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An extension of Mazur's theorem on Gateaux differentiability to the class of strongly $\alpha(\cdot)$-paraconvex functions

by

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Abstract. Let $(X, \|\cdot\|)$ be a separable real Banach space. Let f be a real-valued strongly $\alpha(\cdot)$-paraconvex function defined on an open convex subset $\Omega \subset X$, i.e. such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \min[t, (1-t)]\alpha(\|x-y\|).$$

Then there is a dense G_δ-set $A_{\mathrm{G}} \subset \Omega$ such that f is Gateaux differentiable at every point of A_{G}.

Let $(X, \|\cdot\|)$ be a real Banach space. Let f be a real-valued convex continuous function defined on an open convex subset $\Omega \subset X$, i.e.

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1}$$

We recall that a set $B \subset \Omega$ of second Baire category is called *residual* if its complement $\Omega \setminus B$ is of the first Baire category. Mazur (1933) proved that if X is separable, then there is a residual subset A_{G} such that f is Gateaux differentiable on A_{G}. In this note we extend this result to larger (than convex) classes of functions called strongly $\alpha(\cdot)$-paraconvex functions.

Let $\alpha : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{2}$$

Let, as before, $(X, \|\cdot\|)$ be a real Banach space. Let f be a real-valued continuous function defined on an open convex subset $\Omega \subset X$. We say that f is $\alpha(\cdot)$-*paraconvex* if for all $x, y \in \Omega$ and $0 \leq t \leq 1$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \alpha(\|x-y\|). \tag{3}$$

For $\alpha(t) = t^2$ this definition was introduced in Rolewicz (1979a) and the t^2-paraconvex functions were called simply *paraconvex.* In Rolewicz (1979b)

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the notion was extended to the case of $\alpha(t) = t^\gamma$, $1 \le \gamma \le 2$, and the t^γ-paraconvex functions were called *γ-paraconvex.*

We say that f is *strongly $\alpha(\cdot)$-paraconvex* if for all $x, y \in \Omega$ and $0 \le t \le 1$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \min[t, (1-t)]\alpha(\|x-y\|). \tag{4}$$

Of course every strongly $\alpha(\cdot)$-paraconvex function is also $\alpha(\cdot)/2$-paraconvex. The converse is not true and the conditions warranting the existence C_α such that each $\alpha(\cdot)$-paraconvex is strongly $C_\alpha\alpha(\cdot)$-paraconvex can be found in Rolewicz (2000). In particular the function t^γ, $1 < \gamma \le 2$, satisfies these conditions.

The notion of $\alpha(\cdot)$-paraconvex functions can be treated as a uniformization of the notion of approximate convex functions introduced in the papers of Luc, Ngai and Théra (1999), (2000). We recall that a real-valued function f defined on a convex set $\Omega \subset X$ is called *approximate convex* if for any $x_0 \in \Omega$ and $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, x_0)$ such that for x, y with $\|x - x_0\| < \delta$ and $\|y - x_0\| < \delta$ and $0 \le t \le 1$ we have

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon \min[t, (1-t)]\|x-y\|. \tag{5}$$

We say that a real-valued function f defined on a convex set $\Omega \subset X$ is called *uniformly approximate convex* if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that (5) holds for x, y with $\|x - y\| < \delta$.

It is easy to show that a real-valued continuous function f is uniformly approximate convex if and only if there is $\alpha(\cdot)$ satisfying (2) such that f is strongly $\alpha(\cdot)$-paraconvex (Rolewicz (2001b)).

We now recall the notion of directional derivative.

By the *directional derivative* of a continuous function f at a point x_0 in direction h we mean the number

$$d^+f|_{x_0}(h) = \lim_{t\downarrow 0} \frac{f(x_0 + th) - f(x_0)}{t}. \tag{6}$$

It is easy to see that a strongly $\alpha(\cdot)$-paraconvex function has a directional derivative at any point in any direction (Rolewicz (2005)).

We shall show

PROPOSITION 1. *Let Ω be an open convex set in a Banach space X. Let $f : \Omega \to \mathbb{R}$ be an $\alpha(\cdot)$-paraconvex function. Then for any point $x_0 \in \Omega$ the directional derivative $d^+f|_{x_0}(h)$ is a sublinear* (*i.e. positively homogeneous and subadditive*) *function of the direction h.*

Proof. Positive homogeneity is trivial. Now we shall show subadditivity. Indeed, since f is $\alpha(\cdot)$-paraconvex, for $h_1, h_2 \in X$ and sufficiently small t we have

$$\frac{f(x_0 + t\frac{h_1+h_2}{2}) - f(x_0)}{t} \le \frac{1}{2}\frac{f(x_0+th_1)-f(x_0)}{t} + \frac{1}{2}\frac{f(x_0+th_2)-f(x_0)}{t} + \frac{\alpha(t\|h_1-h_2\|)}{t}.$$

Thus multiplying by 2 and letting $t \to 0$, by (2) and positive homogeneity of $d^+f|_{x_0}(h)$ we get the triangle inequality

$$d^+f|_{x_0}(h_1+h_2) \le d^+f|_{x_0}(h_1) + d^+f|_{x_0}(h_2). \ \blacksquare$$

It is easy to observe that a sublinear function is linear if and only if it is homogeneous, i.e. $p(-h) = -p(h)$.

Recall that a strongly $\alpha(\cdot)$-paraconvex function is always locally Lipschitz (Rolewicz (2000)). Basing on this fact it is not difficult to prove that $d^+f|_{x_0}(h)$ is also a locally Lipschitz function.

Any continuous linear functional $x^* \in X^*$ such that $x^*(h) \le d^+f|_{x_0}$ is called an *approximate subgradient* of f at x_0 (see Ioffe (1984), (1986), (1989), (1990), (2000), Mordukhovich (1976), (1980), (1988)). The set of all approximate subgradients of f at x_0 will be called the *approximate subdifferential* of f at x_0 and denoted, as in the classical case, by $\partial f|_{x_0}$.

It is easy to see that if $\partial f|_{x_0}$ consists of one functional, $\partial f|_{x_0} = \{x^*\}$, then x^* is a continuous linear functional. Since in this case $\partial f|_{x_0}(-h) = -\partial f|_{x_0}(h)$, the function f has Gateaux differential at x_0, i.e. the limit $\lim_{t\to 0}(f(x_0+th)-f(x_0))/t$ exists and is equal to $x^*(h)$.

A linear functional $x^* \in X^*$ such that

$$f(x+h) - f(x) \ge x^*(h) - \alpha(\|h\|) \tag{7}$$

is called a *uniform approximate subgradient* of f at x with modulus $\alpha(\cdot)$ (or briefly an $\alpha(\cdot)$-*subgradient* of f at x). The set of all $\alpha(\cdot)$-subgradients of f at x will be called the $\alpha(\cdot)$-*subdifferential* of f at x and denoted by $\partial_\alpha f|_x$.

The relation between $\alpha(\cdot)$-subdifferentials and directional subdifferentials for strongly $\alpha(\cdot)$-paraconvex function is given by

PROPOSITION 2 (Rolewicz (2001)). *Let Ω be an open convex set in a Banach space X. Let $f : \Omega \to \mathbb{R}$ be a strongly $\alpha(\cdot)$-paraconvex function. Then its $\alpha(\cdot)$-subdifferential is equal to the directional subdifferential*, $\partial_\alpha f|_x = \partial f|_x$.

As a consequence we obtain:

COROLLARY 3. *Let Ω be an open convex set in a Banach space X. Let $f : \Omega \to \mathbb{R}$ be a strongly $\alpha(\cdot)$-paraconvex function. Then f is Gateaux differentiable at x_0 if and only if its $\alpha(\cdot)$-subdifferential at x_0 consists of one functional*, $\partial_\alpha f|_{x_0} = \{x_0^*\}$.

Basing on this fact we are able to prove the following extension of the classical Mazur theorem (Mazur (1933)):

THEOREM 4. *Let Ω be an open convex set in a separable Banach space X. Let $f : \Omega \to \mathbb{R}$ be a strongly $\alpha(\cdot)$-paraconvex function. Then there is a dense G_δ-set $A_{\mathrm{G}} \subset \Omega$ such that f is Gateaux differentiable at every point of A_{G}.*

The proof is based on the following

LEMMA 5. *Let Ω be an open convex set in a Banach space X. Let $f : \Omega \to \mathbb{R}$ be a strongly $\alpha(\cdot)$-paraconvex function. Then the multifunction $\partial_\alpha f|_x : X \to 2^{X^*}$ is upper semicontinuous from X with the norm topology into X^* with the weak* topology. In other words, if $x_n \to x$ and $x_n^* \in \partial_\alpha f|_{x_n}$ is weak*-convergent to x_0^* then $x_0^* \in \partial_\alpha f|_{x_0}$.*

Proof. Since f is locally Lipschitz, the $\alpha(\cdot)$-subdifferentials $\partial_\alpha f|_{x_n}$ are uniformly bounded, i.e. there is $M > 0$ such that $\|z^*\| \le M$ for any $z^* \in \bigcup_n \partial_\alpha f|_{x_n}$. Thus

$$\begin{aligned}(8)\qquad |x_n^*(x_n) - x_0^*(x_0)| &\le |x_n^*(x_n) - x_n^*(x_0)| + |x_n^*(x_0) - x_0^*(x_0)| \\ &\le M\|x_n - x_0\| + |x_n^*(x_0) - x_0^*(x_0)| \to 0.\end{aligned}$$

Take now an arbitrary $z \in X$. Then

$$\begin{aligned}(9)\quad \langle x_0^*, z - x_0\rangle &= \lim_{t\to\infty} \langle x_n^*, z - x_n\rangle \le \lim_{t\to\infty} [f(z) - f(x_n) - \alpha(\|x_n - z\|)] \\ &= f(z) - f(x_0) - \alpha(\|x_0 - z\|),\end{aligned}$$

i.e. $x_0^* \in \partial_\alpha f|_{x_0}$. ∎

Proof of Theorem 4. Let $\{r_n\}$ be a dense set in the unit ball of X. Let $A_{m,n}$, $n, m = 1, 2, \ldots$, denote the set of $x \in \Omega$ such that there are $x^*, y^* \in \partial_\alpha f|_x$ such that

$$(10)\qquad \langle x^* - y^*, r_n\rangle \ge 1/m.$$

By Corollary 3 and the density of $\{r_n\}$ in the unit ball we see that f is Gateaux differentiable at x_0 if and only $x_0 \notin \bigcup_{n,m=1}^\infty A_{n,m}$.

We shall show that for any n, m the sets $A_{n,m}$ are closed. Indeed, let $\{x_n\}$ be a sequence of elements of $A_{n,m}$ tending to $x_0 \in \Omega$. By the definition of $A_{m,n}$ there are $x_n^*, y_n^* \in \partial_\alpha f|_{x_n}$ such that

$$(11)\qquad \langle x_n^* - y_n^*, r_n\rangle \ge 1/m.$$

The space X is separable. Thus closed balls are weak*-compact. Therefore we can find subsequences $\{x_{n_k}^*\}, \{y_{n_k}^*\}$ weak*-convergent to x_0^*, y_0^* respectively. By Lemma 5, $x_0^*, y_0^* \in \partial_\alpha f|_{x_0}$. Passing to the limit in (11) we get

$$\langle x_0^* - y_0^*, r_n\rangle \ge 1/m$$

and by the definition $x_0 \in A_{n,m}$.

Next observe that the sets $A_{n,m}$ are nowhere dense. Indeed, suppose to the contrary that there is an open set $U \subset \Omega$ such that $U \subset \overline{A}_{n,m} = A_{n,m}$. Take any $\widehat{x} \in U$ and take a line $L_n(\widehat{x}) = \{\widehat{x} + tr_n \mid -\infty < t < \infty\}$. The function f restricted to $L_n(\widehat{x}) \cap \Omega$ is strongly $\alpha(\cdot)$-paraconvex. Thus it is Fréchet differentiable on a residual set (Rolewicz (2002)). Therefore we obtain a contradiction with the fact that $U \subset A_{n,m}$.

Since the sets $A_{n,m}$ are nowhere dense and closed the function f is Gateaux differentiable on a dense G_δ-set. ∎

There are non-separable Banach spaces $C(T)$ in which the norms are not Gateaux differentiable at any point (Coban and Kenderov (1985)). Phelps (1989) showed that the function $p(x) = \limsup_n |x_n|$ defined on the space ℓ^∞ has this property. There is, however, a class of non-separable Banach spaces in which every convex function is Gateaux differentiable on a dense G_δ-set. It is the class of weakly compactly generated spaces (Phelps (1989)). We recall that a Banach space X is *weakly compactly generated* if there is a weakly compact set $K \subset X$ whose linear span is dense in X. Thus there is a natural question:

PROBLEM 5. *Let X be a weakly compactly generated Banach space*, *and let $\Omega \subset X$ be a convex open set. Let $f : \Omega \to \mathbb{R}$ be a strongly $\alpha(\cdot)$-paraconvex function. Is f Gateaux differentiable on a dense G_δ-set?*

References

M. Coban and P. S. Kenderov (1985), *Dense Gateaux differentiability of the sup-norm in $C(T)$ and the topological properties of T*, C. R. Bulg. Acad. Sci. 38, 1603–1604.

A. D. Ioffe (1984), *Approximate subdifferentials and applications I*, Trans. Amer. Math. Soc. 281, 389–416.

A. D. Ioffe (1986), *Approximate subdifferentials and applications II*, Mathematika 33, 111–128.

A. D. Ioffe (1989), *Approximate subdifferentials and applications III*, ibid. 36, 1–38.

A. D. Ioffe (1990), *Proximal analysis and approximate subdifferentials*, J. London Math. Soc. 41, 175–192.

A. D. Ioffe (2000), *Metric regularity and subdifferential calculus*, Uspekhi Mat. Nauk 55, no. 3, 104–162 (in Russian).

D. T. Luc, H. V. Ngai and M. Théra (1999), *On ε-convexity and ε-monotonicity*, in: Calculus of Variations and Differential Equations, A. Ioffe, S. Reich and I. Shapiro (eds.), Chapman & Hall, 82–100.

D. T. Luc, H. V. Ngai and M. Théra (2000), *Approximate convex functions*, J. Nonlinear Convex Anal. 1, 155–176.

S. Mazur (1933), *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4, 70–84.

B. S. Mordukhovich (1976), *Maximum principle in the optimal control problems with non-smooth constraints*, Prikl. Mat. Mekh. 40, 1014–1023 (in Russian).

B. S. Mordukhovich (1980), *Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems*, Dokl. Akad. Nauk SSSR 254, 1072–1076 (in Russian): English transl.: Soviet Math. Dokl. 22, 526–530.

B. S. Mordukhovich (1988), *Approximation Methods in Problems of Optimization and Control*, Nauka, Moscow (in Russian).

R. R. Phelps (1989), *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer.

S. Rolewicz (1979a), *On paraconvex multifunctions*, Oper. Research Verf. (Methods Oper. Res.) 31, 540–546.

S. Rolewicz (1979b), *On γ-paraconvex multifunctions*, Math. Japonica 24, 293–300.

S. Rolewicz (2000), *On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions*, Control Cybernet. 29, 367–377.

S. Rolewicz (2001a), *On the coincidence of some subdifferentials in the class of $\alpha(\cdot)$-paraconvex functions*, Optimization 50, 353–360.

S. Rolewicz (2001b), *On uniformly approximate convex and strongly $\alpha(\cdot)$-paraconvex functions*, Control Cybernet. 30, 323–330.

S. Rolewicz (2002), *$\alpha(\cdot)$-monotone multifunctions and differentiability of strongly $\alpha(\cdot)$-paraconvex functions*, ibid. 31, 601–619.

S. Rolewicz (2005), *Paraconvex analysis*, ibid. 34, 951–965.

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