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Distortion and spreading models in modified mixed Tsirelson spaces

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Abstract. The results of the first part concern the existence of higher order ℓ_1 spreading models in asymptotic ℓ_1 Banach spaces. We sketch the proof of the fact that the mixed Tsirelson space $T[(\mathcal{S}_n, \theta_n)_n]$, $\theta_{n+m} \geq \theta_n\theta_m$ and $\lim_n \theta_n^{1/n} = 1$, admits an ℓ_1^ω spreading model in every block subspace. We also prove that if X is a Banach space with a basis, with the property that there exists a sequence $(\theta_n)_n \subset (0,1)$ with $\lim_n \theta_n^{1/n} = 1$, such that, for every $n \in \mathbb{N}$, $\|\sum_{k=1}^m x_k\| \geq \theta_n \sum_{k=1}^m \|x_k\|$ for every $\mathcal{S}_n$-admissible block sequence $(x_k)_{k=1}^m$ of vectors in X, then there exists $c > 0$ such that every block subspace of X admits, for every n, an ℓ_1^n spreading model with constant c. Finally, we give an example of a Banach space which has the above property but fails to admit an ℓ_1^ω spreading model.

In the second part we prove that under certain conditions on the double sequence $(k_n, \theta_n)_n$ the modified mixed Tsirelson space $T_M[(\mathcal{S}_{k_n}, \theta_n)_n]$ is arbitrarily distortable. Moreover, for an appropriate choice of $(k_n, \theta_n)_n$, every block subspace admits an ℓ_1^ω spreading model.

1. Introduction. A Banach space X with a basis $(e_i)_i$ is an *asymptotic ℓ_1 space* if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for every block sequence $(x_i)_{i=1}^n$ supported after n,

$$\Big\|\sum_{i=1}^n x_i\Big\| \geq \frac{1}{C}\sum_{i=1}^n \|x_i\|.$$

Tsirelson's famous space [33] was the first nontrivial example of such a space. Mixed Tsirelson spaces, introduced in [5], and their variants offer a large class of examples of asymptotic ℓ_1 spaces.

This paper consists of two independent parts. The first part concerns the existence of higher order ℓ_1 spreading models in asymptotic ℓ_1 spaces. The second part concerns the problem of distortion on these spaces. In particular, we prove the following.

THEOREM A. *For an appropriate sequence $(k_j, \theta_j)_{j=1}^\infty$, the modified mixed Tsirelson space $T_M[(\mathcal{S}_{k_j}, \theta_j)_{j=1}^\infty]$ is arbitrarily distortable.*

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We recall that a Banach space $(X, \|\cdot\|)$ is said to be λ-*distortable*, $\lambda > 1$, if there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that

$$\inf_Y \sup\{|\!|\!|x|\!|\!|/|\!|\!|y|\!|\!| : x, y \in S_Y\} \geq \lambda$$

where the infimum is taken over all infinite-dimensional subspaces Y of X. Moreover, X is said to be *distortable* if it is λ-distortable for some λ, and *arbitrarily distortable* if it is λ-distortable for every $\lambda > 1$. R. C. James [18] proved that c_0 and ℓ_1 are not distortable. V. D. Milman [24] showed that if a Banach space X does not have a distortable subspace then it contains an almost isometric copy of either c_0 or ℓ_p for some $1 \leq p < \infty$ (see also [28]). Much later E. Odell and Th. Schlumprecht [26] settled the famous Distortion Problem, by proving that the spaces ℓ_p, $1 < p < \infty$, are arbitrarily distortable. It remains an open problem whether there exists a distortable but not arbitrarily distortable Banach space. In view of the results of B. Maurey [23], V. Milman and N. Tomczak-Jaegermann [25] and N. Tomczak-Jaegermann [32], the search for such a space has focused on asymptotic ℓ_1 spaces with an unconditional basis. It is unknown whether Tsirelson's space is such an example.

The first example of an arbitrarily distortable asymptotic ℓ_1 Banach space was a mixed Tsirelson space [5]. We recall the definition of this class of spaces and their modified versions. Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of compact families of finite subsets of $\mathbb{N}$, and $(\theta_n)_{n\in\mathbb{N}}$ a sequence of numbers in $(0,1)$ decreasing to 0. The *mixed Tsirelson space* $T[(\mathcal{M}_n, \theta_n)_n]$ and its modified version $T_M[(\mathcal{M}_n, \theta_n)_n]$ are the Banach spaces whose norms are defined implicitly as follows: For $x \in c_{00}$ (the space of finitely supported sequences),

$$\|x\|_\star = \max\Big\{\|x\|_\infty, \sup_n \sup \theta_n \sum_{i=1}^p \|E_i x\|_\star\Big\}$$

where the inner supremum is taken over all families $\{E_1, \ldots, E_p\}$, $p \in \mathbb{N}$, of finite subsets of $\mathbb{N}$ such that:

(i) In the case of the mixed Tsirelson norm,

$$\forall i = 1, \ldots, p-1 \quad \max E_i < \min E_{i+1} \quad \text{and} \quad (\min E_i)_{i=1}^p \in \mathcal{M}_n.$$

Such a family $(E_i)_{i=1}^p$ is said to be $\mathcal{M}_n$-*admissible.*

(ii) In the case of the modified mixed Tsirelson norm,

$$E_1, \ldots, E_p \text{ are pairwise disjoint} \quad \text{and} \quad (\min E_i)_{i=1}^p \in \mathcal{M}_n.$$

We call such a family $(E_i)_{i=1}^p$ $\mathcal{M}_n$-*allowable.*

Not all spaces included in this general definition are asymptotic ℓ_1. This depends on the sequence $(\mathcal{M}_n)_n$. There are two sequences $(\mathcal{M}_n)_n$ which give the fundamental examples of mixed Tsirelson spaces: the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ where $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}$, and the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of the generalized Schreier families. A typical representative of mixed Tsirelson spaces defined by $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is Schlumprecht's space $S = T[(\mathcal{A}_n, 1/\log_2(n+1))_{n=1}^{\infty}]$ [31], while for the spaces defined by the Schreier sequence, typical representatives are the spaces $T[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}]$ with the sequence $(\theta_n)_n$ satisfying the Androulakis–Odell conditions [2]. It follows immediately from the definition that all mixed Tsirelson spaces defined by the Schreier sequence are asymptotic ℓ_1.

In the literature, the term "mixed Tsirelson spaces" is often used exclusively for the spaces defined by the Schreier sequence $(\mathcal{S}_n)_n$ (or, more generally, $(\mathcal{S}_{\xi_n})_n$ for some sequence $(\xi_n)_n$ of countable ordinals). However, the main results concerning these spaces are completely analogous in the two cases $T[(\mathcal{A}_n, 1/\log_2(n+1))_{n=1}^{\infty}]$ and $T[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}]$. This justifies putting all these spaces in the same class. This similarity disappears when one looks at the modified versions of these two main classes. Indeed, as was shown by Th. Schlumprecht, the modified space $T_M[(\mathcal{A}_n, 1/\log_2(n+1))_{n=1}^{\infty}]$ contains isomorphically the space ℓ_1 (unpublished result, see also [21] for related results). On the other hand, if for some $n \in \mathbb{N}$, $\mathcal{M}_n$ contains the Schreier family $\mathcal{S}$, then the space $T_M[(\mathcal{M}_n, \theta_n)_{n=1}^{\infty}]$ is reflexive [6]. This fact is not easily explained, since in the second case the local ℓ_1 structure of the space is richer than in the first case.

Let us also recall that the modified version of Tsirelson's space, defined by W. B. Johnson [19], is isomorphic to the original one [13]. On the other hand, the spaces $T[(\mathcal{S}_n, \theta_n)_n]$ and $T_M[(\mathcal{S}_n, \theta_n)_n]$ are totally incomparable in the case $\lim_n \theta_n^{1/n} = 1$; this can be seen by the fact (shown in [6]) that c_0 is finitely disjointly representable in every block subspace of $T[(\mathcal{S}_n, \theta_n)_n]$, which clearly is not true in the modified space.

In [6] a "boundedly modified" version of mixed Tsirelson spaces was considered. It was proved that for appropriate sequences (n_j) and (θ_j), the boundedly modified mixed Tsirelson space defined by $(\mathcal{S}_{n_j}, \theta_j)_j$ is arbitrarily distortable. The proof presented there was rather complicated.

We proceed to describe the contents of this paper. Theorem A is presented in Section 4. Its proof is along the same lines as that of the corresponding result for ordinary mixed Tsirelson spaces [5]. As in that case, we prove that the space $T_M[(\mathcal{S}_{k_j}, \theta_j)_{j=1}^{\infty}]$ has an asymptotic biorthogonal system. We recall that $(C_j, A_j)_{j=1}^{\infty}$ is an *asymptotic biorthogonal system* in the Banach space X if $C_j \subset S_X$, $A_j \subset B_{X^*}$ for every $j \in \mathbb{N}$, and there exist a constant $c > 0$ and a sequence $(\varepsilon_j)_j$ decreasing to 0 such that for every j:

(i) $(C_j + \varepsilon B_X) \cap Y \neq \emptyset$ for every $\varepsilon > 0$ and every infinite-dimensional subspace Y of X.

(ii) For every $y \in C_j$ there exists $y^* \in A_j$ such that $y^*(y) \geq c$.

(iii) For every $i \neq j$, every $x \in C_i$ and $y^* \in A_j$, $|y^*(x)| \leq \varepsilon_{\min\{i,j\}}$.

In the space $X_M = T_M[(\mathcal{S}_{k_j},\theta_j)_{j=1}^{\infty}]$ each set C_j consists of normalized (θ_j^2,k_j)-rapidly increasing special convex combinations. These classes of vectors played a similar part in the corresponding result of [5]. The set A_j consists of functionals of the form $f=\theta_j\sum_{r=1}^d f_r$, where $f_r\in B_{X_M^*}$ for all $r=1,\dots,d$ and $(\operatorname{supp} f_r)_{r=1}^d$ is $\mathcal{S}_{k_j}$-allowable.

The key point that distinguishes the behavior of $T_M[(\mathcal{S}_n,\theta_n)_n]$ from that of $T_M[(\mathcal{A}_n,1/\log_2(n+1))_n]$ is the following (Lemma 4.9).

Lemma. *Let $X=T_M[(\mathcal{S}_n,\theta_n)_n]$, $j\in\mathbb{N}$, $\varepsilon<\theta_j$ and let $\sum_{k=1}^m\alpha_kx_k$ be an (ε,j)-special convex combination with $\|x_k\|\le 1$ for all $k=1,\dots,m$. Then, for every $l<j$ and every finite sequence $(f_i)_{i=1}^d$ in B_{X^*} such that $(\operatorname{supp} f_i)_{i=1}^d$ is $\mathcal{S}_l$-allowable, we have*

$$\Big|\sum_{i=1}^d f_i\Big(\sum_{k=1}^m\alpha_kx_k\Big)\Big|\le\frac{1}{\theta_1}+1.$$

This is a variation of a result holding for both $S=T[(\mathcal{A}_n,1/\log_2(n+1))_n]$ and $T[(\mathcal{S}_n,\theta_n)_n]$. However, in the modified Schlumprecht space $T_M[(\mathcal{A}_n,1/\log_2(n+1))_n]$, an analogous result is no longer true.

In Section 3 we study asymptotic ℓ_1 Banach spaces with respect to their higher order ℓ_1 spreading models. We start with the following:

Definition. Let $(x_k)_k$ be a seminormalized sequence in a Banach space X and let ξ be a countable ordinal. The sequence $(x_k)_k$ has an ℓ_1^ξ *spreading model* if there exists $c>0$ such that for every $F\in\mathcal{S}_\xi$ and $(\lambda_k)_{k\in F}\subset\mathbb{R}$,

$$\Big\|\sum_{k\in F}\lambda_kx_k\Big\|\ge c\sum_{k\in F}|\lambda_k|.$$

It is easy to see that every subspace of an asymptotic ℓ_1 space admits an ℓ_1^k spreading model for every $k\in\mathbb{N}$. We prove here that the spaces $T[(\mathcal{S}_n,\theta_n)_n]$ with $(\theta_n)_n$ satisfying the Androulakis–Odell conditions admit an ℓ_1^ω spreading model in every subspace with the same constant c. We obtain this as a consequence of the fact that c_0 is finitely representable in every subspace. This result, as well as its proof, should be compared to the result of D. Kutzarova and P. K. Lin [20] that Schlumprecht's space admits an ℓ_1 spreading model.

A recent result of I. Gasparis [15] includes another method for constructing sequences which have an ℓ_1^ω spreading model, without the use of the finite representability of c_0. This depends on a careful choice of the sequence $(k_n,\theta_n)_n$. Using this method we show in Section 4 that if the sequence $(k_n,\theta_n)_n$ satisfies what we call the Gasparis conditions, then every block subspace of the modified mixed Tsirelson space $T_M[(\mathcal{S}_{k_n},\theta_n)_n]$ admits an ℓ_1^ω spreading model with constant $c\ge 1/64$. We note (see Remark 3.2) that c_0 is not finitely representable in the modified mixed Tsirelson spaces $T_M[(\mathcal{S}_{k_n},\theta_n)_n]$.

In Proposition 3.3 we show that if X is an asymptotic ℓ_1 space with a basis and there exists a sequence $(\theta_k)_{k=1}^\infty$ with $\lim_k \theta_k^{1/k} = 1$ such that, for all $n<\omega$ and all $\mathcal{S}_n$-admissible block sequences $(x_i)_{i=1}^d$, $\|\sum_{i=1}^d x_i\| \geq \theta_n \sum_{i=1}^d \|x_i\|$, then there exists $c>0$ such that every block subspace of X admits, for every k, an ℓ_1^k spreading model with constant c.

Then we proceed to give an example of a Banach space X falling in the previous class which does not admit any ℓ_1^ω spreading model. The norm of the space X is defined implicitly in the following manner. For appropriate sequences $(n_j)_{j=1}^\infty$ in $\mathbb{N}$ and $(\theta_j)_{j=1}^\infty$ in $(0,1)$, the norm $\|\cdot\|$ of X satisfies the following equation: For $x\in c_{00}$,

$$\|x\| = \max\Big\{\|x\|_\infty, \sup\Big\{\sum_{k=1}^n \|x|_{[n,\infty)}\|_{j_k} : n\in\mathbb{N},\ j_1<\ldots<j_n\Big\}\Big\},$$

where $\|x\|_j = \theta_j \sup\{\sum_{l=1}^d \|E_l x\| : d\in\mathbb{N},\ (E_l)_{l=1}^d \text{ is } \mathcal{S}_{n_j}\text{-admissible}\}$. Our construction is similar to the example of E. Odell and Th. Schlumprecht [27] of a Banach space with no ℓ_p $(1\leq p<\infty)$ or c_0 spreading model. A construction of this type was first employed by W. T. Gowers [16] to provide an example of a Banach space which does not contain c_0, ℓ_1 or a reflexive subspace.

The structure of asymptotic ℓ_1 Banach spaces has been studied in [29], where some results which relate the distortion problem with spreading models are included. In this direction the third named author has recently obtained the following result [22]: Let $c>0$ and let X be a Banach space with a bimonotone shrinking basis (e_i) such that X does not admit any ℓ_1^ω spreading model, but every block subspace of X admits, for every $k<\omega$, an ℓ_1^k spreading model with constant c. Then every subspace of X contains an arbitrarily distortable subspace. This implies in particular that the space X of our last mentioned example has an arbitrarily distortable subspace.

Although the present work concerns mainly Banach spaces with an unconditional basis, let us mention that spreading models have also been used in the study of hereditarily indecomposable (H.I.) Banach spaces. It is well known that if a Banach space X does not contain ℓ_1 then there exists a unique $\xi<\omega_1$ such that X admits an ℓ_1^ζ spreading model for all $\zeta<\xi$, but does not admit any ℓ_1^ξ spreading model. This is used in [10] to show that every separable Banach space Z not containing ℓ_1 is a quotient of a hereditarily indecomposable asymptotic ℓ_1 Banach space X, and moreover Z^* is complemented in X^*.

In another direction, spreading models are employed for the construction of strictly singular noncompact operators on H.I. spaces. Recall that

W. T. Gowers [17] first established the existence of a strictly singular noncompact operator from a subspace of the Gowers–Maurey space to the whole space. Next S. Argyros and V. Felouzis [7], using interpolation techniques, proved that there are H.I. spaces admitting strictly singular noncompact operators. Also G. Androulakis and Th. Schlumprecht [4] proved that a strictly singular noncompact operator exists on the Gowers–Maurey space, using the fact that the spreading model of the unit vector basis of this space is the unit vector basis of Schlumprecht's space. Another result in this direction which is related to our work was obtained by I. Gasparis [15]. He proves that, under certain conditions on the H.I. space X, the existence of a c_0^ω spreading model in X^* implies that X admits a strictly singular noncompact operator. See also [3] for related results.

2. Preliminaries

NOTATION. Let $(e_i)_{i=1}^\infty$ be the standard basis of the linear space c_{00} of finitely supported sequences. For $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$, the *support* of x is the set $\operatorname{supp} x = \{i \in \mathbb{N} : a_i \neq 0\}$. The *range* of x, written $\operatorname{range}(x)$, is the smallest interval of $\mathbb{N}$ containing the support of x. For finite subsets E, F of $\mathbb{N}$, $E < F$ means $\max E < \min F$ or either E or F is empty. For $n \in \mathbb{N}$ and $E \subset \mathbb{N}$, $n < E$ (resp. $E < n$) means $n < \min E$ (resp. $\max E < n$). For x, y in c_{00}, $x < y$ means $\operatorname{supp} x < \operatorname{supp} y$. For $n \in \mathbb{N}$ and $x \in c_{00}$, we write $n < x$ (resp. $x < n$) if $n < \operatorname{supp} x$ (resp. $\operatorname{supp} x < n$). We say that the sets $E_i \subset \mathbb{N}$, $i = 1, \ldots, n$, are *successive* if $E_1 < \ldots < E_n$. Similarly, the vectors x_i, $i = 1, \ldots, n$, are *successive* if $x_1 < \ldots < x_n$. For $x = \sum_{i=1}^\infty a_i e_i$ and $E \subset \mathbb{N}$, we denote by Ex the vector $\sum_{i \in E} a_i e_i$. For an infinite subset M of $\mathbb{N}$ we denote by $[M]$ the class of infinite subsets of M, and by $[M]^{<\omega}$ the class of finite subsets of M.

The proofs of the first part of the paper rely essentially on the infinite Ramsey theorem (F. Galvin and K. Prikry, J. Silver, E. E. Ellentuck). We recall the statement of this theorem. Here $[\mathbb{N}]$ is endowed with the topology of pointwise convergence.

THEOREM 2.1. *Let A be an analytic subset of* $[\mathbb{N}]$. *For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that either $[L] \subset A$ or $[L] \subset [M] \setminus A$.*

The *generalized Schreier families* $(\mathcal{S}_\xi)_{\xi<\omega_1}$, introduced in [1], are defined by transfinite induction as follows:

$$\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}.$$

Suppose that the families $\mathcal{S}_\alpha$ have been defined for all $\alpha < \xi$. If $\xi = \zeta + 1$, we set

$$\mathcal{S}_\xi = \Big\{F \in [\mathbb{N}]^{<\omega} : F = \bigcup_{i=1}^n F_i,\ n \in \mathbb{N},\ \ F_i \in \mathcal{S}_\zeta \text{ for } i = 1, \ldots, n \text{ and } n \le F_1 < \ldots < F_n\Big\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal, let $(\xi_n+1)_n$ be a sequence of successor ordinals which strictly increases to ξ. We set

$$\mathcal{S}_\xi = \{F \in [\mathbb{N}]^{<\omega} : \text{for some } n \in \mathbb{N},\ n \le \min F \text{ and } F \in \mathcal{S}_{\xi_n+1}\}.$$

For $\xi < \omega_1$ and $M = (m_i)_{i=1}^\infty \in [\mathbb{N}]$, we denote by $\mathcal{S}_\xi(M)$ the family

$$\mathcal{S}_\xi(M) = \{(m_i)_{i\in F} : F \in \mathcal{S}_\xi\}.$$

We next pass to the definition of the *repeated averages hierarchy* introduced in [9]. We let (e_n) denote the standard basis of c_{00}. For every countable ordinal ξ and every $M \in [\mathbb{N}]$, we define a convex block sequence $(\xi_n^M)_{n=1}^\infty$ of (e_n) by transfinite induction on ξ in the following manner: If $\xi = 0$ and $M = (m_n)_{n=1}^\infty$, then $\xi_n^M = e_{m_n}$ for all $n \in \mathbb{N}$. Assume that $(\eta_n^M)_{n=1}^\infty$ has been defined for all $\eta < \xi$ and $M \in [\mathbb{N}]$. Let $\xi = \zeta + 1$. We set

$$\xi_1^M = \frac{1}{m_1}\sum_{i=1}^{m_1} \zeta_i^M,$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \ldots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max\operatorname{supp} \xi_n^M\}, \qquad k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = \frac{1}{k_n}\sum_{i=1}^{k_n} \zeta_i^{M_n} = \xi_1^{M_n}.$$

If ξ is a limit ordinal, let $(\xi_n+1)_n$ be the sequence of ordinals associated to ξ. Define

$$\xi_1^M = [\xi_{m_1}+1]_1^M,$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \ldots < \xi_n^M$ have been defined. Again, let $M_n = \{m \in M : m > \max\operatorname{supp}\xi_n^M\}$ and $k_n = \min M_n$. Set

$$\xi_{n+1}^M = [\xi_{k_n}+1]_1^{M_n}.$$

The inductive definition of $(\xi_n^M)_{n=1}^\infty, M \in [\mathbb{N}]$, is now complete. We note that $\operatorname{supp}\xi_n^M \in \mathcal{S}_\xi$ for all $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $n \in \mathbb{N}$.

Definition 2.2. Let $\xi < \omega_1$ and $\delta > 0$. A seminormalized sequence (x_n) in a Banach space has an ℓ_1^ξ *spreading model with constant* δ if

$$\Big\|\sum_{i\in F}\alpha_i x_i\Big\| \ge \delta \sum_{i\in F}|\alpha_i|$$

for every $F \in \mathcal{S}_\xi$ and all choices of scalars $(\alpha_i)_{i\in F}$. We say that (x_n) has an ℓ_1^ξ *spreading model* if it has an ℓ_1^ξ spreading model with constant δ for some $\delta > 0$.

A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called *hereditary* if, for every $G \in \mathcal{F}$ and $F \subset G$, we have $F \in \mathcal{F}$.

For every vector $x = \sum_{i=1}^{n} a_i e_i \in c_{00}$ and every finite subset F of $\mathbb{N}$, we set $\langle x, F\rangle = \sum_{i\in F} a_i$.

We next state the definition of *large families* and a result from [9] (see also [8]) which is the main tool for our proof of Proposition 3.3.

Definition 2.3. Let $M \in [\mathbb{N}]$, $\xi < \omega_1$, $\delta > 0$ and $n \in \mathbb{N}$. A hereditary family $\mathcal{F}$ is called (M, ξ, δ)-*large* provided that for all $N \in [M]$ there exists $F \in \mathcal{F}$ such that $\langle \xi_1^N, F\rangle \geq \delta$.

Proposition 2.4 ([9] and [8]). *Let $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ be a hereditary family, $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $\varepsilon > 0$. If $\mathcal{F}$ is (M, ξ, ε)-large then there exists $N \in [M]$ with $\mathcal{S}_\xi(N) \subseteq \mathcal{F}$.*

Definition 2.5. (a) Let $k \in \mathbb{N}$. A finite sequence $(E_i)_{i=1}^m$ of successive subsets of $\mathbb{N}$ is said to be $\mathcal{S}_k$-*admissible* if $(\min E_i)_{i=1}^m \in \mathcal{S}_k$. A finite block sequence $(x_i)_{i=1}^m$ in c_{00} is said to be $\mathcal{S}_k$-*admissible* if $(\operatorname{supp} x_i)_{i=1}^m$ is $\mathcal{S}_k$-admissible.

(b) Let $(k_n)_n$ be an increasing sequence of integers and $(\theta_n) \subset (0,1)$ such that $\theta_n \searrow 0$. The *mixed Tsirelson space* $X = T[(\mathcal{S}_{k_n}, \theta_n)_{n=1}^\infty]$ is the completion of c_{00} under the norm which satisfies the implicit equation

$$\|x\| = \max\Big\{\|x\|_\infty, \sup_n \theta_n \Big\{\sup \sum_{i=1}^m \|E_i x\|\Big\}\Big\},$$

where the inner supremum is taken over all $\mathcal{S}_{k_n}$-admissible families $(E_i)_{i=1}^m$, $m \in \mathbb{N}$.

An essential role in our proofs is played by the following special vectors.

Definition 2.6. (a) Let $n \geq 1$, $\varepsilon > 0$ and $F \subseteq \mathbb{N}$, $F \in \mathcal{S}_n$. A convex combination $\sum_{j\in F} a_j e_j$ is called an (ε, n)-*basic special convex combination* (basic s.c.c.) if $\sum_{j\in G} a_j < \varepsilon$ for every $G \in \mathcal{S}_{n-1}$.

(b) Let $\varepsilon > 0$, $n \in \mathbb{N}$ and suppose that $(z_j)_{j=1}^m$ is a finite block sequence in c_{00} with the property that there exist integers $(l_j)_{j=1}^m$ with $2 < z_1 \leq l_1 < z_2 \leq l_2 < \ldots \leq l_{m-1} < z_m \leq l_m$ such that a convex combination $\sum_{j=1}^m a_j e_{l_j}$ is an (ε, n)-basic s.c.c. Then the corresponding convex combination of the z_j's, $x = \sum_{j=1}^m a_j z_j$, is called an (ε, n)-*s.c.c.* of $(z_j)_{j=1}^m$.

An (ε, j)-s.c.c. $x = \sum_{j=1}^m a_j z_j$ of *unit* vectors $(z_j)_{j=1}^m$ in a Banach space is said to be *seminormalized* if $\|x\| \geq 1/2$.

It is proved in [5, Lemma 1.6] that for every $\varepsilon > 0$, $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$, there exists an (ε, n)-basic s.c.c. $\sum_{j\in F} a_j e_j$ with $F \subset M$. In fact, it is not hard to see that the average n_1^L is a $(3/\min L, n)$-basic s.c.c. for every $L \in [M]$.

Lemma 2.7. *Let $(\theta_n)_n$, $0 < \theta_n < 1$, be a decreasing sequence. Let X be a Banach space with a basis with the following property: For every n and every $\mathcal{S}_n$-admissible block sequence $(x_i)_{i=1}^d$ we have $\|\sum_{i=1}^d x_i\| \geq \theta_n \sum_{i=1}^d \|x_i\|$.*

Suppose that for some $n \in \mathbb{N}$, $x = \sum_{j=1}^m a_j x_j$ is an (ε, n)-s.c.c in X, where $\varepsilon < \theta_n$. Let $i < n$ and suppose that $(E_r)_{r=1}^s$ is an $\mathcal{S}_i$-admissible family of intervals. Then

$$\sum_{r=1}^{s} \|E_r x\| \le (1 + \varepsilon/\theta_i) \max_{1 \le j \le m} \|x_j\| \le 2 \max_{1 \le j \le m} \|x_j\|.$$

The proof is the same as that of [6, Lemma 1.13], so we omit it.

3. ℓ_1^ω spreading models. In this section we present an example of a Banach space X with the following properties:

(1) There exists a constant $\delta > 0$ such that for every $k < \omega$, every block subspace of X admits an ℓ_1^k spreading model with constant δ.

(2) The space X does not admit any ℓ_1^ω spreading model.

As we shall show in Proposition 3.3, (1) is true in a large class of asymptotic ℓ_1 spaces. On the other hand, our next proposition shows that the original mixed Tsirelson spaces admit in addition ℓ_1^ω spreading models.

PROPOSITION 3.1. *Let $(\theta_n)_n$ be a sequence in $(0,1)$ such that $\theta_{m+n} \ge \theta_m \theta_n$, $\theta_n \searrow 0$ and $\lim_n \theta_n^{1/n} = 1$, and let $X = T[(\mathcal{S}_n, \theta_n)_{n\in\mathbb{N}}]$ be the corresponding mixed Tsirelson space. Then there exists a constant $K > 0$ such that every block subspace Y of X has a block sequence which has an ℓ_1^ω spreading model with constant $1/K$.*

The proof of this fact is influenced by the result of [20] that Schlumprecht's space has an ℓ_1 spreading model. To present a complete proof of the proposition we would have to almost copy some proofs from [6], so we only give an outline of the proof.

Sketch of the proof. A refinement of the proof of [6, Theorem 1.6] implies that there exists a lacunary sequence $(j_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\frac{\theta_{k+j_1+\ldots+j_k+1}}{\theta_{j_1+\ldots+j_k}} \ge \frac{1}{2} \quad \text{for all } k,$$

and with the following property: in every block subspace Y of X there exists an infinite block sequence $(z_i)_i$ of seminormalized s.c.c.'s such that for every $n \in \mathbb{N}$, we can choose a finite nested sequence $(x_k^n)_{k=1}^n = (y_k^n/\|y_k^n\|)_{k=1}^n$ satisfying:

(1) $y_k^n = \sum_{i\in F_k^n} \alpha_i z_i$ is a $j_1 + \ldots + j_k$-rapidly increasing s.c.c. (r.i.s.c.c.) of $(z_i)_i$ for every $k = 1, \ldots, n$ ([6, Definition 1.14] and our Definition 4.11).

(2) $F_k^n < F_j^{n+1}$ for all $n \in \mathbb{N}$, $k \le n$ and $j \le n+1$. That is, the sequence $(\sum_{k=1}^n x_k^n)_n$ is a block sequence.

(3) $\|\sum_{k=1}^n x_k^n\| \le 2$ for all $n \in \mathbb{N}$.

It follows from [6, Proposition 1.15] that there exists a constant $C > 0$ such that $\frac{1}{2}\theta_{j_1+\ldots+j_k+1} \le \|y_k^n\| \le C\theta_{j_1+\ldots+j_k}$ for all n and k. We set $w^n = \sum_{k=1}^n x_k^n$. Then $(w^n)_{n\in\mathbb{N}}$ is an ℓ_1^ω spreading model with constant $1/(4C)$. Indeed, let $k \in \mathbb{N}$ and $G \in \mathcal{S}_k$ with $\min G \ge k$. Then, for all $(\beta_n)_{n\in G}$,

$$\Big\|\sum_{n\in G}\beta_n w^n\Big\| \ge \Big\|\sum_{n\in G}\beta_n x_k^n\Big\|. \tag{3.1}$$

By the definition of a $j_1 + \ldots + j_k$-r.i.s.c.c., it follows that for all n the family $\{z_i : i \in F_k^n\}$ is $j_1 + \ldots + j_k + 1$-admissible. Also $\{\min F_k^n : n \in G\}$ is $\mathcal{S}_k$-admissible, so $\{z_i : i \in \bigcup_{n\in G} F_k^n\}$ is $k + j_1 + \ldots + j_k + 1$-admissible. Together with (3.1) and the fact that $\|y_k^n\| = \|\sum_{i\in F_k^n}\alpha_i z_i\| \le C\theta_{j_1+\ldots+j_k}$, this implies that

$$\begin{aligned}\Big\|\sum_{n\in G}\beta_n w^n\Big\| &\ge \theta_{k+j_1+\ldots+j_k+1}\sum_{n\in G}|\beta_n|\frac{\sum_{i\in F_k^n}\alpha_i\|z_i\|}{\|y_k^n\|}\\ &\ge \frac{1}{2C}\,\frac{\theta_{k+j_1+\ldots+j_k+1}}{\theta_{j_1+\ldots+j_k}}\sum_{n\in G}|\beta_n| \ge \frac{1}{4C}\sum_{n\in G}|\beta_n|. \ \blacksquare\end{aligned}$$

Remark 3.2. The proof of Proposition 3.1 is based on the fact that c_0 is finitely disjointly representable in the spaces $T[(\mathcal{S}_n,\theta_n)_{n\in\mathbb{N}}]$ with $\lim_{n\to\infty}\theta_n^{1/n} = 1$. Note that, on the contrary, c_0 is not finitely representable in the modified spaces $T_M[(\mathcal{S}_n,\theta_n)_{n\in\mathbb{N}}]$. However, in Section 4, we shall show that under certain conditions on the sequence $(k_n, m_n)_n$, the modified mixed Tsirelson space $T_M[(\mathcal{S}_{k_n}, 1/m_n)_n]$ contains an ℓ_1^ω spreading model in every block subspace.

The fact that c_0 is not finitely representable in the modified spaces is implied by the following theorem of A. Pełczyński and H. Rosenthal [30], as stated in [19].

Theorem (see [30]). *For every $n \in \mathbb{N}$ there is an $N = N(n)$ with the following property: Let X be a Banach space with a 1-unconditional basis (e_i), and F an n-dimensional subspace of X. Then F is contained in an N-dimensional subspace of X which is 2-isomorphic to the span of N disjointly supported vectors.*

A proof of this theorem can be found in [12]. Let us see how this result implies that c_0 is not finitely representable in $X_M = T_M[(\mathcal{S}_n,\theta_n)_n]$. Suppose that c_0 is finitely representable in X_M. Then there exists $C > 0$ such that for every $n \in \mathbb{N}$ there exist n normalized vectors $(y_i)_{i=1}^n$ in X_M with $\operatorname{supp} y_i \ge N = N(n)$ for all $i = 1,\ldots,n$, which are C-equivalent to the unit vector basis of ℓ_∞^n. It follows from the theorem that there exist N vectors disjointly supported after N, $(z_i)_{i=1}^N$, and an into 2-isomorphism $S : \operatorname{span}\{y_i : 1 \le i \le n\} \to \operatorname{span}\{z_i : 1 \le i \le N\}$. Since the vectors $(z_i)_{i=1}^N$ are disjointly supported after N we find that $\operatorname{span}\{z_i : 1 \le i \le N\}$ is $1/\theta_1$-isomorphic to ℓ_1^N. For $i = 1, \ldots, n$ let $x_i = S(y_i)$. Then $\|x_i\| \ge 1/2$, and since ℓ_1 has cotype 2, it follows that

$$C \ge \frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^n \varepsilon_i y_i \Big\| \ge \frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \frac{1}{2} \Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|$$
$$\ge \frac{A_1 \theta_1}{2} \Big(\sum_{i=1}^n \|x_i\|^2 \Big)^{1/2} \ge \frac{\theta_1 A_1}{4} \sqrt{n},$$

where A_1 is the cotype-2 constant of ℓ_1. This yields a contradiction for large n. ∎

Proposition 3.3. *Let X be a Banach space with a basis $(e_i)_i$ satisfying the following: There exists a sequence $(\theta_k)_{k\in\mathbb{N}}$ such that $\lim_k \theta_k^{1/k} = 1$ and, for every $k \in \mathbb{N}$, every $\mathcal{S}_k$-admissible block sequence $(z_j)_{j=1}^d$ of $(e_i)_i$ satisfies*

$$\Big\| \sum_{j=1}^d z_j \Big\| \ge \theta_k \sum_{j=1}^d \|z_j\|.$$

Then there exists $c > 0$ such that for every $k \in \mathbb{N}$, every block sequence $(x_i)_i$ of $(e_i)_i$ has a further block sequence which has an ℓ_1^k spreading model with constant c.

Proof. We shall use the repeated averages hierarchy. Let $k \in \mathbb{N}$ and let $\vec{x} = (x_i)_{i=1}^\infty$ be a normalized block sequence in X. For each $i \in \mathbb{N}$, let

$$l_i = \min \operatorname{supp} x_i$$

and set $L = (l_i)_{i=1}^\infty$. For $P \in [\mathbb{N}]$, we set $S = \{l_p : p \in P\} \subseteq L$. Let k_1^S be the first k-average with respect to the set S and suppose that $k_1^S = \sum_{j\in G} \alpha_j e_{l_j}$, where $G \subseteq P$, $\alpha_j \ge 0$ and $\sum_{j\in G} \alpha_j = 1$. Then we set $\alpha(k, \vec{x}, P) = \sum_{j\in G} \alpha_j x_j$. Since $\operatorname{supp} k_1^S \in \mathcal{S}_k$, it is clear that the family $(x_j)_{j\in G}$ is $\mathcal{S}_k$-admissible. Since $\alpha(k, \vec{x}, P)$ is determined by an initial segment of P, the set $\mathcal{A} = \{P \in [\mathbb{N}] : \|\alpha(k, \vec{x}, P)\| \ge 1/2\}$ is open. Therefore, applying the infinite Ramsey theorem, we see that either

(i) there exists $M \in [\mathbb{N}]$ such that $\|\alpha(k, \vec{x}, P)\| \ge 1/2$ for all $P \in [M]$, or
(ii) there exists $M \in [\mathbb{N}]$ such that $\|\alpha(k, \vec{x}, P)\| < 1/2$ for all $P \in [M]$.

Suppose that (i) holds. We shall show that $(x_i)_i$ has a subsequence $(y_i)_i$ which has an ℓ_1^k spreading model with constant $c = 1/4$ if (x_i) is unconditional, and $c = 1/512$ in the general case. Let

$$\mathcal{F}_{1/4} = \{F \subseteq L : \exists x_F^* \in B_{X^*} \text{ with } x_F^*(x_j) \ge 1/4 \text{ for every } l_j \in F\}.$$

Set $N = \{l_m : m \in M\} \subseteq L$. Then $\mathcal{F}_{1/4}$ is $(N, k, 1/4)$-large.

Indeed, by our assumption, for every $N' = \{l_m : m \in M'\} \in [N]$, $\|\alpha(k,\vec{x},M')\| = \|\sum_{j\in G}\alpha_j x_j\| \geq 1/2$, so there exists $x^* \in B_{X^*}$ such that $x^*(\sum_{j\in G}\alpha_j x_j) \geq 1/2$. Set $F = \{l_j : j \in G \text{ and } x^*(x_j) \geq 1/4\}$. By definition, $F \in \mathcal{F}_{1/4}$. Also,

$$\begin{aligned}\langle k_1^{N'}, F\rangle &= \sum_{l_j\in F}\alpha_j \geq \sum_{l_j\in F}\alpha_j x^*(x_j)\\ &= x^*\Big(\sum_{j\in G}\alpha_j x_j\Big) - x^*\Big(\sum_{l_j\notin F}\alpha_j x_j\Big) \geq \frac{1}{2}-\frac{1}{4}=\frac{1}{4}.\end{aligned}$$

So, $\mathcal{F}_{1/4}$ is $(N,k,1/4)$-large. It follows from Proposition 2.4 that there exists $Q \in [N]$ such that $\mathcal{S}_k(Q) \subseteq \mathcal{F}_{1/4}$. Suppose first that the basic sequence (x_i) is unconditional. Let $Q = \{l_{s_1}, l_{s_2},\ldots\}$. We set $y_i = x_{s_i}$, $i=1,2,\ldots$ We claim that the sequence $(y_i)_{i=1}^\infty$ has an ℓ_1^k spreading model with constant $1/4$.

Indeed, if $A \in \mathcal{S}_k$ then $\{l_{s_i} : i \in A\} \in \mathcal{S}_k(Q)$, and so $\{l_{s_i} : i \in A\} \in \mathcal{F}_{1/4}$. It follows then there exists $x^* \in B_{X^*}$ such that $x^*(y_i) = x^*(x_{s_i}) \geq 1/4$ for all $i \in A$. So $\|\sum_{i\in A}\beta_i y_i\| \geq 1/4$ for every $(\beta_i)_{i\in A}$ with $\beta_i \geq 0$ and $\sum_{i\in A}\beta_i = 1$, which proves our claim.

If $(x_i)_i$ is not unconditional then the existence of an ℓ_1^k spreading model with constant $c \geq 1/512$ is a consequence of the following result [8, Corollary 3.6]: For a normalized weakly null sequence $(x_i)_i$ and $\xi < \omega_1$, the following are equivalent:

(a) There exists $M \in [\mathbb{N}]$, $M = (m_i)$, so that $(x_{m_i})_i$ has an ℓ_1^ξ spreading model.

(b) There exist $N \in [\mathbb{N}]$ and $\delta > 0$ such that $\mathcal{S}_\xi(N) \subset \mathcal{F}_\delta$.

Suppose now that (ii) holds. We shall show that also in this case, $(x_i)_{i=1}^\infty$ has a block sequence which has an ℓ_1^k spreading model.

Set $N = \{l_m : m \in M\}$ and consider the sequence $k_n^N = \sum_{j\in F_n}\alpha_j e_{l_j}$, $n=1,2,\ldots$ For $n=1,2,\ldots$, we set $y_n^1 = \sum_{j\in F_n}\alpha_j x_j = \alpha(k,\vec{x},M_n)$, where $M_1 = M$ and $M_n = \{m \in M : m > \operatorname{supp} k_{n-1}^M\}$, $n=2,3\ldots$ By our assumption, $\|y_n^1\| < 1/2$ for every n. We now set $w_n^1 = y_n^1/\|y_n^1\|$ for $n \in \mathbb{N}$. We note that for every $\mathcal{S}_k$-admissible sequence $(w_i^1)_{i\in G}$, the family $\{x_j : j \in \bigcup_{i\in G}F_i\}$ is $\mathcal{S}_{2k}$-admissible. So, for any choice of convex coefficients $(\beta_i)_{i\in G}$ we have

$$\Big\|\sum_{i\in G}\beta_i w_i^1\Big\| = \Big\|\sum_{i\in G}\beta_i\frac{y_i^1}{\|y_i^1\|}\Big\| \geq \Big\|\sum_{i\in G}\frac{\beta_i}{\|y_i^1\|}\sum_{j\in F_i}\alpha_j x_j\Big\| \geq 2\theta_{2k}.$$

We again apply the infinite Ramsey theorem, this time to the sequence $\vec{w} = (w_i^1)_{i=1}^\infty$, to conclude that there exists $M \in [\mathbb{N}]$ such that either

(1) $\|\alpha(k,\vec{w},P)\| \geq 1/2$ for all $P \in [M]$, or
(2) $\|\alpha(k,\vec{w},P)\| < 1/2$ for all $P \in [M]$.

If (1) holds, then as in case (i) above, we obtain a subsequence of $(w_i^1)_i$ which has an ℓ_1^k spreading model with constant c.

So suppose that (2) holds. Then, as before, we find a block sequence $(y_i^2)_{i=1}^\infty$ of $(w_i^1)_{i=1}^\infty$ where, for every $i=1,2,\ldots$, y_i^2 is a convex combination of an $\mathcal{S}_k$-admissible sequence $(w_j^1)_{j\in J_i}$, and $\|y_i^2\|<1/2$. Set $w_i^2=y_i^2/\|y_i^2\|$; then for every $\mathcal{S}_k$-admissible sequence $(w_i^2)_{i\in G}$ and every choice of convex coefficients $(\beta_i)_{i\in G}$, $\|\sum_{i\in G}\beta_i w_i^2\|\geq 2^2\theta_{3k}$. Once more we pass to a set $M\in[\mathbb{N}]$ such that for the sequence $\vec{w}=(w_i^2)_{i=1}^\infty$ either (1) or (2) holds. If (1) is true then we are done. If (2) holds, then we proceed in the same way to construct a sequence $(w_i^3)_{i=1}^\infty$ and so on.

Claim. *Let $n\in\mathbb{N}$ be such that $2^{n+1}\theta_{(n+1)k}\geq 1$. Then there exists some $j\leq n$ such that* (1) *holds for the sequence $\vec{w}=(w_i^j)_{i=1}^\infty$.*

Proof of the Claim. Suppose not. Then we can continue the previous construction up to a normalized block sequence $(w_i^n)_{i=1}^\infty$ with the following property: For every $\mathcal{S}_k$-admissible family $(w_i^n)_{i\in G}$, and every choice of convex coefficients $(\beta_i)_{i\in G}$, we have $\|\sum_{i\in G}\beta_i w_i^n\|\geq 2^n\theta_{(n+1)k}\geq 1/2$.

On the other hand, by our assumption, there is a set $M\in[\mathbb{N}]$ for which (2) holds for $\vec{w}=(w_i^n)_{i=1}^\infty$. So $\|\alpha(k,\vec{w},M)\|<1/2$, a contradiction. This completes the proof of the claim. ∎

We have already seen that the claim yields the existence of a block sequence which has an ℓ_1^k spreading model with constant c. ∎

We proceed to give an example of an asymptotic ℓ_1 Banach space X satisfying the assumptions of Proposition 3.3, which does not admit any ℓ_1^ω spreading model.

Definition of the space X. We choose a decreasing sequence $\theta_j\in(0,1)$, $j=1,2,\ldots$, with the property $\sum_{j=1}^\infty\theta_j<1/100$. We also choose a sequence $(n_j)_{j=1}^\infty$ of integers with $n_1=1$ and such that $\lim_{j\to\infty}\theta_j^{1/n_{j-1}}=1$.

Inductively, we construct a sequence $(K_j)_j$ of subsets of $c_{00}(\mathbb{N})$ as follows: Let $K_0=\{\pm e_n:n\in\mathbb{N}\}$. Suppose that for some $j\geq 0$, K_j is defined. For $r=1,2,\ldots$, set

$$A_{j+1}^r=\{\theta_r(f_1+\ldots f_d):(f_i)_{i=1}^d \text{ is } \mathcal{S}_{n_r}\text{-admissible and } f_i\in K_j \text{ for all } i\leq d\}.$$

We set $L_{j+1}=\bigcup_{r=1}^\infty A_{j+1}^r$ and

$$M_{j+1}=\Big\{\sum_{i=1}^n f_i : n\in\mathbb{N},\ n\leq\min\Big(\bigcup_{i=1}^n\operatorname{supp} f_i\Big),\ \forall i=1,\ldots,n,\ f_i\in L_{j+1}$$
$$\text{and } f_i\in A_{j+1}^{r_i} \text{ for some } r_1,\ldots,r_n \text{ with } r_1<\ldots<r_n\Big\}.$$

Note that there is no requirement of disjointness on the supports of the f_i's,

$i=1,\dots,n$. Finally we set $K_{j+1}=K_j\cup L_{j+1}\cup M_{j+1}$. We define

$$L=\bigcup_{j=1}^{\infty}L_j,\qquad M=\bigcup_{j=1}^{\infty}M_j,\qquad K=\bigcup_{j=0}^{\infty}K_j.$$

The norm $\|\cdot\|$ of X is defined on $c_{00}(\mathbb{N})$ by

$$\|x\|=\sup_{f\in K}\langle x,f\rangle.$$

X is the completion of $c_{00}(\mathbb{N})$ under this norm. The following properties are easily established:

(1) $(e_i)_i$ is a 1-unconditional basis of X.

(2) If $x_1<\dots<x_k$ is a block sequence of $(e_i)_i$ which is $\mathcal{S}_{n_r}$-admissible, then $\|\sum_{i=1}^k x_i\|\geq\theta_r\sum_{i=1}^k\|x_i\|$

(3) $\|\cdot\|$ is dominated by the ℓ_1-norm.

Remarks 3.4. 1. The space X is reflexive. This follows from the fact that it is an asymptotic ℓ_1 Banach space with an unconditional basis which does not contain ℓ_1, since, as we shall show, it does not have any ℓ_1^ω spreading model.

2. A characteristic property of the dual of the space X is that we can add functionals which belong to different classes $\mathcal{A}^r=\bigcup_{j=1}^\infty A_j^r$ and get a functional in the unit ball. A similar property holds in the space constructed by W. T. Gowers [16] which does not contain c_0, ℓ_1 or a reflexive subspace, and also in the example of E. Odell and Th. Schlumprecht [27] of a space X without any c_0 or ℓ_p spreading model. In our case, this property does not allow a construction of a bounded sequence similar to the sequence $(w_n)_n$ which had an ℓ_1^ω spreading model in the space $T[(\mathcal{S}_n,\theta_n)_n]$ (Proposition 3.1).

It follows from Proposition 3.3 that for every $k<\omega$, every block sequence in X has a further normalized block sequence which has an ℓ_1^k spreading model with constant $1/4$. The rest of this section is devoted to the proof that X does not admit any ℓ_1^ω spreading model. Assume on the contrary that there exists a sequence $(x_i)_{i=1}^\infty$ in X which has an ℓ_1^ω spreading model with constant $c>0$. We may also assume that $(x_i)_{i=1}^\infty$ is a block sequence. The next lemma shows that, by passing to a further block sequence, we may add the assumption $c=1/2$.

Lemma 3.5. *Let $(x_k)_{k=1}^\infty$ be a normalized block sequence which has an ℓ_1^ω spreading model with constant $\delta<1$. Then, for every $\varepsilon>0$, there exists a block sequence $(y_k)_{k=1}^\infty$ of $(x_k)_{k=1}^\infty$ which has an ℓ_1^ω spreading model with constant $1-\varepsilon$.*

Proof. Suppose that $\delta < 1-\varepsilon$. Assume that the following property holds:

($*$) There exists a strictly increasing sequence $(m_k)_{k=1}^{\infty}$ of integers such that for every k, if $F \in \mathcal{S}_k$ and $F \geq m_k$ then $\|\sum_{i\in F}\alpha_i x_i\| \geq (1-\varepsilon)\sum_{i\in F}|\alpha_i|$ for all real numbers $(\alpha_i)_{i\in F}$.

Then it is easily seen that $(x_{m_k})_{k\in\mathbb{N}}$ has an ℓ_1^{ω} spreading model with constant $1-\varepsilon$.

Assume now that property ($*$) does not hold. Then there exists $k \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there exist $F \in \mathcal{S}_k$ with $m \leq F$ and real numbers $(\alpha_j)_{j\in F}$ such that

$$\Big\|\sum_{j\in F}\alpha_j x_j\Big\| < (1-\varepsilon)\sum_{j\in F}|\alpha_j|.$$

Then inductively we choose $k < F_1 < F_2 < \ldots$ (successive elements of $\mathcal{S}_k$) and real numbers $(\alpha_j)_{j\in F_i}$ such that $\|\sum_{j\in F_i}\alpha_j x_j\| < (1-\varepsilon)\sum_{j\in F_i}|\alpha_j|$ for every $i \in \mathbb{N}$. Set

$$y_i = \frac{\sum_{j\in F_i}\alpha_j x_j}{\|\sum_{j\in F_i}\alpha_j x_j\|} \quad \text{for } i = 1, 2, \ldots$$

Then the sequence $(y_i)_i$ has an ℓ_1^{ω} spreading model with constant $\delta/(1-\varepsilon)$.

Indeed, let $G \in \mathcal{S}_m$ and $G \geq m$ for some $m \in \mathbb{N}$. Then the set $\bigcup_{i\in G}F_i$ belongs to $\mathcal{S}_{k+m}$ and $\bigcup_{i\in G}F_i \geq k+m$. Since $(x_i)_i$ has an ℓ_1^{ω} spreading model with constant δ, it follows that

$$\begin{aligned}\Big\|\sum_{i\in G}\beta_i y_i\Big\| &= \Big\|\sum_{i\in G}\beta_i\,\frac{\sum_{j\in F_i}\alpha_j x_j}{\|\sum_{j\in F_i}\alpha_j x_j\|}\Big\| \\ &\geq \delta\sum_{i\in G}|\beta_i|\,\frac{\sum_{j\in F_i}|\alpha_j|}{\|\sum_{j\in F_i}\alpha_j x_j\|} \geq \frac{\delta}{1-\varepsilon}\sum_{i\in G}|\beta_i|\end{aligned}$$

for all real numbers $(\beta_i)_{i\in G}$.

Let $n \in \mathbb{N}$ be such that $\delta/(1-\varepsilon)^{n+1} \geq 1$. Repeating the above argument at most n times we obtain the result. ∎

PROPOSITION 3.6. *Suppose that the normalized block sequence $\vec{x} = (x_k)_k$ in X has an ℓ_1^{ω} spreading model with constant δ. Set*

$$\mathcal{F}_{\delta/2} = \{F \in [\mathbb{N}]^{<\omega} : \exists x_F^* \in K\ \forall i \in F\ \ x_F^*(x_i) \geq \delta/2\}.$$

Then $\mathcal{F}_{\delta/2}$ is $(\mathbb{N}, \omega, \delta/4)$-large (see Definition 2.3).

Proof. Let $L \in [\mathbb{N}]$. We set $A = \operatorname{supp}\omega_1^L \in \mathcal{S}_\omega$, and $x = \omega_1^L \cdot \vec{x} = \sum_{k\in A}\alpha_k x_k$, where $(\alpha_k)_k$ are nonnegative numbers and $\sum_{k\in A}\alpha_k = 1$.

By our assumption, $\|\sum_{k\in A}\alpha_k x_k\| \geq \delta$, so there exists $x^* \in K$ such that $x^*(\sum_{k\in A}\alpha_k x_k) \geq 3\delta/4$. Let $F = \{k \in A : x^*(x_k) \geq \delta/2\}$. Then, by definition, $F \in \mathcal{F}_{\delta/2}$. We shall show that $\langle\omega_1^L, F\rangle \geq \delta/4$, which will prove that $\mathcal{F}_{\delta/2}$ is $(\mathbb{N},\omega,\delta/4)$-large. We have

$$\langle \omega_1^L, F\rangle = \sum_{k\in F}\alpha_k \geq \sum_{k\in F}\alpha_k x^*(x_k)$$
$$= \sum_{k\in A}\alpha_k x^*(x_k) - \sum_{k\in A\setminus F}\alpha_k x^*(x_k) \geq \frac{3\delta}{4} - \frac{\delta}{2} = \frac{\delta}{4}. \blacksquare$$

Suppose now that $(x_i)_{i=1}^\infty$ has an ℓ_1^ω spreading model in X with constant $1/2$. Combining Propositions 2.4 and 3.6 we find that there exists a set $N = \{n_1, n_2, \ldots\} \subseteq \mathbb{N}$ such that, for $y_i = x_{n_i}$, $i = 1, 2, \ldots$, the following holds:

(3.2) For every $F \in \mathcal{S}_\omega$, there exists $y_F^* \in K$ such that $y_F^*(y_i) \geq 1/4$ $\forall i \in F$.

Notation. For every $r = 1, 2, \ldots$, we set $\mathcal{A}^r = \bigcup_{j=1}^\infty A_j^r \subseteq K$. Let $r, s \in \mathbb{N}$ with $r \leq s$. We set

$$\mathcal{A}[r,s] = \Big\{\phi \in K : \phi = \sum_{i=1}^d f_i,\ d \in \mathbb{N},\ d \leq \min\Big(\bigcup_{i=1}^d \operatorname{supp} f_i\Big) \text{ and}$$
$$\forall i = 1, \ldots, d,\ f_i \in \mathcal{A}^{r_i} \text{ with } r \leq r_1 < \ldots < r_d \leq s\Big\}.$$

Note that $\mathcal{A}^q \subseteq \mathcal{A}[r,s]$ for all $r \leq q \leq s$.

Proposition 3.7. *Let $j_0 \in \mathbb{N}$ and $(y_i)_i$ be a normalized block sequence in X satisfying* (3.2). *Then there exist $i_0 \in \mathbb{N}$ and $s_0 > j_0$ such that for all $i > i_0$, there exists $\phi \in K$ with*

$$\phi \in \mathcal{A}[j_0+1, s_0] \quad \textit{and} \quad \phi(y_i) \geq 1/8.$$

Before presenting the proof of the above proposition, we show how it implies that X does not admit an ℓ_1^ω spreading model.

Theorem 3.8. *The space X does not admit an ℓ_1^ω spreading model.*

Proof. Suppose that X admits an ℓ_1^ω spreading model. Then we can assume that for some normalized block sequence $(y_i)_{i=1}^\infty$, the conclusion of Proposition 3.7 is true. For $j_0 = 2$ there exist i_1 and s_1 such that for every $i > i_1$, there exists $\phi \in K$ with $\phi \in \mathcal{A}[3, s_1]$ and $\phi(y_i) \geq 1/8$. In the same way, there exist $i_2 > i_1$ and $s_2 > s_1$ such that for every $i > i_2$ there exists $\phi \in K$ with $\phi \in \mathcal{A}[s_1+1, s_2]$ and also $\phi(y_i) \geq 1/8$.

Continuing in this manner we find positive integers $s_1 < \ldots < s_9$ and $i_9 \in \mathbb{N}$ such that for all $i > i_9$, there exist $\phi_1, \ldots, \phi_9 \in K$ with $\phi_j \in \mathcal{A}[s_{j-1}+1, s_j]$ and $\phi_j(y_i) \geq 1/8$ for every $j = 1, \ldots, 9$. It only remains to choose $i_0 > i_9$ such that $\min\operatorname{supp} y_{i_0} \geq s_9$. Then by replacing ϕ_j by $\psi_j = \phi_{j|[\min\operatorname{supp} y_{i_0}, \infty)}$, $j = 1, \ldots, 9$, we see that $\psi_1 + \ldots + \psi_9 \in K$. Indeed, this is the sum of a sequence $(f_i)_{i=1}^d$ of elements of L, with $f_i \in \mathcal{A}^{r_i}$ where $r_1 < \ldots < r_d \leq s_9$, which yields $d \leq s_9 \leq \min(\bigcup_{i=1}^d \operatorname{supp} f_i)$.

Furthermore, $(\psi_1 + \ldots + \psi_9)(y_{i_0}) \geq 9/8$, which leads to a contradiction, and the proof of the theorem is complete. ∎

Hence it remains to prove Proposition 3.7.

Proof of Proposition 3.7. Suppose that the result is false. We may assume that $j_0 \geq 3$. Then for all $i_0 \in \mathbb{N}$ and any $s > j_0$ there exists $i > i_0$ such that for all $\phi \in K$, if $\phi \in \mathcal{A}[j_0 + 1, s]$ then $\phi(y_i) < 1/8$.

Let $i_1 = 1$. We choose $j_1 > j_0$ such that

$$\Big(\sum_{r \geq j_1} \theta_r\Big) \|y_{i_1}\|_{\ell_1} < 1/100.$$

Then there exists $i_2 > i_1$ such that for all $\phi \in K$, if $\phi \in \mathcal{A}[j_0 + 1, j_1]$ then $\phi(y_{i_2}) < 1/8$.

We choose $j_2 > j_1$ such that

$$\Big(\sum_{r \geq j_2} \theta_r\Big) \|y_{i_2}\|_{\ell_1} < 1/100$$

and then $i_3 > i_2$ such that for all $\phi \in K$, if $\phi \in \mathcal{A}[j_0+1, j_2]$ then $\phi(y_{i_3}) < 1/8$. Continuing in this way we construct a subsequence $(y_{i_k})_{k=1}^{\infty}$ of $(y_i)_{i=1}^{\infty}$ and $(j_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ with $j_0 < j_1 < \ldots$ with the following properties:

(P.1) For all $k \geq 2$, if $\phi \in K$ and $\phi \in \mathcal{A}[j_0 + 1, j_{k-1}]$ then $\phi(y_{i_k}) < 1/8$.
(P.2) For all $k \geq 1$, $(\sum_{r \geq j_k} \theta_r)\|y_{i_k}\|_{\ell_1} < 1/100$.

(P.2) implies that if $f_i \in \mathcal{A}^{r_i}$, $i = 1, \ldots, n$, with $j_k \leq r_1 < \ldots < r_n$, then $|(\sum_{i=1}^{n} f_i)(y_{i_k})| < 1/100$.

We now set

$$l_k = \max \operatorname{supp} y_{i_k} \quad \text{ for } k = 1, 2, \ldots,$$

and choose $(k_n)_{n=1}^{\infty}$ with $n_{j_0} + 2 \leq i_{k_1} \leq l_{k_1} < \ldots \leq l_{k_{n-1}} < i_{k_n} \leq l_{k_n} < \ldots$

Let $(l_{k_s})_{s\in G} \in \mathcal{S}_{n_{j_0}+1}$ be the support of a $(1/(10j_0), n_{j_0} + 1)$-basic special convex combination, $\sum_{s\in G} \alpha_s e_{l_{k_s}}$, such that $\max_s \alpha_s \leq 1/l^2_{k_{\min G}}$. Then $(i_{k_s})_{s\in G} \in \mathcal{S}_{n_{j_0}+2}$ and $n_{j_0} + 2 \leq i_{k_1}$, so $(i_{k_s})_{s\in G} \in \mathcal{S}_{\omega}$.

This shows that there exists a $(1/(10j_0), n_{j_0} + 1)$-s.c.c. $\sum_{k\in F} \alpha_k y_{i_k}$ of $(y_{i_k})_k$ such that $\{i_k : k \in F\} \in \mathcal{S}_{\omega}$ and $\max_k \alpha_k \leq 1/l^2_{\min F}$. By property (3.2) of the sequence (y_i), we deduce that there exists $x^*_F \in K$ such that

$$x^*_F(y_{i_k}) \geq 1/4 \quad \forall k \in F.$$

We will show that this leads to a contradiction.

Let $F = \{k_1, \ldots, k_n\}$. There are two cases for x^*_F.

Case 1: $x^*_F \in L$. Let $x^*_F \in \mathcal{A}^r$. Suppose $r \leq j_0$. Then, by Lemma 2.7, $|x^*_F(\sum_{k\in F} \alpha_k y_{i_k})| \leq 2\theta_r \leq 1/50$, a contradiction. Now suppose $j_0 < r < j_{k_1}$. Then, by (P.1), $x^*_F(y_{i_{k_2}}) < 1/8$, a contradiction. Finally, suppose $j_{k_1} \leq r$. Then it follows by (P.2) that $|x^*_F(y_{i_{k_1}})| < 1/100$, a contradiction again.

Case 2: $x_F^* \in M \setminus L$. Then $x_F^* = \sum_{q=1}^d f_q$ where $d \le \min(\bigcup_{q=1}^d \operatorname{supp} f_q)$ and for all $q = 1, \ldots, d$, $f_q \in \mathcal{A}^{r_q}$ where $r_1 < \ldots < r_d$. We set

$$A_0 = \{q = 1, \ldots, d : r_q \le j_0\}, \quad A_n = \{q = 1, \ldots, d : j_{k_{n-1}} < r_q\}$$

and for $s = 1, \ldots, n-1$,

$$A_s = \{q = 1, \ldots, d : j_{k_{s-1}} < r_q \le j_{k_s}\},$$

where $k_0 = 0$. Then

$$x_F^* = \sum_{s=0}^{n-1} \sum_{q \in A_s} f_q.$$

We shall show that there are at least $2l_{k_1}$ sets A_s which are not empty. This implies that there are at least $2l_{k_1}$ different f_q's, therefore $2l_{k_1} \le d$. This yields a contradiction, since $x_F^*(y_{i_{k_1}}) \neq 0$, and hence $d \le \min \operatorname{supp} x_F^* \le l_{k_1}$.

So, it remains to prove the following:

Claim. *The cardinality of the set $\{s : A_s \neq \emptyset\}$ is greater than or equal to $2l_{k_1}$.*

Proof of the Claim. Consider the set $A_0 = \{q = 1, \ldots, d : r_q \le j_0\}$. Then obviously, $\#A_0 \le j_0$. For each $q \in A_0$, $f_q = \theta_{r_q}(\sum_{s=1}^p g_s)$, where $g_s \in K$ for all $s = 1, \ldots, p$, the family $(g_s)_{s=1}^p$ is r_q-admissible and $r_q \le j_0$. For $k \in F$, we say that f_q *splits* y_{i_k} if

$$\operatorname{supp} g_s \cap \operatorname{supp} y_{i_k} \neq \emptyset \quad \text{for at least two different } g_s.$$

We set $J_q = \{k \in F : y_{i_k} \text{ is split by } f_q\}$ and note that $\{l_k : k \in J_q\} \in \mathcal{S}_{j_0}$. So, $\sum_{k \in J_q} \alpha_k < 1/(10 j_0)$. We now let $J = \bigcup_{q \in A_0} J_q$ be the set of indices $k \in F$ such that y_{i_k} is split by some f_q with $r_q \le j_0$. We get

$$\sum_{k \in J} \alpha_k \le \sum_{q \in A_0} \sum_{k \in J_q} \alpha_k \le 1/10.$$

Letting now

$$I = \{k \in F : y_{i_k} \text{ is not split by any } f_q \text{ with } r_q \le j_0\},$$

we get $\sum_{k \in I} \alpha_k \ge 9/10$. So,

$$9/10 \le \sum_{k \in I} \alpha_k \le \max \alpha_k \cdot (\#I) \le \frac{1}{l_{k_1}^2} (\#I).$$

Thus $\#I \ge \frac{9}{10} l_{k_1}^2 > 2l_{k_1}$.

We can now prove that for each $k_s \in I$, the set A_s is nonempty. Indeed, let $k_s \in I$. Since $y_{i_{k_s}}$ is not split by any f_q with $q \in A_0$,

$$\Big|\Big(\sum_{q \in A_0} f_q\Big)(y_{i_{k_s}})\Big| \le \sum_{r_q \le j_0} \theta_{r_q} \le 1/100.$$

Also,

$$\Big|\Big(\sum_{t=1}^{s-1}\sum_{q\in A_t} f_q\Big)(y_{i_{k_s}})\Big|\le\frac{1}{8} \text{ by (P.1)},\quad \Big|\Big(\sum_{t\ge s+1}\sum_{q\in A_t} f_q\Big)(y_{i_{k_s}})\Big|\le\frac{1}{100} \text{ by (P.2)}.$$

Since $(\sum_{q=1}^{d} f_q)(y_{i_{k_s}})\ge 1/4$ it follows that there exists $q\in A_s$ with $f_q(y_{i_{k_s}})\neq 0$, hence $A_s\neq\emptyset$. ∎

4. Distortion of modified mixed Tsirelson spaces. The modified Tsirelson space T_M was introduced by W. B. Johnson [19]. Later, P. Casazza and E. Odell [13] and S. Bellenot [11] proved that T_M is naturally 2-isomorphic to T. The situation is different with mixed Tsirelson spaces. The modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n,\theta_n)_n]$ were introduced in [6], where it was proved that these spaces are reflexive, and totally incomparable to the original ones in the case $\lim\theta_n^{1/n}=1$. In this section we prove that if we choose a sequence $(\theta_n)_n$ of reals with $\theta_n\searrow 0$ and $\theta_{n+1}\le\theta_n^3$ and an appropriate subsequence $(\mathcal{S}_{k_n})_n$ of the Schreier sequence $(\mathcal{S}_n)_n$, then the modified mixed Tsirelson space $X_M=T_M[(\mathcal{S}_{k_n},\theta_n)_n]$ is arbitrarily distortable. This is established by proving the existence of an asymptotic biorthogonal system in X_M.

Moreover, assuming some additional properties for the double sequence $(k_n,\theta_n)_n$, which we call the Gasparis conditions (Definition 4.14), we prove that every block subspace of X_M admits an ℓ_1^ω spreading model.

Before we give the definition of the space X_M let us recall the definition of the modified sequence $(\mathcal{S}_n^M)_n$ and state a lemma.

Lemma 4.1. *For $n<\omega$ define the family $\mathcal{S}_n^M$ inductively as follows*:

$$\mathcal{S}_0^M=\mathcal{S}_0=\{\{n\}:n\in\mathbb{N}\}\cup\{\emptyset\}.$$

$$\mathcal{S}_{n+1}^M=\Big\{\bigcup_{i=1}^{k}A_i : k\in\mathbb{N},\ A_i\in\mathcal{S}_n^M \textit{ for } i=1,\dots,k,\ A_i\cap A_j=\emptyset \\ \textit{for } i\neq j \textit{ and } k\le\min A_1<\dots<\min A_k\Big\}\cup\{\emptyset\}.$$

Then $\mathcal{S}_n^M=\mathcal{S}_n$ for all n.

The proof can be found in [6, Lemma 1.2].

Definition 4.2. Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$.

(a) A finite sequence $(E_i)_{i=1}^k$ of finite subsets of $\mathbb{N}$ is said to be $\mathcal{M}$-*allowable* if the set $(\min E_i)_{i=1}^k$ belongs to $\mathcal{M}$ and $E_i\cap E_j=\emptyset$ for all $i,j=1,\dots,k$, $i\neq j$.

(b) A finite sequence $(x_i)_{i=1}^k$ of vectors in c_{00} is $\mathcal{M}$-*allowable* if the sequence $(\operatorname{supp} x_i)_{i=1}^k$ is $\mathcal{M}$-allowable.

We now pass to the definition of the space X_M. We choose a sequence $(m_j)_{j=1}^{\infty}$ of integers such that $m_1 = 2$ and $m_j \geq m_{j-1}^3$ for $j = 2, 3, \ldots,$ We choose inductively a subsequence $(\mathcal{S}_{k_j})_{j=1}^{\infty}$ of $(\mathcal{S}_n)_n$ as follows: We set $k_1 = 1$. Suppose that k_j, $j = 1, \ldots, n-1$, have been chosen. Let t_n be such that $2^{t_n} \geq m_n^2$. We set $k_n = t_n(k_{n-1} + 1)$, $\mathcal{M}_j = \mathcal{S}_{k_j}$ for $j = 1, 2, \ldots,$ and

$$X_M = T_M[(\mathcal{M}_j, 1/m_j)_{j=1}^{\infty}].$$

The norm of X_M is defined by the following implicit equation:

$$\|x\| = \max\Big\{ \|x\|_{\infty}, \sup_j \frac{1}{m_j} \sup\Big\{ \sum_{i=1}^{d} \|E_i x\| : (E_i)_{i=1}^{d} \text{ is } \mathcal{M}_j\text{-allowable}\Big\}\Big\}.$$

We shall also make use of the following alternative definition of the norm of X_M. Inductively, we define a subset $K = \bigcup_{n=0}^{\infty} K^n$ of $B_{X_M^*}$ as follows: For $j = 1, 2, \ldots,$ we set $K_j^0 = \{\pm e_n : n \in \mathbb{N}\}$. Assume that K_j^n, $j = 1, 2, \ldots,$ have been defined. We set $K^n = \bigcup_{j=1}^{\infty} K_j^n$, and for $j = 1, 2, \ldots,$

$$K_j^{n+1} = K_j^n \cup \{m_j^{-1}(f_1 + \ldots + f_d) : d \in \mathbb{N},\ f_i \in K^n \text{ for } i = 1, \ldots, d, \\ \text{and } (f_i)_{i=1}^{d} \text{ is } \mathcal{M}_j\text{-allowable}\}.$$

Let $K = \bigcup_{n=0}^{\infty} K^n$. Then K is a norming set for X_M, that is,

$$\|x\| = \sup_{f \in K} \langle x, f\rangle \quad \text{ for } x \in X_M.$$

For $j = 1, 2, \ldots,$ we set $\mathcal{A}_j = \bigcup_{n=1}^{\infty}(K_j^n \setminus K^0)$. If $f \in K \setminus K^0$ and we have fixed a j with $f \in \mathcal{A}_j$, then we write

$$w(f) = 1/m_j.$$

It is not hard to see that the space X_M is an asymptotic ℓ_1 Banach space and the natural basis $(e_n)_n$ is a 1-unconditional basis for X_M.

Remark 4.3. All our results about this space remain valid, with the same proofs, if we replace the condition $k_n = t_n(k_{n-1}+1)$ by $k_n \geq t_n(k_{n-1}+1)$, where $2^{t_n} \geq m_n^2$. This remark will be used in the proof of Theorem 4.15.

In what follows, by a *tree* $\mathcal{T}$ we shall mean a finite set of finite sequences of positive integers, partially ordered by the relation

$$\alpha \prec \beta \quad \text{iff} \quad \alpha \text{ is an initial segment of } \beta,$$

and with $\{\beta : \beta \prec \alpha\} \subseteq \mathcal{T}$ for every $\alpha \in \mathcal{T}$. The elements of $\mathcal{T}$ are called *nodes*. $\mathcal{T}$ has a unique root, the empty sequence, which we denote by 0. The length of a sequence $\alpha \in \mathcal{T}$ is denoted by $|\alpha|$. The *height* of $\mathcal{T}$ is the maximum length of the maximal nodes of $\mathcal{T}$. If $\alpha \in \mathcal{T}$ we define $S_\alpha = \{\beta \in \mathcal{T} : \alpha \prec \beta \text{ and } |\beta| = |\alpha| + 1\}$.

Definition 4.4. Let $m \in \mathbb{N}$ and $\phi \in K^m$. An *analysis* of ϕ is a subset $(f_\alpha)_{\alpha \in \mathcal{T}}$ of K indexed by a tree $\mathcal{T}$ of height m such that:

(1) $\phi = f_0$.

(2) For every $0 \le s \le m$, the elements of $\{f_\alpha : \alpha \in \mathcal{T} \text{ and } |\alpha| = s\}$ are disjointly supported and $\bigcup_{|\alpha|=s} \operatorname{supp} f_\alpha \subset \operatorname{supp} \phi$.

(3) For every $\beta \in \mathcal{T}$, either $f_\beta = e_{k_\beta}$ for some $k_\beta \in \mathbb{N}$, if β is a maximal element of $\mathcal{T}$, or for some $j \in \mathbb{N}$, $f_\beta = m_j^{-1} \sum_{\alpha \in S_\beta} f_\alpha$, and the set $\{f_\alpha : \alpha \in S_\beta\}$ is $\mathcal{M}_j$-allowable.

It is easy to see that every $\phi \in K$ has an analysis, not necessarily unique. For example, consider $\phi = m_j^{-1}(\sum_{k\in A_1} e_k^* + \sum_{k\in A_2} m_k^{-1} \sum_{i \in F_k} e_i^*) \in K^2$, where, for each $k \in A_2$, $F_k \in \mathcal{S}_k$ and the family $\{\{k\} : k \in A_1\} \cup \{F_k : k \in A_2\}$ is $\mathcal{S}_j$-allowable. Then an analysis of ϕ consists of the following three levels:

$$\{\phi\}, \quad \{e_k^* : k \in A_1\} \cup \Big\{m_k^{-1} \sum_{i \in F_k} e_i^* : k \in A_2\Big\}, \quad \Big\{e_i^* : i \in \bigcup_{k \in A_2} F_k\Big\}.$$

Let $j \in \mathbb{N}$, $\varepsilon > 0$, and $x = \sum_{k=1}^n a_k z_k$ be an (ε, k_j)-s.c.c. in X_M (Definition 2.6) where $\|z_k\| = 1$ for all $k = 1, \dots, n$. Then $\|x\| \ge 1/(2m_j)$. Indeed, if $f_k \in B_{X_M^*}$ are chosen so that $f_k(z_k) = \|z_k\| = 1$, $\operatorname{supp} f_1 \subset (2, l_1]$, and $\operatorname{supp} f_k \subset (l_{k-1}, l_k]$ for $k = 2, \dots, n$, then the family $(f_k)_k$ is $\mathcal{S}_{k_j+1} = \mathcal{S}_1[\mathcal{S}_{k_j}]$-allowable. This implies that the functional $\varphi = (2m_j)^{-1} \sum_{k=1}^n f_k$ belongs to $B_{X_M^*}$, hence $\|x\| \ge \varphi(x) \ge 1/(2m_j)$.

Recall that an (ε, k_j)-s.c.c. $x = \sum_{k=1}^n a_k z_k$ of *unit* vectors $(z_k)_{k=1}^n$ is said to be *seminormalized* if $\|x\| \ge 1/2$.

The following lemma states that every block subspace Y of X contains, for every $\varepsilon > 0$ and $j \ge 2$, a seminormalized (ε, k_j)-s.c.c. Its proof is completely analogous to the proof of the corresponding result proved in [5] for mixed Tsirelson spaces.

LEMMA 4.5. *Let $j \in \mathbb{N}$, $\varepsilon > 0$ and let $(z_k)_{k=1}^\infty$ be a block sequence in X. There exists $n \in \mathbb{N}$ and normalized blocks y_k, $k = 1, \dots, n$, of the sequence $(z_k)_{k=1}^\infty$ such that a convex combination $x = \sum_{k=1}^n a_k y_k$ is a seminormalized (ε, k_j)-s.c.c.*

Proof. We may assume that the vectors z_k, $k = 1, 2, \dots$, are normalized. Choose an infinite block sequence $(x_l^1)_{l=1}^\infty$ of $(z_k)_{k=1}^\infty$ such that, for each l, $x_l^1 = \sum_{k \in A_l} a_k z_k$ is an (ε, k_j)-s.c.c. of $(z_k)_{k \in A_l}$.

If $\|x_l^1\| \ge 1/2$ for some l, then we are done. If not, we set $y_l^1 = x_l^1/\|x_l^1\|$ and as before, choose an infinite sequence $(x_l^2)_l$ of (ε, k_j)-s.c.c.'s of $(y_l^1)_{l=1}^\infty$.

Notice that for each l, the family $\{z_k : \operatorname{supp} z_k \subset \operatorname{supp} x_l^2\}$ is $\mathcal{S}_{2(k_j+1)}$-allowable (since $\mathcal{S}_{2(k_j+1)} = \mathcal{S}_{k_j+1}[\mathcal{S}_{k_j+1}]$), and so x_l^2 is a combination of the form $x_l^2 = \sum b_k(\mu_k z_k)$ where $\sum b_k = 1$, $\mu_k \ge 2$, and (z_k) is an $\mathcal{S}_{2(k_j+1)}$-allowable family. This gives $\|x_l^2\| \ge 2/m_{j+1}$. If $\|x_l^2\| \ge 1/2$ for some l, then we are done. If not, then $1/m_{j+1} \le \left\|\frac{1}{2}x_l^2\right\| < 1/2^2$, $l = 1, 2, \ldots$ We set $y_l^2 = x_l^2/\|x_l^2\|$ and continue as before.

Repeating the procedure t_{j+1} times (recall that the sequence $(t_j)_j$ is given in the definition of X_M), if we never get some (ε, k_j)-s.c.c. x_l^k with $\|x_l^k\| \ge 1/2$, $1 \le k \le t_{j+1}$, then we arrive at a $x_l^{t_{j+1}}$ of the form $x_l^{t_{j+1}} = \sum_{i\in S} \alpha_i \mu_i z_i$ where $(\operatorname{supp} z_i)_{i\in S}$ is $\mathcal{M}_{j+1} = \mathcal{S}_{t_{j+1}(k_j+1)}$-allowable, $\sum_{i\in S}\alpha_i = 1$, and $\mu_i \ge 2^{t_{j+1}-1}$ for all $i \in S$. Then

$$\frac{1}{m_{j+1}} \le \frac{1}{2^{t_{j+1}-1}}\,\|x_l^{t_{j+1}}\| < \frac{1}{2^{t_{j+1}}}.$$

This leads to a contradiction which completes the proof. ∎

Notation. Let $X_{(n)} = T_M[(\mathcal{M}_j, 1/m_j)_{j=1}^n]$ and let $K_{(n)}$ be the norming set of $X_{(n)}$. We denote by $\|\cdot\|_n$ the norm of $X_{(n)}$ and by $\|\cdot\|_n^*$ the corresponding dual norm.

Let us briefly outline the arguments which we shall use to prove the existence of an asymptotic biorthogonal system $(C_j, \mathcal{A}_j)_j$ in X_M. For every j, the set C_j is the asymptotic set consisting of vectors of the form $z = y/\|y\|$, where y is a $(1/m_j^2, k_j)$-rapidly increasing special convex combination (r.i.s.c.c., Definition 4.11) and $\mathcal{A}_j = \bigcup_{n=1}^\infty (K_j^n \setminus K^0)$.

In order to estimate the action of the different functionals of K on an (ε, k_j)-r.i.s.c.c., we reduce it to the action of analogous functionals on a certain (ε, k_j)-basic s.c.c. So, our first step is to estimate the action of the different functionals on (ε, k_j)-basic special convex combinations (Lemma 4.8).

Our next step is to prove the following useful result (Lemma 4.9), about modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n, \theta_n)_n]$: If x is a (θ_j, j)-s.c.c. of normalized vectors and $(E_r)_r$ is any $\mathcal{S}_i$-allowable family of sets where $i < j$, then

$$\sum_r \|E_r x\| \le \frac{1}{\theta_1} + 1.$$

This lemma is crucial for our estimates. The analogous lemma for mixed Tsirelson spaces $T[(\mathcal{S}_n, \theta_n)_n]$ was also very useful in dealing with the problem of distortion on these spaces ([2], [5], [14]).

In Lemma 4.10 we prove that if $x = \sum_{i=1}^n b_i x_i$ is a $(1/m_j^2, k_j)$-s.c.c. of normalized vectors in X_M and $\phi \in K_{(j-1)}$, then $|\phi(x)| \le 5/m_j$. This result is used in the proof of Proposition 4.12 where we estimate the action of the functionals of K on a $(1/m_j^2, k_j)$-r.i.s.c.c. y. We get the following bounds:

$$|\phi(y)| \le \begin{cases} 14/(m_i m_j) & \text{if } \phi \in \mathcal{A}_i,\ i < j,\\ 8/m_i & \text{if } \phi \in \mathcal{A}_i,\ i = j, j+1,\\ 8/m_j^2 & \text{if } \phi \in \mathcal{A}_i,\ i \ge j+2.\end{cases}$$

In particular, $1/(4m_j) \le \|y\| \le 8/m_j$. These estimates imply that the sequence $(C_j, \mathcal{A}_j)_j$ is an asymptotic biorthogonal system.

Estimates on the basis. Before we estimate the action of the functionals on (ε, j)-basic s.c.c.'s we prove an auxiliary lemma.

LEMMA 4.6. *Let $n \in \mathbb{N}$, $\phi \in K$ and $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of ϕ. Let*

$$F = \Big\{\alpha \in \mathcal{T} : \prod_{\beta\prec\alpha} w(f_\beta) > 1/m_n^2 \text{ and } w(f_\beta) \ge 1/m_{n-1} \text{ for all } \beta \prec \alpha\Big\}$$

and let G be a subset of F consisting of incomparable nodes. Then the set $\{f_\alpha : \alpha \in G\}$ is $\mathcal{S}_{k_n-1}$-allowable.

Proof. We recall that from the definition of the space X_M we have $k_n = t_n(k_{n-1}+1)$, where t_n is such that $2^{t_n} \ge m_n^2$.

Since $w(f) \le 1/2$ for $f \in K \setminus K^0$, it follows that if $\alpha \in G$ and $|\alpha| = k$ then $1/m_n^2 < \prod_{\beta\prec\alpha} w(f_\beta) \le 1/2^{k-1}$. Hence $2^{k-1} < m_n^2 \le 2^{t_n}$. Therefore, $|\alpha| \le t_n$ for every $\alpha \in G$.

The result will be an immediate consequence of the following

SUBLEMMA 4.7. *Let $\phi \in K$ and $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of ϕ. For $\alpha \in \mathcal{T}$, α not maximal, let $f_\alpha = m_\alpha^{-1}\sum_{\gamma\in S_\alpha} f_\gamma$, where the sequence $(f_\gamma)_{\gamma\in S_\alpha}$ is $\mathcal{S}_{k_\alpha}$-allowable. If G is a subset of $\mathcal{T}$ consisting of incomparable nodes, then the set $\{f_\alpha : \alpha \in G\}$ is $\mathcal{S}_l$-allowable, where $l = \max\{\sum_{\beta\prec\alpha} k_\beta : \alpha \in G\}$.*

Proof. By induction on $j \le \text{height}(\mathcal{T})$ we shall show that the set $A_j = \{\alpha \in G : |\alpha| \le j\}$ is $\mathcal{S}_{l_j}$-allowable, where $l_j = \max\{\sum_{\beta\prec\alpha} k_\beta : \alpha \in A_j\}$.

For $j = 1$ this is trivial. Assume that it holds for some $j < \text{height}(\mathcal{T})$. We write $A_{j+1} = \bigcup_{|\alpha|=1} G_\alpha$, where $G_\alpha = \{\beta \in A_{j+1} : \alpha \preceq \beta\}$, with some G_α possibly empty. It is evident that the sets G_α consist of pairwise incomparable nodes. Therefore, since the height of each $\mathcal{T}_\alpha = \{\beta \in \mathcal{T} : |\beta| \le j+1,\ \alpha \preceq \beta\}$ is less than or equal to j, it follows from the inductive hypothesis that each family $\{f_\beta : \beta \in G_\alpha\}$ with $|\alpha| = 1$ and $G_\alpha \ne \emptyset$ is at most $\max\{\sum_{\alpha\preceq\gamma\prec\beta} k_\gamma : \beta \in G_\alpha\}$-allowable. Therefore $\bigcup_{|\alpha|=1}\{f_\beta : \beta \in G_\alpha\}$ is at most $k_0+\max\{\sum_{\alpha\preceq\gamma\prec\beta} k_\gamma : \beta \in G_\alpha,\ |\alpha| = 1\} = \max\{\sum_{\beta\prec\alpha} k_\beta : \alpha \in A_{j+1}\}$-allowable. ∎

To complete the proof of the lemma, we observe that $\sum_{\beta\prec\alpha} k_\beta \le t_n k_{n-1} < k_n$ for every node $\alpha \in G$. ∎

REMARK. Sublemma 4.7 is taken from [14]. Our original proof of the above lemma without the use of the sublemma was less elegant.

LEMMA 4.8. *Let $j \ge 2$, $0 < \varepsilon \le 1/m_j^2$, and let $x = \sum_{k=1}^m b_k e_{n_k}$ be an (ε, k_j)-basic s.c.c. Then*:

(a) *For $\varphi \in \bigcup_{s=1}^\infty \mathcal{A}_s$,*

$$\Big|\varphi\Big(\sum_{k=1}^m b_k e_{n_k}\Big)\Big| \le \begin{cases} 1/m_s & \text{if } \varphi \in \mathcal{A}_s,\ s \ge j,\\ 2/(m_s m_j) & \text{if } \varphi \in \mathcal{A}_s,\ s < j.\end{cases}$$

(b) $\| \sum_{k=1}^{m} b_k e_{n_k} \|_{j-1} \leq 2/m_j^2$.

Proof. (a) If $s \geq j$ then the estimate is obvious. Let $s < j$ and $\phi = m_s^{-1} \sum_{i=1}^{d} f_i$. We may assume that $\phi(e_{n_k}) \geq 0$ for all k. We set

$$D = \Big\{ n_k : \sum_{i=1}^{d} f_i(e_{n_k}) > 1/m_j \Big\}, \qquad g_i = f_i|_D, \qquad i = 1, \ldots, d.$$

Then $m_s^{-1} \sum_{i=1}^{d} g_i \in K_{(j-1)}$ and for every $n \in D$ we have

$$\frac{1}{m_s} \sum_{i=1}^{d} g_i(e_n) > \frac{1}{m_s} \frac{1}{m_j} > \frac{1}{m_j^2}.$$

Therefore, by Lemma 4.6, $D = \operatorname{supp}(m_s^{-1} \sum_{i=1}^{d} g_i) \in \mathcal{S}_{k_j - 1}$. So

$$\frac{1}{m_s} \sum_{i=1}^{d} g_i \Big(\sum_{k=1}^{m} b_k e_{n_k} \Big) \leq \sum_{n_k \in D} b_k \leq \frac{1}{m_j^2}.$$

On the other hand,

$$\frac{1}{m_s} \sum_{i=1}^{d} f_{i|D^c} \Big(\sum_{k=1}^{m} b_k e_{n_k} \Big) \leq \frac{1}{m_s} \frac{1}{m_j}.$$

Hence

$$\phi\Big(\sum_{k=1}^{m} b_k e_{n_k} \Big) \leq \frac{1}{m_s} \frac{1}{m_j} + \frac{1}{m_j^2} \leq \frac{2}{m_s m_j}.$$

(b) We let $\phi \in K_{(j-1)}$ and assume again that ϕ is positive. We set $L = \{ n_k : \phi(e_{n_k}) > 1/m_j^2 \}$. Then $\phi_{|L^c}(\sum_k b_k e_{n_k}) \leq 1/m_j^2$. On the other hand, Lemma 4.6 shows that $\operatorname{supp} \phi_{|L} \in \mathcal{S}_{k_j - 1}$, so $\phi_{|L}(\sum_k b_k e_{n_k}) \leq 1/m_j^2$. Therefore, $|\phi(\sum_k b_k e_{n_k})| \leq 2/m_j^2$. ∎

Estimates on block sequences. Our first lemma is true in any modified mixed Tsirelson space $T_M[(\mathcal{S}_{k_n}, \theta_n)_n]$.

LEMMA 4.9. *Let $X = T_M[(\mathcal{S}_{k_n}, \theta_n)_n]$ be a modified mixed Tsirelson space, $j \in \mathbb{N}$, $0 < \varepsilon \leq \theta_j$, and let $(x_k)_{k=1}^{m}$ be a normalized block sequence in X such that $x = \sum_{k=1}^{m} b_k x_k$ is an (ε, k_j)-s.c.c. Then, for every $q < k_j$ and every $\mathcal{S}_q$-allowable family $(f_i)_{i=1}^{d}$ in B_{X^*},*

$$\Big| \sum_{i=1}^{d} f_i \Big(\sum_{k=1}^{m} b_k x_k \Big) \Big| \leq \frac{1}{\theta_1} + 1.$$

Proof. We may assume that $\operatorname{supp} \phi \cap \operatorname{supp} x_k \neq \emptyset$ for every $1 \leq k \leq m$. Let

$$A_1 = \{ k \in \{1, \ldots, m\} : \min \operatorname{supp} f_i \notin \operatorname{range}(x_k) \text{ for all } 1 \leq i \leq d \},$$
$$A_2 = \{1, \ldots, m\} \setminus A_1.$$

CLAIM 1.

$$\Big|\sum_{i=1}^{d} f_i\Big(\sum_{k\in A_2} b_k x_k\Big)\Big| \le \frac{\varepsilon}{\theta_j} \le 1.$$

Proof of Claim 1. Let $k \in A_2$ and $l_k = \max\operatorname{supp} x_k$. Then there exists at least one $1 \le i \le d$ such that $\min\operatorname{supp} f_i \in \operatorname{range}(x_k)$. We set

$$i_k = \max\{i \in \{1,\ldots,d\} : \min\operatorname{supp} f_i \in \operatorname{range}(x_k)\}.$$

Then $\min\operatorname{supp} f_{i_k} \le l_k$ for $k \in A_2$. The correspondence $k \mapsto i_k$, $k \in A_2$, is one-to-one. It follows that $\{l_k : k \in A_2\} \in \mathcal{S}_q$, so $\sum_{k\in A_2} b_k < \varepsilon$.

On the other hand, the family $(f_i)_{i=1}^d$ is $\mathcal{S}_q$-allowable, so $\mathcal{S}_{k_j}$-allowable, and $\|x_k\| \le 1$ for all k. It follows that $|\sum_{i=1}^d f_i(x_k)| \le 1/\theta_j$ for all $k \in A_2$. So

$$\Big|\sum_{i=1}^{d} f_i\Big(\sum_{k\in A_2} b_k x_k\Big)\Big| \le \frac{1}{\theta_j}\sum_{k\in A_2} b_k \le \frac{\varepsilon}{\theta_j} \le 1. \ \blacksquare$$

CLAIM 2.

$$\Big|\sum_{i=1}^{d} f_i\Big(\sum_{k\in A_1} b_k x_k\Big)\Big| \le \frac{1}{\theta_1}.$$

Proof of Claim 2. Let $k \in A_1$. If $\operatorname{supp} f_i \cap \operatorname{supp} x_k \ne \emptyset$ for some $1 \le i \le d$, then $\min\operatorname{supp} f_i < \min\operatorname{supp} x_k$. Hence, for every $k \in A_1$, the set

$$I_k = \{i \le d : \operatorname{supp} f_i \cap \operatorname{supp} x_k \ne \emptyset\}$$

has less than $\min\operatorname{supp} x_k$ elements. It follows that $\{f_i|_{[\min\operatorname{supp} x_k,\infty)} : i \in I_k\}$ is $\mathcal{S}$-allowable for every $k \in A_1$, and therefore $\theta_1 \sum_{i\in I_k} f_i|_{[\min\operatorname{supp} x_k,\infty)} \in B_{X^*}$. Hence

$$\begin{aligned}\Big|\sum_{i=1}^{d} f_i\Big(\sum_{k\in A_1} b_k x_k\Big)\Big| &= \Big|\sum_{k\in A_1} b_k\Big(\sum_{i=1}^{d} f_i\Big)x_k\Big| \\ &\le \sum_{k\in A_1} b_k\Big|\Big(\sum_{i\in I_k} f_i\Big)(x_k)\Big| \le \frac{1}{\theta_1}\sum_{k\in A_1} b_k \le \frac{1}{\theta_1}. \ \blacksquare\end{aligned}$$

Combining the two estimates above we obtain

$$\Big|\sum_{i=1}^{d} f_i\Big(\sum_{k=1}^{m} b_k x_k\Big)\Big| \le \frac{1}{\theta_1} + 1. \ \blacksquare$$

Our next lemma refers to the particular space X_M that we consider.

LEMMA 4.10. *Let $j \in \mathbb{N}$ and let $(x_k)_{k=1}^m$ be a normalized block sequence in X_M such that $x = \sum_{k=1}^m b_k x_k$ is a $(1/m_j^2, k_j)$-s.c.c. If $\phi \in K_{(j-1)}$ then $|\phi(\sum_{k=1}^m b_k x_k)| \le 5/m_j$.*

Proof. Let $(f_\beta)_{\beta\in\mathcal{T}}$ be an analysis of ϕ. In order to estimate ϕ on $\sum_k b_k x_k$ we give the following definition: Let $\beta\in\mathcal{T}$, and let f_β be the corresponding functional. We say that f_β *partially covers* x_k if the following hold:

(1) $\operatorname{supp} f_\beta \cap \operatorname{supp} x_k \neq \emptyset$.
(2) $\operatorname{supp} f_\beta \cap \operatorname{supp} x_j = \emptyset$ for all $j\neq k$.
(3) If $\beta\in S_\alpha$ then $\operatorname{supp} f_\alpha \cap \operatorname{supp} x_j \neq \emptyset$ for some $j\neq k$.

We set

$$A=\Big\{\beta\in\mathcal{T} : f_\beta \text{ partially covers some } x_k \text{ and } \prod_{\alpha\prec\beta} w(f_\alpha) > 1/m_j^2\Big\},$$

$$B=\Big\{\beta\in\mathcal{T} : f_\beta \text{ partially covers some } x_k \text{ and } \prod_{\alpha\prec\beta} w(f_\alpha) \le 1/m_j^2\Big\}.$$

Note that if both f_β and $f_{\beta'}$ partially cover x_k and $\beta\neq\beta'$ then $\operatorname{supp} f_\beta \cap \operatorname{supp} f_{\beta'}=\emptyset$. Also $A\cap B=\emptyset$ and $\operatorname{supp}\phi\cap\operatorname{supp} x_k=\bigcup_{\beta\in A\cup B}\operatorname{supp} f_\beta\cap\operatorname{supp} x_k$ for each k. We set $\phi_A=\phi|_{\bigcup_{\beta\in A}\operatorname{supp} f_\beta}$ and $\phi_B=\phi|_{\bigcup_{\beta\in B}\operatorname{supp} f_\beta}$. Note that $\phi(x_k)=(\phi_A+\phi_B)(x_k)$ for every $k=1,\dots,m$. We denote by $\mathcal{T}_A$ (resp. $\mathcal{T}_B$) the subtree of $\mathcal{T}$ which has as maximal nodes the elements of A (resp. B).

Claim 1.

$$\Big|\phi_B\Big(\sum_{k=1}^m b_k x_k\Big)\Big| \le \frac{1}{m_j}.$$

Proof of Claim 1. Let $\beta\in B$ and let $\alpha_\beta\prec\beta$ be such that $\prod_{\gamma\prec\alpha_\beta} w(f_\gamma) > 1/m_j^2$ and $\prod_{\gamma\preceq\alpha_\beta} w(f_\gamma)\le 1/m_j^2$. Note that if $\beta,\beta'\in B$ then either $\alpha_\beta,\alpha_{\beta'}$ are incomparable nodes or $\alpha_\beta=\alpha_{\beta'}$. Let $\mathcal{R}=\{\alpha_\beta : \beta\in B\}$ be the set of such different nodes. For every $\beta\in B$ there exists $\alpha_\beta\in\mathcal{R}$ with $\alpha_\beta\prec\beta$, hence $\operatorname{supp} f_\beta\subset\operatorname{supp} f_{\alpha_\beta}$. Also, since $\phi\in K_{(j-1)}$, we have $w(f_{\alpha_\beta})\ge 1/m_{j-1}$, so

$$\prod_{\gamma\prec\alpha_\beta} w(f_\gamma)=\frac{1}{w(f_{\alpha_\beta})}\prod_{\gamma\preceq\alpha_\beta} w(f_\gamma)\le\frac{m_{j-1}}{m_j^2}.$$

Therefore,

$$\begin{aligned}\Big|\phi_B\Big(\sum_{k=1}^m b_k x_k\Big)\Big| &\le \sum_{\alpha_\beta\in\mathcal{R}}\Big(\prod_{\gamma\prec\alpha_\beta} w(f_\gamma)\Big)\Big|f_{\alpha_\beta}\Big(\sum_{k=1}^m b_k x_k\Big)\Big|\\ &\le \frac{m_{j-1}}{m_j^2}\sum_{\alpha_\beta\in\mathcal{R}}\Big|f_{\alpha_\beta}\Big(\sum_{k=1}^m b_k x_k\Big)\Big|.\end{aligned}$$

By Lemma 4.6 the family $\{f_{\alpha_\beta} : \alpha_\beta\in\mathcal{R}\}$ is $\mathcal{S}_{k_j-1}$-allowable. Therefore, by

Lemma 4.9,

$$\sum_{\alpha_\beta \in \mathcal{R}} \Big| f_{\alpha_\beta}\Big(\sum_{k=1}^m b_k x_k\Big)\Big| \leq 3.$$

We conclude that

$$\Big|\phi_B\Big(\sum_{k=1}^m b_k x_k\Big)\Big| \leq \frac{3m_{j-1}}{m_j^2} \leq \frac{3}{m_j}. \tag{4.1}$$

For $k = 1, \ldots, m$, let $l_k = \max \operatorname{supp} x_k$. Then we have

Claim 2.

$$\Big|\phi_A\Big(\sum_{k=1}^m b_k x_k\Big)\Big| \leq m_j \Big\| \sum_{k=1}^m b_k e_{l_k}\Big\|_{j-1}.$$

Proof of Claim 2. For each $\alpha \in \mathcal{T}_A$, we set $h_\alpha = f_\alpha|_{\bigcup_{\beta\in A} \operatorname{supp} f_\beta}$. That is, $h_\beta = f_\beta$ for $\beta \in A$, and for $\alpha \in \mathcal{T}_A \setminus A$ with $f_\alpha = m_q^{-1}\sum_{\beta \in S_\alpha} f_\beta$,

$$h_\alpha = \frac{1}{m_q} \sum_{\substack{\beta \in S_\alpha \\ \beta \in \mathcal{T}_A}} h_\beta.$$

For every $\alpha \in \mathcal{T}_A$ we set

$$D_\alpha = \{1 \leq k \leq m : \exists \beta \succ \alpha \text{ such that } \beta \in A \text{ and } f_\beta \text{ partially covers } x_k\}.$$

Inductively we define, for every $\alpha \in \mathcal{T}_A$ with $D_\alpha \neq \emptyset$, a functional g_α with the following properties:

(1) $\operatorname{supp} g_\alpha = \{l_k : k \in D_\alpha\}$ where $l_k = \max \operatorname{supp} x_k$ for $k = 1, \ldots, m$.
(2) $g_\alpha \in K_{(j-1)}$.
(3) $|h_\alpha(x_k)| \leq (\sum_{\gamma \succ \alpha,\, \gamma \in A} |f_\gamma(x_k)|) g_\alpha(e_{l_k})$ for every $k \in D_\alpha$.

Assume that g_γ has been defined for all $\gamma \in \mathcal{T}_A$ with $|\gamma| = s$ and let $\alpha \in \mathcal{T}_A$ with $|\alpha| = s-1$. Let $h_\alpha = m_q^{-1}\sum_{\beta\in S_\alpha} h_\beta$ and suppose that $D_\alpha \neq \emptyset$. We set

$$I = \{\beta \in S_\alpha : D_\beta \neq \emptyset\}, \qquad R = \{\beta \in S_\alpha : \beta \in A\}.$$

Then $I \cap R = \emptyset$. Also, for every $\beta \in I$, g_β has been defined. For every $1 \leq k \leq m$, we set $\Gamma_k = \{\beta \in R : f_\beta \text{ partially covers } x_k\}$. Then the sets Γ_k, $1 \leq k \leq m$, are disjoint.

Let $G = \{k \in D_\alpha : \Gamma_k \neq \emptyset\}$ be the set of all k for which x_k is partially covered by some $\beta \in R$. For every $k \in D_\alpha \setminus G$ we choose a node $\beta_k \in I$ such that

$$g_{\beta_k}(e_{l_k}) = \max\{g_\beta(e_{l_k}) : \beta \in I\}.$$

For every $\beta \in I$ we define $g'_\beta = g_\beta|_{\{l_k : k \in D_\beta \setminus G \text{ and } \beta = \beta_k\}}$. It follows that the functionals g'_β, $\beta \in I$, and e_{l_k}, $k \in G$, are disjointly supported. We now set

$$g_\alpha = \frac{1}{m_q}\Big(\sum_{k\in G} e_{l_k} + \sum_{\beta\in I} g'_\beta\Big).$$

We need to show that $g_\alpha \in K_{(j-1)}$. By the inductive hypothesis $g'_\beta \in K_{(j-1)}$ for all $\beta \in I$. Also $q \le j-1$, since $\phi \in K_{(j-1)}$.

It remains to show that the family $\{e_{l_k} : k \in G\} \cup \{g'_\beta : \beta \in I\}$ is $\mathcal{S}_{k_q}$-allowable. Since $\operatorname{supp} g'_\beta \subseteq \{l_k : k \in D_\beta\}$ for $\beta \in I$, we have $\min\operatorname{supp} h_\beta \le \min\operatorname{supp} g'_\beta$. Also, $\min\bigcup\{\operatorname{supp} f_\beta : \beta \in \Gamma_k\} \le l_k$ for $k \in G$. It follows that $\{l_k : k \in G\} \cup \{\min\operatorname{supp} g'_\beta : \beta \in I\} \in \mathcal{S}_{k_q}$. This establishes property (2) for g_α. Property (1) is easily checked. It remains to show that property (3) holds.

Case 1: $k \in G$. Then

$$\begin{aligned}|h_\alpha(x_k)| &= \frac{1}{m_q}\Big|\sum_{\beta\in\Gamma_k} f_\beta(x_k) + \sum_{\beta\in I} h_\beta(x_k)\Big| \\ &\le \frac{1}{m_q}\Big(\sum_{\beta\in\Gamma_k} |f_\beta(x_k)| + \sum_{\beta\in I}\sum_{\substack{\gamma\succ\beta\\ \gamma\in A}} |f_\gamma(x_k)|\Big) \\ &\le \frac{1}{m_q}\Big(\sum_{\substack{\beta\succ\alpha\\ \beta\in A}} |f_\beta(x_k)|\Big) e^*_{l_k}(e_{l_k}) = \sum_{\substack{\beta\succ\alpha\\ \beta\in A}} |f_\beta(x_k)| g_\alpha(e_{l_k}).\end{aligned}$$

Case 2: $k \in D_\alpha \setminus G$. Then by the inductive hypothesis

$$\begin{aligned}|h_\alpha(x_k)| &= \frac{1}{m_q}\Big|\sum_{\beta\in I} h_\beta(x_k)\Big| \le \frac{1}{m_q}\sum_{\beta\in I} |h_\beta(x_k)| \\ &\le \frac{1}{m_q}\sum_{\beta\in I}\Big(\sum_{\substack{\gamma\in A\\ \gamma\succ\beta}} |f_\gamma(x_k)| g_\beta(e_{l_k})\Big) \\ &\le \frac{1}{m_q}\max_{\beta\in I} g_\beta(e_{l_k}) \sum_{\beta\in I}\sum_{\substack{\gamma\in A\\ \gamma\succ\beta}} |f_\gamma(x_k)| \\ &= \frac{1}{m_q} g'_{\beta_k}(e_{l_k}) \sum_{\substack{\gamma\in A\\ \gamma\succ\alpha}} |f_\gamma(x_k)| = \Big(\sum_{\substack{\gamma\in A\\ \gamma\succ\alpha}} |f_\gamma(x_k)|\Big) g_\alpha(e_{l_k}),\end{aligned}$$

since $f_\beta(x_k) = 0$ for $\beta \in R$.

This completes the proof of property (3) and the inductive construction. It follows that for every k,

$$|\phi_A(x_k)| \le \Big(\sum_{\gamma\in A} |f_\gamma(x_k)|\Big) g_0(e_{l_k}).$$

Using Lemma 4.6 and the definition of the set A, we find that the family $\{f_\gamma : \gamma \in A\}$ is $\mathcal{S}_{k_j-1}$-allowable. It follows that $m_j^{-1}\sum_{\gamma\in A} \pm f_\gamma \in K$ for every choice of signs, so $\sum_{\gamma\in A}|f_\gamma(x_k)| \le m_j$ for every k. Hence $|\phi_A(x_k)| \le m_j g_0(e_{l_k})$ for all k, so

$$\Big|\phi_A\Big(\sum_{k=1}^m b_k x_k\Big)\Big| \le m_j \Big\|\sum_{k=1}^m b_k e_{l_k}\Big\|_{j-1}.$$

This completes the proof of Claim 2. ∎

Using Lemma 4.8(b) we get $|\phi_A(\sum_{k=1}^n b_k x_k)| \le 2/m_j$. Combining this with Claim 1 we get $|\phi(\sum_{k=1}^n b_k x_k)| \le 5/m_j$. ∎

Definition 4.11. Let $j \ge 2$, $\varepsilon > 0$. An (ε, k_j)-special convex combination $\sum_{n=1}^m b_n x_n$ is called an (ε, k_j)-*rapidly increasing special convex combination* (r.i.s.c.c.) if there exist integers $(j_n)_{n=1}^m$ with $j+2 < j_1 < \ldots < j_m$ such that:

(1) Each x_n is a seminormalized $(1/m_{j_n}^2, k_{j_n})$-s.c.c.
(2) $\|x_n\|_{\ell_1} \le m_{j_{n+1}}/m_{j_n}$ for all $n = 1, \ldots, m-1$.

Proposition 4.12. *Let $\sum_{k=1}^m b_k x_k$ be a $(1/m_j^2, k_j)$-r.i.s.c.c. and $\phi \in K$ with $w(\phi) = 1/m_s$. Then*

$$\Big|\phi\Big(\sum_{k=1}^m b_k x_k\Big)\Big| \le \begin{cases} 14/(m_s m_j) & \text{if } s < j, \\ 8/m_s & \text{if } s = j, j+1, \\ 8/m_j^2 & \text{if } j+2 \le s. \end{cases}$$

In particular, $1/(4m_j) \le \|\sum_{k=1}^m b_k x_k\| \le 8/m_j$.

Proof. Let $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of ϕ. First we partition the support of each x_k, $1 \le k \le m$, as follows: We set

$$\overline{\overline{x}}_k = x_{k|\bigcup\{\operatorname{supp} f_\alpha : \alpha\in\mathcal{T},\, \operatorname{supp} f_\alpha \cap \operatorname{supp} x_k \neq \emptyset \text{ and } w(f_\alpha)\le 1/m_{j_{k+1}}\}}.$$

Then the definition of the r.i.s.c.c. shows that

$$|\phi(\overline{\overline{x}}_k)| \le \frac{1}{m_{j_{k+1}}}\|x_k\|_{\ell_1} \le \frac{m_{j_{k+1}}}{m_{j_{k+1}} m_{j_k}} = \frac{1}{m_{j_k}} \quad \text{for all } 1 \le k \le m.$$

It follows that

$$\Big|\phi\Big(\sum_{k=1}^m b_k \overline{\overline{x}}_k\Big)\Big| \le \sum_k \frac{b_k}{m_{j_k}} \le \frac{2}{m_{j_1}} \max_k b_k. \tag{4.2}$$

We now set $\overline{x}_k = x_k - \overline{\overline{x}}_k$. Abusing notation we denote by x_k the vector $\overline{x}_k$. This means that from now on we assume the following:

$(*)$ If $\operatorname{supp} x_k \cap \operatorname{supp} f_\alpha \neq \emptyset$ for some $\alpha \in \mathcal{T}$, then $w(f_\alpha) > 1/m_{j_{k+1}}$.

We make the following definition: Let $\alpha \in \mathcal{T}$ and $k = 1, \ldots, m$. We say that f_α *partially estimates* x_k if:

(1) $\operatorname{supp} f_\alpha \cap \operatorname{supp} x_k \neq \emptyset$.
(2) $w(f_\alpha) \leq 1/m_{j_k}$.
(3) $w(f_\beta) > 1/m_{j_k}$ for all $\beta \prec \alpha$.

Suppose that f_α partially estimates x_k for some $1 \leq k \leq m$. The definition of x_r (which actually denotes $\overline{x}_r$) shows that $\operatorname{supp} f_\alpha \cap \operatorname{supp} x_r = \emptyset$ for all $r < k$. This implies that a given functional f_α can partially estimate at most one x_k. Also, if f_α partially estimates x_k and $\beta \succ \alpha$ then f_β does not partially estimate any x_r with $r \leq k$. In particular, if f_α and f_β partially estimate the same x_k then $\operatorname{supp} f_\alpha \cap \operatorname{supp} f_\beta = \emptyset$.

Once more, we partition the support of each vector x_k as follows:

$$x_k^1 = x_{k|\bigcup\{\operatorname{supp} f_\alpha \,:\, f_\alpha \text{ partially estimates } x_k\}}, \quad x_k^2 = x_k - x_k^1 \quad \text{for all } 1 \leq k \leq m.$$

For $\beta \in \mathcal{T}$, if $\operatorname{supp} f_\beta \cap \operatorname{supp} x_k^2 \neq \emptyset$ for some k, then $w(f_\beta) > 1/m_{j_k}$.

Indeed, suppose that $\operatorname{supp} f_\beta \cap \operatorname{supp} x_k^2 \neq \emptyset$ and $w(f_\beta) \leq 1/m_{j_k}$. Let γ_0 be the minimum element of $\{\gamma \in \mathcal{T} : \gamma \preceq \beta \text{ and } w(f_\gamma) \leq 1/m_{j_k}\}$ under $\prec$. Then $\operatorname{supp} f_\beta \subset \operatorname{supp} f_{\gamma_0}$ and f_{γ_0} partially estimates x_k. Therefore, $\operatorname{supp} f_\beta \cap \operatorname{supp} x_k \subseteq \operatorname{supp} x_k^1$, which leads to a contradiction.

It follows that $\phi_{|\operatorname{supp} x_k^2} \in K_{(j_k-1)}$ and therefore, by Lemma 4.10, $|\phi(x_k^2)| \leq 5/m_{j_k}$ for all $1 \leq k \leq m$. Hence

$$\Big|\phi\Big(\sum_{k=1}^m b_k x_k^2\Big)\Big| \leq \sum_{k=1}^m b_k \frac{5}{m_{j_k}} < \frac{10}{m_{j_1}} \max_k b_k. \tag{4.3}$$

It remains to estimate ϕ on $\sum_k b_k x_k^1$. For every k with $x_k^1 \neq 0$ there exists $\alpha \in \mathcal{T}$ such that f_α partially estimates x_k. We partition the set of nodes which partially estimate each x_k into two sets A_k, B_k as follows:

$$A_k = \Big\{\alpha \in \mathcal{T} : f_\alpha \text{ partially estimates } x_k \text{ and } \prod_{\beta \prec \alpha} w(f_\beta) > 1/m_{j_k}^2\Big\},$$
$$B_k = \Big\{\alpha \in \mathcal{T} : f_\alpha \text{ partially estimates } x_k \text{ and } \prod_{\beta \prec \alpha} w(f_\beta) \leq 1/m_{j_k}^2\Big\}.$$

As already noted, if $\alpha \in A_k$ and $\beta \in B_k$, then $\operatorname{supp} f_\alpha \cap \operatorname{supp} f_\beta = \emptyset$. For every $k = 1, \ldots, m$, we set

$$y_k^1 = x^1_{k|\bigcup\{\operatorname{supp} f_\alpha \,:\, \alpha \in A_k\}}, \qquad y_k^2 = x_k^1 - y_k^1 = x^1_{k|\bigcup\{\operatorname{supp} f_\alpha \,:\, \alpha \in B_k\}}.$$

Claim 1.

$$\Big|\phi\Big(\sum_{k=1}^m b_k y_k^2\Big)\Big| < \frac{6}{m_{j_1}} \max_k b_k.$$

Proof of Claim 1. We shall estimate ϕ separately on each y_k^2, to show that $|\phi(y_k^2)| \leq 3/m_{j_k}$. The proof is similar to that of Lemma 4.10.

For every $\alpha \in B_k$, let $R_\alpha = \{\beta \in \mathcal{T} : \beta \prec \alpha\}$. Choose $\beta_\alpha \in R_\alpha$ such that $\prod_{\gamma \prec \beta_\alpha} w(f_\gamma) > 1/m_{j_k}^2$ and $\prod_{\gamma \preceq \beta_\alpha} w(f_\gamma) \leq 1/m_{j_k}^2$. Note that since $w(f_{\beta_\alpha}) > 1/m_{j_k}$, we have $\prod_{\gamma\prec\beta_\alpha} w(f_\gamma) < 1/m_{j_k}$. It is easy to check that if $\alpha,\alpha' \in B_k$ then either $\beta_\alpha = \beta_{\alpha'}$ or $\beta_\alpha,\beta_{\alpha'}$ are incomparable. Let $\mathcal{R} = \{\beta_\alpha : \alpha \in B_k\}$ be the set of all such different nodes. Since $\beta_\alpha \prec \alpha$ for all $\alpha \in B_k$, we have $\operatorname{supp} f_\alpha \subseteq \operatorname{supp} f_{\beta_\alpha}$. Therefore,

$$|\phi(y_k^2)| \le \sum_{\beta_\alpha\in\mathcal{R}} \Big(\prod_{\gamma\prec\beta_\alpha} w(f_\gamma)\Big)|f_{\beta_\alpha}(y_k^2)|.$$

By Lemma 4.6, the family $\{f_{\beta_\alpha} : \beta_\alpha \in \mathcal{R}\}$ is $\mathcal{S}_{k_j-1}$-allowable. Since y_k is a $(1/m_{j_k}^2, k_{j_k})$-s.c.c., by Lemma 4.9 we get $\sum_{\beta_\alpha\in R} |f_{\beta_\alpha}(y_k^2)| \le 3$. So

$$|\phi(y_k^2)| \le 3 \max_{\beta_\alpha\in R} \prod_{\gamma\prec\beta_\alpha} w(f_\gamma) \le \frac{3}{m_{j_k}}. \tag{4.4}$$

Claim 2.

$$\Big|\phi\Big(\sum_{k=1}^m b_k y_k^1\Big)\Big| \le \begin{cases} 6/m_j^2 & \textit{if } \phi\in\mathcal{A}_s,\ s\ge j+2,\\ 6/m_s & \textit{if } \phi\in\mathcal{A}_s,\ s=j,j+1,\\ 12/(m_s m_j) & \textit{if } \phi\in\mathcal{A}_s,\ s<j.\end{cases} \tag{4.5}$$

Proof of Claim 2. For $k=1,\dots,m$, we let $l_k = \max\operatorname{supp} x_k$. For $\alpha\in\mathcal{T}$, we set

$$D_\alpha = \{1\le k\le m : \exists\beta\succeq\alpha \text{ such that } f_\beta \text{ partially estimates } x_k \text{ and } \beta\in A_k\}.$$

For every $k=1,\dots,m$, we set $T_k(\alpha) = \{\beta\succeq\alpha : \beta\in A_k\}$. Inductively, for every $\alpha\in\mathcal{T}$ with $D_\alpha\ne\emptyset$, we define a functional g_α with the following properties:

(1) $g_\alpha \in \operatorname{co}(K)$.
(2) $\operatorname{supp} g_\alpha = \{l_k : k\in D_\alpha\}$.
(3) For every $k\in D_\alpha$,

$$|f_\alpha(y_k^1)| \le \Big(2\sum_{\beta\in T_k(\alpha)} |f_\beta(y_k^1)|\Big) g_\alpha(e_{l_k}).$$

(4) Either $w(g_\alpha) = w(f_\alpha)$ or g_α is of the form $g_\alpha = \frac12(e_{l_k} + g'_\alpha)$ where $w(g'_\alpha) = w(f_\alpha)$.

Assume that g_γ has been defined for $\gamma\in\mathcal{T}$ with $|\gamma|\ge s+1$ and $D_\gamma\ne\emptyset$. Let $\alpha\in\mathcal{T}$ with $|\alpha| = s$ be such that $D_\alpha\ne\emptyset$ and let $f_\alpha = m_q^{-1}\sum_{\beta\in S_\alpha} f_\beta$. We distinguish two cases.

Case 1: f_α partially estimates some x_{k_0}. Let $I = \{\beta\in S_\alpha : D_\beta\ne\emptyset\}$. Then, as we have noted, no $k\le k_0$ is in $\bigcup_{\beta\in I} D_\beta$. Let $k>k_0$ with $k\in D_\alpha$. By the inductive hypothesis,

$$|f_\alpha(y_k^1)| \le \frac{1}{m_q}\sum_{\beta\in I} |f_\beta(y_k^1)| \le \frac{1}{m_q}\sum_{\beta\in I}\Big(2\sum_{\gamma\in T_k(\beta)} |f_\gamma(y_k^1)|\Big) g_\beta(e_{l_k}).$$

For every $k > k_0$ with $k \in D_\alpha$ we choose $\beta_k \in I$ such that $g_{\beta_k}(e_{l_k}) = \max_{\beta\in I} g_\beta(e_{l_k})$. For every $\beta \in I$ we set $g'_\beta = g_{\beta|\{l_k\,:\,k\in D_\beta \text{ and } \beta=\beta_k\}}$.

Then, for $k > k_0$, $k \in D_\alpha$,

$$\begin{aligned}|f_\alpha(y_k^1)| &\le \frac{1}{m_q}\, g'_{\beta_k}(e_{l_k})\, 2\sum_{\beta\in I}\sum_{\gamma\in T_k(\beta)} |f_\gamma(y_k^1)| \\ &= \frac{1}{m_q}\Big(2\sum_{\gamma\in T_k(\alpha)} |f_\gamma(y_k^1)|\Big) g'_{\beta_k}(e_{l_k}).\end{aligned} \tag{4.6}$$

It is clear that the functionals $e_{l_{k_0}}, g'_\beta$, $\beta \in I$, are disjointly supported. Also since $\{l_k : k = 1, \dots, m\} \in \mathcal{S}_{k_j}$ and $\operatorname{supp} g'_\beta \subseteq \{l_k : k = 1, \dots, m\}$ for all $\beta \in I$, it is clear that the family $\{g'_\beta : \beta \in I\}$ is $\mathcal{S}_{k_j}$-allowable.

Since f_α partially estimates x_{k_0}, we have $1/m_q = w(f_\alpha) \le 1/m_{j_{k_0}}$, so $q \ge j_{k_0} > j+1$. It follows that the family $\{g'_\beta : \beta \in I\}$ is $\mathcal{S}_{q-1}$-allowable.

We define

$$g_\alpha = \frac{1}{2}\Big(e_{l_{k_0}} + \frac{1}{m_{q-1}}\sum_{\beta\in I} g'_\beta\Big).$$

Then $g_\alpha \in \operatorname{co}(K)$ and for every $k \in D_\alpha$,

$$|f_\alpha(y_k^1)| \le 2\Big(\sum_{\gamma\in T_k(\alpha)} |f_\gamma(y_k^1)|\Big) g_\alpha(e_{l_k}).$$

Case 2: f_α does not partially estimate any x_k. Let $I = \{\beta \in S_\alpha : D_\beta \ne \emptyset\}$. We repeat the procedure of Case 1: For every $k \in D_\alpha = \bigcup_{\beta\in I} D_\beta$, we choose $\beta_k \in I$ such that $g_{\beta_k}(e_{l_k}) = \max_{\beta\in I} g_\beta(e_{l_k})$. For every $\beta \in I$ we set $g'_\beta = g_{\beta|\{l_k\,:\,k\in D_\beta \text{ and } \beta=\beta_k\}}$. Then for every $k \in D_\alpha$, by the inductive hypothesis,

$$\begin{aligned}|f_\alpha(y_k^1)| &\le \frac{1}{m_q}\sum_{\beta\in I}|f_\beta(y_k^1)| \le \frac{1}{m_q}\sum_{\beta\in I}\Big(2\sum_{\gamma\in T_k(\beta)} |f_\gamma(y_k^1)|\Big) g_\beta(e_{l_k}) \\ &\le \frac{1}{m_q}\, g'_{\beta_k}(e_{l_k})\Big(2\sum_{\gamma\in T_k(\alpha)} |f_\gamma(y_k^1)|\Big).\end{aligned}$$

The functionals g'_β, $\beta \in I$, are disjointly supported. Also, since $\min\operatorname{supp} f_\beta \le \min\operatorname{supp} g_\beta \le \min\operatorname{supp} g'_\beta$, the family $\{g'_\beta : \beta \in I\}$ is $\mathcal{S}_{k_q}$-allowable. We define $g_\alpha = m_q^{-1}\sum_{\beta\in I} g'_\beta$. It is easy to verify properties (1)–(4). This completes the inductive construction.

For the functional $\phi = f_0$ we get, for $k = 1, \dots, m$,

$$|\phi(y_k^1)| \le 2\sum_{\beta\in A_k} |f_\beta(x_k)| g_0(e_{l_k}).$$

The family $\{f_\beta : \beta \in A_k\}$ satisfies the assumptions of Lemma 4.6 with $n = j_k$. Therefore, it is $\mathcal{S}_{k_{j_k}-1}$-allowable. It follows from Lemma 4.9 that

$\sum_{\beta\in A_k} |f_\beta(x_k)| \le 3$. We conclude that

$$\Big|\phi\Big(\sum_{k=1}^m b_k y_k^1\Big)\Big| \le 6g_0\Big(\sum_{k=1}^m b_k e_{l_k}\Big).$$

By the form of g_0, using Lemma 4.8, we get

$$\Big|\phi\Big(\sum_{k=1}^m b_k y_k^1\Big)\Big| \le \begin{cases} 3\beta_{k_0} + 3/m_{s-1} \le 6/m_j^2 & \text{if } \phi\in\mathcal{A}_s,\ s\ge j+2,\\ 6/m_s & \text{if } \phi\in\mathcal{A}_s,\ s=j,j+1,\\ 12/(m_s m_j) & \text{if } \phi\in\mathcal{A}_s,\ s<j.\end{cases}$$

This completes the proof of Claim 2. ∎

Combining Claims 1, 2 and relations (4.2), (4.3) we get the desired estimate. ∎

Let $C_j = \{z/\|z\| : z \text{ is a } (1/m_j^2, k_j)\text{-r.i.s.c.c.}\}$ for $j\in\mathbb{N}$. Then Lemma 4.5 implies that each C_j is *asymptotic*, i.e., $S_Y\cap C_j \ne \emptyset$ for every block subspace Y of X_M. Let

$$\mathcal{A}_j = \Big\{ f = \frac{1}{m_j}\sum_{r=1}^d f_r : f_r\in K \text{ for all } r \text{ and } (\operatorname{supp} f_r)_{r=1}^d \text{ is } \mathcal{S}_{k_j}\text{-allowable}\Big\}.$$

From the definition, it follows that $\mathcal{A}_j \subset B_{X_M^*}$.

THEOREM 4.13. *The sequence $(C_j,\mathcal{A}_j)_j$ is an asymptotic biorthogonal system in X_M. In particular, the space X_M is arbitrarily distortable.*

Proof. For every $j\in\mathbb{N}$ let $\varepsilon_j = 56/m_j$. The sequence $(\varepsilon_j)_j$ strictly decreases to 0. Since the sets C_j are asymptotic and $\mathcal{A}_j\subset B_{X_M^*}$ for all j, it suffices to prove that

(1) $\sup_{f\in\mathcal{A}_j} f(y) \ge 1/32$ for every $y\in C_j$,
(2) $|f(y)| \le \varepsilon_{\min\{i,r\}}$ for all $i\ne r$, $f\in\mathcal{A}_i$ and $y\in C_r$.

To prove (1), let $z=\sum_{k=1}^n b_k x_k$ be a $(1/m_j^2,k_j)$-r.i.s.c.c. and $y = z/\|z\|$. Then, by Proposition 4.12, $\|z\|\le 8/m_j$. For every $k=2,\dots,n$, we can choose $f_k\in K$ with $f_k(x_k)\ge 1/3$ and $\operatorname{supp} f_k\subset (l_{k-1}, l_k]$. Then the family $(f_k)_{k=2}^n$ is $\mathcal{S}_{k_j}$-allowable, so $\phi = m_j^{-1}\sum_{k=2}^n f_k \in \mathcal{A}_j$ and

$$\phi(z) \ge \frac{1}{m_j}\,\frac{1}{3}\sum_{k=2}^n b_k \ge \frac{1}{4m_j}.$$

It follows that

$$\phi(y) \ge \frac{m_j}{8}\,\frac{1}{4m_j} = \frac{1}{32}.$$

To prove (2), let $y = z/\|z\|\in C_r$ and $f\in\mathcal{A}_i$. We distinguish two cases.

CASE 1: $i<r$. Then Proposition 4.12 shows that $|f(z)|\le 14/(m_i m_r)$, since z is a $(1/m_r^2,k_r)$-r.i.s.c.c. Dividing by $\|z\|$ we get $|f(z/\|z\|)| \le 56/m_i = \varepsilon_i = \varepsilon_{\min\{r,i\}}$.

Case 2: $i > r$. Then Proposition 4.12 yields $|f(z)| \le 8/m_r^2$. Dividing by $\|z\|$ we get $|f(z/\|z\|)| \le 32/m_r < \varepsilon_{\min\{r,i\}}$. ∎

We now prove that if the sequence $(k_i, m_i)_i$ satisfies certain additional conditions, then every block subspace of $T_M[(\mathcal{S}_{k_i}, 1/m_i)_i]$ admits an ℓ_1^ω spreading model.

Definition 4.14. Let $(m_i), (k_i), (t_i)$ be strictly increasing sequences of positive integers satisfying the following *Gasparis conditions*:

(1) $m_1 = 2$, and there exists an increasing sequence $(s_i)_{i=1}^\infty$ of positive integers, with $s_1 > 2$, so that $m_i = \prod_{j<i} m_j^{s_j}$ for every $i \ge 2$.

(2) $t_1 > 4$ and $2^{t_i} \ge m_i^2$ for every $i \ge 1$.

(3) $t_i(r_i + 1) < k_i$ for every $i \ge 1$, where (r_i) is defined as follows: $r_1 = 1$ and for every $i \ge 2$,

$$r_i = \max\Big\{\sum_{j<i} \alpha_j k_j : \forall j < i,\ \alpha_j \in \mathbb{N} \cup \{0\} \text{ and } \prod_{j<i} m_j^{\alpha_j} < m_i^3\Big\}.$$

We set $Y_M = T_M[(\mathcal{S}_{k_i}, 1/m_i)_i]$.

The above conditions appeared in an early version of [15], where it was proved that the dual X^* of the "conditional version" X of the mixed Tsirelson space $T[(\mathcal{S}_{k_i}, 1/m_i)_i]$ admits a c_0^ω spreading model in every subspace.

The sequence $(m_i, k_i, t_i)_i$ satisfies $k_i \ge t_i(k_{i-1} + 1)$ and $2^{t_i} \ge m_i^2$ for all i, and this ensures that all the results of this section also hold for the space Y_M; in particular, Y_M is arbitrarily distortable (see Remark 4.3). In order to show that every subspace admits an ℓ_1^ω spreading model, we shall work with $(1/m_i^2, p_i)$-r.i.s.c.c.'s, where $p_i = \sum_{j<i} s_j k_j$, instead of $(1/m_i^2, k_i)$-r.i.s.c.c.'s that we used for the distortion. We shall show the following.

Theorem 4.15. *Every block subspace of Y_M admits an ℓ_1^ω spreading model with constant $c \ge 1/64$.*

We shall need the following arithmetical lemma from [15].

Lemma 4.16. *Assume that $(\alpha_j)_{j=0}^{i-1}$ are positive integers satisfying $\prod_{j<i} m_j^{\alpha_j} < m_i$. Then $\sum_{j<i} \alpha_j k_j < \sum_{j<i} s_j k_j$.*

Proof of Theorem 4.15. First we shall construct a sequence having an ℓ_1^ω spreading model starting from the basis $(e_i)_i$, and next we shall use the estimates on rapidly increasing sequences to deduce the existence of an ℓ_1^ω spreading model in every block subspace of Y_M.

For every $i \in \mathbb{N}$, we set $p_i = \sum_{j<i} s_j k_j$.

Lemma 4.17. *Let $i \ge 2$ and $x = \sum_{k\in F} \beta_k e_k$ be a $(1/m_i^2, p_i)$-basic special convex combination. Then $1/m_i \le \|x\| \le 2/m_i$.*

Proof. The lower estimate is obvious, since $p_i \le k_i$. For the upper estimate, let $\phi \in K$. If $w(\phi) = 1/m_r \le 1/m_i$ it follows immediately that $|\phi(x)| \le 1/m_i$. If $w(\phi) > 1/m_i$, let $D = \{k \in \operatorname{supp} x : |\phi(e_k)| > 1/m_i\}$. Then $|\phi_{|D^c}(x)| \le 1/m_i$.

CLAIM. $\operatorname{supp} \phi_{|D} \in \mathcal{S}_{p_i-1}$.

Once we prove the Claim, it follows that $|\phi_{|D}(x)| \le 1/m_i^2$, and this completes the proof.

Proof of the Claim. Let $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of ϕ. Sublemma 4.7 shows that the set $\{f_\alpha : \alpha \text{ a terminal node}\} = \{e_{l_\alpha} : \alpha \text{ a terminal node}\}$ is at most

$$\ell = \max\Big\{\sum_{\beta\prec\alpha} k_\beta : \alpha \text{ a terminal node}\Big\}\text{-allowable.}$$

For each terminal node α of $\mathcal{T}$, $\sum_{\beta\prec\alpha} k_\beta$ is of the form $\sum_{j<i} \varrho_j k_j$, $\varrho_j \in \mathbb{N}\cup\{0\}$ $(j<i)$, and

$$\frac{1}{m_i} < \phi(e_{l_\alpha}) \le \prod_{\beta\prec\alpha} \frac{1}{m_\beta} = \prod_{j<i} \frac{1}{m_j^{\varrho_j}}.$$

It follows from Lemma 4.16 that $\sum_{j<i} \varrho_j k_j < \sum_{j<i} s_j k_j = p_i$. Therefore $\operatorname{supp} \phi_{|D}$ is at most $\mathcal{S}_{p_i-1}$-allowable, and the proof of the Claim is complete. ∎

The Gasparis conditions imply the following key property of the space Y_M. A $(1/m_i^2, p_i)$-basic special convex combination, $x = \sum_{k\in F} b_k e_k$, can be normed by a functional x^* which belongs to various different classes $\mathcal{A}_j$.

Indeed, the functional $x^* = m_i^{-1} \sum_{k\in F} e_k^*$, which obviously belongs to $\mathcal{A}_i$, can also be written, for every $j<i$, in the form $x^* = m_j^{-1} \sum_{s\in G} f_s$ where the family $(f_s)_{s\in G}$ is $\mathcal{S}_{p_j}$-admissible. This is a consequence of the relations $m_i = \prod_{j<i} m_j^{s_j}$ and $p_i = \sum_{j<i} s_j k_j$, and the fact that $\mathcal{S}_n[\mathcal{S}_m] = \mathcal{S}_{n+m}$ for every $n, m \in \mathbb{N}$.

Let $(y_i)_{i=1}^\infty$ be a block sequence such that $y_i = \frac{1}{2} m_i x_i$ for $i = 1, 2, \ldots$, where $x_i = \sum_{l\in F_i} \alpha_l e_l$ is a $(1/m_i^2, p_i)$-basic special convex combination. Then $1/2 \le \|y_i\| \le 1$. We claim that the sequence $(y_i)_i$ has an ℓ_1^ω spreading model with constant $1/2$.

Indeed, let $F \in \mathcal{S}_r$ with $r \le \min F$. Then $(y_i)_{i\in F}$ is $\mathcal{S}_r$-admissible. For each $k \in F$, we consider the norming functional of x_k, $x_k^* = m_k^{-1} \sum_{l\in F_k} e_l^* = m_r^{-1} \sum_{i\in G_k} f_i$, where $(f_i)_{i\in G_k}$ is $\mathcal{S}_{p_r}$-admissible. Then the family $\{f_i : i \in \bigcup_{k\in F} G_k\}$ is $\mathcal{S}_r[\mathcal{S}_{p_r}] = \mathcal{S}_{r+p_r}$-admissible. Since $r + p_r < k_r$, it follows that the functional $f = m_r^{-1} \sum_{k\in F} \sum_{i\in G_k} f_i$ belongs to the norming set of Y_M. Hence

$$f\Big(\sum_{k\in F} b_k y_k\Big) = \sum_{k\in F} \frac{1}{m_r} \sum_{i\in G_k} f_i(b_k y_k) \ge \frac{1}{2} \sum_{k\in F} b_k.$$

Let now Z be a block subspace of Y_M and $(y_i)_{i\in\mathbb{N}}$ be a rapidly increasing sequence of $(1/m_{n_i}^2, k_{n_i})$-seminormalized s.c.c.'s in Z. Inductively we choose a block sequence (z_k) such that for every k, $z_k = \sum_{j\in F_k} b_j y_j$ is a $(1/m_k^2, p_k)$-rapidly increasing special convex combination of the sequence (y_i). Quoting step by step the proof of Proposition 4.12, and using Lemma 4.17 for the estimate of the norm of a $(1/m_k^2, p_k)$-basic s.c.c., we get $1/(4m_k) \le \|z_k\| \le 14/m_k$.

We set $y_k = (m_k/14)z_k$, $k\in\mathbb{N}$. Using the previous estimates, in the same manner as above we conclude that the sequence $(y_k)_{k\in\mathbb{N}}$ has an ℓ_1^ω spreading model with constant $c \ge 1/64$. ∎

REMARKS 4.18. 1. It is not clear whether in the general mixed Tsirelson space $T[(\mathcal{S}_n,\theta_n)_n]$ with $\theta_{m+n} \ge \theta_n\theta_m$, one can find normalized functionals which belong simultaneously to various different classes $\mathcal{A}_j$, as it happens when the Gasparis conditions are satisfied.

2. It is easy to see that if a block sequence $(y_i)_i$ in a Banach space has a c_0^ξ spreading model, then any sequence biorthogonal to it in the dual space has an ℓ_1^ξ spreading model. The dual of this statement is not always true. For example, consider the sequence $(w^n)_n = (\sum_{k=1}^n x_k^n)_n$ in the space $T[(\mathcal{S}_n,\theta_n)_n]$ which appeared in the proof of Proposition 3.1. Recall that $x_k^n = y_k^n/\|y_k^n\|$ and $y_k^n = \sum_{i\in F_k^n} b_i z_i$ is an $(\varepsilon_k^n, j_1+\ldots+j_k)$-r.i.s.c.c. for every $k\le n$ and $n\in\mathbb{N}$. As proved in Proposition 3.1, $(w^n)_n$ has an ℓ_1^ω spreading model. Let $(z_i^*)_i$ be a normalized sequence in the dual with $z_i^*(z_i)\ge 1/2$ and $\operatorname{supp} z_i^* \subseteq \operatorname{supp} z_i$. Then, for fixed k_0, the sequence of functionals $w_{n,k_0}^* = \theta_{j_1+\ldots+j_{k_0}+1}\sum_{i\in F_{k_0}^n} z_i^*$ is almost biorthogonal to $(w^n)_n$ (recall that $\|y_k^n\| \approx \theta_{j_1+\ldots+j_{k_0}}$). However, $(w_{n,k_0}^*)_n$ fails to have a c_0^ω spreading model in the dual space. Indeed, for $r\in\mathbb{N}$, let $y = \sum_{n\in F}\lambda_n y_{k_0}^n = \sum_{n\in F}\lambda_n\sum_{i\in F_{k_0}^n} b_i z_i$ be an $(\varepsilon, r+j_1+\ldots+j_{k_0})$-r.i.s.c.c. of (z_i). Then, by [6, Proposition 1.15], $\|y\|\approx \theta_{r+j_1+\ldots+j_{k_0}}$. It follows that $\sum_{n\in F} w_{n,k_0}^*(y/\|y\|) \approx \theta_{j_1+\ldots+j_{k_0+1}}/\theta_{r+j_1+\ldots+j_{k_0}}$, so $\|\sum_{n\in F} y_{n,k_0}^*\|$ tends to infinity with r.

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