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RUC systems in rearrangement invariant spaces

by

P. G. DODDS (Bedford Park), E. M. SEMENOV (Voronezh)
and F. A. SUKOCHEV (Bedford Park)

To Professor Aleksander Pełczyński
on the occasion of his 70th birthday

Abstract. We present necessary and sufficient conditions for a rearrangement invariant function space to have a complete orthonormal uniformly bounded RUC system.

1. Introduction and preliminaries. The familiar *Haar system* is a complete orthonormal system in $L_2[0,1]$ which is an unconditional basis in each space $L_p[0,1]$, $1 < p < \infty$. On the other hand ([KS, Chapter 1]), it is well known that if $p \neq 2$ and $1 < p < \infty$, then the space $L_p[0,1]$ has no orthonormal unconditional basis that is uniformly bounded. In this paper, we study uniformly bounded orthonormal systems in rearrangement invariant Banach function spaces on $[0,1]$ for which the expansion of every element converges for almost all choices of signs. Such systems are said to be *randomly unconditionally convergent* or *RUC systems*. While each unconditional basic sequence in any Banach space is necessarily an RUC system, an RUC system need not be unconditional. For example, while the trigonometric system is not an unconditional basis for any space $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, it does form an RUC system in $L_p[0,1]$, $2 < p$. See, for example, [BKPS, Corollary 1.4 and Remark V following Corollary 2.2].

The principal result of our paper (Theorem 2.8) characterizes those separable rearrangement invariant Banach function spaces E on $[0,1]$ which have the property that each orthonormal uniformly bounded system is necessarily an RUC system. This property is shown to be equivalent to the existence of a complete orthonormal uniformly bounded system which is a complete RUC system in E. In turn, this is shown to be equivalent to the

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validity of the continuous embeddings $G \subseteq E \subseteq L_2[0,1]$. Here G is the "separable" part of the Orlicz space $L_M[0,1]$ defined via the Orlicz function $M(t) = (e^{t^2} - 1)/(e-1)$. It is not without interest to observe that, by a theorem of Rodin and Semenov ([RS], [LT2]), the embedding $G \subseteq E$ is itself equivalent to the assertion that the Khinchin inequalities are valid in E, or alternatively, that the usual Rademacher system $\{r_n\}_{n=1}^{\infty}$ in E is equivalent to the unit vector basis of l_2. Of course, the Rademacher system is itself a uniformly bounded, orthonormal system which is an unconditional basic sequence in every rearrangement invariant space, and is therefore an RUC system. However, the Rademacher system is not complete in $L_2[0,1]$.

We now gather some basic terminology. A Banach space $(E, \|\cdot\|_E)$ of real-valued measurable functions on the interval $[0,1]$ is called a *rearrangement invariant space* if for every $y \in E$ and any measurable function x on $[0,1]$ with $x^* \le y^*$, we have $x \in E$ and $\|x\|_E \le \|y\|_E$. Here, x^* denotes the non-increasing, right-continuous rearrangement of x given by

$$x^*(t) = \inf\{s \ge 0 : m(\{|x| > s\}) \le t\}, \qquad t > 0.$$

For basic properties of rearrangement invariant spaces we refer to the monographs [BS], [KPS], [LT2]. Let us note explicitly that the continuous embeddings

$$L_\infty[0,1] \subseteq E \subseteq L_1[0,1]$$

are valid for any rearrangement invariant Banach function space E on $[0,1]$.

We denote by $[x_n]_{n=1}^{\infty}$ the closed linear span of the sequence $\{x_n\}_{n=1}^{\infty}$ in the Banach space X. The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be *complete* (or *fundamental*) in X if $[x_n]_{n=1}^{\infty} = X$. Following [BKPS], a biorthogonal system (x_j, x_j^*) in $X \times X^*$, where X^* denotes the Banach dual of X, is said to be an *RUC system* in X if for every $x \in [x_n]_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty} r_n(t) x_n^*(x) x_n$ converges for almost all $t \in [0,1]$. Here $\{r_n\}_{n=1}^{\infty}$ denotes the usual Rademacher sequence given by

$$r_n(t) = \operatorname{sign} \sin 2^n \pi t \quad \text{ for } 0 \le t \le 1 \ (n = 1, 2, \ldots).$$

Equivalently, the biorthogonal system (x_j, x_j^*) in the Banach space X is an RUC system if and only if there exists a constant $K > 0$ such that

$$\Big\| \sum_{j=1}^{n} r_j(\cdot) c_j x_j \Big\|_{L_1([0,1],X)} \le K \Big\| \sum_{j=1}^{n} c_j x_j \Big\|_X \tag{1.1}$$

for all scalars $c_1, \ldots, c_n$, $n = 1, 2, \ldots$ It is shown further in [BKPS, Corollary 1.1] that (1.1) is equivalent to saying that there exists a constant $K > 0$ such that

$$\int_0^1 \sup_n \Big\| \sum_{j=1}^{n} x_j^*(x) x_j r_k(s) \Big\|_E \, ds \le K \|x\|_E \tag{1.2}$$

for all $x \in [x_n]_{n=1}^{\infty}$.

2. Main results. We begin this section with the result whose proof is a variant of the arguments in [BKPS, Example 1.3].

Let (Ω, μ) be a probability space and let M be the Orlicz function given by setting

$$M(t) := \frac{e^{t^2} - 1}{e - 1}, \qquad t \in \mathbb{R}.$$

The Orlicz space $L_M(\Omega)$ is the space of all measurable functions f on Ω such that

$$\int_\Omega M(|f(\omega)|/\varrho)\, d\omega < \infty$$

for some $\varrho > 0$, equipped with the norm

$$\|f\|_M = \inf\Big\{\varrho > 0 : \int_\Omega M(|f(\omega)|/\varrho)\, d\omega \le 1\Big\}.$$

Throughout this paper, we shall denote by G the closure of $L_\infty[0,1]$ in the Orlicz space $L_M[0,1]$.

Proposition 2.1. *Let E be a separable rearrangement invariant Banach function space on $[0,1]$. If $G \subseteq E \subseteq L_2$ with continuous embeddings, then each orthonormal uniformly bounded system in E is an RUC system.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a uniformly bounded orthonormal system and set

$$\sup_{n\in\mathbb{N}} \|f_n\|_\infty = C < \infty.$$

Via (1.1) and using the continuity of the embeddings $E \subseteq L_2$ and $L_1([0,1], G) \subseteq L_1([0,1], E)$ it will suffice to show the existence of a constant $K' > 0$ such that

$$\Big\|\sum_{j=1}^n r_j(\cdot) c_j f_j\Big\|_{L_1([0,1],G)} \le K' \Big\|\sum_{j=1}^n c_j f_j\Big\|_2$$

for arbitrary scalars $c_1, \ldots, c_n$ and for $n = 1, 2, \ldots$ Using further the fact that there exists a constant K'' such that

$$\Big\|\sum_{j=1}^n r_j(\cdot) c_j f_j\Big\|_{L_1([0,1],L_M[0,1])} \le K'' \Big\|\sum_{j=1}^n c_j r_j(\cdot) \otimes f_j(\cdot)\Big\|_{L_M([0,1]\times[0,1])}$$

for arbitrary scalars $c_1, \ldots, c_n$ and for $n = 1, 2, \ldots$ (see [S, Proposition 2.4 and Definition 2.1]), it suffices to show that there exists $K > 0$ such that

$$\Big\|\sum_{j=1}^n c_j r_j(\cdot) \otimes f_j(\cdot)\Big\|_{L_M([0,1]\times[0,1])} \le K \Big\|\sum_{j=1}^n c_j f_j\Big\|_2 \tag{2.1}$$

for arbitrary scalars $c_1, \ldots, c_n$ and for $n = 1, 2, \ldots$ To this end, fix $\varrho > Ce(e-1)^{-1/2}$, $n \in \mathbb{N}$, and scalars $c_1, \ldots, c_n$ such that $\sum_{j=1}^n |c_j|^2 \le 1$, and set for brevity

$$\Omega := [0,1] \times [0,1], \qquad w(t,s) := \sum_{j=1}^{n} c_j f_j(t) r_j(s), \quad (t,s) \in \Omega.$$

Using the Khinchin inequality in the form

$$\int_0^1 \Big| \sum_{n=1}^{m} a_n r_n(t) \Big|^{2k} dt \le k^k \Big(\sum_{n=1}^{m} |a_n|^2 \Big)^k$$

for arbitrary scalars $\{a_j\}$ and $k = 1, 2, \ldots$ (see, for example, [Z, Theorem V.8.4] or [LT1, proof of Theorem 2.b.3]) we obtain

$$\begin{aligned}
&\int_\Omega M(|w(t,s)|\varrho^{-1})\,dt\,ds \\
&\quad = \int_0^1 \Big(\int_0^1 M\Big(\Big| \sum_{j=1}^{n} c_j f_j(t) r_j(s) \Big| \varrho^{-1} \Big)\, dt \Big)\, ds \\
&\quad = (e-1)^{-1} \int_0^1 \Big(\int_0^1 \sum_{k=1}^{\infty} (k!)^{-1} \Big| \sum_{j=1}^{n} c_j f_j(t) r_j(s) \Big|^{2k} \varrho^{-2k}\, dt \Big)\, ds \\
&\quad = (e-1)^{-1} \int_0^1 \sum_{k=1}^{\infty} (k!)^{-1} \varrho^{-2k} \Big(\int_0^1 \Big| \sum_{j=1}^{n} c_j f_j(t) r_j(s) \Big|^{2k} ds \Big)\, dt \\
&\quad \le (e-1)^{-1} \int_0^1 \sum_{k=1}^{\infty} (k!)^{-1} \varrho^{-2k} k^k \Big(\sum_{j=1}^{n} |c_j f_j(t)|^2 \Big)^k dt \\
&\quad \le (e-1)^{-1} \int_0^1 \sum_{k=1}^{\infty} (k!)^{-1} \varrho^{-2k} k^k C^{2k} \Big(\sum_{j=1}^{n} |c_j|^2 \Big)^k dt \\
&\quad \le (e-1)^{-1} \sum_{k=1}^{\infty} (k!)^{-1} \varrho^{-2k} k^k C^{2k}.
\end{aligned}$$

From the elementary inequality $k^k/k! \le e^k$, $k \in \mathbb{N}$, it now follows that

$$\begin{aligned}
\int_\Omega M(|x(t,s)|\varrho^{-1})\,dt\,ds &\le (e-1)^{-1} \sum_{k=1}^{\infty} \Big(\frac{C^2 e}{\varrho^2} \Big)^k \\
&= (e-1)^{-1} \frac{C^2 e}{\varrho^2} \Big(1 - \frac{C^2 e}{\varrho^2} \Big)^{-1} < 1.
\end{aligned}$$

By definition of the norm in the Orlicz space $L_M([0,1] \times [0,1])$ it follows that

$$\|w(t,s)\|_{L_M([0,1]\times[0,1])} \le \varrho,$$

and this suffices to complete the proof of (2.1) and of Proposition 1.1. ∎

In what follows, our principal aim is to show that the converse of Proposition 2.1 is valid, that is, if each uniformly bounded orthonormal system in E is an RUC system, then necessarily $G \subseteq E \subseteq L_2$ with continuous embeddings. We shall base our proof of this assertion on the following key technical lemma. We recall that a sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is said to be *semi-normalized* if

$$0 < \liminf_{n\to\infty} \|x_n\|_X \leq \limsup_{n\to\infty} \|x_n\|_X < \infty.$$

LEMMA 2.2. *Let E be a rearrangement invariant Banach function space on $[0,1]$ and let $\{g_n\}_{n=1}^{\infty} \subseteq E$ be an orthonormal sequence which is complete in $L_2[0,1]$, which is a complete RUC system in E and which is bounded in $L_p[0,1]$ for some $p > 2$. If $\{g_n\}_{n=1}^{\infty}$ is semi-normalized and weakly null in E, then there exists a subsequence $\{f_n\}_{n=1}^{\infty} \subseteq \{g_n\}_{n=1}^{\infty}$ and a sequence of signs $\varepsilon_n = \pm 1$, $n = 1, 2, \ldots$, such that*

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \Big\| \sum_{k=1}^{n} \varepsilon_{n(k)} f_{n(k)} \Big\|_E < \infty$$

for every subsequence $\{\varepsilon_{n(k)} f_{n(k)}\}_{k=1}^{\infty} \subseteq \{\varepsilon_n f_n\}_{n=1}^{\infty}$.

The proof of the lemma will be based on the following well known results, which we state for convenience of reference. The first is a generalization to general orthonormal systems of the de Leeuw–Katznelson–Kahane Theorem [LKK], as given in [KS, Theorem 5 of Chapter 9].

PROPOSITION 2.3. *If $\{g_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L_2[0,1]$ which is bounded in $L_p[0,1]$ for some $p > 2$, then for every sequence $\{a_n\}_{n=1}^{\infty} \in l_2$ with $a_n \geq 0$, $n \geq 1$, there exists a continuous function $\phi(t)$, $t \in [0,1]$, such that*

$$|c_n(\phi)| \geq a_n, \qquad n \in \mathbb{N},$$

where

$$c_n(\phi) := \int_0^1 \phi(t) g_n(t)\, dt, \qquad n \in \mathbb{N},$$

and a constant $C > 0$, depending on M and p only, such that

$$\|\phi\|_\infty \leq C \Big(\sum_{n=1}^{\infty} a_n^2 \Big)^{1/2}.$$

The second ingredient that we shall need is due to Brunel and Sucheston [BrS1], [BrS2].

PROPOSITION 2.4. *Let $\{x_j\}_{j=1}^{\infty}$ be a semi-normalized weakly null sequence in a Banach space X, and let $\varepsilon > 0$. Then there exists a subsequence $\{y_j\}_{j=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ and a Banach space $\mathcal{X}$ with a 2-unconditional semi-normalized basis $\{e_j\}_{j=1}^{\infty}$ such that*

(i) $\{e_j\}_{j=1}^{\infty}$ *is isometrically equivalent to all of its subsequences*, *and*

(ii) *for all* k *large enough*, *and any* $k \le j_1 < \ldots < j_{2^k}$, $\{y_{j_i}\}_{i=1}^{2^k}$ *is* 2-*equivalent to* $(e_1, \ldots, e_{2^k})$.

Proof of Lemma 2.2. Since $\{g_n\}_{n=1}^{\infty}$ is semi-normalized and weakly null in E, there exists a subsequence $\{g_{n(k)}\}_{k=1}^{\infty} \subseteq \{g_n\}_{n=1}^{\infty}$ which satisfies the assertion of Proposition 2.4 with $\varepsilon = 1$. For $m = 1, 2, \ldots$, define the sequence $\{a_k^m\}_{k=1}^{\infty} \in l_2$ by setting

$$a_k^m := \begin{cases} 1/\sqrt{m} & \text{if } k = n(j),\ 1 \le j \le m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

It follows from Proposition 2.3 and the continuity of the embedding $L_\infty \subseteq E$ that there exists a constant $C_1 > 0$ and a sequence $\{\phi_m\}_{m=1}^{\infty}$ such that

$$\|\phi_m\|_E \le C_1, \qquad |c_k(\phi_m)| \ge a_k^m \tag{2.3}$$

for all $k, m \in \mathbb{N}$, where $\{c_k(\phi_m)\}_{k=1}^{\infty}$ denotes the sequence of Fourier coefficients of the function ϕ_m with respect to the orthonormal sequence $\{g_k\}_{k=1}^{\infty}$ for all $m \in \mathbb{N}$. Since $\{g_k\}_{k=1}^{\infty}$ is a complete RUC system in E, it follows from (1.2) that there exists a constant C_2 such that

$$\int_0^1 \sup_n \Big\| \sum_{k=1}^{n} c_k(\phi_m) r_k(s) g_k \Big\|_E \, ds \le C_2 \|\phi_m\|_E$$

for all $m \in \mathbb{N}$. Consequently, we obtain

$$\int_0^1 \Big\| \sum_{k=1}^{n(m)} c_k(\phi_m) r_k(s) g_k \Big\|_E \, ds \le C_2 C_1 \tag{2.4}$$

for all $m, n \in \mathbb{N}$. Since the sequence $\{r_n(\cdot) g_n\}_{n=1}^{\infty}$ is a 1-unconditional basic sequence in $L_1([0,1], E)$, it follows from (2.3) and [LT1, Proposition 1.c.7] that

$$\int_0^1 \Big\| \sum_{k=1}^{n(m)} a_k^m r_k(s) g_k \Big\|_E \, ds \le 2 \int_0^1 \Big\| \sum_{k=1}^{n(m)} c_k(\phi_m) r_k(s) g_k \Big\|_E \, ds$$

for all $m \in \mathbb{N}$. From the preceding inequality, together with (2.4) and (2.2) it now follows that

$$\frac{1}{\sqrt{m}} \int_0^1 \Big\| \sum_{k=1}^{m} r_{n(k)}(s) g_{n(k)} \Big\|_E \, ds \le 2 C_2 C_1 \tag{2.5}$$

for all $m \in \mathbb{N}$. If we now set $f_k = g_{n(k)}$, $k \in \mathbb{N}$, then it follows from (2.5) that there exists a sequence $\varepsilon_{m,k} = \pm 1$, $1 \le k \le m$, $m \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{m}} \Big\| \sum_{k=1}^{m} \varepsilon_{m,k} f_k \Big\|_E \, ds \le 2 C_2 C_1 \tag{2.6}$$

for all $m \in \mathbb{N}$. We now construct the desired sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of signs by induction. We set $\varepsilon_j := \varepsilon_{2,j}$, $j = 1, 2$. Suppose that $\varepsilon_1, \dots, \varepsilon_{2^{2^{i-1}}}$, $i \geq 1$, have already been chosen. Using the same notation as in (2.6), we set

$$\varepsilon_j := \varepsilon_{2^{2^i},j}, \qquad j = 2^{2^{i-1}} + 1, 2^{2^{i-1}} + 2, \dots, 2^{2^i},$$

and this completes the construction. To establish the assertion of the lemma, let us first observe that by (2.6),

$$\begin{aligned}(2.7)\quad & \frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=1}^{2^{2^n}} \varepsilon_k f_k\Big\|_E \\ &\leq \frac{1}{2^{2^{n-1}}}\Big(\Big\|\sum_{k=1}^{2^{2^n}} \varepsilon_{2^{2^n},k} f_k\Big\|_E + \Big\|\sum_{k=1}^{2^{2^{n-1}}} \varepsilon_k f_k\Big\|_E + \Big\|\sum_{k=1}^{2^{2^{n-1}}} \varepsilon_{2^{2^n},k} f_k\Big\|_E\Big) \\ &\leq 2C_1C_2 + \frac{2}{2^{2^{n-1}}}\sum_{k=1}^{2^{2^{n-1}}} \|f_k\|_E \\ &\leq 2(C_1C_2 + M)\end{aligned}$$

for every $n \in \mathbb{N}$. Consider now an arbitrary subsequence $\{\varepsilon_{j_k} f_{j_k}\}_{k=1}^{\infty}$ of $\{\varepsilon_k f_k\}_{k=1}^{\infty}$. We have

$$\begin{aligned}(2.8)\quad \frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=1}^{2^{2^n}} \varepsilon_{j_k} f_{j_k}\Big\|_E &\leq \frac{1}{2^{2^{n-1}}}\Big(\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}} \varepsilon_{j_k} f_{j_k}\Big\|_E + \Big\|\sum_{k=1}^{2^{2^{n-1}}} \varepsilon_{j_k} f_{j_k}\Big\|_E\Big) \\ &\leq \frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}} \varepsilon_{j_k} f_{j_k}\Big\|_E + M\end{aligned}$$

for every $n \in \mathbb{N}$. We note further that for $k = 2^{2^{n-1}}$ we have $j_k \geq k$ and $2^{2^n} - k \leq 2^k$. Thus, by our earlier choice of the sequence $\{f_n\}_{n=1}^{\infty}$, for all sufficiently large n, the basic sequence $\{f_{j_i}\}_{i=k}^{2^{2^n}}$ (respectively, the basic sequence $\{f_i\}_{i=k}^{2^{2^n}}$) is 2-equivalent to the 2-unconditional semi-normalized (finite) basic sequence $\{e_i\}_{i=1}^{2^{2^n}-k}$ of a Banach space $\mathcal{X}$. Therefore, the basic sequence $\{f_{j_i}\}_{i=k}^{2^{2^n}}$ is 4-equivalent to the basic sequence $\{f_i\}_{i=k}^{2^{2^n}}$, in particular

$$\frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}} f_{j_k}\Big\|_E \leq \frac{4}{2^{2^{n-1}}}\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}} f_k\Big\|_E.$$

Further, since the sequence $\{e_i\}_{i=1}^{2^{2^n}-k}$ is 2-unconditional, we see that both sequences $\{f_{j_i}\}_{i=k}^{2^{2^n}}$ and $\{f_i\}_{i=k}^{2^{2^n}}$ are 4-unconditional. In particular it follows from the preceding inequality that

$$\frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}}\varepsilon_{j_k}f_{j_k}\Big\|_E\le\frac{4\cdot4\cdot4}{2^{2^{n-1}}}\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}}\varepsilon_k f_k\Big\|_E,$$

for arbitrary "signs" $\varepsilon_i=\pm1$, $i=1,\dots,j_{2^{2^n}}$. Applying (2.7) we further get

$$\begin{aligned}(2.9)\quad \frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=2^{2^{n-1}}}^{2^{2^n}}\varepsilon_{j_k}f_{j_k}\Big\|_E&\le\frac{2^6}{2^{2^{n-1}}}\Big(\Big\|\sum_{k=1}^{2^{2^n}}\varepsilon_k f_k\Big\|_E+\Big\|\sum_{k=1}^{2^{2^{n-1}}}\varepsilon_k f_k\Big\|_E\Big)\\&\le 2^7(C_1C_2+M)+2^6M.\end{aligned}$$

Combining (2.9) and (2.8) we finally get

$$\frac{1}{2^{2^{n-1}}}\Big\|\sum_{k=1}^{2^{2^n}}\varepsilon_{j_k}f_{j_k}\Big\|_E\le 2^7C_1C_2+193M$$

for all sufficiently large n, and this completes the proof of Lemma 2.2. ∎

The following central limit type theorem for orthonormal systems is due to V. Gaposhkin, and follows from [G, Theorems 1.5.4, 1.5.3, 1.5.1].

PROPOSITION 2.5 (see [G]). *If $\{g_n\}_{n=1}^\infty\subseteq L_2[0,1]$ is an orthonormal system which is bounded in $L_p[0,1]$ for some $p>2$, then there exists a subsequence $\{f_n\}_{n=1}^\infty\subseteq\{g_n\}_{n=1}^\infty$ for which the following conditions are satisfied*:

(i) *$\{f_n^2\}_{n=1}^\infty$ converges weakly in $L_1[0,1]$ to a non-negative function $g\in L_1[0,1]$ such that $\int_0^1 g(t)\,dt=1$;*

(ii) *we have*

$$\lim_{n\to\infty}\operatorname{mes}\Big\{t:\frac{1}{\sqrt n}\Big|\sum_{k=1}^n f_n(t)\Big|\ge\tau\Big\}\ge\frac{1}{\sqrt{2\pi}}\int_0^1\int_{\tau/\sqrt{g(t)}}^\infty e^{-u^2/2}\,du\,dt\qquad\forall\tau>0.$$

Finally, we need the following lemma whose proof is easily extracted from the argument of [LT2, Theorem 2.b.4(i)] and is therefore omitted.

LEMMA 2.6. *Let E be a separable rearrangement invariant Banach function space on $[0,1]$ and suppose that the sequence $\{f_n\}_{n=1}^\infty$ satisfies*

$$\lim_{n\to\infty}\operatorname{mes}\Big\{t:\frac{1}{\sqrt n}\Big|\sum_{k=1}^n f_k(t)\Big|\ge\tau\Big\}\ge Ce^{-\tau^2/b},\qquad\tau>0,$$

for some constants $b,C>0$. If

$$\liminf_{n\to\infty}\Big\|\frac{1}{\sqrt n}\sum_{k=1}^n f_k\Big\|_E<\infty$$

then G embeds continuously into E.

We may now state one of the principal results of the paper.

THEOREM 2.7. *Let E be a separable rearrangement invariant Banach function space on $[0,1]$. Let $\{g_n\}_{n=1}^\infty$ be an orthonormal system which is a complete RUC system in E and which is bounded in L_p for some $p>2$.*

(i) *If $\{g_n\}_{n=1}^\infty$ is weakly null in E, then G is continuously embedded in E.*

(ii) *If $\{g_n\}_{n=1}^\infty$ is complete in L_2, then E is continuously embedded in L_2.*

Proof. (i) Since $\{g_n\}_{n=1}^\infty$ is weakly null in E, it follows that $\{g_n\}_{n=1}^\infty$ is bounded in E. Let us observe that $\liminf_{n\to\infty}\|g_n\|_E>0$. In fact, if this is not the case, then it may be assumed, by passing to a subsequence and relabelling if necessary, that $\|g_n\|_1\to 0$, using the fact that E embeds continuously into L_1. Passing to a further subsequence and relabelling if necessary, and applying Egorov's theorem, it may be assumed that there exists a sequence $\{e_n\}_{n=1}^\infty$ of measurable subsets of $[0,1]$ such that $\mathrm{mes}(e_n)\to 0$ and $\|g_n\chi_{[0,1]\setminus e_n}\|_2\to 0$ as $n\to\infty$. If we now observe that

$$\|g_n\chi_{e_n}\|_2\le (\sup_n\|g_n\|_p)\|\chi_{e_n}\|_1^{1/2-1/p}$$

it follows simply that $\|g_n\|_2\to 0$ (since $p>2$) and this contradicts the fact that $\|g_n\|_2=1$, $n=1,2\ldots$ Consequently, the sequence $\{g_n\}_{n=1}^\infty$ is weakly null and semi-normalized in E and it now follows from Lemma 2.2 that there exists a subsequence $\{f_n\}_{n=1}^\infty\subseteq\{g_n\}_{n=1}^\infty$ and a sequence of signs $\varepsilon_n=\pm1$, $n\in\mathbb{N}$, such that

$$\liminf_{n\to\infty}\frac{1}{\sqrt{n}}\Big\|\sum_{k=1}^n\varepsilon_{n(k)}f_{n(k)}\Big\|_E<\infty \tag{2.10}$$

for all subsequences $\{\varepsilon_{n(k)}f_{n(k)}\}_{k=1}^\infty\subseteq\{\varepsilon_nf_n\}_{n=1}^\infty$. Applying Proposition 2.5 to the sequence $\{\varepsilon_nf_n\}_{n=1}^\infty$ and relabelling if necessary we may assume that the sequence

$$\{(\varepsilon_nf_n)^2\}_{n=1}^\infty=\{f_n^2\}_{n=1}^\infty$$

converges weakly in $L_1[0,1]$ to a non-negative function $g\in L_1[0,1]$ and that

$$\lim_{m\to\infty}\mathrm{mes}\Big\{t:\frac{1}{\sqrt{n}}\Big|\sum_{k=1}^n\varepsilon_kf_k(t)\Big|\ge\tau\Big\}=\frac{1}{\sqrt{2\pi}}\int_0^1\int_{\tau/\sqrt{g(t)}}^\infty e^{-u^2/2}\,du\,dt\qquad\forall\tau>0.$$

Since $g\ge 0$ and $\|g\|_1=1$ there exists a measurable set $A\subseteq[0,1]$ and constants $b,c>0$ such that

$$g(t)>b\quad\forall t\in A,\quad\text{and}\quad\mathrm{mes}(A)=c>0.$$

Further,

$$\int_0^1 \int_{\tau/\sqrt{g(t)}}^{\infty} e^{-u^2/2}\,du\,dt \ge \int_A \int_{\tau/\sqrt{b}}^{\infty} e^{-u^2/2}\,du\,dt$$
$$\ge c \int_{\tau/\sqrt{b}}^{\infty} e^{-u^2/2}\,du \ge ce^{-(\tau/\sqrt{b}+1)^2/2} \ge ce^{-\tau^2/b-1}.$$

Consequently,

$$\lim_{m\to\infty} \operatorname{mes}\Big\{t : \frac{1}{\sqrt{n}}\Big|\sum_{k=1}^{n} \varepsilon_k f_k(t)\Big| \ge \tau\Big\} \ge \frac{c}{e\sqrt{2\pi}}\, e^{-\tau^2/b} \qquad \forall \tau > 0.$$

The assertion of (i) now follows from the estimate given by (2.10) and Lemma 2.6.

(ii) It is sufficient to show that $E \subseteq L_2$. If it is not the case, then since $\{g_k\}_{k=1}^{\infty} \subseteq L_2[0,1]$ is complete in $L_2[0,1]$, there exists $x \in E$ such that

$$\sum_{k=1}^{\infty} c_k^2(x) = \infty. \tag{2.11}$$

Via (1.2), there exists a constant $C_2 > 0$ such that

$$\int_0^1 \sup_n \Big\|\sum_{k=1}^{n} c_k(x) r_k(s) g_k\Big\|_E\, ds \le C_2\|x\|_E.$$

Consequently (see e.g. [VTC, Ch. V, Proposition 5.1]),

$$\int_0^1 \Big\|\sum_{k=1}^{\infty} c_k(x) r_k(s) g_k\Big\|_E\, ds \le 2C_2\|x\|_E.$$

Using the Khinchin inequality and the fact that E embeds continuously into $L_1[0,1]$, we obtain

$$\int_0^1 \Big(\sum_{k=1}^{\infty} (c_k(x) g_k(t))^2\Big)^{1/2} dt \le \sqrt{2}\int_0^1 \int_0^1 \Big|\sum_{k=1}^{\infty} c_k(x) r_k(x) g_k(t)\Big|\, ds\, dt$$
$$\le \sqrt{2}\int_0^1 \Big\|\sum_{k=1}^{\infty} c_k(x) r_k(s) g_k\Big\|_E\, ds$$
$$\le 2\sqrt{2}\, C_2\|x\|_E.$$

Consequently, for every $\varepsilon > 0$, there exists a measurable subset $e_\varepsilon \subseteq [0,1]$ such that $\operatorname{mes}(e_\varepsilon) \ge 1-\varepsilon$ and the function $\sum_{k=1}^{\infty} c_k^2(x) g_k^2(t)$ is bounded on e_ε. This implies, in particular, that

$$\sum_{k=1}^{\infty} c_k^2(x)\|g_k\chi_{e_\varepsilon}\|_{L_2}^2 = \int_0^1 \chi_{e_\varepsilon}(t)\sum_{k=1}^{\infty} c_k^2(x) g_k^2(t)\,dt < \infty, \tag{2.12}$$

where $\chi_{e_\varepsilon}(\cdot)$ is the indicator function of e_ε. By the Hölder inequality and the given assumptions on the sequence $\{g_k\}_{k=1}^{\infty}$ it follows that for all $k \in \mathbb{N}$,

$$\begin{aligned}\|g_k\chi_{[0,1]\setminus e_\varepsilon}\|_{L_2} &\le \|g_k\|_{L_p}\cdot\|\chi_{[0,1]\setminus e_\varepsilon}\|_{L_1}^{1/2-1/p}\\ &\le M\,\mathrm{mes}([0,1]\setminus e_\varepsilon)^{1/2-1/p} \le M\varepsilon^{1/2-1/p}.\end{aligned}$$

In particular, if

$$\varepsilon := (2M)^{2p/(2-p)},$$

then

$$\|g_k\chi_{[0,1]\setminus e_\varepsilon}\|_{L_2} \le M((2M)^{2p/(2-p)})^{1/2-1/p} = 1/2,$$

whence

$$\|g_k\chi_{e_\varepsilon}\|_{L_2} \ge 1/2, \qquad k \in \mathbb{N}.$$

Combining these estimates with (2.12) we arrive at a contradiction to (2.11). This completes the proof of Theorem 2.7. ∎

We may now state the principal result of the paper, which characterizes those separable rearrangement invariant spaces in which every uniformly bounded orthonormal system is an RUC system.

THEOREM 2.8. *If E is a separable rearrangement invariant Banach function space on* $[0,1]$, *then the following statements are equivalent*:

(i) *each orthonormal uniformly bounded system is an RUC system in* E;

(ii) *there exists a complete orthonormal uniformly bounded system which is a complete RUC system in* E;

(iii) *there exists a complete orthonormal system which is a weakly null and complete RUC system in E and which is bounded in L_p for some* $p > 2$;

(iv) *the continuous embeddings $G \subseteq E \subseteq L_2$ hold.*

Proof. The implication (i)⇒(ii) follows from the fact that the trigonometric system is a complete system in any separable rearrangement invariant space (see [K, Theorem I.2.11]). The implication (ii)⇒(iii) is clear since any orthonormal uniformly bounded system in E is automatically weakly null in E. The implication (iii)⇒(iv) is a consequence of Theorem 2.7, and the implication (iv)⇒(i) is simply the assertion of Proposition 2.1. ∎

3. Concluding remarks. (i) The assumption $p > 2$ in Theorem 2.8(iii) is essential and may not be replaced with the assumption $p \ge 2$. Consider, for example, the familiar (complete) orthonormal Haar system (see e.g. [KPS, Chapter II.9.3]) and set $E = L_p[0,1]$, $1 < p < 2$. It is well known that if the Boyd indices of a rearrangement invariant space (see [LT2, Definition 2.b.1]) are non-trivial, then the Haar system is an unconditional basis in this space (see [LT2, Theorem 2.c.6]). Consequently, the Haar system is a complete RUC system in every $L_p[0,1]$, $1 < p < 2$, and is clearly bounded in $L_2[0,1]$. However, $L_p[0,1]$ is not continuously embedded in $L_2[0,1]$ if $1 \leq p < 2$. This example shows further that the embedding $E \subseteq L_2$ is not necessary for the existence in E of a complete RUC system. In addition, we note that the embedding $G \subseteq E$ is also not a necessary condition for the existence in E of a complete RUC system. In fact, it is not difficult to construct a separable rearrangement invariant space E which contains a copy of c_0 but which fails to contain G. However, by a result of Wojtaszczyk [W], such a space E necessarily contains a complete RUC system.

(ii) Suppose that a separable rearrangement invariant space E on $[-\pi,\pi]$ satisfies assumption (iv) of Theorem 2.8. Fix an arbitrary element $x \in E$ and suppose that there exists an ordering $\{e_n\}_{n=1}^{\infty}$ of the trigonometric system such that the Fourier series

$$\sum_{n=1}^{\infty} c_n(x)e_n \tag{3.1}$$

(conditionally) converges to x in E. Let $\Sigma_{\{c_n(x)e_n\}}$ be the set of all $y \in E$ such that the series $\sum_{n=1}^{\infty} c_{\pi(n)}(x)e_{\pi(n)}$ converges to y in E for some permutation $\pi : \mathbb{N} \to \mathbb{N}$. Let A be the set of all $z \in E$ such that for each $f \in E^*$ there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ (which may depend on f) such that $f(z) = \sum_{n=1}^{\infty} f(c_{\pi(n)}(x)e_{\pi(n)})$. It is obvious that $\Sigma_{\{c_n(x)e_n\}} \subseteq A$. We say that a conditionally converging series (3.1) *satisfies the Steinitz theorem* if $\Sigma_{\{c_n(x)e_n\}} = A$. It follows from Theorem 2.8 combined with the main result of [Ch] that the series (3.1) satisfies the Steinitz theorem. Since E is separable, the space E^* coincides with the Köthe dual E', thus for every $f \in E^*$ and $z \in E$ we have $f(z) = \int_{-\pi}^{\pi} f(s)z(s)\,ds$. Therefore, if $z \in E$ and if, for every $f \in E'$, the equality $f(z) = \sum_{n=1}^{\infty} c_{\pi(n)}(x)c_{\pi(n)}(f)$ holds for some some permutation $\pi : \mathbb{N} \to \mathbb{N}$, then $z \in \Sigma_{\{c_n(x)e_n\}}$.

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Department of Mathematics and Statistics
School of Informatics and Engineering
The Flinders University of South Australia
Bedford Park, 5042, SA, Australia
E-mail: peter@ist.flinders.edu.au
sukochev@ist.flinders.edu.au

Department of Mathematics
Voronezh State University
Universitetskaya pl. 1
Voronezh 394693, Russia
E-mail: root@func.vsu.ru

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