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n-supercyclic operators

by

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Abstract. We show that there are linear operators on Hilbert space that have n-dimensional subspaces with dense orbit, but no $(n-1)$-dimensional subspaces with dense orbit. This leads to a new class of operators, called the n-supercyclic operators. We show that many cohyponormal operators are n-supercyclic. Furthermore, we prove that for an n-supercyclic operator, there are n circles centered at the origin such that every component of the spectrum must intersect one of these circles.

1. Introduction. If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a separable Hilbert space and $C \subseteq \mathcal{H}$, then the orbit of C under T is $\bigcup\{C, T(C), T^2(C), \ldots\}$. An operator T is said to be *hypercyclic* if there is a vector with dense orbit. The first example of a hypercyclic operator on a Hilbert space was given by Rolewicz [11] in 1969. He showed that if B is the backward shift, then λB is hypercyclic for any scalar $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In 1974 Hilden and Wallen [8] introduced the class of *supercyclic* operators as those operators that have a vector whose scaled orbit is dense. That is, T is supercyclic if there is a vector x such that $\{\alpha T^n x : n \geq 0, \alpha \in \mathbb{C}\}$ is dense. Hilden and Wallen showed, among other things, that any unilateral backward weighted shift is supercyclic. Hypercyclic and supercyclic operators have received considerable attention recently, especially since they arise in familiar classes of operators, such as weighted shifts [12], [13], composition operators [3], adjoints of multiplication operators on spaces of analytic functions [6] and adjoints of subnormal and hyponormal operators [5]. For a general survey of hypercyclicity, see [7].

Notice that an operator is supercyclic if and only if it has a one-dimensional subspace with dense orbit. We shall say that an operator T is n-*supercyclic* $(1 \leq n < \infty)$ if there is an n-dimensional subspace whose orbit under T is dense.

In this paper we shall prove that for every $n \geq 2$, there are very natural operators (adjoints of multiplication operators) that are n-supercyclic but

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not $(n-1)$-supercyclic. Thus (for $n = 2$) there are operators that have a "plane" with dense orbit, but no "line" with dense orbit.

We shall say that T is *infinitely supercyclic*, denoted by ∞-supercyclic, if there exists a proper closed subspace $\mathcal{M}$ such that (1) the orbit of $\mathcal{M}$ under T is dense, (2) for each $n \geq 1$, $\bigcup\{\mathcal{M}, T(\mathcal{M}), \ldots, T^n(\mathcal{M})\}$ is not dense, and (3) $\mathcal{M}$ does not contain any nonzero invariant subspaces for T. While condition (1) is of interest in its own right, the reason for conditions (2) and (3) is to rule out certain trivialities. We will prove that there are ∞-supercyclic operators that are not n-supercyclic for any $n < \infty$.

The outline of the paper is as follows: In Section 2 we state some preliminary results about hypercyclicity and supercyclicity that are needed throughout the paper. In Section 3 we present sufficient conditions for operators to be n-supercyclic; Theorem 3.7 is one of the main results of this section.

THEOREM 3.7 (Direct sums are n-supercyclic). *If $T_1, \ldots, T_n$, $1 \leq n < \infty$, are supercyclic operators and each satisfies the supercyclicity criterion with respect to the same sequence $\{n_k\}$, then $\bigoplus_{k=1}^n T_k$ is n-supercyclic.*

In Section 4 some necessary conditions for an operator to be n-supercyclic are given, the main result—which was unexpected—is Theorem 4.1. This result shows that there really is some structure to these operators.

THEOREM 4.1 (The Circle Theorem). *If T is n-supercyclic, then there are n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \geq 0$, $i = 1, \ldots, n$, such that for every invariant subspace $\mathcal{M}$ of T^*, we have $\sigma(T^*|\mathcal{M}) \cap \bigcup_{i=1}^n \Gamma_i \neq \emptyset$. In particular, every component of the spectrum of T intersects $\bigcup_{i=1}^n \Gamma_i$.*

It is also proven in Section 4 that normal operators on infinite-dimensional spaces cannot be n-supercyclic. It is left open as to whether a subnormal or a hyponormal operator can be n-supercyclic, although the author expects not. One may easily check that bilateral weighted shifts are ∞-supercyclic, thus a subnormal operator, and even a unitary operator, may be ∞-supercyclic.

Example 4.8 is a sharp example giving necessary and sufficient spectral conditions for a class of cosubnormal operators to be n-supercyclic.

EXAMPLE 4.8 (Important example). *If $\{\Delta_j : 1 \leq j < \infty\}$ is a bounded collection of open disks, $S_j = M_z$ on $L^2_{\mathrm{a}}(\Delta_j)$, and $S = \bigoplus_{j=1}^\infty S_j$, then S^* is n-supercyclic if and only if there are k circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \geq 0$, $i = 1, \ldots, k$, $k \leq n$, and a side associated with each circle (inside, outside or either) such that for every disk Δ_j, there exists a circle Γ_i so that $\operatorname{cl} \Delta_j$ intersects Γ_i and the assigned side of Γ_i. Furthermore, the total number of sides is n.*

In Section 5 we use local spectral theory techniques to take a first step towards characterizing the cohyponormal operators that are n-supercyclic. Corollaries 5.7 and 5.8 are the main results here.

COROLLARY 5.7 (Cohyponormal operators with eigenvectors). *If S is a pure hyponormal operator and there exists a $\varrho > 0$ such that for each $\varepsilon > 0$,*

$$\operatorname{span}\{\ker(S^* - \lambda) : \varrho - \varepsilon < |\lambda| < \varrho + \varepsilon\}$$

is dense, then S^* is 2-supercyclic.

COROLLARY 5.8 (Cohyponormal operators). *Suppose that S is a pure hyponormal operator and there exists a circle $\Gamma = \{z : |z| = r\}$, $r > 0$, such that for every hyperinvariant subspace $\mathcal{M}$ of S, $\sigma(S|\mathcal{M}) \cap \Gamma \neq \emptyset$. Then S^* is 2-supercyclic.*

Finally in Section 6, several open questions are stated about this new class of operators. The most important being: If T is n-supercyclic and T^* has no eigenvalues, then is T necessarily cyclic?

2. Preliminaries. In what follows, $\mathcal{H}$ will denote a separable complex Hilbert space; although most of what follows can be done on Banach spaces or even more general spaces, we will mainly work on Hilbert spaces, as there are many unanswered questions there.

There are a number of different "criteria" for an operator to be supercyclic. The first criterion was given by Salas [13] in 1999. In [5] Feldman, Miller and Miller gave an inner and an outer version of Salas' criteria. Fortunately, it has recently been shown by Bermúdez, Bonilla and Peris [1] that all these criteria are equivalent. However, each criterion has its own advantages and may be easier to apply in a given setting.

THEOREM 2.1 (The Supercyclicity Criterion (Salas)). *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that there is a sequence $n_k \to \infty$ and dense sets X and Y and functions $B_{n_k} : Y \to \mathcal{H}$ such that:*

(1) *If $y \in Y$, then $T^{n_k} B_{n_k} y \to y$ as $k \to \infty$.*
(2) *If $x \in X$ and $y \in Y$, then $\|T^{n_k} x\| \cdot \|B_{n_k} y\| \to 0$ as $k \to \infty$.*

Then T is supercyclic.

THEOREM 2.2 (An Outer Supercyclicity Criterion). *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that there is a sequence $n_k \to \infty$, a dense linear subspace Y, and for every $y \in Y$ a dense linear subspace X_y such that:*

(1) *There exist functions $B_{n_k} : Y \to \mathcal{H}$ such that $T^{n_k} B_{n_k} y \to y$ for all $y \in Y$.*
(2) *If $y \in Y$ and $x \in X_y$, then $\|T^{n_k} x\| \cdot \|B_{n_k} y\| \to 0$ as $k \to \infty$.*

Then T is supercyclic.

Theorem 2.3 (An Inner Supercyclicity Criterion). *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that there is a sequence $n_k \to \infty$, a dense linear subspace Y, and for every $y \in Y$ a dense linear subspace X_y such that*:

(1) *There exist functions $B_{y,n_k} : X_y \to \mathcal{H}$ such that $T^{n_k} B_{y,n_k} x \to x$ for all $x \in X_y$.*

(2) *If $y \in Y$ and $x \in X_y$, then $\|T^{n_k} y\| \cdot \|B_{y,n_k} x\| \to 0$ as $k \to \infty$.*

Then T is supercyclic.

Note that the functions B_n and $B_{y,n}$, which are approximate right inverses of T^n, are nothing more than well defined functions; they may be, and usually are, discontinuous.

The following corollary follows easily from the above (inner & outer) criteria (see Feldman, Miller, and Miller [5]).

Corollary 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$.*

(1) (inner) *If there exists a number $\varrho > 0$ such that for every $\varepsilon > 0$,*

$$\operatorname{span}\{\ker(T - \lambda) : \varrho - \varepsilon < |\lambda| < \varrho\}$$

is dense in X, then T is supercyclic.

(2) (outer) *If there exists a number $\varrho \geq 0$, such that for every $\varepsilon > 0$,*

$$\operatorname{span}\{\ker(T - \lambda) : \varrho < |\lambda| < \varrho + \varepsilon\}$$

is dense in X, then T is supercyclic.

The following results from Feldman, Miller and Miller [5] characterize the cohyponormal operators that are hypercyclic or supercyclic.

Theorem 2.5. *If T is a hyponormal operator on a separable Hilbert space $\mathcal{H}$, then T^* is hypercyclic if and only if for every hyperinvariant subspace $\mathcal{M}$ of T,*

$$\sigma(T|\mathcal{M}) \cap \{z : |z| < 1\} \neq \emptyset \quad \textit{and} \quad \sigma(T|\mathcal{M}) \cap \{z : |z| > 1\} \neq \emptyset.$$

Theorem 2.6. *If T is a pure hyponormal operator, then T^* is supercyclic if and only if there exists a circle $\Gamma_\varrho = \{z : |z| = \varrho\}$, $\varrho \geq 0$, such that either*:

(a) (inner) *for every hyperinvariant subspace $\mathcal{M}$ of T,*

$$\sigma(T|\mathcal{M}) \cap \Gamma_\varrho \neq \emptyset \quad \textit{and} \quad \sigma(T|\mathcal{M}) \cap \{z : |z| < \varrho\} \neq \emptyset, \quad \textit{or}$$

(b) (outer) *for every hyperinvariant subspace $\mathcal{M}$ of T,*

$$\sigma(T|\mathcal{M}) \cap \Gamma_\varrho \neq \emptyset \quad \textit{and} \quad \sigma(T|\mathcal{M}) \cap \{z : |z| > \varrho\} \neq \emptyset.$$

3. Sufficient conditions for n-supercyclicity. In this section we give two different conditions for an operator to be n-supercyclic and present some examples of n-supercyclic operators.

PROPOSITION 3.1. *If $n \in \mathbb{Z}^+ \cup \{\infty\}$ and $\{T_k : 1 \le k \le n\}$ is a bounded collection of operators such that there exist constants $\{c_k : 1 \le k \le n\}$ such that $\bigoplus_{k=1}^n c_k T_k$ is supercyclic, then $\bigoplus_{k=1}^n T_k$ is n-supercyclic.*

Proof. Set $T := \bigoplus_{k=1}^n c_k T_k$ and $S := \bigoplus_{k=1}^n T_k$. Let $x = (x_1, \dots, x_n)$ be a supercyclic vector for T. Then $\{\alpha T^i x : \alpha \in \mathbb{C},\ i \ge 0\}$ is dense. Now for $1 \le j \le n$, let $e_j = (0, \dots, 0, x_j, 0, \dots, 0)$ where x_j is in the jth coordinate. Then $\alpha T^i x = (\alpha c_1^i T_1^i x_1, \dots, \alpha c_n^i T_n^i x_n) = S^i(\beta_1^{(i)} e_1 + \dots + \beta_n^{(i)} e_n)$ where $\beta_j^{(i)} = \alpha c_j^i$. Thus $\{S^i(\beta_1^{(i)} e_1 + \dots + \beta_n^{(i)} e_n) : i \ge 0,\ \beta_j^{(i)} \in \mathbb{C}\} \supseteq \{\alpha T^i x : \alpha \in \mathbb{C}, i \ge 0\}$ and hence the orbit of $\mathcal{M} := \operatorname{span}\{e_1, \dots, e_n\}$ under S is dense. So, S is n-supercyclic. If $n = \infty$, then one can show that the remaining conditions are satisfied; see the proof of Proposition 3.6. ■

A similar proof to that above establishes the following result.

PROPOSITION 3.2. *If $n \in \mathbb{Z}^+ \cup \{\infty\}$ and $\{T_k : 1 \le k \le n\}$ is a bounded collection of operators such that there exist constants $\{c_k : 1 \le k \le n\}$ such that $\bigoplus_{k=1}^n c_k T_k$ is m-supercyclic, $m < \infty$, then $\bigoplus_{k=1}^n T_k$ is nm-supercyclic.*

For a bounded open set $G \subseteq \mathbb{C}$, let $L^2_{\mathrm{a}}(G)$ denote the Bergman space of all analytic functions on G that belong to $L^2(G, dA)$ where dA denotes the area measure on G.

EXAMPLE 3.3. Let $S_i = M_z$ on $L^2_{\mathrm{a}}(\Delta_i)$, $i = 1, 2$, where Δ_i is a disk, and set $S = S_1 \oplus S_2$. If $\Delta_1 \subseteq \{z : |z| \le 1\}$ and $\Delta_2 \subseteq \{z : |z| \ge 1\}$, then S^* is 2-supercyclic but not supercyclic.

Proof. The fact that S^* is 2-supercyclic follows from Proposition 3.1; see also the proof of the next example. That S^* is not supercyclic follows from Theorem 2.6; see also Theorem 4.1. ■

EXAMPLE 3.4. Let $n \in \mathbb{Z}^+ \cup \{\infty\}$. If $\{\Delta_k : 1 \le k \le n\}$ is any bounded collection of open disks, $S_k = M_z$ on $L^2_{\mathrm{a}}(\Delta_k)$, and $S = \bigoplus_{k=1}^n S_k$, then S^* is n-supercyclic.

Proof. There exist positive scalars $\{c_k\}$ such that the interior of the spectrum of $c_k S_k$ intersects the unit circle. Then it follows easily, say from Theorem 2.5, that $\bigoplus_{k=1}^n c_k S_k^*$ is hypercyclic. Thus by Proposition 3.1, S^* is n-supercyclic. ■

We now present a general method used to show that the direct sum of operators is n-supercyclic.

PROPOSITION 3.5. *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $1 \le i \le n < \infty$. If for any two nonempty open subsets $U, V \subseteq \bigoplus_{i=1}^n \mathcal{H}_i$ there exist $(x_1, \dots, x_n) \in U$, $k \ge 0$, and $\{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{C}$ such that $(\alpha_1 T_1^k x_1, \dots, \alpha_n T_n^k x_n) \in V$, then $T_1 \oplus \dots \oplus T_n$ is n-supercyclic.*

Proof. Let $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ and for $h \in \mathcal{H}$, let h_i denote the ith component of h. Suppose also that $\{V_j\}_{j=1}^\infty$ is a basis of open sets for $\mathcal{H}$. Let

$$G_j = \{h \in \mathcal{H} : \exists \alpha_1, \dots, \alpha_n \in \mathbb{C} \text{ and } k \ge 0 \text{ such that } (\alpha_1 T_1^k h_1, \dots, \alpha_n T_n^k h_n) \in V_j\}.$$

Then G_j is an open set, which, by hypothesis, is dense in $\mathcal{H}$. Thus the Baire Category Theorem implies that $\bigcap_{j=1}^\infty G_j$ is a dense G_δ.

Let $x \in \bigcap_{j=1}^\infty G_j$. Then one easily checks that the n-dimensional subspace of $\mathcal{H}$ spanned by $(x_1, 0, \dots, 0), (0, x_2, 0, \dots, 0), \dots, (0, \dots, 0, x_n)$ has dense orbit under $T_1 \oplus \dots \oplus T_n$, where $x = (x_1, \dots, x_n)$. ∎

PROPOSITION 3.6. *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $1 \le i < \infty$ be a uniformly bounded sequence of operators. If for any two open subsets $U, V \subseteq \bigoplus_{i=1}^\infty \mathcal{H}_i$ there exists an $n \ge 1$ and $(x_1, \dots, x_n, 0, 0, \dots) \in U$, an integer $k \ge 1$, and $\{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{C}$ such that $(\alpha_1 T_1^k x_1, \dots, \alpha_n T_n^k x_n, 0, 0, \dots) \in V$, then $\bigoplus_{i=1}^\infty T_i$ is ∞-supercyclic.*

Proof. Proceeding as above, let $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$ and for $h \in \mathcal{H}$, let h_i denote the ith component of h. Suppose also that $\{V_j\}_{j=1}^\infty$ is a basis of open sets for $\mathcal{H}$. Let

$$G_j = \{h \in \mathcal{H} : \exists n \ge 1,\ \alpha_1, \dots, \alpha_n \in \mathbb{C} \text{ and } k \ge 0 \text{ such that } (\alpha_1 T_1^k h_1, \dots, \alpha_n T_n^k h_n, 0, 0, \dots) \in V_j\}.$$

Then G_j is an open set, which, by hypothesis, is dense in $\mathcal{H}$. Thus the Baire Category Theorem implies that $\bigcap_{j=1}^\infty G_j$ is a dense G_δ.

Let $(x_1, \dots, x_n, \dots) \in \bigcap_{j=1}^\infty G_j$ where $x_i \in \mathcal{H}_i$, and let $e_n = (0, \dots, 0, x_n, 0, 0, \dots)$ be the vector with x_n in the nth coordinate and zeros elsewhere. Then let $\mathcal{M}$ be the closed linear span of $\{e_n : n \ge 1\}$. It follows from the fact that $(x_1, \dots, x_n, \dots) \in \bigcap_{j=1}^\infty G_j$ that the orbit of $\mathcal{M}$ under $T := \bigoplus_{i=1}^\infty T_i$ is dense in $\mathcal{H}$.

Furthermore, for any $n \ge 1$, if one considers $E_n := \bigcup_{k=0}^n T^k(\mathcal{M})$, then the projection of E_n onto any of the coordinate spaces $\mathcal{H}_i$ is simply a finite union of one-dimensional subspaces, hence cannot be dense. Thus E_n is not dense for any n.

To see that $\mathcal{M}$ contains no invariant subspace for T, we show that for every nonzero $x \in \mathcal{M}$, $Tx \notin \mathcal{M}$. If $x \in \mathcal{M} \setminus \{0\}$, then $x = \sum_n \alpha_n e_n / \|e_n\|$ where $\{\alpha_n\} \in \ell^2$. However, if $Tx \in \mathcal{M}$, then x_n is necessarily an eigenvector for T_n whenever $\alpha_n \neq 0$. But since each x_n is a supercyclic vector for T_n, none of the x_n's are eigenvectors. Thus, $\alpha_n = 0$ for every n, hence $x = 0$. It follows that T is ∞-supercyclic. ∎

We now present our main result for constructing examples of n-supercyclic operators. If $\{n_k\}$ is a sequence of integers satisfying $n_k \to \infty$, then we say that an operator T satisfies the *supercyclicity criterion with respect to* $\{n_k\}$ if T satisfies the hypothesis of Theorem 2.1 with respect to the sequence $\{n_k\}$. Recall that Theorems 2.1, 2.2, and 2.3 are all equivalent (see [1]).

THEOREM 3.7. *If $T_1, \ldots, T_n$, $1 \le n < \infty$, are supercyclic operators and each satisfy the supercyclicity criterion with respect to the same sequence $\{n_k\}$, then $\bigoplus_{k=1}^{n} T_k$ is n-supercyclic.*

Proof. Suppose that $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \{1, \ldots, n\}$. For $1 \le i \le n$, since T_i satisfies the supercyclicity criterion (Theorem 2.1) there are dense sets $X^{(i)}$ and $Y^{(i)}$, and functions $B_{n_k}^{(i)}$ satisfying the conditions in Theorem 2.1 (we have used the superscript (i) to denote the dependence of the sets on the operator T_i).

We want to apply Proposition 3.5. So, suppose that U, V are two nonempty open sets in $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$. Let $a = (a_1, \ldots, a_n) \in U$ and $b = (b_1, \ldots, b_n) \in V$. Choose an $\varepsilon > 0$ such that the closed balls $\operatorname{cl} B(a, \varepsilon) \subseteq U$ and $\operatorname{cl} B(b, \varepsilon) \subseteq V$.

Now for each $1 \le i \le n$, since $X^{(i)}$ and $Y^{(i)}$ are dense we may choose $x_i \in X^{(i)}$ such that $\|x_i - a_i\| < \varepsilon/(2n)$ and $y_i \in Y^{(i)}$ such that $\|y_i - b_i\| < \varepsilon/(4n)$.

So, for each $i \in \{1, \ldots, n\}$ we have chosen vectors x_i, y_i. By the supercyclicity criterion, we have, for $i \in \{1, \ldots, n\}$,

$$\|T_i^{n_k} x_i\| \cdot \|B_{n_k}^{(i)} y_i\| \to 0 \quad \text{and} \quad T_i^{n_k} B_{n_k}^{(i)} y_i \to y_i \quad \text{as } k \to \infty.$$

So choose k large enough such that

$$\|T_i^{n_k} x_i\| \cdot \|B_{n_k}^{(i)} y_i\| < \frac{\varepsilon^2}{4n^2} \quad \text{and} \quad \|T_i^{n_k} B_{n_k}^{(i)} y_i - y_i\| < \frac{\varepsilon}{4n}$$

for all $i \in \{1, \ldots, n\}$. Now that k has been chosen, for each $i \in \{1, \ldots, n\}$ let

$$\alpha_i = \frac{2n}{\varepsilon} \|B_{n_k}^{(i)} y_i\| \quad \text{and} \quad z_i = x_i + \frac{1}{\alpha_i} B_{n_k}^{(i)} y_i.$$

Hence, $z = (z_1, \ldots, z_n) \in \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$.

We claim that $z \in U$ and $(\alpha_1 T_1^{n_k} z_1, \ldots, \alpha_n T_n^{n_k} z_n) \in V$. Notice that

$$\begin{aligned}\|z-a\| &= \Big(\sum_{i=1}^{n}\|z_i-a_i\|^2\Big)^{1/2} \le \sum_{i=1}^{n}\|z_i-a_i\| \\ &= \sum_{i=1}^{n}\Big\|x_i+\frac{1}{\alpha_i}B^{(i)}_{n_k}y_i-a_i\Big\| \\ &\le \sum_{i=1}^{n}\Big(\|x_i-a_i\|+\Big\|\frac{1}{\alpha_i}B^{(i)}_{n_k}y_i\Big\|\Big) \le \sum_{i=1}^{n}\Big(\frac{\varepsilon}{2n}+\frac{\varepsilon}{2n}\Big)=\sum_{i=1}^{n}\frac{\varepsilon}{n}=\varepsilon.\end{aligned}$$

Thus $\|z-a\|\le\varepsilon$, so $z\in\operatorname{cl}B(a,\varepsilon)\subseteq U$.

Similarly, let $w=(\alpha_1T_1^{n_k}z_1,\ldots,\alpha_nT_n^{n_k}z_n)$. We want $w\in V$, so we will show that $\|w-b\|\le\varepsilon$. To see this, notice that

$$\begin{aligned}\|w-b\| &= \Big(\sum_{i=1}^{n}\|\alpha_iT_i^{n_k}z_i-b_i\|^2\Big)^{1/2} \le \sum_{i=1}^{n}\|\alpha_iT_i^{n_k}z_i-b_i\| \\ &= \sum_{i=1}^{n}\|\alpha_iT_i^{n_k}x_i+T_i^{n_k}B^{(i)}_{n_k}y_i-b_i\| \le \sum_{i=1}^{n}(\|\alpha_iT_i^{n_k}x_i\|+\|T_i^{n_k}B^{(i)}_{n_k}y_i-b_i\|) \\ &\le \sum_{i=1}^{n}(\|\alpha_iT_i^{n_k}x_i\|+\|T_i^{n_k}B^{(i)}_{n_k}y_i-y_i\|+\|y_i-b_i\|) \le \sum_{i=1}^{n}\Big(\|\alpha_iT_i^{n_k}x_i\|+\frac{\varepsilon}{2n}\Big) \\ &= \frac{\varepsilon}{2}+\sum_{i=1}^{n}\frac{2n}{\varepsilon}\|B^{(i)}_{n_k}y_i\|\cdot\|T_i^{n_k}x_i\| \le \frac{\varepsilon}{2}+\sum_{i=1}^{n}\Big(\frac{2n}{\varepsilon}\cdot\frac{\varepsilon^2}{4n^2}\Big)=\frac{\varepsilon}{2}+\sum_{i=1}^{n}\frac{\varepsilon}{2n}=\varepsilon.\end{aligned}$$

Thus, $\|w-b\|\le\varepsilon$, hence $w\in\operatorname{cl}B(b,\varepsilon)\subseteq V$. It now follows from Proposition 3.5 that $T_1\oplus\ldots\oplus T_n$ is n-supercyclic. ∎

THEOREM 3.8. *If $\{T_k : 1\le k<\infty\}$ is a uniformly bounded sequence of supercyclic operators and they all satisfy the supercyclicity criterion with respect to the same sequence $\{n_j\}$, then $\bigoplus_{k=1}^{\infty}T_k$ is ∞-supercyclic.*

Proof. Suppose that $T_k\in\mathcal{B}(\mathcal{H}_k)$ and set $\mathcal{H}=\bigoplus_{k=1}^{\infty}\mathcal{H}_k$. We wish to apply Proposition 3.6. So, suppose that U,V are two nonempty open sets in $\mathcal{H}$. Let $a=(a_1,\ldots,a_n,0,0,\ldots)\in U$ and $b=(b_1,\ldots,b_n,0,0,\ldots)\in V$. Choose an $\varepsilon>0$ such that the closed balls $\operatorname{cl}B(a,\varepsilon)\subseteq U$ and $\operatorname{cl}B(b,\varepsilon)\subseteq V$. Since $T_1,\ldots,T_n$ each satisfy the supercyclicity criteria, it follows from the proof of Theorem 3.7 that there exists $(x_1,\ldots,x_n)\in\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_n$, a $k\ge0$, and scalars $\{\alpha_1,\ldots,\alpha_n\}$ such that $\|(x_1,\ldots,x_n)-(a_1,\ldots,a_n)\|\le\varepsilon$ and $\|(\alpha_1T_1^kx_1,\ldots,\alpha_nT_n^kx_n)-(b_1,\ldots,b_n)\|\le\varepsilon$. It follows that $(x_1,\ldots,x_n,0,0,\ldots)\in U$ and $(\alpha_1T_1^kx_1,\ldots,\alpha_nT_n^kx_n,0,0,\ldots)\in V$. Thus, by Proposition 3.6, $\bigoplus_{k=1}^{\infty}T_k$ is ∞-supercyclic. ∎

COROLLARY 3.9. *Suppose $T\in\mathcal{B}(\mathcal{H})$ and $\{\mathcal{M}_1,\ldots,\mathcal{M}_n\}$ are closed invariant subspaces for T such that $\mathcal{M}_1+\ldots+\mathcal{M}_n$ is dense in $\mathcal{H}$. If there is a sequence of integers $n_j\to\infty$ such that for each $k\in\{1,\ldots,n\}$, $T|\mathcal{M}_k$ satisfies the supercyclicity criterion with respect to $\{n_j\}$, then T is n-supercyclic.*

Proof. For $k \in \{1, \dots, n\}$, let $T_k = T|\mathcal{M}_k$. Since each T_k satisfies the supercyclicity criterion with respect to $\{n_j\}$, Theorem 3.7 implies that $\bigoplus_{k=1}^n T_k$ is n-supercyclic. Now define $A : \bigoplus_{k=1}^n \mathcal{M}_k \to \mathcal{H}$ by $A(x_1, \dots, x_n) = x_1 + \dots + x_n$. By assumption A has dense range. Since $A(\bigoplus_{k=1}^n T_k) = TA$ it follows that T is n-supercyclic. ∎

THEOREM 3.10. *Let $n \in \mathbb{N} \cup \{\infty\}$. If $\{S_i : 1 \le i \le n\}$ is a uniformly bounded collection of pure hyponormal operators such that for each i, S_i^* is supercyclic, then $\bigoplus_{i=1}^n S_i^*$ is n-supercyclic.*

Proof. In [5] it was shown that if S is a pure hyponormal operator and S^* is supercyclic, then S^* satisfies the supercyclicity criteria (either Theorem 2.2 or 2.3) with respect to the sequence $n_k = k$. ∎

EXAMPLE 3.11. (1) If $\{S_1, \dots, S_n\}$ are irreducible cyclic subnormal operators, then $(\bigoplus_{k=1}^n S_k)^*$ is n-supercyclic.

(2) If $\{T_1, \dots, T_n\}$ are coanalytic Toeplitz operators, with nonconstant symbols, on $H^2(\mathbb{D})$, then $\bigoplus_{k=1}^n T_k$ is n-supercyclic.

If Γ is a circle, then let $\operatorname{int} \Gamma$ and $\operatorname{ext} \Gamma$ denote the interior and exterior of Γ.

EXAMPLE 3.12. Suppose that $S_n = M_z$ on $L^2_{\mathrm{a}}(\Delta_n)$ where $\{\Delta_n : n \ge 1\}$ is a bounded sequence of open disks. Let $S = \bigoplus_{n=1}^\infty S_n$.

If there are n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \ge 0$, $1 \le i \le n$, such that for every $k \ge 1$, $\operatorname{cl} \Delta_k$ intersects $\bigcup_{i=1}^n \Gamma_i$, then S^* is $2n$-supercyclic.

Proof. Let $\mathcal{I}_k$ ($\mathcal{I}$ for inner) be the collection of all the disks Δ_i such that $\operatorname{cl} \Delta_i \cap \Gamma_k \ne \emptyset$ and $\Delta_i \cap \operatorname{int} \Gamma_k \ne \emptyset$. Also, let $\mathcal{O}_k$ ($\mathcal{O}$ for outer) be the collection of all the disks Δ_i such that $\operatorname{cl} \Delta_i \cap \Gamma_k \ne \emptyset$ and $\Delta_i \cap \operatorname{ext} \Gamma_k \ne \emptyset$. Let $A_k := \bigoplus_{i \in \mathcal{I}_k} S_i$ and $B_k := \bigoplus_{i \in \mathcal{O}_k} S_i$. Then A_k^* is supercyclic (Theorem 2.6) and B_k^* is supercyclic (Theorem 2.6), thus by Theorem 3.10, $T := (\bigoplus_{k=1}^n A_k^*) \oplus (\bigoplus_{k=1}^n B_k^*)$ is $2n$-supercyclic. Now, either $S^* \cong T$ or S^* may be obtained by restricting T to a reducing subspace; either way S^* is $2n$-supercyclic. ∎

4. Necessary conditions. The main result in this section is an analogue of the "Circle Theorem" for supercyclic operators (see [5]). That is, we prove that if T is n-supercyclic, $n \in \mathbb{N}$, then there are n circles centered at the origin and every "part of the spectrum" of T^* must intersect one of these circles. It follows that every component of the spectrum of T must intersect one of these circles.

THEOREM 4.1. *If T is n-supercyclic, then there are n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \ge 0$, $i = 1, \dots, n$, such that for every invariant subspace $\mathcal{M}$ of T^*, we have $\sigma(T^*|\mathcal{M}) \cap \bigcup_{i=1}^n \Gamma_i \ne \emptyset$. In particular, every component of the spectrum of T intersects $\bigcup_{i=1}^n \Gamma_i$.*

This necessary condition allows us to construct operators that are n-supercyclic, but not $(n-1)$-supercyclic. A few preliminary results are needed in order to prove Theorem 4.1.

LEMMA 4.2. *Let T be a bounded linear operator on Hilbert space $\mathcal{H}$.*

(a) *If $\sigma(T) \subseteq \{z : |z| < \beta\}$, then there exists a constant $C > 0$ such that $\|T^n x\| \le C\beta^n \|x\|$ for every $x \in \mathcal{H}$.*

(b) *If $\sigma(T) \subseteq \{z : |z| > \alpha\}$, then there exists a constant $C > 0$ such that $\|T^n x\| \ge C\alpha^n \|x\|$ for every $x \in \mathcal{H}$.*

The above estimates are well known and follow either from estimates from the Riesz Functional Calculus or from the spectral radius formula.

PROPOSITION 4.3. *Suppose that $T_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, and $\sigma(T_1) \subseteq \{z : |z| < \varrho\}$ and $\sigma(T_2) \subseteq \{z : |z| > \varrho\}$ for some $\varrho > 0$. Suppose also that $T = T_1 \oplus T_2$ is n-supercyclic on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and that $\mathcal{M} \subseteq \mathcal{H}$ is an n-dimensional subspace with dense orbit under T. If $\{v_i : 1 \le i \le n\} \subseteq \mathcal{M}$ is a linearly independent set and if $v_i = (x_i, y_i)$ with $x_i \in \mathcal{H}_1$ and $y_i \in \mathcal{H}_2$, then $y_1, \ldots, y_n$ are linearly dependent.*

Proof. If $y_1, \ldots, y_n$ are linearly independent, then $\operatorname{span}\{y_1, \ldots, y_n\}$ is an n-dimensional subspace of $\mathcal{H}_2$. Hence the linear map $\mathbb{C}^n \to \operatorname{span}\{y_1, \ldots, y_n\}$ that sends $(a_1, \ldots, a_n)$ to $a_1 y_1 + \ldots + a_n y_n$ is invertible. Thus by the continuity of the inverse of this map, there is an $\varepsilon > 0$ such that $\|a_1 y_1 + \ldots + a_n y_n\| \ge \varepsilon(|a_1|^2 + \ldots + |a_n|^2)^{1/2}$ for any $(a_1, \ldots, a_n) \in \mathbb{C}^n$.

Now let $(e, f) \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be any vector. Since the orbit of $\mathcal{M}$ under T is dense, there exist $n_k \to \infty$ and $a_{i,k} \in \mathbb{C}$ such that $T^{n_k}(a_{1,k} v_1 + \ldots + a_{n,k} v_n) \to (e, f)$ as $k \to \infty$. Thus

$$\begin{bmatrix} T_1^{n_k}(a_{1,k} x_1 + \ldots + a_{n,k} x_n) \\ T_2^{n_k}(a_{1,k} y_1 + \ldots + a_{n,k} y_n) \end{bmatrix} \to \begin{bmatrix} e \\ f \end{bmatrix} \quad \text{as } k \to \infty.$$

Now, let $C = \max\{\|x_1\|, \ldots, \|x_n\|\}$ and let $K, K' > 0$ be such that $\|T_1^j x\| \le K\varrho^j \|x\|$ and $\|T_2^j y\| \ge (1/K')\varrho^j \|y\|$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, and $j \ge 0$ (see Lemma 4.2). Given this, we have the following:

$$\begin{aligned}
\|e\| &= \lim_{k\to\infty} \|T_1^{n_k}(a_{1,k} x_1 + \ldots + a_{n,k} x_n)\| \le \liminf_{k\to\infty} K\varrho^{n_k} \|(a_{1,k} x_1 + \ldots + a_{n,k} x_n)\| \\
&\le \liminf_{k\to\infty} KC\varrho^{n_k}(|a_{1,k}| + \ldots + |a_{n,k}|) \\
&\le \liminf_{k\to\infty} nKC\varrho^{n_k}(|a_{1,k}|^2 + \ldots + |a_{n,k}|^2)^{1/2} \\
&\le \liminf_{k\to\infty} n\varrho^{n_k} KC(1/\varepsilon)\|a_{1,k} y_1 + \ldots + a_{n,k} y_n\| \\
&\le \liminf_{k\to\infty} nCKK'(1/\varepsilon)\|T_2^{n_k}(a_{1,k} y_1 + \ldots + a_{n,k} y_n)\| = nCKK'(1/\varepsilon)\|f\|.
\end{aligned}$$

Thus, $\|e\| \le nCKK'(1/\varepsilon)\|f\|$. Now since $nCKK'(1/\varepsilon)$ depends only on x_i and y_i and not on e and f, it follows that not every vector (e, f) is in the closure of the orbit of $\mathcal{M}$ (any vector where $\|e\|$ is significantly larger than $\|f\|$ will not be). However this contradicts our assumption that $\mathcal{M}$ has dense orbit under T. It follows then that $y_1, \ldots, y_n$ are linearly dependent. ∎

If we assume that T_i is cohyponormal, then we may relax the assumptions that the spectra are contained in open sets and allow them to intersect the circle $\{z : |z| = \varrho\}$. That is, we have the following result as well.

PROPOSITION 4.4. *Suppose that $T_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, are cohyponormal operators and $\sigma(T_1) \subseteq \{z : |z| \le \varrho\}$ and $\sigma(T_2) \subseteq \{z : |z| \ge \varrho\}$ for some $\varrho > 0$. Suppose also that $T = T_1 \oplus T_2$ is n-supercyclic on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and that $\mathcal{M} \subseteq \mathcal{H}$ is an n-dimensional subspace with dense orbit under T. If $\{v_i : 1 \le i \le n\} \subseteq \mathcal{M}$ is a linearly independent set and if $v_i = (x_i, y_i)$ with $x_i \in \mathcal{H}_1$ and $y_i \in \mathcal{H}_2$, then $y_1, \ldots, y_n$ are linearly dependent.*

Proof. The proof is similar to Proposition 4.3 except that estimates given by Lemma 4.2 now follow immediately from the hyponormality of T^* (in fact $K = K' = 1$). ∎

PROPOSITION 4.5. *Suppose that $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $0 \le i \le n < \infty$ and let $T = \bigoplus_{k=0}^{n} T_k$. If there exist numbers $0 < \varrho_1 < \ldots < \varrho_n$ such that*

$$\begin{aligned} &\sigma(T_0) \subseteq \{z : |z| < \varrho_1\}, \\ &\sigma(T_i) \subseteq \{z : \varrho_i < |z| < \varrho_{i+1}\} \quad \textit{for } 1 \le i \le n-1, \\ &\sigma(T_n) \subseteq \{z : |z| > \varrho_n\}, \end{aligned}$$

then T is not n-supercyclic.

Proof. We shall proceed by induction. The result is known for $n = 1$ (see [5]). So our induction hypothesis is that whenever T is the direct sum of n operators whose spectra can be separated by $n - 1$ circles centered at the origin, then T is not $(n - 1)$-supercyclic.

Suppose that $T = \bigoplus_{k=0}^{n} T_k$ and the spectra of T_k are separated by n circles as stated in the proposition. Suppose also that T is n-supercyclic and that $\mathcal{M}$ is an n-dimensional subspace of $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ with dense orbit under T. Let $v_1, \ldots, v_n$ be a basis for $\mathcal{M}$.

Let $S := T_1 \oplus \ldots \oplus T_n$, so $T = T_0 \oplus S$ on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{K}$ where $\mathcal{K} := \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$. Write the basis $\{v_1, \ldots, v_n\}$ as vectors in $\mathcal{H}_0 \oplus \mathcal{K}$, say $v_i = (x_i, y_i)$ where $x_i \in \mathcal{H}_0$ and $y_i \in \mathcal{K}$.

It then follows from Proposition 4.3 that $y_1, \ldots, y_n$ are linearly dependent. Hence $\operatorname{span}\{y_1, \ldots, y_n\}$ is at most $(n-1)$-dimensional. Thus, it easily follows that S is $(n - 1)$-supercyclic. However, since S is the direct sum of n operators with spectra separated by $n - 1$ circles, the induction hypothesis says that S cannot be $(n-1)$-supercyclic. Hence we have a contradiction. Thus T is not n-supercyclic, and now the result follows by induction. ∎

The next lemma says that for a collection $\mathcal{C}$ of compact connected sets in the complex plane, either there exist n circles such that every set in $\mathcal{C}$ intersects at least one of the circles or there are $n+1$ sets in $\mathcal{C}$ that can be separated by n circles. The $n=1$ case was used in [5, Theorem 6.2] for establishing the existence of the supercyclicity circle for supercyclic operators.

Lemma 4.6. *If $\mathcal{C}$ is a collection of compact connected sets in $\mathbb{C}$, then either there exist n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \geq 0$, such that for every $K \in \mathcal{C}$, $K \cap \bigcup_{i=1}^n \Gamma_i \neq \emptyset$, or there exist $n+1$ sets $\{K_0, \ldots, K_n\} \subseteq \mathcal{C}$ and n radii $\varrho_i > 0$, $i = 1, \ldots, n$, such that $K_0 \subseteq \{z : |z| < \varrho_1\}$, $K_i \subseteq \{z : \varrho_i < |z| < \varrho_{i+1}\}$ for $1 \leq i \leq n-1$, and $K_n \subseteq \{z : |z| > \varrho_n\}$.*

The author would like to thank Paul Bourdon for the following proof; although it is not constructive, it is shorter and cleaner than the author's original inductive proof. We leave the $n=1$ case to the reader.

Proof. Let $f(z) = |z|$. Consider the collection $\mathcal{C}'$ of compact intervals (possibly degenerating to a point) given by $\mathcal{C}' = \{f(K) : K \in \mathcal{C}\}$. Now, for a given $n \geq 1$, the problem may be stated as follows: either (a) there exist n numbers $\{r_1, \ldots, r_n\}$ such that every interval in $\mathcal{C}'$ intersects $\{r_1, \ldots, r_n\}$ or (b) there are $n+1$ pairwise disjoint intervals in $\mathcal{C}'$. Now consider those subsets $\mathcal{A} \subseteq \mathcal{C}'$ with the property that any two intervals in $\mathcal{A}$ have nonempty intersection. If $\mathcal{X}$ is the set of all such subsets $\mathcal{A}$, partially ordered by inclusion, then Zorn's Lemma implies that $\mathcal{X}$ must have maximal elements. If $\mathcal{X}$ has at most n maximal elements, then condition (a) above holds, otherwise condition (b) holds. ∎

Remark. In Lemma 4.6, it may be necessary to have $r_i = 0$ for some value of i. For example if $\mathcal{C}$ includes the closed disks $\{B(0, 1/n) : n \geq 1\}$, then we must have $r_i = 0$ for some i.

Proof of Theorem 4.1. Let $S = T^*$. We need to show that there are n circles $\Gamma_1, \ldots, \Gamma_n$ such that for every invariant subspace $\mathcal{M}$ of S, $\sigma(S|\mathcal{M}) \cap \bigcup_{i=1}^n \Gamma_i \neq \emptyset$. We know that S^* is n-supercyclic, and hence $(S|\mathcal{M})^*$ is also n-supercyclic for every invariant subspace $\mathcal{M}$ of S.

Suppose the result is not true. Then by Lemma 4.6 (with $\mathcal{C}$ being the collection of all connected components of sets of the form $\sigma(S|\mathcal{M})$ where $\mathcal{M}$ is an invariant subspace for S, [5, Theorem 6.2]) there exist $n+1$ invariant subspaces $\mathcal{M}_0, \ldots, \mathcal{M}_n$ for S and n positive radii $\varrho_1, \ldots, \varrho_n$ such that $\sigma(S|\mathcal{M}_0) \subseteq \{z : |z| < \varrho_1\}$, $\sigma(S|\mathcal{M}_i) \subseteq \{z : \varrho_i < |z| < \varrho_{i+1}\}$ for $1 \leq i \leq n-1$, and $\sigma(S|\mathcal{M}_n) \subseteq \{z : |z| > \varrho_n\}$.

It follows from Lemma 4.2 that $\mathcal{M}_i \cap (\sum_{j\neq i} \mathcal{M}_j) = (0)$. Thus the operator $A : \mathcal{M}_0 \oplus \ldots \oplus \mathcal{M}_n \to \mathcal{H}$ defined as $A(x_1, \ldots, x_n) = x_1 + \ldots + x_n$ is one-to-one. Furthermore, if we set $S_i = S|\mathcal{M}_i$, then A intertwines $S_0 \oplus \ldots \oplus S_n$ with S. Thus A^* intertwines S^* with $S_0^* \oplus \ldots \oplus S_n^*$. Since A^* has dense range and S^* is n-supercyclic, it follows that $S_0^* \oplus \ldots \oplus S_n^*$ is also n-supercyclic. However, this contradicts Proposition 4.5. Hence the n circles exist. ∎

Example 4.7. Let $n \in \mathbb{Z}^+$. If $\{\Delta_k : 1 \le k \le n\}$ is the collection of open disks where Δ_k is centered at k and has radius $1/4$, then letting $S_k = M_z$ on $L^2_{\mathrm{a}}(\Delta_k)$, and $S = \bigoplus_{k=1}^n S_k$, we see that S^* is n-supercyclic, but not $(n-1)$-supercyclic.

Or, if for each $n \ge 1$, we have a disk Δ_n centered at $1/n$ and choose the radii small enough so that the disks are pairwise disjoint, then $S = \bigoplus_{n=1}^\infty S_n$ is subnormal and S^* is ∞-supercyclic, but not n-supercyclic for any $n < \infty$.

Using Proposition 4.4 instead of Proposition 4.3 we may prove a sharper version of Proposition 4.5 for cohyponormal operators (where strict inequalities are replaced by inequalities, in the separations) and thus also a sharper version of Theorem 4.1 for cohyponormal operators; this will be illustrated in Example 4.8.

We want to discuss compact sets intersecting a circle $\Gamma = \{z : |z| = r\}$ and one "side" of Γ. The possible sides are the inside, outside or either. Thus suppose Γ is a circle with a side assigned to it. For a compact set $K \subseteq \mathbb{C}$, when we say that "K intersects Γ and the assigned side of Γ" we mean that $K \cap \Gamma \neq \emptyset$ and one of the following holds: If the assigned side is inside, then $K \cap \{z : |z| < r\} \neq \emptyset$; if the assigned side is outside, then $K \cap \{z : |z| > r\} \neq \emptyset$. Finally, if the assigned side to Γ is "either", then K must intersect either the inside or the outside of Γ (or both).

Finally, if we have a collection of circles each with an assigned side and we are counting the number of sides, then a circle whose assigned side is "inside" or "outside" contributes one side to the total number of sides counted. But a circle whose assigned side is "either" contributes two sides to the total number of sides. With this terminology we give the following example.

Example 4.8. If $\{\Delta_j : 1 \le j < \infty\}$ is a bounded collection of open disks, $S_j = M_z$ on $L^2_{\mathrm{a}}(\Delta_j)$, and $S = \bigoplus_{j=1}^\infty S_j$, then S^* is n-supercyclic if and only if there are k circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \ge 0$, $i = 1, \ldots, k$, $k \le n$, and a side associated with each circle (inside, outside or either) such that for every disk Δ_j, there exists a circle Γ_i so that $\mathrm{cl}\,\Delta_j$ intersects Γ_i and the assigned side of Γ_i. Furthermore, the total number of sides is n.

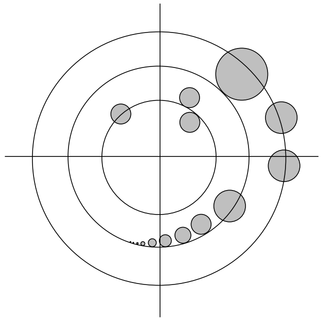

Fig. 1. In this figure we illustrate an arrangement of disks such that S^* is 4-supercyclic. Notice that the assigned side of the innermost circle is "either", furthermore this circle's radius is unique. For the middle circle, there is one disk intersecting both sides of the circle as well as an infinite number of disks internally tangent whose radii are going to zero. The assigned side of the middle circle is "inside" and the circle and its assigned side are uniquely determined (the one disk that is externally tangent to the middle circle is not important since it also hits the outside circle). As for the largest circle, we have a choice, the circle itself is not unique and neither is its assigned side. Its assigned side could be "inside" or it could be "outside". We do not want the assigned side of the outer circle to be "either" since we want to minimize the total number of sides.

Remark. In the previous example there is nothing special about the Bergman space or the fact that Δ_j is an open disk. In fact we could let $S_j = M_z$ on any Hilbert space $\mathcal{H}$ of analytic functions on a bounded open connected set Δ_j, provided that the norm equals the spectral radius, $\|M_z\| = r(M_z)$.

In [2] Bourdon proves that a hyponormal operator cannot be supercyclic. Here we give the first step towards such a result for n-supercyclicity by proving that normal operators cannot be n-supercyclic.

Theorem 4.9. *If $n \in \mathbb{N}$, then a normal operator on an infinite-dimensional space cannot be n-supercyclic.*

Proof. First consider the special case of a unitary operator U. Suppose that $\mathcal{M}$ is an n-dimensional subspace with dense orbit under U. We may suppose that U has the form M_ϕ on $L^2(\mu)$ for some measure μ, where $\phi \in L^\infty(\mu)$ and $|\phi(z)| = 1$, μ-a.e. So, for every $g \in L^2(\mu)$ there is a sequence $n_k \to \infty$ and $f_k \in \mathcal{M}$ such that $U^{n_k} f_k \to g$. That is, $\phi^{n_k} f_k \to g$ in $L^2(\mu)$. Hence $\|f_k\| = \|\phi^{n_k} f_k\| \to \|g\|$. Thus, $\{\|f_k\|\}$ is bounded, hence there is a convergent subsequence (because $\mathcal{M}$ is finite-dimensional). Therefore we may assume that $f_k \to h$ for some $h \in \mathcal{M}$. Thus, by passing to a subsequence, for μ-almost every z we have $|g(z)| = \lim |\phi(z)^{n_k} f_k(z)| = \lim |f_k(z)| = |h(z)|$. Hence

(∗) for every $g \in L^2(\mu)$ there is an $h \in \mathcal{M}$ such that $|g| = |h|$ μ-almost everywhere.

Now, since $\dim L^2(\mu) \geq n+1$ we can find $n+1$ disjoint sets with positive μ-measure, say $\{F_1, \ldots, F_{n+1}\}$. Let $g_i = \chi_{F_i}$ be the characteristic function of F_i. By (∗), for each i, there is an $h_i \in \mathcal{M}$ such that $|g_i| = |h_i|$. Thus, each h_i is nonzero and the functions $\{h_i : 1 \leq i \leq n+1\}$ are pairwise orthogonal, since they are carried by disjoint sets. Thus $\mathcal{M}$ is at least $(n+1)$-dimensional, a contradiction. Hence U is not n-supercyclic.

General case: Suppose N is normal and n-supercyclic. We may assume that $N = M_\phi$ on $L^2(\mu)$. Since N is n-supercyclic, by Theorem 4.1 the essential range of ϕ is a subset of n circles centered at the origin. Since N must have dense range, each circle has a positive radius. If μ_k is μ restricted to the inverse image (under ϕ) of the kth circle, then we may write $N = \bigoplus N_k$ where $N_k = M_\phi$ on $L^2(\mu_k)$. Clearly, each N_k is also n-supercyclic, as is $c_k N_k$ for any nonzero scalar c_k. Choose $c_k = \|N_k\|^{-1}$ and let $U = \bigoplus c_k N_k$. Then U is a unitary operator that is, by Proposition 3.2, n^2-supercyclic; a contradiction. ∎

The result above naturally leads to the question of whether or not a subnormal operator S can be n-supercyclic. With this question in mind, Theorem 4.1 naturally raises the following question about subnormal operators.

QUESTION 4.10. If S is a pure subnormal operator, $a \in \sigma(S^*)$ and $\varepsilon > 0$, then does there exist an invariant subspace $\mathcal{M}$ for S^* such that $\sigma(S^*|\mathcal{M}) \subseteq B(a, \varepsilon)$?

If the above question has an affirmative answer for a subnormal operator S, then S cannot be n-supercyclic for any $n < \infty$. In particular, if the eigenvalues for S^* have nonempty interior, then S cannot be n-supercyclic. This applies, for example, to show that any multiplication operator on a Hilbert space of analytic functions cannot be n-supercyclic (since its adjoint has lots of eigenvalues).

5. Cohyponormal operators & local spectral theory. In this section we will give a local spectral theory condition for an operator to be 2-supercyclic. In particular this applies nicely to adjoints of subnormal and hyponormal operators. Everything here could be done in a Banach space, but we are mainly interested in Hilbert space operators. Thus for simplicity $\mathcal{H}$ will continue to denote a separable complex Hilbert space.

If $T \in \mathcal{B}(\mathcal{H})$ and $K \subseteq \mathbb{C}$ is a compact set, then define (as in [9, p. 32], the glocal analytic subspaces) $\mathcal{H}_T(K)$ to be all those vectors $x \in \mathcal{H}$ such that there exists an analytic function $f : \mathbb{C} \setminus K \to \mathcal{H}$ such that $(T - z)f(z) = x$

for all $z \notin K$. Now for an open set $U \subseteq \mathbb{C}$, define $\mathcal{H}_T(U) = \bigcup\{\mathcal{H}_T(K) : K$ is compact and $K \subseteq U\}$.

An operator $T \in \mathcal{B}(\mathcal{H})$ has the *decomposition property* (δ) provided that for any open cover $\{U_1, \ldots, U_n\}$ of $\sigma(T)$, the space $\mathcal{H}$ can be written as the sum of the analytic subspaces: $\mathcal{H} = \mathcal{H}_T(\operatorname{cl} U_1) + \ldots + \mathcal{H}_T(\operatorname{cl} U_n)$.

It is known that $T \in \mathcal{B}(\mathcal{H})$ has the decomposition property (δ) if and only T^* is subdecomposable [9, Theorem 2.4.4, Theorem 2.5.18]. In particular, the adjoint of every subnormal and every hyponormal operator has the decomposition property (δ) (see [10]).

Here is the main theorem of this section.

Theorem 5.1. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ has the decomposition property (δ). If there exists a number $\varrho > 0$ such that for every $\varepsilon > 0$,*

$$\mathcal{H}_T(\{z \in \mathbb{C} : \varrho - \varepsilon < |z| < \varrho + \varepsilon \text{ and } |z| \neq \varrho\})$$

is dense in $\mathcal{H}$, then T is 2-supercyclic.

The above theorem should be contrasted and compared with the following result that appears in Feldman, Miller and Miller [5].

Theorem 5.2. *Let $T \in \mathcal{B}(\mathcal{H})$.*

(1) *If $\mathcal{H}_T(\{z : |z| < 1\})$ and $\mathcal{H}_T(\{z : |z| > 1\})$ are dense, then T is hypercyclic.*

(2) *If there exists a number $\varrho \geq 0$ satisfying either:*

(a) *for every $\varepsilon > 0$, $\mathcal{H}_T(\{z \in \mathbb{C} : \varrho < |z| < \varrho + \varepsilon\})$ is dense, or*

(b) *for every $\varepsilon > 0$, $\mathcal{H}_T(\{z \in \mathbb{C} : \varrho - \varepsilon < |z| < \varrho\})$ is dense,*

then T is supercyclic.

If (a) holds, we say T is *ϱ-outer* or *outer with respect* to $\Gamma_\varrho := \{z : |z| = \varrho\}$, and if (b) holds, then we say T is *ϱ-inner* or *inner with respect* to Γ_ϱ. The proof of Theorem 5.2 follows by verifying that the hypercyclicity criterion and the inner/outer supercyclicity criteria hold for the sequence $n_k = k$.

The next two results are needed for the proof of Theorem 5.1.

Proposition 5.3. *If $T \in \mathcal{B}(\mathcal{H})$, $x \in \mathcal{H}$ and $V \subseteq \mathbb{C}$ is open and there exists an analytic function $f : V \to \mathcal{H}$ such that $x = (T - z)f(z)$ for all $z \in V$, then for each $z_0 \in V$, there exists an analytic function $g : V \to \mathcal{H}$ such that $f(z_0) = (T - z)g(z)$ for all $z \in V$.*

Thus, if $x \in \mathcal{H}_T(U)$, then there exists a compact set $K \subseteq U$ and an analytic function $f : \mathbb{C} \setminus K \to \mathcal{H}_T(U)$ such that $x = (T - z)f(z)$ for $z \notin K$.

The important point of the above proposition, for us, is the last sentence, and in particular the fact that f takes values in $\mathcal{H}_T(U)$. For the proof see [9, Lemma 1.2.14].

The next theorem is one reason property (δ) is important. For an operator $S \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$ we will use $\sigma_S(x)$ for the local spectrum of S at x [9, p. 16].

THEOREM 5.4 [9, Proposition 2.5.14]. *If $T \in \mathcal{B}(\mathcal{H})$ has the decomposition property (δ), then for every open set $U \subseteq \mathbb{C}$ we have $\mathcal{H}_T(U)^{\perp} = \mathcal{H}_{T^*}(\mathbb{C} \setminus U) = \{x \in \mathcal{H} : \sigma_{T^*}(x) \subseteq \mathbb{C} \setminus U\}$.*

Proof of Theorem 5.1. Suppose that $\mathcal{M} := \operatorname{cl} \mathcal{H}_T(\{z \in \mathbb{C} : |z| < \varrho\})$ and $\mathcal{N} := \operatorname{cl} \mathcal{H}_T(\{z \in \mathbb{C} : |z| > \varrho\})$. Then $\mathcal{M}$ and $\mathcal{N}$ are closed invariant subspaces for T, $\mathcal{M} + \mathcal{N}$ is dense in $\mathcal{H}$, thus by Corollary 3.9, it suffices to show that $T|\mathcal{M}$ and $T|\mathcal{N}$ each satisfy the supercyclicity criterion with respect to the same sequence $\{n_k\}$. For this, it suffices to verify the conditions in (a) or (b) of Theorem 5.2 for $T|\mathcal{M}$ and $T|\mathcal{N}$. We will do this for $T|\mathcal{M}$, the other case being similar. We must show that for every $\varepsilon > 0$, $\mathcal{M}_T(\{z : \varrho - \varepsilon < |z| < \varrho\})$ is dense in $\mathcal{M}$.

For convenience, let $U_\varepsilon = \{z \in \mathbb{C} : \varrho - \varepsilon < |z| < \varrho\}$. First notice that for each $\varepsilon > 0$, $\mathcal{H}_T(U_\varepsilon)$ is a dense subspace of $\mathcal{M}$. That $\mathcal{H}_T(U_\varepsilon)$ is a subspace of $\mathcal{M}$ is clear. To see that $\mathcal{H}_T(U_\varepsilon)$ is dense in $\mathcal{M}$, suppose that $x \perp \mathcal{H}_T(U_\varepsilon)$; then we must show that $x \perp \mathcal{M}$. Since $x \perp \mathcal{H}_T(U_\varepsilon)$, by Theorem 5.4 either $\sigma_{T^*}(x) \subseteq \{z : |z| \le \varrho - \varepsilon\}$ or $\sigma_{T^*}(x) \subseteq \{z : |z| \ge \varrho\}$. In the first case, by Theorem 5.4, we must have $x \perp \mathcal{H}_T(\{z \in \mathbb{C} : \varrho - \varepsilon < |z| < \varrho + \varepsilon$ and $|z| \neq \varrho\})$, which we are assuming is dense, thus $x = 0$. In the second case, Theorem 5.4 implies that $x \perp \mathcal{M}$. Thus, for every $\varepsilon > 0$, $\mathcal{H}_T(U_\varepsilon)$ is dense in $\mathcal{M}$.

Now, by Proposition 5.3, it follows that $\mathcal{M}_T(U_\varepsilon) = \mathcal{H}_T(U_\varepsilon)$. However we have just shown that this latter set is dense in $\mathcal{M}$. Thus, $T|\mathcal{M}$ satisfies the conditions in Theorem 5.2 for supercyclicity. ∎

COROLLARY 5.5. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ has the decomposition property (δ). If there exists a number $\varrho > 0$ such that for every $\varepsilon > 0$,*

$$\operatorname{span}\{\ker(T - \lambda) : \varrho - \varepsilon < |\lambda| < \varrho + \varepsilon \textit{ and } |\lambda| \neq \varrho\}$$

is dense in $\mathcal{H}$, then T is 2-supercyclic.

Proof. This follows immediately from Theorem 5.1 since $\ker(T - \lambda) \subseteq \mathcal{H}_T(U)$ whenever $\lambda \in U$. ∎

COROLLARY 5.6. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ has the decomposition property (δ) and that none of the local spectra for T^* are contained in a circle centered at the origin. If there exists a circle $\Gamma = \{z : |z| = \varrho\}$, $\varrho > 0$, such that for every nonzero $x \in \mathcal{H}$, $\sigma_{T^*}(x) \cap \Gamma \neq \emptyset$, then T is 2-supercyclic.*

Proof. This also follows easily from Theorems 5.1 and 5.4. ∎

Corollary 5.7. *If S is a pure hyponormal operator and there exists a* $\varrho > 0$ *such that for each* $\varepsilon > 0$, $\operatorname{span}\{\ker(S^* - \lambda) : \varrho - \varepsilon < |\lambda| < \varrho + \varepsilon\}$ *is dense*, *then* S^* *is* 2*-supercyclic.*

Proof. The assumption guarantees that $\mathcal{H}_{S^*}(\{z : \varrho - \varepsilon < |z| < \varrho + \varepsilon\})$ is dense in $\mathcal{H}$. In view of Theorem 5.4, this guarantees that each local spectrum for S intersects the circle $\Gamma := \{z : |z| = \varrho\}$. Since S is a pure hyponormal operator, each of its local spectra must have positive area, thus Corollary 5.6 applies. ∎

Corollary 5.8. *Suppose that S is a pure hyponormal operator and there exists a circle* $\Gamma = \{z : |z| = r\}$, $r > 0$, *such that for every hyperinvariant subspace* $\mathcal{M}$ *of* S, $\sigma(S|\mathcal{M}) \cap \Gamma \neq \emptyset$. *Then* S^* *is* 2*-supercyclic.*

These last two corollaries should be compared with Corollary 2.4 and Theorem 2.6.

6. Final remarks and questions. The author believes that the hyponormal operators whose adjoints are n-supercyclic should be characterized by the same condition that appears in Example 4.8. In particular the following should have an affirmative answer:

Question 6.1. If T is a pure hyponormal operator and if there are n circles centered at the origin with the property that $\sigma(T|\mathcal{M})$ intersects at least one of these circles for every hyperinvariant subspace $\mathcal{M}$ of T, then is T^* $2n$-supercyclic?

One reason for considering n-supercyclicity is because it is related to the general (open) question of whether or not every pure cohyponormal operator is cyclic (see [4]). Since we can prove that certain cohyponormal operators are n-supercyclic, we need to answer the following question.

Question 6.2. If T is an n-supercyclic operator, $n \in \mathbb{N}$, and T^* has no eigenvalues, then is T cyclic?

Question 6.3. If S is a pure subnormal operator, then is S^* ∞-supercyclic?

Question 6.4. Can a pure subnormal (hyponormal) operator be n-supercyclic?

Question 6.5. Does Corollary 5.5 hold for all operators? That is, can we remove the hypothesis that T has property (δ)?

Question 6.6. For $n \geq 2$, is there a bilateral weighted shift that is n-supercyclic and not $(n-1)$-supercyclic? If so, can we characterize the n-supercyclic weighted shifts?

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