$L^2_h$-domains of holomorphy and the Bergman kernel

by

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Abstract. We give a characterization of $L^2_h$-domains of holomorphy with the help of the boundary behavior of the Bergman kernel and geometric properties of the boundary, respectively.

For $\lambda_0 \in \mathbb{C}$, $r > 0$ we define $\triangle(\lambda_0, r) := \{ \lambda \in \mathbb{C} : |\lambda - \lambda_0| < r \}$. We also put $E := \triangle(0, 1)$. Moreover, the set of all plurisubharmonic (respectively, subharmonic) functions on an open set $D \subset \mathbb{C}^n$ is denoted by PSH($D$) (respectively, SH($D$)). We allow the (pluri)subharmonic functions to be identically $-\infty$ on connected components of $D$.

Following [Kli] for a domain $D \subset \mathbb{C}^n$ define

$$g_D(p; z) := \sup\{u(z)\}, \quad p, z \in D,$$

where the supremum is taken over all negative $u \in$ PSH($D$) such that $u(\cdot) - \log \| \cdot - p \|$ is bounded from above near $p$. We call the function $g_D(p, \cdot)$ the pluricomplex Green function (with the logarithmic pole at $p$). We also define

$$A_D(p; X) := \limsup_{\lambda \to 0} \frac{\exp(g_D(p, p + \lambda X))}{|\lambda|}, \quad p \in D, \ X \in \mathbb{C}^n.$$

Following [Jar-Pfl] the function $A_D$ is called the Azukawa pseudometric.

For a boundary point $w$ of a bounded domain $D \subset \mathbb{C}$ we introduce the notion of regularity. Namely, we say that $D$ is regular at $w$ if there exist a neighborhood $U$ of $w$ and a subharmonic function $u$ on $U \cap D$ with $u < 0$ on $U \cap D$ and $\lim_{U \cap D \ni \lambda \to w} u(\lambda) = 0$.

A set $P \subset \mathbb{C}^n$ is called pluripolar if for any point $z \in P$ there exist a connected neighborhood $U = U(z)$ and a function $u \in$ PSH($U$), $u \not\equiv -\infty$, such that $P \cap U \subset \{ z \in U : u(z) = -\infty \}$. In case $n = 1$ we call such a

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set $P$ polar. It is well known (cf. [Kli], Josefson theorem) that a set $P \subset \mathbb{C}^n$ is pluripolar if and only if there is a function $u \in \text{PSH}(\mathbb{C}^n)$, $u \neq -\infty$, such that $P \subset \{ z \in \mathbb{C}^n : u(z) = -\infty \}$.

A bounded domain $D \subset \mathbb{C}^n$ is said to be hyperconvex if there exists a negative and continuous plurisubharmonic exhaustion function of $D$.

Denote the class of square integrable holomorphic functions on an open set $D$ by $L^2_h(D)$. It is a Hilbert space with the standard scalar product induced from $L^2(D)$. Let us recall the definition of the Bergman kernel:

$$K_D(z) := \sup \left\{ \frac{|f(z)|^2}{\|f\|_{L^2_h(D)}^2} : f \neq 0, f \in L^2_h(D) \right\}.$$  

If $D$ is a bounded domain then $\log K_D$ is smooth and strictly plurisubharmonic. Therefore, for a bounded domain $D$ one may define the Bergman metric $\beta_D$:

$$\beta_D(z; X) := \sqrt{\sum_{j,k=1}^n \frac{\partial^2 \log K_D(z)}{\partial z_j \overline{z}_k} X_j X_k}, \quad z \in D, \ X \in \mathbb{C}^n,$$

and set

$$b_D(w, z) := \inf \{ L_{\beta_D}(\alpha) \}, \quad w, z \in D,$$

where $L_{\beta_D}(\alpha) = \int_0^1 \beta_D(\alpha(t); \alpha'(t)) \, dt$ and the infimum is taken over all piecewise $C^1$-curves $\alpha : [0, 1] \to D$ such that $\alpha(0) = w$, $\alpha(1) = z$. We call $b_D$ the Bergman distance. If $(D, b_D)$ is a complete metric space we say that $D$ is Bergman complete.

A domain $D \subset \mathbb{C}^n$ is called a domain (resp. an $L^2_h$-domain) of holomorphy if there are no domains $D_0, D_1 \subset \mathbb{C}^n$ with $\emptyset \neq D_0 \subset D_1 \cap D$, $D_1 \not\subset D$ such that for any $f \in \mathcal{O}(D)$ (resp. $f \in L^2_h(D)$) there exists an $\tilde{f} \in \mathcal{O}(D_1)$ with $\tilde{f} = f$ on $D_0$.

Let us recall several results concerning the above-mentioned notions, which show a close relationship between the theory of square integrable holomorphic functions and pluripotential theory.

For a bounded pseudoconvex domain $D$ consider the following properties:

(1) $D$ is hyperconvex,

(2) for any $w \in \partial D$, $\lim_{D \ni z \to w} K_D(z) = \infty$,

(3) $D$ is Bergman complete,

(4) $D$ is an $L^2_h$-domain of holomorphy.

All the relations between the properties (1)–(4) are known. Namely, (1)$\Rightarrow$(2) (see [Ohs 1]), (1)$\Rightarrow$(3) (see [Blo-Pfl], [Her]), and (3)$\Rightarrow$(4). The implication (2)$\Rightarrow$(1) does not hold in general (take the Hartogs triangle in $\mathbb{C}^2$ or consider some one-dimensional Zalcman-type domains—see [Ohs 1]). The one-dimensional counterexample to the implication (3)$\Rightarrow$(1) is given in [Chen 1].
Recall that any bounded pseudoconvex fat domain is an $L^2_h$-domain of holomorphy (see [Pfl]). Thus the Hartogs triangle is an $L^2_h$-domain of holomorphy in $\mathbb{C}^2$ which is not Bergman complete. Moreover, there also exists a fat domain in the complex plane that is not Bergman complete (see [Jar-Pfl-Zwo]). Thus, the implication $(4)\Rightarrow(3)$ does not hold even for fat pseudoconvex domains. In dimension one the implication $(2)\Rightarrow(3)$ does hold (see [Chen 2]) but in higher dimensions this is no longer the case (take the Hartogs triangle once more). As far as $(3)\Rightarrow(2)$ is concerned one may find a counterexample already in dimension one (see [Zwo 2]).

Let us have a closer look at the last example. The counterexamples belong to the following class of domains:

$$D := E \setminus \left( \bigcup_{j=1}^{\infty} \overline{\Delta}(z_j, r_j) \cup \{0\} \right),$$

where $z_j \to 0$, $r_j > 0$, $\overline{\Delta}(z_j, r_j) \subset E \setminus \{0\}$, $\overline{\Delta}(z_j, r_j) \cap \overline{\Delta}(z_k, r_k) = \emptyset$, $j \neq k$.

It is easy to see that for any $w \in \partial D$, $w \neq 0$, we have $\lim_{D \ni z \to w} K_D(z) = \infty$. The point is that the sequences can be chosen so that $\liminf_{D \ni z \to 0} K_D(z) < \infty$ and the domain is still Bergman complete. On the other hand one may easily see that $\limsup_{z \to 0} K_D(z) = \infty$. So the natural problem arises whether one may construct an example of a Bergman complete domain such that for some $w \in \partial D$ we have $\limsup_{z \to w} K_D(z) < \infty$. Below we show that this is impossible. Let us write down explicitly the condition we are interested in (as some kind of complement to properties (1)–(4)):

$$(5) \quad \text{for any } w \in \partial D \text{ we have } \limsup_{D \ni z \to w} K_D(z) = \infty.$$ 

The main aim of this paper is to present the following characterizations of $L^2_h$-domains of holomorphy.

**Theorem 1.** Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then (4) is equivalent to (5), i.e. $D$ is an $L^2_h$-domain of holomorphy if and only if for any $w \in \partial D$ we have $\limsup_{D \ni z \to w} K_D(z) = \infty$.

Making use of Theorem 1 and a result of A. Sadullaev we also get the following characterization of bounded $L^2_h$-domains of holomorphy.

**Theorem 2.** Let $D$ be a bounded pseudoconvex domain. Then $D$ is an $L^2_h$-domain of holomorphy if and only if for any $w \in \partial D$ and for any neighborhood $U$ of $w$ the set $U \setminus D$ is not pluripolar.

Before proving Theorem 1 let us recall some properties of the notions just defined that we need in what follows.

We shall start by considering $L^2_h$-domains of holomorphy in $\mathbb{C}$ ($n = 1$). First we list a number of properties of polar sets in $\mathbb{C}$ that we shall use (see [Ran], [Con]).
Let $D$ be an open set in $\mathbb{C}$ and let $K \subset D$ be a polar set relatively closed in $D$. Then:

- if $D$ is additionally connected then so is $D \setminus K$,
- for any $\lambda \in D$ and for any $0 < s$ with $\Delta(\lambda, s) \subset D$ there is an $s < r$ with $\Delta(\lambda, r) \subset D$ and $\partial \Delta(\lambda, r) \cap K = \emptyset$,
- for any $f \in L^2_h(D \setminus K)$ there is an $\tilde{f} \in \mathcal{O}(D)$ such that $\tilde{f}|_{D \setminus K} = f$.

There is also a precise description of $L^2_h$-domains of holomorphy in $\mathbb{C}$.

**Theorem 3** (see [Con], Theorem 9.9, p. 351). Let $D$ be a bounded domain in $\mathbb{C}$ and let $z \in \partial D$. Then there is an open neighborhood $U$ of $z$ such that any $f \in L^2_h(D)$ extends holomorphically to $D \cup U$ if and only if there is a neighborhood $V$ of $z$ such that the set $V \setminus D$ is polar.

One may easily get from Theorem 3 the following description of $L^2_h$-domains of holomorphy in $\mathbb{C}$.

**Theorem 4.** Let $D$ be a bounded domain in $\mathbb{C}$. Then $D$ is an $L^2_h$-domain of holomorphy iff for any $w \in \partial D$ and for any neighborhood $U$ of $w$ the set $U \setminus D$ is not polar.

Note that Theorem 2 is the exact higher-dimensional counterpart of Theorem 4.

Let us now recall some basic properties of regular points and the Green function. For a domain $D \subset \mathbb{C}^n$ we have $g_D(p, \cdot) \in \text{PSH}(D)$, $g_D(p, \cdot) < 0$. A bounded domain $D$ is hyperconvex iff $g_D(p, \cdot)$ is a continuous exhaustive function of $D$.

In the case of bounded planar domains it is well known that the Green function is symmetric (as a function of two variables) and $g_D(p, \cdot)$ is harmonic on $D \setminus \{p\}$. Moreover, a point $w \in \partial D$ is regular iff for some (any) $p \in D$, $g_D(p, \lambda) \to 0$ as $D \ni \lambda \to w$. Consequently, a bounded domain $D \subset \mathbb{C}$ is hyperconvex iff any point of its boundary is regular. The set of irregular points of any bounded domain in $\mathbb{C}$ is polar.

Below we shall need some estimate for the Bergman kernel in the one-dimensional case that will enable us to prove Theorem 1 in dimension one.

**Theorem 5** (see [Ohs 2]). Let $D$ be a domain in $\mathbb{C}$. Then there is a positive constant $C$ such that

$$\sqrt{K_D(z)} \geq CA_D(z; 1), \quad z \in D.$$

Our first aim is to obtain the following exhaustion property of the Bergman kernel at regular points.

**Proposition 6.** Let $D$ be a bounded domain in $\mathbb{C}$. Assume that $w \in \partial D$ is a regular point. Then $K_D(z) \to \infty$ as $D \ni z \to w$. 

Proof. In view of Theorem 5 it is sufficient to show that

$$\limsup_{z \to w} K_D(z) < \infty,$$

which easily finishes the proof.

Remark 7. In view of property (6) it follows from the estimates in [Die-Her] that for any bounded domain in \(\mathbb{C}\) the convergence \(\beta_D(z;1) \to \infty\) as \(z \to w \in \partial D\) holds for any regular point \(w \in \partial D\).

Lemma 8. Let \(D\) be a bounded domain in \(\mathbb{C}\), \(w \in \partial D\). Then the following conditions are equivalent:

1. \(\limsup_{z \to w} K_D(z) < \infty\),
2. there is an open neighborhood \(U\) of \(w\) such that the set \(U \setminus D\) is polar.

Proof. Let us first make a general remark: \(U \setminus D\) being polar is equivalent to \(U \cap \partial D\) being polar.

(2)\(\Rightarrow\)(1). If \(U\) satisfies (2) then without loss of generality one may assume that \(K := U \cap \partial D \subset U\). So there is a domain \(\tilde{D}\) with \(D = \tilde{D} \setminus K\), \(w \in \tilde{D}\), where \(K\) is a compact polar set. Then

\[ L^2_h(D) = L^2_h(\tilde{D}) |_{D} \]

and, consequently, \(K_D = K_{\tilde{D}}|_D\), which implies (1).

(1)\(\Rightarrow\)(2). Suppose that for any neighborhood \(U\) of \(w\) the set \(U \cap \partial D\) is not polar. Then there is a sequence \(w_\nu \to w\), \(w_\nu \in \partial D\), such that \(D\) is regular at \(w_\nu\). In view of Proposition 6 we have \(K_D(z) \to \infty\) as \(z \to w_{\nu}\), which easily finishes the proof.

We are now able to study the situation in \(\mathbb{C}^n\) \((n > 1)\).

Lemma 9. Let \(D\) be a domain in \(\mathbb{C}^n\), \(n \geq 2\). Fix \(0 < r < t\). For any \(z' \in \mathbb{C}^{n-1}\) define \(A(z') := \{z_n \in tE : (z', z_n) \in D\} = tE \setminus K(z')\). Assume that \(K(0')\) is polar and there is a neighborhood \(0' \in V\) such that for almost any \(z' \in V\) (with respect to the \((2n-2)\)-dimensional Lebesgue measure) the
set $K(z')$ is polar. Then there is a neighborhood $0' \in V' \subset V$ such that for any $f \in L^2_h(D)$ there exists a function $F \in \mathcal{O}(V' \times sE)$ with $F = f$ on $(V' \times sE) \cap D$.

Proof. Because $K(0')$ is polar there is an $s$ with $0 < r < s < t$ such that $K(0') \cap \partial(sE) = \emptyset$. Then there is a neighborhood $0' \in V' \subset V$ such that for any $\zeta' \in V'$ we have $K(\zeta') \cap \partial(sE) = \emptyset$.

Define

$$F(\zeta', z_n) := \frac{1}{2\pi i} \int_{\partial(sE)} \frac{f(\zeta', \lambda)}{\lambda - z_n} \, d\lambda, \quad (\zeta', z_n) \in V' \times sE.$$ 

Then $F$ is a holomorphic function on $V' \times sE$.

On the other hand by the square integrability of $f$, the Fubini theorem and the assumptions of the lemma, for almost all $\zeta' \in V'$ (with respect to the $(2n - 2)$-dimensional Lebesgue measure) the function $f(\zeta', \cdot)$ is in $L^2_h(tE \setminus K(\zeta'))$ and $K(\zeta')$ is polar. Since closed polar sets are removable for $L^2_h$-functions, for almost all $\zeta' \in V'$ the function $f(\zeta', \cdot)$ extends to a holomorphic function on $tE$. So the Cauchy formula applies and we obtain the equality $f(\zeta', z_n) = F(\zeta', z_n)$, $(\zeta', z_n) \in (V' \times sE) \cap D$, for almost all $\zeta' \in V'$. Since the equality holds on a dense subset of $(V' \times sE) \cap D$, it holds on the whole set. $lacksquare$

Before we start the proof of Theorem 1 let us formulate, in the form that we need, the most powerful tool we shall use, namely the Ohsawa–Takegoshi extension theorem.

**Theorem 10** (see [Ohs-Tak]). Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $L$ be a complex line. Then there is a constant $C > 0$ such that for any $f \in L^2_h(D \cap L)$ there is an $F \in L^2_h(D)$ with $\|F\|_{L^2_h(D)} \leq C\|f\|_{L^2_h(D \cap L)}$ and $F|_{D \cap L} = f$.

Note that Theorem 10 directly leads to the following inequality for the Bergman kernel:

$$K_{D \cap L}(z) \leq C^2 K_D(z), \quad z \in D \cap L.$$ 

This inequality will often be used below. Note only that the set $D \cap L$ on the left-hand side is open (as a subset of $\mathbb{C}$) but not necessarily connected.

We now prove our main result.

**Proof of Theorem 1.** First note that the result for $n = 1$ follows from Theorem 4 and Lemma 8, so assume that $n \geq 2$.

$(5) \Rightarrow (4)$. Suppose that $D$ is not an $L^2_h$-domain of holomorphy. Then there are a polydisc $P \subset D$ with $\partial P \cap \partial D \neq \emptyset$ and a polydisc $\tilde{P} \supset \supset P$,
\[ \tilde{P} \not\subset D, \text{ such that for every function } f \in L^2_h(D) \text{ there is a function } \hat{f} \in H^\infty(\tilde{P}) \text{ with } f = \hat{f} \text{ on } P. \]

We claim that for any \( z \in P \) and for any complex line \( L \) passing through \( z \) we have

\[ L \cap D \cap \tilde{P} = (L \cap \tilde{P}) \setminus K(z), \text{ where } K(z) \text{ is a polar set.} \]

Suppose that \( L \cap D \cap \tilde{P} = (L \cap \tilde{P}) \setminus K(z), \) where \( K(z) \) is not a polar set. Choose a compact non-polar set \( K' \subset K(z) \subset (L \cap \tilde{P}) \setminus D \) such that \( V_0 = L \setminus \hat{K}' \) (where \( \hat{K}' \) denotes the polynomial hull of \( K' \)) contains \( L \cap P \).

Then there is a function \( f \in L^2_h(V_0) \) which does not extend holomorphically through \( \hat{K}' \) (cf. Theorem 3). Let \( \{V_j\}_{j=1}^N \), where \( 0 \leq N \leq \infty \), be the family of bounded components of \( L \setminus K' \). Additionally, we let \( f \) be identically 0 on \( \bigcup_{j=1}^N V_j \).

In view of the Ohsawa–Takegoshi extension theorem there exists an \( F \in L^2_h(D) \) such that \( F|_{L \cap D} = f|_{L \cap D} \). But then there is an \( \hat{F} \in H^\infty(\tilde{P}) \) such that \( \hat{F}|_P = F|_P \). Consequently, \( \hat{F}|_{L \cap \tilde{P}} \) is a holomorphic extension of \( f|_{L \setminus K'} \), through \( \hat{K}' \), a contradiction.

It follows from the above claim that \( \tilde{P} \cap D \) is connected. Consequently, for any function \( f \in L^2_h(D) \) its (unique) extension \( \hat{f} \in H^\infty(\tilde{P}) \) satisfies the equality \( f = \hat{f} \) on \( D \cap \tilde{P} \).

Consider the space

\[ A := \{(f, \hat{f}) : f \in L^2_h(D)\} \subset L^2_h(D) \times H^\infty(\tilde{P}) \]

with the norm \( \|(f, \hat{f})\| := \|f\|_{L^2_h(D)} + \|\hat{f}\|_{H^\infty(\tilde{P})} \). It is easily seen that \( A \) is a Banach space. Consider the mapping \( \pi : A \ni (f, \hat{f}) \mapsto f \in L^2_h(D) \). Then \( \pi \) is a one-to-one surjective continuous linear mapping. Hence, in view of the Banach open mapping theorem, \( \pi^{-1} \) is a continuous linear mapping. In other words, there is a constant \( C > 1 \) such that

\[ \|(f, \hat{f})\| \leq C\|f\|_{L^2_h(D)}, \quad f \in L^2_h(D); \]

in particular, \( \|\hat{f}\|_{H^\infty(\tilde{P})} \leq C\|f\|_{L^2_h(D)}. \) Consequently,

\[ \sup_{z \in \tilde{P} \cap D} K_D(z) = \sup \left\{ \frac{|f(z)|^2}{\|f\|_{L^2_h(D)}^2} : z \in \tilde{P} \cap D, f \neq 0, f \in L^2_h(D) \right\} \leq C^2, \]

which contradicts (5) for any \( w \in \partial P \cap \partial D \neq \emptyset. \)

(4) \( \Rightarrow \) (5). Fix \( w \in \partial D \). First consider the case \( w \notin \text{int}(D) \). Then there is a sequence \( z_\nu \to w \) with \( z_\nu \notin D \). Let \( B_\nu \) be the largest open ball centered at \( z_\nu \) disjoint from \( D \). Choose \( w_\nu \in \partial B_\nu \cap \partial D \). Obviously, \( w_\nu \to w. \)
Note that for any $\nu$, $D$ satisfies at $w_\nu$ the “outer cone condition” (see [Pfl]). Therefore, for any $\nu$ we have $\lim_{D \ni z \rightarrow w_\nu} K_D(z) = \infty$ (see [Pfl]), which easily implies (5).

Assume now that $w \in \text{int}(\overline{D})$. Suppose that (5) does not hold at $w$. Then there is a polydisc $P$ with center at $w$ such that $\sup\{K_D(z) : z \in D \cap P\} < \infty$. Without loss of generality we may assume that $P \subset \subset \text{int}(\overline{D})$. Consider any complex line $L$ intersecting $P$. We claim that $L \cap P \cap D$ is equal to $(L \cap P) \setminus K$, where $K$ is a polar set or $K = L \cap P$. In fact if this were not the case then $\sup_{z \in L \cap P \cap D} K_{L \cap D}(z) = \infty$ (the Bergman kernel is here understood as that of a one-dimensional set) (use Lemma 8) and, consequently, in view of the Ohsawa–Takegoshi extension theorem we would get $\sup_{z \in L \cap P \cap D} K_D(z) = \infty$, a contradiction.

Note that there is a complex line $L$ passing through $w$ such that $L \cap P \cap D$ is not empty. Assume that $w = 0$. Making a linear change of coordinates and shrinking $P$ if necessary we may assume that $P = E^n$ and that $\{\lambda \in E : (0, \ldots, 0, \lambda) \in D\}$ is not empty.

Therefore, the assumptions of Lemma 9 are satisfied (with some neighborhood $V \subset E^{n-1}$ of $0' \in \mathbb{C}^{n-1}$) and there is a neighborhood $0' \in V' \subset E^{n-1}$ such that for any $f \in L^2(D)$ there is a function $F \in \mathcal{O}(V' \times \frac{1}{2}E)$ with $F = f$ on $(V' \times \frac{1}{2}E) \cap D$, a contradiction.

**Proof of Theorem 2.** Because of Theorem 4 we may assume that $n \geq 2$.

$(\Rightarrow)$ Suppose that for some $w \in \partial D$ there is a polydisc $P$ such that $P \setminus D$ is pluripolar. Let $u \in \text{PSH}(P)$ be such that $u \not\equiv -\infty$ and $P \setminus D \subset \{u = -\infty\}$. Take a non-empty open set $U \subset D \cap P$ and consider all complex lines connecting $w$ to some point from $U$. It is easy to see that there is a complex line $L$ such that $u \not\equiv -\infty$ on $L \cap P$. Assume that $w = 0$. Making a linear change of coordinates and shrinking $P$ if necessary, we may assume that $P = E^n$ and $\{z_n \in E : (0', z_n) \not\in D\}$ is polar. Because of the local integrability of $u$, for almost any $z' \in E^{n-1}$ (with respect to the $(2n - 2)$-dimensional Lebesgue measure) the function $u(z', \cdot)$ is not identically $-\infty$ on $E$. Consequently, for almost every $z' \in E^{n-1}$ the set $\{z_n \in E : (z', z_n) \not\in D\}$ is polar. Applying Lemma 9 we obtain the existence of an open set $0 \in Q$ such that for any $f \in L^2(D)$ there exists an $F \in \mathcal{O}(Q)$ with $f = F$ on $D \cap Q$, a contradiction.

$(\Leftarrow)$ Suppose that the implication does not hold, so in view of Theorem 1 there is a $w \in \partial D$ such that $\limsup_{D \ni z \rightarrow w} K_D(z) < \infty$. In other words there is a polydisc $P$ with center at $w$ such that $\sup_{z \in D \cap P} K_D(z) < \infty$.

First note that for any complex line $L$ with $L \cap P \neq \emptyset$ we have $L \cap P \cap D = \emptyset$ or $L \cap P \cap D = (L \cap P) \setminus K$, where $K$ is a polar set. Actually, if there were $L$ such that $L \cap P \cap D = (L \cap P) \setminus K$, with $K \neq L \cap P$ and $K$ not polar, then for some $U \subset L \cap P$, $\sup_{z \in U \cap D} K_{D \cap L}(z) = \infty$ (use Lemma 8). Therefore,
in view of the Ohsawa–Takegoshi theorem, \( \sup_{z \in U \cap D} K_D(z) = \infty \), a contradiction.

Consequently, one may apply a result of A. Sadullaev (see [Sad 2] and also [Sad 1]) to deduce that the set \( P \setminus D \) is pluripolar, a contradiction. \( \blacksquare \)

It follows from the reasoning in the proofs of Theorems 1 and 2 that the following higher-dimensional counterpart of Lemma 8 holds.

**Lemma 11.** Let \( D \) be a bounded pseudoconvex domain and let \( w \in \partial D \). Then \( \limsup_{D \ni z \to w} K_D(z) < \infty \) if and only if for any neighborhood \( U \) of \( w \) the set \( U \setminus D \) is pluripolar.

The known examples of \( L^2_h \)-domains of holomorphy include bounded pseudoconvex fat domains and bounded pseudoconvex balanced domains. The characterization of \( L^2_h \)-domains of holomorphy given by us yields many examples of such domains. Below we give an example of a new class of domains having this property.

For a bounded pseudoconvex domain \( D \subset \mathbb{C}^n \) we define the following Hartogs domain with \( m \)-dimensional balanced fibers:

\[
G_D := \{(w, z) \in \mathbb{C}^{n+m} : H(z, w) < 1\},
\]

where \( \log H \) is plurisubharmonic on \( D \times \mathbb{C}^m \), \( H(z, \lambda w) = |\lambda| H(z, w) \), \( (z, w) \in D \times \mathbb{C}^m \), \( \lambda \in \mathbb{C} \), and \( G_D \) is bounded (i.e. \( H(z, w) \geq C\|w\| \) for some \( C > 0 \), \( (z, w) \in D \times \mathbb{C}^m \)). Then \( G_D \) is a bounded pseudoconvex domain.

**Proposition 12.** Let \( D \) be a bounded \( L^2_h \)-domain of holomorphy. Then \( G_D \) is an \( L^2_h \)-domain of holomorphy.

**Proof.** Take \( (z^0, w^0) \in \partial G_D \). If \( z^0 \in D \) then

\[
\lim_{G_D \ni (z, w) \to (z^0, w^0)} K_{G_D}(z, w) = \infty
\]

(use Theorem 3.1(i) from [Jar-Pfl-Zwo]).

Assume now that \( z^0 \in \partial D \). Let \( V \) be any neighborhood of \((z^0, w^0)\). In view of Lemma 11 and Theorem 1 it is sufficient to show that \( V \setminus G_D \) is not pluripolar. We may assume that \( V = V_1 \times V_2 \subset \mathbb{C}^{n+m} \). Because \( D \) is an \( L^2_h \)-domain of holomorphy Theorem 2 applies and \( V_1 \setminus D \) is not pluripolar. Since \( V \setminus G_D \supset (V_1 \setminus D) \times V_2 \) and the latter set is not pluripolar, the proof is finished. \( \blacksquare \)

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After the paper had been finished the authors learnt about the existence of a paper of J. Siciak (see [Sic]) in which a similar result to that of Lemma 9 was proven (but with other methods).
References


