=1}^m E_A(\varDelta'_j) \Big(\sum_{i=1}^n \chi_{J_i}(j) h_i \Big) \Big\|^2 \\
&= \sum_{j=1}^m \langle E_A(\varDelta'_j) h'_j, h'_j \rangle.
\end{aligned}$$

Similar reasoning leads to

$$\text{(3.2)} \qquad \Big\| \sum_{i=1}^n E_B(\varDelta_i) T h_i \Big\|^2 = \sum_{j=1}^m \langle E_B(\varDelta'_j) T h'_j, T h'_j \rangle.$$

Since $E_A \leq T^* E_B T$, we infer from (3.1) and (3.2) that

$$\text{(3.3)} \qquad \Big\| \sum_{i=1}^n E_A(\varDelta_i) h_i \Big\|^2 \leq \Big\| \sum_{i=1}^n E_B(\varDelta_i) T h_i \Big\|^2$$

for all finite systems $\varDelta_1, \ldots, \varDelta_n \in \mathfrak{B}(\mathbb{C})$ and $h_1, \ldots, h_n \in \mathcal{H}$. Define the space $\mathcal{N}$ by (2) $\mathcal{N} = \bigvee_{\varDelta \in \mathfrak{B}(\mathbb{C})} E_B(\varDelta) \mathcal{R}(T)$. It is clear that $\mathcal{R}(T) \subseteq \mathcal{N}$ and $\mathcal{N}$ reduces E_B. It follows from (3.3) that there exists a unique contraction $V_0 \in \boldsymbol{B}(\mathcal{N}, \mathcal{H})$ such that $V_0 E_B(\varDelta) T h = E_A(\varDelta) h$ for all $\varDelta \in \mathfrak{B}(\mathbb{C})$ and $h \in \mathcal{H}$. Define $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ by $Vf = V_0 P f$ for $f \in \mathcal{K}$, where $P \in \boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{N}$. Then V is a contractive linear extension of V_0 such that

$$\text{(3.4)} \qquad V E_B(\varDelta) T = E_A(\varDelta), \qquad \varDelta \in \mathfrak{B}(\mathbb{C}).$$

Substituting $\varDelta = \mathbb{C}$ into (3.4), we get $VT = I$. It follows from (3.4) that

$$\begin{aligned}
V E_B(\varDelta)(E_B(\varDelta') T h) &= V E_B(\varDelta \cap \varDelta') T h \\
&= E_A(\varDelta \cap \varDelta') h = E_A(\varDelta) V (E_B(\varDelta') T h)
\end{aligned}$$

(2) $\bigvee_{\varDelta \in \mathfrak{B}(\mathbb{C})} E_B(\varDelta) \mathcal{R}(T)$ stands for the closure of the linear span of the set $\bigcup_{\varDelta \in \mathfrak{B}(\mathbb{C})} E_B(\varDelta) \mathcal{R}(T)$.

for all $h \in \mathcal{H}$ and $\varDelta, \varDelta' \in \mathfrak{B}(\mathbb{C})$. Hence $VE_B(\varDelta)|_{\mathcal{N}} = E_A(\varDelta)V|_{\mathcal{N}}$ for all $\varDelta \in \mathfrak{B}(\mathbb{C})$. Since $\mathcal{N}$ reduces E_B, we deduce that $VE_B(\varDelta)|_{\mathcal{K}\ominus\mathcal{N}} = 0 = E_A(\varDelta)V|_{\mathcal{K}\ominus\mathcal{N}}$ for all $\varDelta \in \mathfrak{B}(\mathbb{C})$. Hence $VE_B = E_AV$, which implies $VB = AV$.

(ii)⇒(ii′). Evident.

(ii′)⇒(iii). Let $V = U|V|$ be the polar decomposition of V. Then $U \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ is a partial isometry with initial space $\overline{\mathcal{R}(|V|)}$ and final space $\mathcal{H}$ (because $\overline{\mathcal{R}(V)} = \mathcal{H}$). By the Putnam–Fuglede theorem, $VB = AV$ implies $V^*A = BV^*$. This yields

$$|V|^2B = V^*VB = V^*AV = BV^*V = B|V|^2.$$

By the square root theorem, $|V|B = B|V|$ and thus $\overline{\mathcal{R}(|V|)}$ reduces B. Hence

$$UB(|V|f) = U|V|Bf = VBf = AVf = AU(|V|f), \qquad f \in \mathcal{K},$$

which yields $UB|_{\overline{\mathcal{R}(|V|)}} = AU|_{\overline{\mathcal{R}(|V|)}}$. Since $UB|_{\mathcal{K}\ominus\overline{\mathcal{R}(|V|)}} = 0 = AU|_{\mathcal{K}\ominus\overline{\mathcal{R}(|V|)}}$, we get $UB = AU$. Set $W = U^*$. Then W is an isometry, $W^*BW = UBU^* = AUU^* = A$ and $\mathcal{R}(W) = \mathcal{R}(U^*) = \overline{\mathcal{R}(|V|)}$, which shows that $\mathcal{R}(W)$ reduces B.

(iii)⇒(iv). Since W is an isometry, the space $\mathcal{R}(W)$ is closed. Define the operator $\widehat{W} \in \boldsymbol{B}(\mathcal{H}, \mathcal{R}(W))$ by $\widehat{W}h = Wh$ for $h \in \mathcal{H}$. Then $\mathcal{R}(W)$ reduces B, $\widehat{W}$ is a unitary operator and $A = \widehat{W}^*(B|_{\mathcal{R}(W)})\widehat{W}$. This implies that $\mathcal{R}(W)$ reduces E_B, and $E_A = \widehat{W}^*(E_B|_{\mathcal{R}(W)})\widehat{W}$. Hence $E_A = W^*E_BW$, as is easily verified.

The implications (iv)⇒(iv′) and (iv′)⇒(i) are obvious.

We now proceed to the proof of the "moreover" part. If T is as in (i), then in view of the proof of (i)⇒(ii) the same T can be chosen in (ii). Conversely, if T is as in (ii), then by the Putnam–Fuglede theorem $VE_B = E_AV$, and therefore

$$\begin{aligned}\langle E_A(\varDelta)h, h\rangle &= \|E_A(\varDelta)VTh\|^2 = \|VE_B(\varDelta)Th\|^2 \\ &\le \|E_B(\varDelta)Th\|^2 = \langle E_B(\varDelta)Th, Th\rangle, \qquad h \in \mathcal{H},\ \varDelta \in \mathfrak{B}(\mathbb{C}).\end{aligned}$$

In turn, if W is as in (iii), then by the proof of (iii)⇒(iv) the same W can be chosen in (iv). Conversely, if W is as in (iv), then obviously the same W works for (iii). This completes the proof. ∎

COROLLARY 3.3. *If $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators such that $E_A = T^*E_BT$ for some $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ and $\bigvee_{\varDelta \in \mathfrak{B}(\mathbb{C})} E_B(\varDelta)\mathcal{R}(T) = \mathcal{K}$, then T is a unitary operator and $A \cong B$.*

Proof. Taking a quick look at the proof of the implication (i)⇒(ii) of Theorem 3.1, we conclude that $V = V_0$ is a unitary operator, $VT = I$ and $VB = AV$. This implies that $T = V^{-1}$ and therefore T is a unitary operator. ∎

Corollary 3.4. *Let* $A \in \boldsymbol{B}(\mathcal{H})$ *and* $B \in \boldsymbol{B}(\mathcal{K})$ *be normal operators and* $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$. *Suppose that* $A \ll_T B$. *If any of the following two conditions is satisfied, then* $A \cong B$:

(i) T *has dense range.*
(ii) $\mathcal{H}$ *is finite-dimensional.*

Proof. (i) Let V be as in Theorem 3.1(ii). Since $T^*T \geq I$, the range of T is closed, and so $\mathcal{R}(T) = \mathcal{K}$. As a consequence, we see that T is a bijection. It follows from $VT = I$ that V is a bijection as well. Since $VB = AV$, the normal operators A and B are similar, and so they are automatically unitarily equivalent (cf. [14]).

(ii) It follows from $T^*T \geq I$ that T is injective and thus surjective. Applying the previous case completes the proof. ∎

A normal operator $A \in \boldsymbol{B}(\mathcal{H})$ is said to be *star-cyclic* if there exists a vector $e \in \mathcal{H}$, called a *star-cyclic vector* of A, such that the closure of the linear span of the vectors $\{A^m A^{*n} e \colon m, n \geq 0\}$ is equal to $\mathcal{H}$.

Corollary 3.5. *If* $A \in \boldsymbol{B}(\mathcal{H})$ *and* $B \in \boldsymbol{B}(\mathcal{K})$ *are normal operators such that* $A \ll B$ *and* B *is star-cyclic, then* A *is star-cyclic.*

Proof. Let V be as in Theorem 3.1(ii) and let e be a star-cyclic vector of B. It follows from $VB = AV$ and the Putnam–Fuglede theorem that $VB^m B^{*n} e = A^m A^{*n} V e$ for all $m, n \geq 0$. This and $\mathcal{R}(V) = \mathcal{H}$ imply that Ve is a star-cyclic vector (3) of A. ∎

Remark 3.6. Note that Corollary 3.3 is also a direct consequence of the following fact whose proof goes through as for Theorem 3.1.

> *If* $A \in \boldsymbol{B}(\mathcal{H})$ *and* $B \in \boldsymbol{B}(\mathcal{K})$ *are normal operators and* $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$, *then* $E_A = T^* E_B T$ *if and only if there exists a* (*unique*) *partial isometry* $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ *with initial space* $\bigvee_{\Delta \in \mathfrak{B}(\mathbb{C})} E_B(\Delta)\mathcal{R}(T)$ *and final space* $\mathcal{H}$ *such that* $VT = I$ *and* $VB = AV$.

By Proposition 7.3, if $E_A = T^* E_B T$, then $\bigvee_{\Delta \in \mathfrak{B}(\mathbb{C})} E_B(\Delta)\mathcal{R}(T) = \mathcal{R}(T)$.

4. Variations on $\ll$**.** In this section we analyze the connections between the relations $E_A \leq T^* E_B T$ and $S^* E_A S \leq E_B$. We begin with the following simple observation.

Proposition 4.1. *If* $A \in \boldsymbol{B}(\mathcal{H})$ *and* $B \in \boldsymbol{B}(\mathcal{K})$ *are normal operators and* $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ *is a bijection such that* $S^* E_A S \leq E_B$, *then* $A \cong B$.

Proof. It follows from $S^* E_A S \leq E_B$ that $A \ll_{S^{-1}} B$. Hence, by Corollary 3.4, the operators A and B are unitarily equivalent. ∎

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(3) We can also show that if e is a star-cyclic vector of B and W is as in Theorem 3.1(iii), then $W^* e$ is a star-cyclic vector of A.

It turns out that if the assumption on the bijectivity of S in Proposition 4.1 is weakened to the requirement that S have dense range, then $A \ll B$.

THEOREM 4.2. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be normal operators. Then the following two conditions are equivalent:*

(i) $A \ll B$,
(ii) *there exists $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ with dense range such that $S^* E_A S \leq E_B$.*

Moreover, the operator S in (ii) *can always be chosen to be a surjection.*

Proof. (i)⇒(ii). Take $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ as in Theorem 3.1(iv). Then the operator $S := W^*$ is a surjection. Since $\mathcal{R}(W)$ reduces B, and WW^* is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}(W)$, we deduce that WW^* commutes with B and thus with E_B. This fact combined with $E_A = W^* E_B W$ gives

$$\langle S^* E_A(\varDelta) S f, f\rangle = \|WW^* E_B(\varDelta) f\|^2 \leq \langle E_B(\varDelta) f, f\rangle, \quad f \in \mathcal{K},\ \varDelta \in \mathfrak{B}(\mathbb{C}).$$

(ii)⇒(i). As in the proof of the implication (i)⇒(ii) of Theorem 3.1 we show that

$$\Big\| \sum_{i=1}^{n} E_A(\varDelta_i) S f_i \Big\|^2 \leq \Big\| \sum_{i=1}^{n} E_B(\varDelta_i) f_i \Big\|^2$$

for all finite systems $\varDelta_1, \ldots, \varDelta_n \in \mathfrak{B}(\mathbb{C})$ and $f_1, \ldots, f_n \in \mathcal{K}$. Hence, there exists a unique contraction $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ such that

$$V E_B(\varDelta) f = E_A(\varDelta) S f, \quad f \in \mathcal{K},\ \varDelta \in \mathfrak{B}(\mathbb{C}). \tag{4.1}$$

This implies that $Vf = VE_B(\mathbb{C})f = E_A(\mathbb{C})Sf = Sf$ for all $f \in \mathcal{K}$, which means that $V = S$. As a consequence of (4.1), we have $SE_B(\varDelta) = E_A(\varDelta)S$ for all $\varDelta \in \mathfrak{B}(\mathbb{C})$, which by the Putnam–Fuglede theorem is equivalent to $SB = AS$. As $\overline{\mathcal{R}(S)} = \mathcal{H}$, we infer from Theorem 3.1(ii′) that $A \ll B$. ∎

It follows from the proof of the implication (i)⇒(ii) of Theorem 4.2 that if $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ is an isometry such that $A = W^* B W$ and $\mathcal{R}(W)$ reduces B (cf. Theorem 3.1(iii)), then $E_A = W^* E_B W$ and $W E_A W^* \leq E_B$.

The relation $S^* E_A S \leq E_B$, when considered in a general setting, can be characterized as follows.

PROPOSITION 4.3. *Assume that $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators and $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$. Then the following conditions are equivalent:*

(i) $S^* E_A S \leq E_B$,
(ii) *S is a contraction and $SB = AS$.*

Moreover, $S^* E_A S = E_B$ if and only if S is an isometry and $SB = AS$.

Proof. (i)⇒(ii). Argue as in the proof of the implication (ii)⇒(i) of Theorem 4.2.

(ii)$\Rightarrow$(i). By the Putnam–Fuglede theorem, we have $SE_B(\varDelta) = E_A(\varDelta)S$ and $S^*E_A(\varDelta) = E_B(\varDelta)S^*$ for all $\varDelta \in \mathfrak{B}(\mathbb{C})$. Hence

$$S^*SE_B(\varDelta) = S^*E_A(\varDelta)S = E_B(\varDelta)S^*S, \quad \varDelta \in \mathfrak{B}(\mathbb{C}).$$

This fact combined with $\|S\| \le 1$ yields

$$\begin{aligned}\langle S^*E_A(\varDelta)Sf, f\rangle &= \langle S^*SE_B(\varDelta)f, f\rangle = \|(S^*S)^{1/2}E_B(\varDelta)f\|^2 \\ &\le \|E_B(\varDelta)f\|^2 = \langle E_B(\varDelta)f, f\rangle, \quad f \in \mathcal{K},\ \varDelta \in \mathfrak{B}(\mathbb{C}),\end{aligned}$$

which means that $S^*E_AS \le E_B$.

The proof of the "moreover" part proceeds along the same lines as that of (i)$\Leftrightarrow$(ii) (see also Appendix). ∎

As a consequence of Proposition 4.3, we see that if $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ and $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ are such that V is a contraction, $VB = AV$ and $VT = I$ (cf. Theorem 3.1(ii)), then $E_A \le T^*E_BT$ and $V^*E_AV \le E_B$.

Corollary 4.4. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be normal operators.*

(i) *If $S^*E_AS \le E_B$ for some $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$, then there exist closed linear subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ and $\mathcal{K}$, respectively, such that $\mathcal{M}$ reduces A, $\mathcal{N}$ reduces B, $A|_{\mathcal{M}} \cong B|_{\mathcal{N}}$ and* $\dim \mathcal{M} = \dim \mathcal{N} = \dim \overline{\mathcal{R}(S)}$.

(ii) *If there exist closed linear subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ and $\mathcal{K}$, respectively, such that $\mathcal{M}$ reduces A, $\mathcal{N}$ reduces B, and $A|_{\mathcal{M}} \cong B|_{\mathcal{N}}$, then $S^*E_AS \le E_B$ for some partial isometry $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ with initial space $\mathcal{N}$ and final space $\mathcal{M}$; in particular,* $\dim \mathcal{M} = \dim \mathcal{N} = \dim \mathcal{R}(S)$.

Proof. (i) It follows from Proposition 4.3 that $SB = AS$. Hence, by [5, Lemma 4.1], the spaces $\mathcal{M} := \overline{\mathcal{R}(S)}$ and $\mathcal{N} := \mathcal{K} \ominus \mathcal{N}(S)$ have all the required properties.

(ii) By assumption, there exists a unitary operator $U \in \boldsymbol{B}(\mathcal{N}, \mathcal{M})$ such that $B|_{\mathcal{N}} = U^*A|_{\mathcal{M}}U$. This implies that $\mathcal{M}$ reduces E_A, $\mathcal{N}$ reduces E_B, and $E_B|_{\mathcal{N}} = U^*E_A|_{\mathcal{M}}U$. Define $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ by $Sf = UPf$ for $f \in \mathcal{K}$, where $P \in \boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{N}$. Clearly, the operator S is a partial isometry with initial space $\mathcal{N}$ and final space $\mathcal{M}$. Since $\mathcal{N}$ reduces E_B, we have $PE_B(\varDelta) = E_B(\varDelta)P$ for all $\varDelta \in \mathfrak{B}(\mathbb{C})$. Hence

$$\begin{aligned}&\langle S^*E_A(\varDelta)Sf, f\rangle = \langle U^*E_A(\varDelta)|_{\mathcal{M}}UPf, Pf\rangle = \langle E_B(\varDelta)Pf, Pf\rangle \\ &\quad = \|PE_B(\varDelta)f\|^2 \le \|E_B(\varDelta)f\|^2 = \langle E_B(\varDelta)f, f\rangle, \quad f \in \mathcal{K},\ \varDelta \in \mathfrak{B}(\mathbb{C}),\end{aligned}$$

which means that $S^*E_AS \le E_B$. This completes the proof. ∎

In analogy with the relation $\ll$, we can define a binary relation $\precsim$ for normal operators $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ by: $A \precsim B$ if $S^*E_AS \le E_B$ for some $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$. However, this relation is uninteresting because $A \precsim B$ for all normal operators $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$.

The following corollary should be compared with Theorem 4.2.

COROLLARY 4.5. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be normal operators. Then $S^*E_AS = E_B$ for some $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ if and only if $B \ll A$.*

Proof. The "only if" part is obvious. In turn, the "if" part follows from Theorem 3.1(iv′). ∎

5. Antisymmetry of the relation $\ll$. We start by stating a result originally proved by Ernest (cf. [6, Theorem 1.3]) and recently rediscovered by the present authors. It can be thought of as an operator analogue of the Cantor–Bernstein theorem from elementary set theory. We include its proof (which is different from that given by Ernest) to keep the exposition as self-contained as possible.

THEOREM 5.1. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be arbitrary operators. Suppose there exist closed linear subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ and $\mathcal{K}$, respectively, such that $\mathcal{M}$ reduces A, $\mathcal{N}$ reduces B, $A \cong B|_{\mathcal{N}}$ and $B \cong A|_{\mathcal{M}}$. Then $A \cong B$.*

Proof. We split the proof into two steps.

STEP 1. Let $A \in \boldsymbol{B}(\mathcal{H})$ be an arbitrary operator, $U \in \boldsymbol{B}(\mathcal{H})$ be an isometry and $\mathcal{N}$ be a closed linear subspace of $\mathcal{H}$ such that $\mathcal{R}(U) \subseteq \mathcal{N}$, $\mathcal{R}(U)$ and $\mathcal{N}$ reduce A, and $A = U^*AU$. Then A and $A|_{\mathcal{N}}$ are unitarily equivalent.

Indeed, it follows from Proposition 7.1 in the Appendix that $UA = AU$ and $UA^* = A^*U$. Hence

$$U^nA = AU^n \quad \text{and} \quad U^nA^* = A^*U^n \quad \text{for every integer } n \geq 0. \tag{5.1}$$

This implies that $\mathcal{R}(U^n)$ reduces A for all $n \geq 0$.

Set $\mathcal{H}_{\mathrm{u}} = \bigcap_{n=0}^{\infty} \mathcal{R}(U^n)$ and $\mathcal{M}_n = U^n(\mathcal{H} \ominus \mathcal{R}(U))$ for $n \geq 0$. By the von Neumann–Wold decomposition theorem [10, Theorem 4.7.1], $\mathcal{H} \ominus \mathcal{H}_{\mathrm{u}} = \bigoplus_{n=0}^{\infty} \mathcal{M}_n$. Since $\mathcal{M}_n = \mathcal{R}(U^n) \ominus \mathcal{R}(U^{n+1})$ for all $n \geq 0$ and $\mathcal{R}(U^m)$ reduces A for all $m \geq 0$, we deduce that $\mathcal{H}_{\mathrm{u}}$ and $\mathcal{M}_n$ reduce A for all $n \geq 0$. Set $A_{\mathrm{u}} = A|_{\mathcal{H}_{\mathrm{u}}}$ and $A_n = A|_{\mathcal{M}_n}$ for $n \geq 0$. It follows from (5.1) that

$$A_nU^nh = AU^nh = U^nAh = U^nA_0h, \quad h \in \mathcal{M}_0,\ n \geq 0. \tag{5.2}$$

By assumption, $\mathcal{P} := \mathcal{N} \ominus \mathcal{R}(U)$ reduces A, and $\mathcal{P} \subseteq \mathcal{M}_0$. Hence $\mathcal{P}$ and $\mathcal{Q} := \mathcal{M}_0 \ominus \mathcal{P}$ reduce A_0, and $\mathcal{M}_0 = \mathcal{P} \oplus \mathcal{Q}$. Since $\mathcal{R}(U) = \mathcal{H}_{\mathrm{u}} \oplus \{0\} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$, we get $\mathcal{N} = \mathcal{P} \oplus \mathcal{R}(U) = \mathcal{H}_{\mathrm{u}} \oplus \mathcal{P} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$. Define $X \in \boldsymbol{B}(\mathcal{N}, \mathcal{H})$ by

$$\begin{aligned} X(h_{\mathrm{u}} &\oplus U^0p_0 \oplus U^1(p_1 \oplus q_1) \oplus U^2(p_2 \oplus q_2) \oplus \cdots) \\ &= h_{\mathrm{u}} \oplus U^0(p_0 \oplus q_1) \oplus U^1(p_1 \oplus q_2) \oplus U^2(p_2 \oplus q_3) \oplus \cdots \end{aligned}$$

for $h_{\mathrm{u}} \in \mathcal{H}_{\mathrm{u}}$, $p_n \in \mathcal{P}$ and $q_n \in \mathcal{Q}$. Then, by (5.2), X is unitary and $AX = X(A|_{\mathcal{N}})$.

STEP 2. Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be arbitrary operators. Suppose there exist isometries $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ and $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ such that $A = W^*BW$, $B = V^*AV$, $\mathcal{R}(W)$ reduces B and $\mathcal{R}(V)$ reduces A. Then $A \cong B$.

Indeed, set $U = VW$. The operator U is an isometry on $\mathcal{H}$. It follows from Proposition 7.1 that $WA = BW$, $WA^* = B^*W$, $VB = AV$ and $VB^* = A^*V$. Hence $UA = VBW = AU$ and $UA^* = VB^*W = A^*U$, which implies that $\mathcal{R}(U)$ reduces A. Moreover, $A = W^*BW = W^*V^*AVW = U^*AU$. Applying Step 1 with $\mathcal{N} = \mathcal{R}(V)$, we get $A \cong A|_{\mathcal{R}(V)}$. Since $B = \widehat{V}^*(A|_{\mathcal{R}(V)})\widehat{V}$, where $\widehat{V} \in \boldsymbol{B}(\mathcal{K}, \mathcal{R}(V))$ is the unitary operator defined by $\widehat{V}f = Vf$ for $f \in \mathcal{K}$, we conclude that $A \cong B$.

Using Step 2 and Remark 3.2 completes the proof. ∎

Applying Theorem 3.1(iii) and Step 2 of the proof of Theorem 5.1, we obtain the main result of this paper.

THEOREM 5.2. *If $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators such that $A \ll B$ and $B \ll A$, then A and B are unitarily equivalent.*

COROLLARY 5.3. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be normal operators and let $S \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ and $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ be operators with dense ranges such that $S^*E_AS \leq E_B$ and $T^*E_BT \leq E_A$. Then A and B are unitarily equivalent.*

Proof. Employ Theorems 4.2 and 5.2. ∎

COROLLARY 5.4. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be normal operators. Suppose that there exist operators $V \in \boldsymbol{B}(\mathcal{K}, \mathcal{H})$ and $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ with dense ranges such that $VB = AV$ and $WA = BW$. Then A and B are unitarily equivalent.*

Proof. Apply Theorems 3.1(ii$'$) and 5.2. ∎

6. Examples. In this section $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are assumed to be bounded sequences of complex numbers such that $\lambda_n \neq 0$ for all $n \geq 1$. Let $A \in \boldsymbol{B}(\ell^2)$ and $B \in \boldsymbol{B}(\ell^2)$ be diagonal operators with diagonals $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$, respectively, and let $T \in \boldsymbol{B}(\ell^2)$ be a weighted shift with weights $\{\lambda_n\}_{n=1}^{\infty}$ (i.e., $Ae_n = \alpha_n e_n$, $Be_n = \beta_n e_n$ and $Te_n = \lambda_n e_{n+1}$ for $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ stands for the standard orthonormal basis of ℓ^2). The operators A, B are known to be normal. Their spectral measures are given by

$$E_A(\varDelta) = (\mathrm{SOT}) \sum_{\substack{n \geq 1 \\ \alpha_n \in \varDelta}} e_n \otimes e_n, \qquad \varDelta \in \mathfrak{B}(\mathbb{C}), \tag{6.1}$$

$$E_B(\varDelta) = (\textsc{sot}) \sum_{\substack{n\geq 1\\ \beta_n\in\varDelta}} e_n \otimes e_n, \quad \varDelta \in \mathfrak{B}(\mathbb{C}), \tag{6.2}$$

where both series in (6.1) and (6.2) are convergent in the strong operator topology. Since in general $X(u\otimes v)Y = (Xu)\otimes(Y^*v)$ for $u, v \in \mathcal{H}$ and $X, Y \in \boldsymbol{B}(\mathcal{H})$, we get

$$T^*E_B(\varDelta)T = (\textsc{sot}) \sum_{\substack{n\geq 1\\ \beta_{n+1}\in\varDelta}} |\lambda_n|^2 e_n \otimes e_n, \quad \varDelta \in \mathfrak{B}(\mathbb{C}), \tag{6.3}$$

$$TE_A(\varDelta)T^* = (\textsc{sot}) \sum_{\substack{n\geq 1\\ \alpha_n\in\varDelta}} |\lambda_n|^2 e_{n+1} \otimes e_{n+1}, \quad \varDelta \in \mathfrak{B}(\mathbb{C}). \tag{6.4}$$

All the examples that follow come from the above triplet (A, B, T) by specifying the sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\lambda_n\}_{n=1}^\infty$. We begin with two examples illustrating Theorem 2.3.

Example 6.1. Fix an integer $\varkappa \geq 2$ and a complex number $\gamma \neq 0$, and set

$$\begin{aligned}
&\alpha_1 = \cdots = \alpha_{\varkappa-1} = 0, &&\alpha_n = \gamma \quad \text{for } n \geq \varkappa,\\
&\beta_1 = \cdots = \beta_\varkappa = 0, &&\beta_n = \gamma \quad \text{for } n \geq \varkappa + 1,\\
&\lambda_n = 1 \quad \text{for } n \geq \varkappa.
\end{aligned}$$

We claim that A, B, T satisfy (2.3). Take $\varDelta \in \mathfrak{B}(\mathbb{C})$ and consider four possible cases relating 0 and γ to $\varDelta$. If $0 \notin \varDelta$ and $\gamma \notin \varDelta$, then the equality in (2.3) follows from (6.1) and (6.3) directly. If $0 \notin \varDelta$ and $\gamma \in \varDelta$, then by (6.1) and (6.3) we have

$$E_A(\varDelta) = T^*E_B(\varDelta)T = (\textsc{sot}) \sum_{n=\varkappa}^{\infty} e_n \otimes e_n,$$

which means that the equality in (2.3) is valid. In turn, if $0 \in \varDelta$ and $\gamma \notin \varDelta$, then

$$E_A(\varDelta) - T^*E_B(\varDelta)T \overset{(6.1)\&(6.3)}{=} \sum_{n=1}^{\varkappa-1} (1 - |\lambda_n|^2) e_n \otimes e_n \overset{(6.3)}{=} I - T^*T. \tag{6.5}$$

Finally, if $0 \in \varDelta$ and $\gamma \in \varDelta$, then the equality in (2.3) holds automatically. This proves our claim. The present example can be summarized as follows:

- the operators A, B, T satisfy (2.3),
- A and B are not unitarily equivalent (as $\dim \mathcal{N}(A) < \dim \mathcal{N}(B) < \infty$),
- $\sigma(A) = \sigma(B) = \{0, \gamma\}$,
- any of the two possibilities $A \ll_T B$ and $A \not\ll_T B$ may occur (cf. (6.5)),
- for fixed A and B, there exist infinitely many operators T satisfying (2.3).

Employing (6.2) and (6.4), we can also show that

- $E_B(\Delta) = TE_A(\Delta)T^* + \chi_\Delta(0)(I - TT^*)$ for all $\Delta \in \mathfrak{B}(\mathbb{C})$,
- $I - TT^* = \sum_{n=1}^{\varkappa}(1 - |\lambda_{n-1}|^2)e_n \otimes e_n$ with $\lambda_0 = 0$, and so the relation $B \ll_{T^*} A$ never holds.

Note that if triplets (A_n, B_n, T_n), $n \geq 1$, are as in Example 6.1 (diagonals of A_n, B_n and weights of T_n may vary with n) and $\sup_{n\geq 1}(\|A_n\| + \|B_n\| + \|T_n\|) < \infty$, then the operators $A := \bigoplus_{n=1}^{\infty} A_n$, $B := \bigoplus_{n=1}^{\infty} B_n$ and $T := \bigoplus_{n=1}^{\infty} T_n$ satisfy (2.3), $A \cong B$ and any of the two possibilities $A \ll_T B$ and $A \not\ll_T B$ may occur. Moreover, if $\{T_n\}_{n=1}^{\infty}$ is suitably chosen, then the operator $I - T^*T$ may not be compact.

Example 6.2. Fix a complex number $\gamma \neq 0$, and set

$$\begin{aligned} &\alpha_n = 0 \quad \text{for odd } n \geq 1, && \alpha_n = \gamma \quad \text{for even } n \geq 1,\\ &\beta_n = 0 \quad \text{for even } n \geq 1, && \beta_n = \gamma \quad \text{for odd } n \geq 1,\\ &\lambda_n = 1 \quad \text{for even } n \geq 1. \end{aligned}$$

Now we can verify that

- the operators A, B, T satisfy (2.3),
- A and B are unitarily equivalent (indeed, the operator $U \in \boldsymbol{B}(\ell^2)$ defined by $Ue_{2n} = e_{2n-1}$ and $Ue_{2n-1} = e_{2n}$ for $n \geq 1$ is unitary and $U^*AU = B$),
- $\sigma(A) = \sigma(B) = \{0, \gamma\}$,
- $I - T^*T = (\text{SOT})\sum_{n\geq 1,\, n \text{ odd}}(1 - |\lambda_n|^2)e_n \otimes e_n$, and so any of the two possibilities $A \ll_T B$ and $A \not\ll_T B$ may occur with noncompact $I - T^*T$.

We conclude this section with yet another example showing that $A \ll_T B$ may not imply $A \cong B$.

Example 6.3. Suppose that $\{\beta_n\}_{n=1}^{\infty}$ is a (bounded) strictly decreasing sequence of real numbers, $\alpha_n = \beta_{n+1}$ for all $n \geq 1$, and $\inf_{n\geq 1}|\lambda_n| \geq 1$. It follows from (6.1) and (6.3) that $E_A \leq T^*E_BT$, i.e., $A \ll_T B$. Since $\beta_1 \in \sigma(B) \setminus \sigma(A)$, the operators A and B are not unitarily equivalent. Note also that if $\Delta \in \mathfrak{B}(\mathbb{C})$ and $\beta_1 \notin \Delta$, then by (6.2) and (6.4), $E_B(\Delta) \leq TE_A(\Delta)T^*$. Moreover, if $S \in \boldsymbol{B}(\ell^2)$ is a weighted shift with weights $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \setminus \{0\}$ such that $\sup_{n\geq 1}|\mu_n| \leq 1$, then by (6.1) and (6.3), $S^*E_BS \leq E_A$. Finally, if $\lambda_n = 1$ for all $n \geq 1$, then T is an isometry and $E_A = T^*E_BT$ (though $A \ncong B$).

7. Appendix. We begin by proving a general fact which is closely related to Theorem 3.1(iii) and the proofs of Theorems 5.1 and 5.2.

Proposition 7.1. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be arbitrary operators and let $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ be an isometry. Then the following two conditions are equivalent:*

(i) *$A = W^*BW$ and $\mathcal{R}(W)$ is invariant for B,*
(ii) *$WA = BW$.*

Moreover, the following two conditions are equivalent:

(iii) *$A = W^*BW$ and $\mathcal{R}(W)$ reduces B,*
(iv) *$WA = BW$ and $WA^* = B^*W$.*

Proof. (i)⇒(ii). Since $\mathcal{R}(W)$ is invariant for B, and WW^* is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}(W)$, we have $WA = WW^*BW = BW$.

(ii)⇒(i). The equality $WA = BW$ immediately implies that $\mathcal{R}(W)$ is invariant for B. In turn, multiplying $WA = BW$ on the left by W^*, we get $A = W^*BW$.

Applying the equivalence (i)⇔(ii) to the pairs (A, B) and (A^*, B^*), we obtain (iii)⇔(iv). ∎

In general, $WA = BW$ does not imply $WA^* = B^*W$ even if W is an isometry and $A = B$. Indeed, let $W \in \boldsymbol{B}(\ell^2)$ be the unilateral shift and let $A \in \boldsymbol{B}(\ell^2)$ be an operator represented by a lower triangular Toeplitz matrix which is not diagonal (see [1, p. 109] for a necessary and sufficient condition for a formal Toeplitz matrix to represent a bounded linear operator on ℓ^2). Then $WA = AW$ and $WA^* \neq A^*W$. As a consequence, $A = W^*AW$ and $\mathcal{R}(W)$ is an invariant subspace for A that does not reduce A.

Of course, if $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ are normal operators and $WA = BW$, then, by the Putnam–Fuglede theorem, $WA^* = B^*W$ regardless of whether W is an isometry or not. The natural question which now arises is whether the implication

$$A = W^*BW \;\Rightarrow\; \mathcal{R}(W) \text{ is invariant for } B$$

holds for all normal operators $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ and all isometries $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$. In general, the answer is no even if $A = B$. Indeed, if $W \in \boldsymbol{B}(\ell^2)$ is the unilateral shift and $A \in \boldsymbol{B}(\ell^2)$, then the equality $A = W^*AW$ holds if and only if the operator A is represented by a Toeplitz matrix (cf. [1, Proposition 4.2.3]). Hence, if A is represented by a symmetric Toeplitz matrix which is not diagonal (it is enough to consider a band matrix), then $A = W^*AW$, A is selfadjoint and $\mathcal{R}(W)$ is not invariant for A. In fact, the operator A can be chosen to be positive and invertible. Indeed, if W is the Hardy shift on the Hardy space H^2 and A is a Toeplitz operator on H^2 with a symbol $\phi \in L^\infty$ such that ϕ is not a scalar multiple of the unit $\mathbf{1}$ of L^∞ and $\phi \geq \delta\mathbf{1}$ for some positive real number δ, then $A = W^*AW$, $\mathcal{R}(W)$ is not invariant for A (because there exists an integer $n \neq 0$ such that the

nth Fourier coefficient of ϕ is nonzero) and $A \geq \delta I$, which implies that A is invertible in the algebra $\boldsymbol{B}(H^2)$. For more information on Toeplitz operators we refer the reader to [1, Section 4.2]).

We now indicate two situations in which the equality $A = W^*BW$ implies that $\mathcal{R}(W)$ reduces B. The first one requires the subspace $\mathcal{R}(W)$ to be invariant for B. The other one does not require this, and is closely related to Theorem 3.1(iv′).

PROPOSITION 7.2. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be normal operators and let $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ be an isometry. If $A = W^*BW$ and $\mathcal{R}(W)$ is invariant for B, then $\mathcal{R}(W)$ reduces B.*

Proof. Since $A = \widehat{W}^*(B|_{\mathcal{R}(W)})\widehat{W}$, where $\widehat{W} \in \boldsymbol{B}(\mathcal{H}, \mathcal{R}(W))$ is the unitary operator defined by $\widehat{W}h = Wh$ for $h \in \mathcal{H}$, we see that $B|_{\mathcal{R}(W)}$ is normal. As B is normal, the space $\mathcal{R}(W)$ reduces B (see, e.g., [16, Corollary 1]). ∎

PROPOSITION 7.3. *Let $A \in \boldsymbol{B}(\mathcal{H})$ and $B \in \boldsymbol{B}(\mathcal{K})$ be orthogonal projections and let $W \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ be an isometry. Then the following two conditions are equivalent:*

(i) $A = W^*BW$,
(ii) $W(\mathcal{R}(A)) \subseteq \mathcal{R}(B)$ *and* $W(\mathcal{H} \ominus \mathcal{R}(A)) \subseteq \mathcal{K} \ominus \mathcal{R}(B)$.

In particular, if $A = W^*BW$, then $\mathcal{R}(W)$ reduces B.

Proof. (i)⇒(ii). Since A and B are orthogonal projections, we get

$$\|Ah\|^2 = \langle Ah, h\rangle = \langle W^*BWh, h\rangle = \|BWh\|^2, \quad h \in \mathcal{H}. \tag{7.1}$$

This yields

$$\|Wh\| = \|h\| = \|Ah\| \overset{(7.1)}{=} \|BWh\| \leq \|Wh\|, \quad h \in \mathcal{R}(A).$$

Hence $Wh \in \mathcal{R}(B)$ for any $h \in \mathcal{R}(A)$, which means that $W(\mathcal{R}(A)) \subseteq \mathcal{R}(B)$. In turn, if $h \in \mathcal{H} \ominus \mathcal{R}(A)$, then $Ah = 0$, which together with (7.1) gives $BWh = 0$. Thus $W(\mathcal{H} \ominus \mathcal{R}(A)) \subseteq \mathcal{K} \ominus \mathcal{R}(B)$. As a consequence,

$$B(\mathcal{R}(W)) = B(W(\mathcal{R}(A))) + B(W(\mathcal{H} \ominus \mathcal{R}(A))) = W(\mathcal{R}(A)) \subseteq \mathcal{R}(W),$$

which shows that $\mathcal{R}(W)$ reduces B.

(ii)⇒(i). Note that if $f \in \mathcal{R}(A)$ and $g \in \mathcal{H} \ominus \mathcal{R}(A)$, then

$$W^*BW(f+g) = W^*BWf + W^*BWg = W^*Wf = f = A(f+g).$$

This completes the proof. ∎

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