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Lower semicontinuity of variational integrals on elliptic complexes

by

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Abstract. We prove a lower semicontinuity result for variational integrals associated with a given first order elliptic complex, extending, in this general setting, a well known result in the case $\mathcal{D}'(\mathbb{R}^n,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\mathbb{R}^n,\mathbb{R}^n) \xrightarrow{\text{curl}} \mathcal{D}'(\mathbb{R}^n,\mathbb{R}^{n\times n})$.

1. Introduction. This paper can be regarded as a sequel to the papers [GV1, GV2] where we discussed variational integrals associated with a given first order elliptic complex of the type

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^N,\mathbf{W}),$$

where $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional inner product spaces. Here $\mathcal{P}$ and $\mathcal{Q}$ are linear differential operators of first order with constant coefficients such that

$$\operatorname{Im}\mathcal{P}(\xi) = \operatorname{Ker}\mathcal{Q}(\xi) \quad \text{for all } \xi \neq 0. \tag{1.1}$$

Such complexes can be viewed, in many ways, as generalizations of the classical exact sequence of the gradient and the curl operator

$$\mathcal{D}'(\mathbb{R}^N,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^N) \xrightarrow{\text{curl}} \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^{N\times N}).$$

There has been some related work concerning differential forms and the exterior derivative operators (see [I, IS, ISS]).

In this general context we are interested in studying lower semicontinuity for functionals of the type

$$(\alpha,\beta) \mapsto \int_\Omega g(\langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle)\,dx, \tag{1.2}$$

where $\alpha \in W^{1,1}(\Omega,\mathbb{R}^d)$, $\beta \in W^{1,1}(\Omega,\mathbb{R}^k)$, $\Omega \subset \mathbb{R}^N$ is a bounded open set and g is a nonnegative convex function. Here $\mathcal{Q}^* : \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^k) \to \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^m)$ denotes the formal adjoint operator of $\mathcal{Q}$ (see Section 2 below).

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The relevance of this general framework has been emphasized by Tartar in the context of continuum mechanics and electromagnetism PDE's.

Recently, in view of applications to materials with nonstandard elastic and magnetic behaviours, many authors have studied lower semicontinuity for functionals defined from a general linear operator of first order (see for instance [BFL, FLM, FM, GV2]).

From a different point of view functionals of the type (1.2) can be viewed as a generalization of the usual polyconvex functionals defined in [B]. In fact, if $N = 2$, taking $\mathcal{P}\alpha = \nabla\alpha$, $\alpha \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $\mathcal{Q}\gamma = \operatorname{curl}\gamma = \partial\gamma^2/\partial x - \partial\gamma^1/\partial y$, $\gamma \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ then one has an elliptic complex and $\langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle$ is equal to the determinant of the matrix whose rows are given by $\nabla\alpha$, $\nabla\beta$. In this case, the lower semicontinuity has been studied in a series of papers (see for instance [AD, ADMM, CD, DM2, DS, FH, G, M1, M2, Ma1, Ma2]).

Here, in the general setting, we are interested in proving the L^1-lower semicontinuity for functionals of the type (1.2) under a suitable equi-integrability assumption on the sequences involved (see Section 3 below).

The paper is organized as follows. In Section 2 we set up notation and preliminaries concerning elliptic complexes. Section 3 is devoted to the proof of the lower semicontinuity result, first in the case of $g(\eta, \xi, \langle \eta, \xi\rangle)$ and then for more general functionals.

2. Notation and preliminaries. Let $\mathbf{U}$ and $\mathbf{V}$ be finite-dimensional vector spaces over the field of real numbers. We assume that both spaces are equipped with inner products, denoted by $\langle\,,\rangle_{\mathbf{U}}$ and $\langle\,,\rangle_{\mathbf{V}}$, respectively. The space of Schwartz distributions on $\mathbb{R}^N$ with values in $\mathbf{U}$ will be denoted by $\mathcal{D}'(\mathbb{R}^N, \mathbf{U})$. Let $\mathcal{L} : \mathcal{D}'(\mathbb{R}^N, \mathbf{U}) \to \mathcal{D}'(\mathbb{R}^N, \mathbf{V})$ be a differential operator of first order with constant coefficients. More precisely,

$$\mathcal{L} = \sum_{k=1}^{N} A_k \frac{\partial}{\partial x_k},$$

where A_k, $k = 1, \ldots, N$, are given linear transformations from $\mathbf{U}$ into $\mathbf{V}$. The formal adjoint $\mathcal{L}^* : \mathcal{D}'(\mathbb{R}^N, \mathbf{V}) \to \mathcal{D}'(\mathbb{R}^N, \mathbf{U})$ is defined by the rule

$$\int_{\mathbb{R}^N} \langle \mathcal{L}^*\beta, \gamma\rangle_{\mathbf{U}} = \int_{\mathbb{R}^N} \langle \beta, \mathcal{L}\gamma\rangle_{\mathbf{V}}$$

for $\gamma \in C_0^\infty(\mathbb{R}^N, \mathbf{U})$ and $\beta \in C^\infty(\mathbb{R}^N, \mathbf{V})$. Thus

$$\mathcal{L}^* = -\sum_{k=1}^{N} A_k^t \frac{\partial}{\partial x_k},$$

where $A_k^t : \mathbf{V} \to \mathbf{U}$ is the transpose of A_k.

The first example that we have in mind is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\right) : \mathcal{D}'(\mathbb{R}^N, \mathbb{R}) \to \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^N)$$

and its adjoint

$$-\operatorname{div} : \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^N, \mathbb{R})$$

defined by

$$\operatorname{div} \alpha = \frac{\partial \alpha^1}{\partial x_1} + \dots + \frac{\partial \alpha^N}{\partial x_N},$$

for $\alpha = (\alpha^1, \dots, \alpha^N)$. More generally, the differential

$$D : \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^m) \to \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^{m\times N})$$

assigns to a mapping $\alpha = (\alpha^1, \dots, \alpha^m)$ its Jacobian matrix

$$D\alpha(x) = \left[\frac{\partial \alpha^j(x)}{\partial x_i}\right] \in \mathbb{R}^{m\times N}.$$

Its formal adjoint $D^* : \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^{m\times N}) \to \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^m)$ is the divergence operator on matrix fields, that is,

$$-D^*\beta = (\operatorname{div} \beta^1, \dots, \operatorname{div} \beta^m),$$

where $\beta^1, \dots, \beta^m$ are the row vectors of the matrix β.

We can also consider the curl operator

$$\operatorname{curl} : \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^{N\times N})$$

defined by

$$\operatorname{curl} \alpha = \left[\frac{\partial \alpha^i}{\partial x_j} - \frac{\partial \alpha^j}{\partial x_i}\right], \qquad i, j = 1, \dots, N,$$

for $\alpha = (\alpha^1, \dots, \alpha^N)$.

Let us point out that another relevant example concerns differential forms. We refer to [ISS] for more details.

In the rest of this section we recall the main concepts related to *elliptic complexes.*

Let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional inner product spaces and consider the sequence of differential operators of first order in N independent variables with constant coefficients

$$\mathcal{D}'(\mathbb{R}^N, \mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^N, \mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^N, \mathbf{W}).$$

More precisely, suppose that

$$\mathcal{P} = \sum_{k=1}^N A_k \frac{\partial}{\partial x_k}, \qquad \mathcal{Q} = \sum_{k=1}^N B_k \frac{\partial}{\partial x_k}, \tag{2.1}$$

where A_k and B_k are linear operators for $k = 1, \dots, N$ belonging to $L(\mathbf{U}, \mathbf{V})$ and $L(\mathbf{V}, \mathbf{W})$ respectively.

The symbols $\mathcal{P} = \mathcal{P}(\xi)$ and $\mathcal{Q} = \mathcal{Q}(\xi)$ are linear functions in $\xi = (\xi_1, \ldots, \xi_N)$ with values in $L(\mathbf{U}, \mathbf{V})$ and $L(\mathbf{V}, \mathbf{W})$ respectively, given explicitly by

$$\mathcal{P}(\xi) = \sum_{k=1}^{N} \xi_k A_k, \qquad \mathcal{Q}(\xi) = \sum_{k=1}^{N} \xi_k B_k. \tag{2.2}$$

The complex (2.1) is said to be *elliptic* if the sequence of symbols

$$\mathbf{U} \xrightarrow{\mathcal{P}(\xi)} \mathbf{V} \xrightarrow{\mathcal{Q}(\xi)} \mathbf{W} \tag{2.3}$$

is *exact*, i.e.

$$\operatorname{Im} \mathcal{P}(\xi) = \operatorname{Ker} \mathcal{Q}(\xi) \quad \text{for all } \xi \neq 0. \tag{2.4}$$

The dual sequence consists of the formal adjoint operators

$$\mathcal{D}'(\mathbb{R}^N, \mathbf{U}) \xleftarrow{\mathcal{P}^*} \mathcal{D}'(\mathbb{R}^N, \mathbf{V}) \xleftarrow{\mathcal{Q}^*} \mathcal{D}'(\mathbb{R}^N, \mathbf{W}), \tag{2.5}$$

$$\mathcal{P}^* = -\sum_{k=1}^{N} A_k^* \frac{\partial}{\partial x_k}, \qquad \mathcal{Q}^* = -\sum_{k=1}^{N} B_k^* \frac{\partial}{\partial x_k}. \tag{2.6}$$

The dual spaces $\mathbf{U}^*, \mathbf{V}^*$ and $\mathbf{W}^*$ are identified with $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$, since $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ are equipped with inner products. It is immediate that the dual complex is elliptic if the original complex is.

If Ω is any domain in $\mathbb{R}^N$, $N \geq 2$, the Sobolev space on Ω with values in $\mathbf{U}$ will be denoted by $W^{1,p}(\Omega, \mathbf{U})$, $1 \leq p < \infty$.

A pair

$$\mathcal{F} = [A, B] = [\mathcal{P}\alpha, \mathcal{Q}^*\beta], \tag{2.7}$$

where $\alpha \in W^{1,p}_{\text{loc}}(\Omega, \mathbf{U})$, $\beta \in W^{1,p}_{\text{loc}}(\Omega, \mathbf{W})$, is said to be an *elliptic couple* associated to the elliptic complex (2.1). The norm and the Jacobian of the elliptic couple $\mathcal{F} = [A, B]$ are defined by

$$|\mathcal{F}(x)|^2 = |A(x)|^2 + |B(x)|^2, \qquad J(x, \mathcal{F}) = \langle A(x), B(x) \rangle_{\mathbf{V}} = \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta \rangle.$$

From now on we will consider variational integrals defined on elliptic couples. They are of the form

$$I[\mathcal{F}] = \int_{\mathbb{R}^N} f(X, Y) \quad \text{for } \mathcal{F} = [X, Y] \in L^p(\mathbb{R}^N, \mathbf{V} \times \mathbf{V}),$$

where the integrand $f : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is at least continuous.

Let us recall the following definition given in [GV1].

Definition 2.1. The function f is said to be *polyconvex* if it can be expressed as

$$f(X, Y) = g(X, Y, \langle X, Y \rangle), \tag{2.8}$$

where $g : \mathbf{V} \times \mathbf{V} \times \mathbb{R} \to \mathbb{R}$ is convex.

3. Lower semicontinuity. In this section we will consider the sequence of differential operators

$$C^\infty(\mathbb{R}^N,\mathbb{R}^d) \xrightarrow{\mathcal{P}} C^\infty(\mathbb{R}^N,\mathbb{R}^m) \xrightarrow{\mathcal{Q}} C^\infty(\mathbb{R}^N,\mathbb{R}^k)$$

and the dual sequence

$$C^\infty(\mathbb{R}^N,\mathbb{R}^d) \xleftarrow{\mathcal{P}^*} C^\infty(\mathbb{R}^N,\mathbb{R}^m) \xleftarrow{\mathcal{Q}^*} C^\infty(\mathbb{R}^N,\mathbb{R}^k).$$

Let Ω be a bounded open subset of $\mathbb{R}^N$.

THEOREM 3.1. *Let $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^m\to[0,\infty)$ be a Carathéodory function such that:*

(i) *for all $x\in\Omega$, $(y,z)\in\mathbb{R}^d\times\mathbb{R}^k$ the function $(\eta,\xi)\mapsto f(x,y,z,\eta,\xi)$ is polyconvex;*

(ii) *for any $(x_0,y_0,z_0)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^k$ and any $\epsilon>0$, there exists $\delta>0$ such that if $|x-x_0|<\delta$, $|(y,z)-(y_0,z_0)|<\delta$ and $\eta,\xi\in\mathbb{R}^m$, then*

$$f(x,y,z,\eta,\xi)\ge(1-\epsilon)f(x_0,y_0,z_0,\eta,\xi);$$

(iii) *there exist $\psi,\varphi:[0,+\infty)\to[0,+\infty)$ satisfying*

$$\lim_{t\to+\infty}\frac{\psi(t)}{t}=\lim_{t\to+\infty}\frac{\varphi(t)}{t}=+\infty$$

such that for all $(x,y,z)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^k$ and all $\eta,\xi\in M^{m\times m}$,

$$\psi(|\eta|)+\varphi(|\xi|)+c|\langle\eta,\xi\rangle|\le f(x,y,z,\eta,\xi).$$

If $\alpha\in W^{1,1}(\Omega,\mathbb{R}^d)$, $\beta\in W^{1,1}(\Omega,\mathbb{R}^k)$ and α_h,β_h are C^1-functions such that $\alpha_h\to\alpha$ and $\beta_h\to\beta$ in L^1, then

$$\int_\Omega f(x,\alpha,\beta,\mathcal{P}\alpha,\mathcal{Q}^*\beta)\,dx\le\liminf_h\int_\Omega f(x,\alpha_h,\beta_h,\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h)\,dx.$$

First of all we will prove a lower semicontinuity result for the functional

$$F(\alpha,\beta)=\int_\Omega g\big(\mathcal{P}\alpha,\mathcal{Q}^*\beta,\langle\mathcal{P}\alpha,\mathcal{Q}^*\beta\rangle\big)\,dx,$$

where g is a convex function.

The following lemma is not difficult to prove.

LEMMA 3.2. *Let $g:\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}\to[0,\infty)$ be a convex function and $\alpha_h,\beta_h,\alpha,\beta$ be Lipschitz functions such that $\alpha_h\to\alpha$ in $L^\infty(\Omega,\mathbb{R}^d)$, $\beta_h\to\beta$ in $L^\infty(\Omega,\mathbb{R}^k)$ and*

$$\sup_h\int_\Omega[|\mathcal{P}\alpha_h|+|\mathcal{Q}^*\beta_h|+|\langle\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h\rangle|]\,dx<\infty.$$

Then $(\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h,\langle\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h\rangle)\overset{*}{\rightharpoonup}(\mathcal{P}\alpha,\mathcal{Q}^*\beta,\langle\mathcal{P}\alpha,\mathcal{Q}^*\beta\rangle)$ in the sense of measures and

$$\int_\Omega g(\mathcal{P}\alpha,\mathcal{Q}^*\beta,\langle\mathcal{P}\alpha,\mathcal{Q}^*\beta\rangle)\,dx\le\liminf_h\int_\Omega g(\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h,\langle\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h\rangle)\,dx. \tag{3.1}$$

Proof. By the assumptions we see immediately that $\mathcal{P}\alpha_h$ converges weakly* to $\mathcal{P}\alpha$ and $\mathcal{Q}^*\beta_h$ converges weakly* to $\mathcal{Q}^*\beta$ in the sense of measures. Passing to a subsequence we may suppose that

$$\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle \overset{*}{\rightharpoonup} \mu \quad \text{for some measure } \mu.$$

Then for any $\eta \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \eta\, d\mu = \lim_h \int_\Omega \eta\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle\, dx = \lim_h \int_\Omega \langle \nabla\eta \otimes \mathcal{P}\alpha_h, \beta_h\rangle\, dx$$
$$= \int_\Omega \langle \nabla\eta \otimes \mathcal{P}\alpha, \beta\rangle\, dx = \int_\Omega \eta\langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle\, dx.$$

The estimate (3.1) follows from Lemma 2.1 of [FH]. ∎

The next proposition is the main ingredient in proving Theorem 3.1.

PROPOSITION 3.3. *Let α, β be Lipschitz functions and α_h, β_h be C^1-functions. Suppose $\alpha_h \to \alpha$, $\beta_h \to \beta$ in L^1, $\sup_h \int_\Omega (|\mathcal{P}\alpha_h| + |\mathcal{Q}^*\beta_h| + |\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle|)\, dx < \infty$ and the sequences $(\mathcal{P}\alpha_h)$ and $(\mathcal{Q}^*\beta_h)$ are equi-integrable. Then there exist sequences $(\gamma_h), (\delta_h)$ of Lipschitz functions such that*

$$\gamma_h \to \alpha, \qquad \delta_h \to \beta \qquad \text{in } L^\infty$$

and $(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle) \overset{*}{\rightharpoonup} (\mathcal{P}\alpha, \mathcal{Q}^*\beta, \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle)$ in the sense of measures. Moreover, if we define $A_h = \{x \in \Omega : (\gamma_h, \delta_h) \neq (\alpha_h, \beta_h)\}$ then

$$\mathcal{L}^N(A_h) \to 0 \quad \textit{and} \quad \int_{A_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|\, dx \to 0.$$

Proof. The construction of such sequences is obtained via a truncation argument as in [M2].

Since $\alpha_h \to \alpha$ and $\beta_h \to \beta$ in L^1, extracting a subsequence if necessary, we may assume that

$$4^h\|\alpha_h - \alpha\|_{L^1} \to 0, \qquad 4^h\|\beta_h - \beta\|_{L^1} \to 0$$

and so

$$(3.2)\qquad |\{x \in \Omega : |\alpha_h - \alpha| 4^h \ge 1\}| \to 0, \qquad |\{x \in \Omega : |\beta_h - \beta| 4^h \ge 1\}| \to 0.$$

Let $C = \sup_h \int_\Omega |\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle|\, dx$. We can find $k_h \in \{h+1, \ldots, 2h\}$ and $l_h \in \{h+1, \ldots, 2h\}$ such that

$$(3.3)\qquad \int_{\{1/2^{k_h} < |\alpha_h - \alpha| < 1/2^{k_h - 1}\}} |\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle|\, dx \le \frac{C}{h},$$

$$(3.4)\qquad \int_{\{1/2^{l_h} < |\beta_h - \beta| < 1/2^{l_h - 1}\}} |\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle|\, dx \le \frac{C}{h}.$$

Let us define

$$(3.5)\qquad \varphi(t)=\begin{cases}1 & \text{if } t\le 1,\\ (2-t)/t & \text{if } 1<t<2,\\ 0 & \text{if } t\ge 2.\end{cases}$$

Denote also

$$\gamma_h(x)=\alpha(x)+\varphi(2^{k_h}|\alpha_h-\alpha|(x))(\alpha_h-\alpha)(x),$$
$$\delta_h(x)=\beta(x)+\varphi(2^{l_h}|\beta_h-\beta|(x))(\beta_h-\beta)(x).$$

Note that

$$\|\gamma_h-\alpha\|_\infty\le\frac{1}{2^{k_h-1}}\le 2^{-h}\to 0,\qquad \|\delta_h-\beta\|_\infty\le\frac{1}{2^{l_h-1}}\le 2^{-h}\to 0$$

as $h\to\infty$. Now defining $C_h=\{1/2^{k_h}<|\alpha_h-\alpha|<1/2^{k_h-1}\}$ and $C'_h=\{1/2^{l_h}<|\beta_h-\beta|<1/2^{l_h-1}\}$ we have, on C_h and C'_h,

$$(3.6)\qquad \mathcal{P}\gamma_h=(1-\varphi)\mathcal{P}\alpha+\varphi\mathcal{P}\alpha_h+\varphi' R_h(\mathcal{P}\alpha_h-\mathcal{P}\alpha),$$
$$(3.7)\qquad \mathcal{Q}^*\delta_h=(1-\varphi)\mathcal{Q}^*\beta+\varphi\mathcal{Q}^*\beta_h+\varphi' R_h^1(\mathcal{Q}^*\beta_h-\mathcal{Q}^*\beta)$$

respectively, where φ,φ' are evaluated in the first case at $2^{k_h}|\alpha_h-\alpha|$ and in the second case at $2^{l_h}|\beta_h-\beta|$, and

$$R_h=2^{k_h}(\alpha_h-\alpha)\otimes\frac{\alpha_h-\alpha}{|\alpha_h-\alpha|},\qquad |R_h|\le 2\quad\text{on } C_h,$$
$$R_h^1=2^{l_h}(\beta_h-\beta)\otimes\frac{\beta_h-\beta}{|\beta_h-\beta|},\qquad |R_h^1|\le 2\quad\text{on } C'_h.$$

Recalling that α,β are Lipschitz functions, let us estimate the terms on $C_h\cap C'_h$. We have

$$(3.8)\qquad \int_{C_h\cap C'_h}(|\mathcal{P}\gamma_h|+|\mathcal{Q}^*\delta_h|+|\langle\mathcal{P}\gamma_h,\mathcal{Q}^*\delta_h\rangle|)\,dx$$
$$\le\int_{C_h\cap C'_h} c[|\mathcal{P}\alpha|+|\mathcal{P}\alpha_h|+|\mathcal{Q}^*\beta|+|\mathcal{Q}^*\beta_h|+|\langle\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h\rangle|+|\langle\mathcal{P}\alpha,\mathcal{Q}^*\beta\rangle|]\,dx.$$

From (3.2) we infer that $|C_h|,|C'_h|\to 0$ and obviously $|C_h\cap C'_h|\to 0$. Moreover, using (3.3), (3.4) and the equi-integrability of $(\mathcal{P}\alpha_h)$ and $(\mathcal{Q}^*\beta_h)$ we conclude that the right hand side goes to 0.

Now, if we set $B_h=\{|\alpha_h-\alpha|>1/2^{k_h-1}\}$, $B'_h=\{|\beta_h-\beta|>1/2^{l_h-1}\}$, $D_h=\{|\alpha_h-\alpha|\le 1/2^{k_h}\}$ and $D'_h=\{|\beta_h-\beta|\le 1/2^{l_h}\}$, we have

$$\begin{aligned}
&\int_{A_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|\,dx \\
&= \int_{B_h\cap B'_h} |(\mathcal{P}\alpha, \mathcal{Q}^*\beta, \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle)|\,dx + \int_{B_h\cap C'_h} |(\mathcal{P}\alpha, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\alpha, \mathcal{Q}^*\delta_h\rangle)|\,dx \\
&\quad + \int_{B_h\cap D'_h} |(\mathcal{P}\alpha, \mathcal{Q}^*\beta_h, \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta_h\rangle)|\,dx + \int_{C_h\cap B'_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\beta, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\beta\rangle)|\,dx \\
&\quad + \int_{C_h\cap C'_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|\,dx + \int_{C_h\cap D'_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\beta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\beta_h\rangle)|\,dx \\
&\quad + \int_{D_h\cap B'_h} |(\mathcal{P}\alpha_h, \mathcal{Q}^*\beta, \langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta\rangle)|\,dx + \int_{D_h\cap C'_h} |(\mathcal{P}\alpha_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\alpha_h, \mathcal{Q}^*\delta_h\rangle)|\,dx.
\end{aligned}$$

Observe that from (3.2) and the fact that $k_h, l_h \le 2h$ we have $|B_h|, |B'_h| \to 0$.

Therefore since α, β are Lipschitz functions, by using the equi-integrability of $(\mathcal{P}\alpha_h)$ and $(\mathcal{Q}^*\beta_h)$ and (3.6)–(3.8) we obtain

$$\int_{A_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|\,dx \to 0.$$

Finally, since

$$\begin{aligned}
&\sup_h \int_{\Omega} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|\,dx \\
&\qquad \le \sup_h \Big[\int_{\Omega\setminus A_h} |(\mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h, \langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle)|\,dx \\
&\qquad\qquad + \int_{A_h} |(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|\,dx \Big] < \infty
\end{aligned}$$

and $\gamma_h \to \alpha$ and $\delta_h \to \beta$ in L^∞, it follows by Lemma 3.2 that

$$(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle) \overset{*}{\rightharpoonup} (\mathcal{P}\alpha, \mathcal{Q}^*\beta, \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle)$$

in the sense of measures. ∎

Theorem 3.4. *Let $g : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to [0,\infty)$ be a convex function. Suppose α, β are Lipschitz functions and α_h, β_h C^1-functions such that $\alpha_h \to \alpha$ and $\beta_h \to \beta$ in L^1,*

$$\sup_h \int_{\Omega} [|\mathcal{P}\alpha_h| + |\mathcal{Q}^*\beta_h| + |\langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle|]\,dx < \infty$$

and $(\mathcal{P}\alpha_h)$ and $(\mathcal{Q}^*\beta_h)$ equi-integrable. Then

$$\int_{\Omega} g(\mathcal{P}\alpha, \mathcal{Q}^*\beta, \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle)\,dx \le \liminf_h \int_{\Omega} g(\mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h, \langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle)\,dx.$$

Proof. Assume that g has linear growth, i.e. $0 \le g(\xi) \le c(1+|\xi|)$. If (γ_h) and (δ_h) are the sequences of Proposition 3.3, then

$$\begin{aligned}\int_\Omega g(\mathcal{P}\alpha, \mathcal{Q}^*\beta, \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle)\,dx &\le \liminf_h \int_\Omega g(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)\,dx \\ &\le \liminf_h \Big[\int_{\Omega\setminus A_h} g(\mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h, \langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle)\,dx \\ &\qquad + c\int_{A_h} [1+|(\mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h, \langle \mathcal{P}\gamma_h, \mathcal{Q}^*\delta_h\rangle)|]\,dx\Big] \\ &\le \liminf_h \int_\Omega g(\mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h, \langle \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h\rangle)\,dx.\end{aligned}$$

The general case is obtained by noting that g can be written as the supremum of an increasing sequence (g_j) of convex functions with linear growth and using the fact that the supremum of lower semicontinuous functionals is lower semicontinuous. ∎

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We can certainly assume that

$$\sup_h \int_\Omega f(x, \alpha_h, \beta_h, \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h)\,dx < \infty.$$

Passing to subsequences if necessary, we obtain the existence of a finite, Radon nonnegative measure μ such that

$$\liminf_{h\to\infty} \int_\Omega f(x, \alpha_h, \beta_h, \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h)\,dx = \lim_{h\to\infty} \int_\Omega f(x, \alpha_h, \beta_h, \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h)\,dx$$

and

$$\mu = w^*\text{-}\lim_{h\to\infty} f(x, \alpha_h, \beta_h, \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h)\mathcal{L}^N. \tag{3.9}$$

Now our purpose is to prove that

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{\rho\to 0^+} \frac{\mu(B(x_0,\rho))}{\mathcal{L}^N(B(x_0,\rho))} \\ &\ge f(x_0, \alpha(x_0), \beta(x_0), \mathcal{P}\alpha(x_0), \mathcal{Q}^*\beta(x_0))\end{aligned} \tag{3.10}$$

for a.e. $x_0 \in \Omega$. Assuming that (3.10) is true, we have

$$\lim_{h\to\infty} \int_\Omega f(x, \alpha_h, \beta_h, \mathcal{P}\alpha_h, \mathcal{Q}^*\beta_h)\,dx \ge \int_\Omega \frac{d\mu}{d\mathcal{L}^N}dx \ge \int_\Omega f(x, \alpha, \beta, \mathcal{P}\alpha, \mathcal{Q}^*\beta)\,dx.$$

It remains to prove (3.10). Let $x_0 \in \Omega$ be such that the limit

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(B(x_0,\rho))}{\mathcal{L}^N(B(x_0,\rho))}$$

exists and is finite and

$$\lim_{\rho\to 0^+}\frac{1}{\rho}\fint_{B(x_0,\rho)}|\alpha(y)-\alpha(x_0)-\mathcal{P}\alpha(x_0)(y-x_0)|\,dy=0, \tag{3.11}$$

$$\lim_{\rho\to 0^+}\frac{1}{\rho}\fint_{B(x_0,\rho)}|\beta(y)-\beta(x_0)-\mathcal{Q}^*\beta(x_0)(y-x_0)|\,dy=0. \tag{3.12}$$

We select $\rho_k\to 0^+$ such that $\mu(\partial B(x_0,\rho_k))=0$. Then

$$\begin{aligned}\lim_{k\to\infty}\frac{\mu(B(x_0,\rho_k))}{\mathcal{L}^N(B(x_0,\rho_k))}&=\lim_{k\to\infty}\lim_{h\to\infty}\fint_{B(x_0,\rho_k)}f(x,\alpha_h,\beta_h,\mathcal{P}\alpha_h,\mathcal{Q}^*\beta_h)\,dx\\&=\lim_{k\to\infty}\lim_{h\to\infty}\fint_B f(x_0+\rho_k y,\alpha(x_0)+\rho_k\alpha_{h,k}(y),\beta(x_0)+\rho_k\beta_{h,k}(y),\\&\qquad\mathcal{P}\alpha_{h,k}(y),\mathcal{Q}^*\beta_{h,k}(y))\,dy\end{aligned} \tag{3.13}$$

where

$$\alpha_{h,k}(y)=\frac{\alpha_h(x_0+\rho_k y)-\alpha(x_0)}{\rho_k},\qquad \beta_{h,k}(y)=\frac{\beta_h(x_0+\rho_k y)-\beta(x_0)}{\rho_k}.$$

Since (3.11), (3.12) it follows that

$$\lim_{k\to\infty}\lim_{h\to\infty}\|\alpha_{h,k}-\mathcal{P}\alpha(x_0)y\|_{L^1(B)}=0,$$
$$\lim_{k\to\infty}\lim_{h\to\infty}\|\beta_{h,k}-\mathcal{Q}^*\beta(x_0)y\|_{L^1(B)}=0.$$

Hence, we may extract subsequences

$$v_k=\alpha_{h,k}\quad\text{and}\quad z_k=\beta_{h,k}$$

with v_k converging to $\mathcal{P}\alpha(x_0)y$ in $L^1(B,\mathbb{R}^d)$ and z_k converging to $\mathcal{Q}^*\beta(x_0)y$ in $L^1(B,\mathbb{R}^k)$ and such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)=\lim_{k\to\infty}\fint_B f\big(x_0+\rho_k y,\alpha(x_0)+\rho_k v_k(y),\beta(x_0)+\rho_k z_k(y),\mathcal{P}v_k(y),\mathcal{Q}^*z_k(y)\big)\,dy.$$

From the growth assumption (iii) we get, in particular, the equi-integrability of the sequences $(\mathcal{P}v_k)$ and $(\mathcal{Q}^*z_k)$. Let us now fix $\epsilon>0$. Therefore, by using the hypothesis (ii) and the properties of the sequences ρ_k, v_k, z_k, from Theorem 3.4 we get

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0)&\geq(1-\epsilon)\liminf_{k\to\infty}\fint_B f\big(x_0,\alpha(x_0),\beta(x_0),\mathcal{P}v_k(y),\mathcal{Q}^*z_k(y)\big)\,dy\\&\geq(1-\epsilon)f\big(x_0,\alpha(x_0),\beta(x_0),\mathcal{P}\alpha(x_0),\mathcal{Q}^*\beta(x_0)\big)\end{aligned}$$

and letting $\epsilon\to 0$ we conclude the proof.

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