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Bounded elements and spectrum in Banach quasi *-algebras

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Abstract. A normal Banach quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ has a distinguished Banach *-algebra $\mathfrak{X}_b$ consisting of bounded elements of $\mathfrak{X}$. The latter *-algebra is shown to coincide with the set of elements of $\mathfrak{X}$ having finite spectral radius. If the family $\mathcal{P}(\mathfrak{X})$ of bounded invariant positive sesquilinear forms on $\mathfrak{X}$ contains sufficiently many elements then the Banach *-algebra of bounded elements can be characterized via a C^*-seminorm defined by the elements of $\mathcal{P}(\mathfrak{X})$.

1. Introduction. A *quasi *-algebra* [15] is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{X}$, and $\mathfrak{X}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$.

Quasi *-algebras were introduced by Lassner [11, 12] with the purpose of providing a reasonable mathematical environment for properly dealing with the thermodynamical limit of local observables of certain quantum statistical models that did not fit into the set-up developed by Haag and Kastler [10]. For this purpose, of course, a topological structure with sufficiently many reasonable properties is needed; in other terms, *locally convex* quasi *-algebras have to be considered [1, 18]. The simplest way to construct such an object consists in taking the completion of a locally convex *-algebra $(\mathfrak{A}_0, \tau)$ where the multiplication is separately but not jointly continuous. Of particular interest is, of course, the case where τ is a norm topology. This situation has however received so far a rather limited attention, in spite of the fact that it covers very familiar examples such as L^p-spaces (both commutative and non-commutative). Some results in this direction have been obtained for the so called *CQ^*-algebras* in a series of papers [3]–[7], [19]–[21].

In this paper we consider the more general case where $(\mathfrak{X}, \mathfrak{A}_0)$ is a *Banach quasi *-algebra*. This means, roughly speaking, that $\mathfrak{X}$ is a Banach space whose norm $\|\cdot\|$ has certain coupling properties related to the *partial* multiplication of $(\mathfrak{X}, \mathfrak{A}_0)$. In Section 2 we study the set $\mathfrak{X}_b$ of *bounded*

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elements of $\mathfrak{X}$, i.e. elements whose associated multiplication operators are bounded linear maps in $\mathfrak{X}$. Then we focus our attention on the class of normal Banach quasi *-algebras: they are characterized by the fact that $\mathfrak{X}_b$ is a Banach *-algebra. If $(\mathfrak{X}, \mathfrak{A}_0)$ is normal, the Banach *-algebra $\mathfrak{X}_b$ turns out to be useful for defining a notion of *spectrum* of an element $x \in \mathfrak{X}$, which enjoys properties analogous to the spectrum of an element of a Banach *-algebra.

In Section 3 we discuss some properties of the family of bounded positive sesquilinear forms on $\mathfrak{X}$ with certain *invariance* properties and, starting from them, we construct two seminorms $\mathfrak{p}$, $\mathfrak{q}$ that emulate the Gel'fand–Naĭmark seminorm on a Banach *-algebra (but $\mathfrak{q}$ is only defined on a domain $D(\mathfrak{q}) \subseteq \mathfrak{X}$; it is actually an *unbounded* C^*-seminorm in the sense of [2]). These seminorms are then used to derive some properties of the spectrum of an element $x \in \mathfrak{X}$, under the assumption that the class $\mathcal{P}(\mathfrak{X})$ of bounded invariant positive sesquilinear forms is rich enough. The outcome is that, in this case, $D(\mathfrak{q})$ exactly equals the *-algebra of bounded elements of $\mathfrak{X}$ (or, equivalently, the set of elements of $\mathfrak{X}$ that have finite spectral radius). Furthermore, it is shown that $(\mathfrak{X}, \mathfrak{A}_0)$ admits a faithful *-representation π and that $D(\mathfrak{q})$ also coincides with the set of elements whose image under π is a bounded operator.

2. Banach quasi *-algebras

2.1. *Basic definitions*

Definition 2.1. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a quasi *-algebra. $(\mathfrak{X}, \mathfrak{A}_0)$ is called a *Banach quasi *-algebra* if a norm $\|\cdot\|$ is defined on $\mathfrak{X}$ with the properties:

(i) $(\mathfrak{X}, \|\cdot\|)$ is a Banach space;
(ii) $\|x^*\| = \|x\|, \ \forall x \in \mathfrak{X}$;
(iii) $\mathfrak{A}_0$ is dense in $\mathfrak{X}$;
(iv) for each $a \in \mathfrak{A}_0$, the map $R_a : x \in \mathfrak{X} \mapsto xa \in \mathfrak{X}$ is continuous in $\mathfrak{X}$.

The continuity of the involution implies that

(iv′) for each $a \in \mathfrak{A}_0$, the map $L_a : x \in \mathfrak{X} \mapsto ax \in \mathfrak{X}$ is continuous in $\mathfrak{X}$.

The *unit* of $(\mathfrak{X}, \mathfrak{A}_0)$ is an element $e \in \mathfrak{A}_0$ such that $xe = ex = x$ for every $x \in \mathfrak{X}$. If $(\mathfrak{X}, \mathfrak{A}_0)$ is a Banach quasi*-algebra with unit e, we will assume (without loss of generality) that $\|e\| = 1$. If $(\mathfrak{X}, \mathfrak{A}_0)$ has no unit, it can always be embedded in a Banach quasi *-algebra with unit e in a standard fashion.

In what follows, we will always assume that if $xa = 0$ for every $a \in \mathfrak{A}_0$, then $x = 0$ (of course, this is automatically true if $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit).

If $(\mathfrak{X}, \mathfrak{A}_0)$ is a Banach quasi*-algebra a norm topology can be defined on $\mathfrak{A}_0$ in the following way. Define

$$\|a\|_L = \sup_{\|x\|\leq 1} \|R_a x\| = \sup_{\|x\|\leq 1} \|xa\|$$

and

$$\|a\|_R = \sup_{\|x\|\leq 1} \|L_a x\| = \sup_{\|x\|\leq 1} \|ax\|$$

and finally

$$\|a\|_0 = \max\{\|a\|, \|a\|_L, \|a\|_R\}.$$

Then

PROPOSITION 2.2. *$(\mathfrak{A}_0, \|\cdot\|_0)$ is a normed *-algebra. Moreover*

$$\|ab\| \leq \|a\|\,\|b\|_0, \qquad \|ba\| \leq \|a\|\,\|b\|_0, \qquad \forall a, b \in \mathfrak{A}_0.$$

The above statements follow immediately from the corresponding properties of algebras of bounded operators on a normed space. The two inequalities come directly from the definitions.

Clearly, $\|b\| \leq \|b\|_0$ for each $b \in \mathfrak{A}_0$.

DEFINITION 2.3. A Banach quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is called a *BQ^*-algebra* if $(\mathfrak{A}_0, \|\cdot\|_0)$ is a Banach *-algebra, and a *proper CQ^*-algebra* if $(\mathfrak{A}_0, \|\cdot\|_0)$ is a C^*-algebra.

2.1.1. *Examples*

EXAMPLE 2.4 (*Banach function spaces*). Many Banach function spaces provide examples of Banach quasi *-algebras since they often contain a dense *-algebra of functions. For instance, if $I = [0,1]$ then $(L^p(I), C(I))$, where $C(I)$ denotes the C^*-algebra of all continuous functions on I and $p \geq 1$, is a Banach quasi *-algebra (more precisely a proper CQ^*-algebra). Similarly $(L^p(\mathbb{R}), C_0^0(\mathbb{R}))$ is a Banach quasi *-algebra without unit (here $C_0^0(\mathbb{R})$ is the *-algebra of continuous functions in $\mathbb{R}$ with compact support). Other examples are easily found among Sobolev spaces, Besov spaces etc.

EXAMPLE 2.5 (*Non-commutative L^p-spaces*). Let $\mathfrak{M}$ be a von Neumann algebra and τ a normal semifinite faithful trace [17] on $\mathfrak{M}$. Then the completion of the *-ideal

$$\mathcal{J}_p = \{X \in \mathfrak{M} : \tau(|X|^p) < \infty\}$$

with respect to the norm

$$\|X\|_p = \tau(|X|^p)^{1/p}, \qquad X \in \mathfrak{M},$$

is usually called $L^p(\tau)$ [13, 16] and is a Banach space consisting of operators affiliated with $\mathfrak{M}$. Then $(L^p(\tau), \mathcal{J}_p)$ is a Banach quasi *-algebra (without unit). If τ is a finite trace then $(L^p(\tau), \mathfrak{M})$ is a BQ^*-algebra.

Example 2.6 (*Hilbert algebras*). A *Hilbert algebra* [14, Section 11.7] is a *-algebra $\mathfrak{A}_0$ which is also a pre-Hilbert space with inner product $\langle\cdot\,|\,\cdot\rangle$ such that

(i) the map $b \mapsto ab$ is continuous with respect to the norm defined by the inner product;
(ii) $\langle ab\,|\,c\rangle = \langle b\,|\,a^*c\rangle$ for all $a, b, c \in \mathfrak{A}_0$;
(iii) $\langle a\,|\,b\rangle = \langle b^*\,|\,a^*\rangle$ for all $a, b \in \mathfrak{A}_0$;
(iv) $\mathfrak{A}_0^2$ is total in $\mathfrak{A}_0$.

Let $\mathcal{H}$ denote the Hilbert space which is the completion of $\mathfrak{A}_0$ with respect to the norm defined by the inner product. The involution of $\mathfrak{A}_0$ extends to the whole of $\mathcal{H}$, since (iii) implies that * is isometric. Then $(\mathcal{H}, \mathfrak{A}_0)$ is a Banach quasi *-algebra.

2.2. *Bounded elements*

Definition 2.7. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra and $x \in \mathfrak{X}$. We say that x is *left bounded* if there exists $\gamma_x > 0$ such that

$$\|xa\| \le \gamma_x \|a\|, \qquad \forall a \in \mathfrak{A}_0.$$

The set of all left bounded elements of $\mathfrak{X}$ is denoted by $\mathfrak{X}_{\blacktriangleright}$. Analogously, we say that x is *right bounded* if there exists $\gamma'_x > 0$ such that

$$\|ax\| \le \gamma'_x \|a\|, \qquad \forall a \in \mathfrak{A}_0.$$

The set of all right bounded elements of $\mathfrak{X}$ is denoted by $\mathfrak{X}_{\blacktriangleleft}$.

The terminology is motivated by the fact that, if x is left bounded, the map

$$a \in \mathfrak{A}_0 \mapsto L_x a = xa$$

is bounded on $\mathfrak{A}_0$ and so it has a bounded extension $\bar{L}_x$ to $\mathfrak{X}$. We put

$$\|x\|_{\blacktriangleright} = \max\{\|x\|, \|\bar{L}_x\|\}.$$

Analogously, we define a norm on $\mathfrak{X}_{\blacktriangleleft}$ by

$$\|x\|_{\blacktriangleleft} = \max\{\|x\|, \|\bar{R}_x\|\}.$$

We put $\mathfrak{X}_{\mathrm{b}} = \mathfrak{X}_{\blacktriangleright} \cap \mathfrak{X}_{\blacktriangleleft}$. Clearly, $\mathfrak{A}_0 \subseteq \mathfrak{X}_{\mathrm{b}}$. On $\mathfrak{X}_{\mathrm{b}}$ we define the norm

$$\|x\|_{\mathrm{b}} = \max\{\|x\|, \|\bar{L}_x\|, \|\bar{R}_x\|\}.$$

Remark 2.8. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then since $\|e\| = 1$, we have $\|\bar{L}_x\| \ge \|x\|$ for every $x \in \mathfrak{X}_{\blacktriangleright}$, and therefore $\|x\|_{\blacktriangleright} = \|\bar{L}_x\|$. Analogous statements hold for $\|\cdot\|_{\blacktriangleleft}$ and $\|\cdot\|_{\mathrm{b}}$.

As usual, we denote by $\mathcal{B}(\mathfrak{X})$ the Banach algebra of bounded operators in the Banach space $\mathfrak{X}$. From the definition it follows that $\mathfrak{X}_{\blacktriangleright}$, as well as $\mathfrak{X}_{\blacktriangleleft}$, can be identified with a subspace of $\mathcal{B}(\mathfrak{X})$.

Let $x \in \mathfrak{X}_{\blacktriangleright}$ and $y \in \mathfrak{X}$. Then we put

$$x \blacktriangleright y = \bar{L}_x y. \tag{2.1}$$

Similarly, if $y \in \mathfrak{X}_{\blacktriangleleft}$ and $x \in \mathfrak{X}$, we put

$$x \blacktriangleleft y = \bar{R}_y x. \tag{2.2}$$

REMARK 2.9. We notice that an element $x \in \mathfrak{X}_{\blacktriangleright}$ is not necessarily right bounded.

If $x, y \in \mathfrak{X}_{\mathrm{b}}$ then both $x \blacktriangleright y$ and $x \blacktriangleleft y$ are well defined, but, in general, $x \blacktriangleright y \neq x \blacktriangleleft y$. Conditions for the equality to hold will be given later.

It is easy to show that if $x, y \in \mathfrak{X}_{\blacktriangleright}$ and $\mu \in \mathbb{C}$ then both $x + y$ and μx belong to $\mathfrak{X}_{\blacktriangleright}$.

PROPOSITION 2.10. *If $(\mathfrak{X}, \mathfrak{A}_0)$ is a Banach quasi *-algebra, then the set $\mathfrak{X}_{\blacktriangleright}$ of all left bounded elements is a Banach algebra with respect to the multiplication $\blacktriangleright$ and the norm $\|\cdot\|_{\blacktriangleright}$.*

Proof. (i) We prove that if $x, y \in \mathfrak{X}_{\blacktriangleright}$ then $x \blacktriangleright y \in \mathfrak{X}_{\blacktriangleright}$ and

$$\|x \blacktriangleright y\|_{\blacktriangleright} \leq \|x\|_{\blacktriangleright} \, \|y\|_{\blacktriangleright}.$$

Indeed, for each $a \in \mathfrak{A}_0$ one has, using the associativity properties of the multiplication in $\mathfrak{X}$,

$$(\bar{L}_x y)a = \lim_{m\to\infty} (x b_m)a = \lim_{m\to\infty} x(b_m a) = \bar{L}_x(ya) = \bar{L}_x(L_y a),$$

where $\{b_m\}$ is a sequence in $\mathfrak{A}_0$, $\|\cdot\|$-converging to y. Therefore,

$$\|(\bar{L}_x y)a\| \leq \|\bar{L}_x\| \, \|\bar{L}_y\| \, \|a\|, \qquad \forall a \in \mathfrak{A}_0.$$

Hence $x \blacktriangleright y \in \mathfrak{X}_{\blacktriangleright}$, $\bar{L}_{x\blacktriangleright y} = \bar{L}_x \bar{L}_y$ and

$$\|\bar{L}_{x\blacktriangleright y}\| \leq \|\bar{L}_x\| \, \|\bar{L}_y\| \leq \|x\|_{\blacktriangleright} \|y\|_{\blacktriangleright}.$$

Since $\|x \blacktriangleright y\| \leq \|x\|_{\blacktriangleright} \|y\|_{\blacktriangleright}$, we finally get

$$\|x \blacktriangleright y\|_{\blacktriangleright} \leq \|x\|_{\blacktriangleright} \, \|y\|_{\blacktriangleright}.$$

Thus, $\mathfrak{X}_{\blacktriangleright}$ endowed with $\|\cdot\|_{\blacktriangleright}$ is a normed algebra. We will now show that $(\mathfrak{X}_{\blacktriangleright}, \|\cdot\|_{\blacktriangleright})$ is complete. Let $\{x_n\}$ be a Cauchy sequence in $(\mathfrak{X}_{\blacktriangleright}, \|\cdot\|_{\blacktriangleright})$. Then $\{\bar{L}_{x_n}\}$ is a Cauchy sequence in $\mathcal{B}(\mathfrak{X})$. Thus there exists $L \in \mathcal{B}(\mathfrak{X})$ such that $\bar{L}_{x_n} \to L$ with respect to the natural norm of $\mathcal{B}(\mathfrak{X})$. Since $\|x_n - x_m\| \to 0$, there exists $x \in \mathfrak{X}$ such that $\|x_n - x\| \to 0$. Since the right multiplication by a is continuous in $\mathfrak{X}$, it follows that $x_n a \to xa = \bar{L}_x a$ in the norm of $\mathfrak{X}$. This implies that $\bar{L}_x = L$. From these facts it follows easily that x is left bounded and $x_n \to x$ with respect to $\|\cdot\|_{\blacktriangleright}$. ■

A similar result can be proved for $\mathfrak{X}_{\blacktriangleleft}$ taking into account the following facts concerning the involution * of $\mathfrak{X}$:

(1*) $x \in \mathfrak{X}_{\blacktriangleright} \Leftrightarrow x^* \in \mathfrak{X}_{\blacktriangleleft}$;

(2*) $\|x^*\|_{\blacktriangleleft} = \|x\|_{\blacktriangleright}$ for every $x \in \mathfrak{X}_{\blacktriangleright}$;
(3*) $(x \blacktriangleright y)^* = y^* \blacktriangleleft x^*$ for every $x, y \in \mathfrak{X}_{\blacktriangleright}$.

Definition 2.7 easily yields

Lemma 2.11.

(i) *If $x \in \mathfrak{X}_{\blacktriangleright}$ and $y \in \mathfrak{X}$, then $\|x \blacktriangleright y\| \le \|x\|_{\blacktriangleright} \|y\|$.*
(ii) *If $y \in \mathfrak{X}_{\blacktriangleleft}$ and $x \in \mathfrak{X}$, then $\|x \blacktriangleleft y\| \le \|x\| \, \|y\|_{\blacktriangleleft}$.*

If $x, y \in \mathfrak{X}_{\mathrm{b}}$ then, as noticed before, both $x \blacktriangleright y$ and $x \blacktriangleleft y$ are well defined, but, in general, $x \blacktriangleright y \neq x \blacktriangleleft y$. We want to analyze this situation more carefully. First of all, if $x, y \in \mathfrak{X}_{\mathrm{b}}$, then $\bar{L}_x, \bar{L}_y \in \mathcal{B}(\mathfrak{X})$. As shown in the proof of Proposition 2.10, $\bar{L}_x \bar{L}_y = \bar{L}_{x \blacktriangleright y}$. Similarly, if $x, y \in \mathfrak{X}_{\mathrm{b}}$, then $\overline{R}_y \overline{R}_x = \overline{R}_{x \blacktriangleleft y}$.

In what follows, we denote by $\mathfrak{X}^\sharp$ the Banach dual space of $(\mathfrak{X}, \|\cdot\|)$. The norm in $\mathfrak{X}^\sharp$ is defined, as usual, by $\|f\|^\sharp = \sup_{\|x\| \le 1} |f(x)|$ for $f \in \mathfrak{X}^\sharp$.

Proposition 2.12. *The following statements are equivalent.*

(i) *$x \blacktriangleright y = x \blacktriangleleft y$ for every $x, y \in \mathfrak{X}_{\mathrm{b}}$.*
(ii) *$x \blacktriangleright y$ is right bounded and $\|x \blacktriangleright y\| \le \|x\| \|y\|_{\blacktriangleleft}$ for every $x, y \in \mathfrak{X}_{\mathrm{b}}$.*
(iii) *$x \blacktriangleleft y$ is left bounded and $\|x \blacktriangleleft y\| \le \|x\|_{\blacktriangleright} \|y\|$ for every $x, y \in \mathfrak{X}_{\mathrm{b}}$.*
(iv) *For any pair $\{a_n\}$, $\{b_n\}$ of sequences of elements of $\mathfrak{A}_0$, $\|\cdot\|$-converging to elements of $\mathfrak{X}_{\mathrm{b}}$, one has*

$$\lim_{n\to\infty} \lim_{m\to\infty} a_n b_m = \lim_{m\to\infty} \lim_{n\to\infty} a_n b_m.$$

(v) *There exists a weak *-dense subspace $\mathcal{M}$ of $\mathfrak{X}^\sharp$ such that for any pair $\{a_n\}$, $\{b_n\}$ of sequences of elements of $\mathfrak{A}_0$, $\|\cdot\|$-converging to elements of $\mathfrak{X}_{\mathrm{b}}$, one has*

$$\lim_{n\to\infty} \lim_{m\to\infty} f(a_n b_m) = \lim_{m\to\infty} \lim_{n\to\infty} f(a_n b_m), \qquad \forall f \in \mathcal{M}.$$

Proof. (i)⇒(ii): Clearly, the equality $x \blacktriangleright y = x \blacktriangleleft y$ implies that $x \blacktriangleright y$ is right bounded and for $x \blacktriangleright y$ the inequality in Lemma 2.11(ii) holds.

(ii)⇔(iii) follows easily by taking *.

(iii)⇒(i): Assume that, for every $x, y \in \mathfrak{X}_{\mathrm{b}}$, $x \blacktriangleleft y$ is left bounded and $\|x \blacktriangleleft y\| \le \|x\|_{\blacktriangleright} \|y\|$. Let $\{b_n\} \subset \mathfrak{A}_0$ be such that $\|y - b_n\| \to 0$ as $n \to \infty$. Then, since $\mathfrak{A}_0 \subseteq \mathfrak{X}_{\mathrm{b}}$ and $\mathfrak{X}_{\mathrm{b}}$ is a vector space, we get

$$\|x \blacktriangleleft y - x b_n\| = \|x \blacktriangleleft y - x \blacktriangleleft b_n\| = \|x \blacktriangleleft (y - b_n)\| \le \|x\|_{\blacktriangleright} \|y - b_n\| \to 0.$$

Hence

$$x \blacktriangleleft y = \lim_{n\to\infty} x b_n = \bar{L}_x y = x \blacktriangleright y.$$

(i)⇒(iv): Let $\{a_n\}, \{b_n\} \subset \mathfrak{A}_0$ with $\|x - a_n\| \to 0$, $\|y - b_n\| \to 0$ and $x, y \in \mathfrak{X}_{\mathrm{b}}$. Then

$$x \blacktriangleright y = \bar{L}_x y = \lim_{m\to\infty} x b_m = \lim_{m\to\infty} \lim_{n\to\infty} a_n b_m.$$

On the other hand,

$$x \blacktriangleleft y = \overline{R}_y x = \lim_{n\to\infty} a_n y = \lim_{n\to\infty}\lim_{m\to\infty} a_n b_m.$$

The equality $x \blacktriangleright y = x \blacktriangleleft y$ then implies that the two iterated limits coincide.

(iv)⇒(v): This is clear.

(v)⇒(i): Assume that (i) fails. Then there exists $f \in \mathfrak{X}^\sharp$ such that $f(x \blacktriangleright y) \neq f(x \blacktriangleleft y)$. Since $\mathcal{M}$ is weak*-dense in $\mathfrak{X}^\sharp$, we may suppose that $f \in \mathcal{M}$. Then, if $\{a_n\}, \{b_n\} \subset \mathfrak{A}_0$ $\|\cdot\|$-converge, respectively, to x and y, we have

$$\lim_{m\to\infty}\lim_{n\to\infty} f(a_n b_m) = f(x \blacktriangleright y) \neq f(x \blacktriangleleft y) = \lim_{n\to\infty}\lim_{m\to\infty} f(a_n b_m).$$

This completes the proof. ∎

If any of the equivalent conditions of Proposition 2.12 holds, we put

$$x \bullet y := x \blacktriangleright y = x \blacktriangleleft y, \qquad x, y \in \mathfrak{X}_{\mathrm{b}}.$$

Definition 2.13. A Banach quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ such that $x \blacktriangleright y = x \blacktriangleleft y$ for every $x, y \in \mathfrak{X}_{\mathrm{b}}$ is called *normal.*

Corollary 2.14.

(i) $(\mathfrak{X}, \mathfrak{A}_0)$ *is normal if, and only if,* $\mathfrak{X}_{\mathrm{b}}$ *is a *-algebra with respect to* $\blacktriangleright$ (*or, equivalently, with respect to* $\blacktriangleleft$).

(ii) *If* $(\mathfrak{X}, \mathfrak{A}_0)$ *is a normal Banach quasi *-algebra, then* $(\mathfrak{X}_{\mathrm{b}}, \|\cdot\|_{\mathrm{b}})$ *is a Banach *-algebra with respect to the multiplication* $\bullet$.

Proof. (i) The fact that if $(\mathfrak{X}, \mathfrak{A}_0)$ is normal, then $\mathfrak{X}_{\mathrm{b}}$ is a *-algebra with respect to $\blacktriangleright$ follows from the previous discussion. On the other hand, assume that $\mathfrak{X}_{\mathrm{b}}$ is a *-algebra with respect to $\blacktriangleright$; then, for every $x, y \in \mathfrak{X}_{\mathrm{b}}$, $x \blacktriangleright y \in \mathfrak{X}_{\mathrm{b}}$ and

$$x \blacktriangleleft y = (y^* \blacktriangleright x^*)^* = x \blacktriangleright y.$$

(ii) follows easily from Proposition 2.10 and from the properties of the involution. ∎

Example 2.15. Assume that for each $x \in \mathfrak{X}_{\mathrm{b}}$ there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that

$$\sup_n \|a_n\|_0 < \infty \quad \text{and} \quad \lim_{n\to\infty} \|x - a_n\| = 0.$$

Then $(\mathfrak{X}, \mathfrak{A}_0)$ is normal. Indeed, in this case, it is easily seen that (ii) or (iii) of Proposition 2.12 holds.

Remark 2.16. If $(\mathfrak{X}, \mathfrak{A}_0)$ is a commutative Banach quasi *-algebra, i.e. $xa = ax$ for all $x \in \mathfrak{X}$ and $a \in \mathfrak{A}_0$, then it is easily seen that each left bounded element x is also right bounded and $x \blacktriangleright y = y \blacktriangleleft x$ for every $y \in \mathfrak{X}$. Thus if $x, y \in \mathfrak{X}_{\mathrm{b}}$ then both $x \blacktriangleright y$ and $x \blacktriangleleft y$ are in $\mathfrak{X}_{\mathrm{b}}$ but they need not be equal. In this case, in general, $\mathfrak{X}_{\mathrm{b}}$ is an algebra with respect to $\blacktriangleright$ (and also with respect to $\blacktriangleleft$). Normality, in the commutative case, is equivalent to $\mathfrak{X}_{\mathrm{b}}$ being also commutative.

EXAMPLE 2.17. For the Banach quasi *-algebra $(L^p(I), C(I))$ considered in Example 2.4, one finds that $(L^p(I))_{\mathrm{b}} = L^\infty(I)$ and the norm $\|\cdot\|_{\mathrm{b}}$ is exactly the L^∞-norm. Since the multiplications $\blacktriangleright$ and $\blacktriangleleft$ both coincide with the ordinary multiplication of functions, $(L^p(I), C(I))$ is normal. This example also shows that, in general, $\mathfrak{A}_0$ is not dense in $\mathfrak{X}_{\mathrm{b}}$ with respect to $\|\cdot\|_{\mathrm{b}}$ since, as is well known, $C(I)$ is not dense in $L^\infty(I)$.

Similarly, $(L^p(\mathbb{R}), C_0^0(\mathbb{R}))$ is a Banach quasi *-algebra without unit. In this case $(L^p(\mathbb{R}))_{\mathrm{b}} = L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ and $(L^p(\mathbb{R}), C_0^0(\mathbb{R}))$ is normal. The norm $\|\cdot\|_{\mathrm{b}}$ is equivalent to $\|\cdot\|_p + \|\cdot\|_\infty$.

For the non-commutative L^p-spaces of Example 2.5 one finds that $(L^p(\tau))_{\mathrm{b}} = \mathcal{J}_p$ if τ is semifinite, while $(L^p(\tau))_{\mathrm{b}} = \mathfrak{M}$ if τ is finite. Normality follows from the fact that the multiplications $\blacktriangleright$ and $\blacktriangleleft$ both coincide with the ordinary multiplication of bounded operators.

EXAMPLE 2.18. In the case of the Banach quasi *-algebra $(\mathcal{H}, \mathfrak{A}_0)$ constructed from a Hilbert algebra $\mathfrak{A}_0$ as in Example 2.6, the set $\mathcal{H}_{\mathrm{b}}$ of bounded elements of $\mathcal{H}$ is the so-called *fulfillment* of $\mathfrak{A}_0$ ($\mathfrak{A}_0$ is called a *full* Hilbert algebra if $\mathcal{H}_{\mathrm{b}} = \mathfrak{A}_0$). $(\mathcal{H}, \mathfrak{A}_0)$ is normal. Indeed, let $x, y \in \mathcal{H}_{\mathrm{b}}$, and let $\{a_n\}$, $\{b_n\}$ be sequences in $\mathfrak{A}_0$, $\|\cdot\|$-converging, respectively, to x and y. Then

$$\langle x \blacktriangleright y \,|\, a\rangle = \lim_{n\to\infty} \langle x b_n \,|\, a\rangle = \lim_{n\to\infty} \langle b_n \,|\, x^* a\rangle = \langle y \,|\, x^* a\rangle, \qquad \forall a \in \mathfrak{A}_0.$$

On the other hand,

$$\langle x \blacktriangleleft y \,|\, a\rangle = \lim_{m\to\infty} \langle a_n y \,|\, a\rangle = \lim_{m\to\infty} \langle y \,|\, a_n^* a\rangle = \langle y \,|\, x^* a\rangle, \qquad \forall a \in \mathfrak{A}_0.$$

This implies that $x \blacktriangleright y = x \blacktriangleleft y$.

LEMMA 2.19. *If* $(\mathfrak{X}, \mathfrak{A}_0)$ *is a normal Banach quasi* **-algebra, then*

$$\overline{L}_x \overline{R}_y = \overline{R}_y \overline{L}_x, \qquad \forall x, y \in \mathfrak{X}_{\mathrm{b}}. \tag{2.3}$$

Proof. Indeed, let $x, y \in \mathfrak{X}_{\mathrm{b}}$, and let $\{a_n\}, \{b_n\} \subset \mathfrak{A}_0$ $\|\cdot\|$-converge, respectively, to x and y. Then, for every $a \in \mathfrak{A}_0$,

$$(\overline{L}_x \overline{R}_y)a = \overline{L}_x(\overline{R}_y a) = \lim_{m\to\infty} x(ab_m) = \lim_{m\to\infty} \lim_{n\to\infty} a_n(ab_m).$$

On the other hand,

$$(\overline{R}_y \overline{L}_x)a = \overline{R}_y(\overline{L}_x a) = \lim_{n\to\infty} (a_n a) y = \lim_{n\to\infty} \lim_{m\to\infty} (a_n a) b_m.$$

The statement then follows from Proposition 2.12(iv). ∎

REMARK 2.20. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit, then (2.3) implies the normality of $(\mathfrak{X}, \mathfrak{A}_0)$.

If $(\mathfrak{X}, \mathfrak{A}_0)$ is a normal Banach quasi *-algebra the products of an element $x \in \mathfrak{X}$ and an element $y \in \mathfrak{X}_{\mathrm{b}}$ are defined via (2.1) and (2.2).

PROPOSITION 2.21. *If* $(\mathfrak{X}, \mathfrak{A}_0)$ *is a normal Banach quasi* *-*algebra*, *then* $(\mathfrak{X}, \mathfrak{X}_{\mathrm{b}})$ *is a BQ*-algebra.*

Proof. We need only check the module associativity rules. Let $x \in \mathfrak{X}$ and $y_1, y_2 \in \mathfrak{X}_{\mathrm{b}}$. Then

$$x \blacktriangleleft (y_1 \bullet y_2) = x \blacktriangleleft (y_1 \blacktriangleleft y_2) = \overline{R}_{y_1 \blacktriangleleft y_2} x = (\overline{R}_{y_2} \overline{R}_{y_1}) x = \overline{R}_{y_2}(\overline{R}_{y_1} x) = (x \blacktriangleleft y_1) \blacktriangleleft y_2.$$

Using (2.3), we also have

$$\begin{aligned}(y_1 \blacktriangleright x) \blacktriangleleft y_2 &= \overline{R}_{y_2}(y_1 \blacktriangleright x) = \overline{R}_{y_2}(\bar{L}_{y_1} x) = (\overline{R}_{y_2} \bar{L}_{y_1}) x = (\bar{L}_{y_1} \overline{R}_{y_2}) x \\ &= \bar{L}_{y_1}(\overline{R}_{y_2} x) = \bar{L}_{y_1}(x \blacktriangleleft y_2) = y_1 \blacktriangleright (x \blacktriangleleft y_2). \ \blacksquare\end{aligned}$$

2.3. *The spectrum.* Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normal Banach quasi *-algebra with unit e and $x \in \mathfrak{X}$. We say that x has a *bounded inverse* if there exists $y \in \mathfrak{X}_{\mathrm{b}}$ such that $\overline{R}_y(x) = \bar{L}_y(x) = e$. From Proposition 2.21 it follows easily that this element y, if any, is unique. If x has a bounded inverse we denote it by x_{b}^{-1}.

DEFINITION 2.22. The *resolvent* $\varrho(x)$ of $x \in \mathfrak{X}$ is the set

$$\varrho(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ has a bounded inverse}\}.$$

The set $\sigma(x) = \mathbb{C} \setminus \varrho(x)$ is called the *spectrum* of x.

PROPOSITION 2.23. *Let* $x \in \mathfrak{X}$. *Then*:

(i) *The resolvent* $\varrho(x)$ *is an open subset of the complex plane.*
(ii) *The resolvent function* $R_\lambda(x) : \lambda \in \varrho(x) \mapsto (x - \lambda e)_{\mathrm{b}}^{-1}$ *is* $\|\cdot\|_{\mathrm{b}}$-*analytic on each connected component of* $\varrho(x)$.
(iii) *For any* $\lambda, \mu \in \varrho(x)$, $R_\lambda(x)$ *and* $R_\mu(x)$ *commute and*

$$R_\lambda(x) - R_\mu(x) = (\mu - \lambda) R_\mu(x) \bullet R_\lambda(x).$$

Proof. (i) Let $\lambda_0 \in \varrho(x)$ and $\lambda \in \mathbb{C}$ be such that $|\lambda - \lambda_0| \leq (\|R_{\lambda_0}(x)\|_{\mathrm{b}})^{-1}$. Then the series

$$\sum_{n=1}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}(x)^n$$

converges in $\mathfrak{X}_{\mathrm{b}}$ with respect to $\|\cdot\|_{\mathrm{b}}$ to an element $S_{\lambda,x}$.

Let now $T_{\lambda,x} := R_{\lambda_0}(x)(e + S_{\lambda,x})$. It is easily checked, using the $\| \, \|$-convergence for the product $T_{\lambda,x}(x - \lambda e)$, that $T_{\lambda,x}$ is a bounded inverse of $x - \lambda e$.

(ii) follows immediately from the proof of (i). The proof of (iii) is straightforward. $\blacksquare$

The classical argument based on Liouville's theorem can be applied to prove the following

PROPOSITION 2.24. *Let* $x \in \mathfrak{X}$. *Then* $\sigma(x)$ *is non-empty.*

Definition 2.25. Let $x \in \mathfrak{X}$. The non-negative number

$$r(x) = \sup_{\lambda \in \sigma(x)} |\lambda|$$

is called the *spectral radius* of x.

Remark 2.26. Of course, if $x \in \mathfrak{X}_{\mathrm{b}}$ then $\sigma(x)$ coincides with the spectrum of x regarded as an element of the Banach *-algebra $\mathfrak{X}_{\mathrm{b}}$. For an arbitrary element x, the set $\sigma(x) \subset \mathbb{C}$, which is closed, could be unbounded. The next proposition shows that $\sigma(x)$ is indeed unbounded if $x \in \mathfrak{X} \setminus \mathfrak{X}_{\mathrm{b}}$.

Proposition 2.27. *Let* $x \in \mathfrak{X}$. *Then* $r(x) < \infty$ *if, and only if,* $x \in \mathfrak{X}_{\mathrm{b}}$.

Proof. The "if" part has been discussed in the previous remark. Assume now that $r(x) < \infty$. Then the function $\lambda \mapsto (x - \lambda e)^{-1}$ is $\|\cdot\|_{\mathrm{b}}$-analytic in the region $|\lambda| > r(x)$. Therefore it has there a $\| \ \|_{\mathrm{b}}$-convergent Laurent expansion

$$(x - \lambda e)^{-1} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}, \qquad |\lambda| > r(x),$$

with $a_k \in \mathfrak{X}_{\mathrm{b}}$ for each $k \in \mathbb{N}$. As usual,

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{(x - \lambda e)^{-1}}{\lambda^{-k+1}}\, d\lambda, \qquad k \in \mathbb{N},$$

where γ is a circle centered in 0 and with radius $R > r(x)$. The integral on the r.h.s. converges with respect to $\|\cdot\|_{\mathrm{b}}$. The $\|\cdot\|$-continuity of multiplication implies that, as in the ordinary case,

$$xa_k = \frac{1}{2\pi i} \int_{\gamma} \frac{x(x - \lambda e)^{-1}}{\lambda^{-k+1}}\, d\lambda = \frac{1}{2\pi i} \int_{\gamma} \frac{(x - \lambda e)^{-1}}{\lambda^{-k}}\, d\lambda = a_{k+1}.$$

In particular, using Cauchy's integral formula, we find $xa_1 = -x$. This implies that $x \in \mathfrak{X}_{\mathrm{b}}$. ∎

Remark 2.28. If $\lambda \in \varrho(x)$ then all powers $(x - \lambda)^{-n}$ exist in $\mathfrak{X}_{\mathrm{b}}$, for every $n \in \mathbb{N}$. This does not imply the existence of $(x - \lambda)^n$ for $n > 1$. As an example, consider the Banach quasi *-algebra $(L^2(I), C(I))$ where $I = [0, 1]$ (cf. Example 2.4). The function $v(x) = x^{-1/4}$ is in $L^2(I)$; obviously, $0 \in \varrho(v)$ since $v^{-1}(x) = x^{1/4} \in C(I)$. We have $v^{-n}(x) = x^{n/4} \in L^2(I)$ for all $n \in \mathbb{N}$, but $v^2(x) = x^{-1/2} \notin L^2(I)$.

3. Representations and seminorms. Families of sesquilinear forms have been shown to play a relevant role in the study of the structure of CQ^*-algebras [6] or more generally Banach C^*-modules [23]. The main reason is that they give rise to *representations* with operators acting in Hilbert space.

3.1. *Representations.* Before going on we recall some definitions. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all linear operators X such that $D(X) = \mathcal{D}$ and $D(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a *partial* **-algebra* [1] with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \lceil \mathcal{D}$ and the (*weak*) partial multiplication $X_1 \,\square\, X_2 = X_1^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, iff $X_2\mathcal{D} \subset D(X_1^{\dagger *})$ and $X_1^*\mathcal{D} \subset D(X_2^*)$ (we write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$). Let

$$\mathcal{L}^\dagger(\mathcal{D}) = \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : X\mathcal{D} \subseteq \mathcal{D},\ X^\dagger\mathcal{D} \subseteq \mathcal{D}\}.$$

Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to $\square$ and $X_1 \,\square\, X_2\xi = X_1(X_2\xi)$ for each $\xi \in \mathcal{D}$ (see [15]).

A **-representation* of the Banach quasi *-algebra $(\mathfrak{X},\mathfrak{A}_0)$ is a *-homomorphism of $\mathfrak{X}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, for some pair $(\mathcal{D},\mathcal{H})$ where $\mathcal{D}$ is a dense subspace of a Hilbert space $\mathcal{H}$, that is, a linear map $\pi : \mathfrak{X} \to \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that (i) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathfrak{X}$, and (ii) if $x \in \mathfrak{X}$ and $a \in \mathfrak{A}_0$ then $\pi(x) \in L^{\mathrm{w}}(\pi(a))$ and $\pi(x) \,\square\, \pi(a) = \pi(xa)$.

A *-representation π of $(\mathfrak{X},\mathfrak{A}_0)$ is called *cyclic* if there exists $\eta \in \mathcal{D}$ such that $\pi(\mathfrak{A}_0)\eta$ is dense in $\mathcal{H}$, and *faithful* if $\pi(x) = 0$ implies $x = 0$.

If π is a *-representation of $(\mathfrak{X},\mathfrak{A}_0)$ in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, then the *closure* $\widetilde{\pi}$ of π is defined, for each $x \in \mathfrak{X}$, as the restriction of $\overline{\pi(x)}$ to the domain $\widetilde{\mathcal{D}}$, which is the completion of $\mathcal{D}$ under the *graph topology* defined by the seminorms $\xi \in \mathcal{D} \mapsto \|\pi(x)\xi\|$, $x \in \mathfrak{X}$ (see [1]). If $\pi = \widetilde{\pi}$ the representation is said to be *closed.*

The Gel'fand–Naĭmark–Segal (GNS) construction for positive linear functionals is one of the most relevant tools when studying the structure of a Banach *-algebra. As customary when a partial multiplication is involved (see [1]), we consider as starting point for the construction a positive sesquilinear form enjoying certain *invariance* properties.

As usual, a sesquilinear form φ on $\mathfrak{X} \times \mathfrak{X}$ is said to be *bounded* if there exists a positive constant γ such that

$$|\varphi(x,y)| \le \gamma\|x\|\,\|y\|, \qquad \forall x,y \in \mathfrak{X}.$$

In this case, we put

$$\|\varphi\| := \sup_{\|x\|=\|y\|=1} |\varphi(x,y)| = \sup_{\|x\|=1} \varphi(x,x).$$

Definition 3.1. Let $\mathcal{P}(\mathfrak{X})$ denote the set of all sesquilinear forms on $\mathfrak{X} \times \mathfrak{X}$ such that

(i) $\varphi(x,x) \ge 0,\ \forall x \in \mathfrak{X}$;
(ii) $\varphi(xa,b) = \varphi(a,x^*b),\ \forall x \in \mathfrak{X},\ a,b \in \mathfrak{A}_0$;
(iii) φ is bounded.

Remark 3.2. We notice that if $\varphi \in \mathcal{P}(\mathfrak{X})$ then an easy limit argument shows that, besides (ii) of Definition 3.1, the following equality holds:

$$\varphi(ax, y) = \varphi(x, a^*y), \qquad \forall x, y \in \mathfrak{X},\ a \in \mathfrak{A}_0.$$

Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Then the positivity of φ implies that:

$$\begin{aligned} \varphi(x, y) &= \overline{\varphi(y, x)}, && \forall x, y \in \mathfrak{X}; \\ |\varphi(x, y)|^2 &\leq \varphi(x, x)\varphi(y, y), && \forall x, y \in \mathfrak{X}. \end{aligned}$$

Hence

$$N_\varphi := \{x \in \mathfrak{X} : \varphi(x, x) = 0\} = \{x \in \mathfrak{X} : \varphi(x, y) = 0,\ \forall y \in \mathfrak{X}\},$$

and so N_φ is a subspace of $\mathfrak{A}$. For each $x \in \mathfrak{X}$, we denote by $\lambda_\varphi(x)$ the coset of $\mathfrak{X}/N_\varphi$ which contains x, and define an inner product $\langle \cdot \mid \cdot \rangle$ on

$$\lambda_\varphi(\mathfrak{X}) = \mathfrak{X}/N_\varphi$$

by

$$\langle \lambda_\varphi(x) \mid \lambda_\varphi(y) \rangle = \varphi(x, y), \qquad x, y \in \mathfrak{X}.$$

We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{X})$. The subspace $\lambda_\varphi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\varphi$. Indeed, if $x \in \mathfrak{X}$, there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $a_n \to x$ in $\mathfrak{X}$. Then

$$\|\lambda_\varphi(x) - \lambda_\varphi(a_n)\|^2 = \varphi(x - a_n, x - a_n) \leq \|\varphi\|^2 \|x - a_n\|^2 \to 0.$$

Proposition 3.3. *Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Put*

$$\pi^\circ_\varphi(x)\lambda_\varphi(a) = \lambda_\varphi(xa), \qquad x \in \mathfrak{X},\ a \in \mathfrak{A}_0. \tag{3.1}$$

Then π°_φ is a *-representation of $\mathfrak{X}$ in $\mathcal{L}^\dagger(\lambda_\varphi(\mathfrak{A}_0), \mathcal{H}_\varphi)$.

If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, the following properties also hold:

(i) $\mathcal{D} = \lambda_\varphi(\mathfrak{A}_0) = \pi(\mathfrak{A}_0)\lambda_\varphi(e)$ (*i.e.* $\lambda_\varphi(e)$ *is ultra-cyclic*);

(ii) $\varphi(x, y) = \langle \pi^\circ_\varphi(x)\lambda_\varphi(e) \mid \pi^\circ_\varphi(y)\lambda_\varphi(e) \rangle$, $\forall x, y \in \mathfrak{X}$.

Proof. First we prove that, for each $x \in \mathfrak{X}$, the map $\pi^\circ_\varphi(x)$ of (3.1) is well defined. Assume that $\lambda_\varphi(a) = 0$ for some $a \in \mathfrak{A}_0$. If $x \in \mathfrak{X}$, we then get $\varphi(a, x^*b) = 0$ for every $b \in \mathfrak{A}_0$. For each $y \in \mathfrak{X}$ there exists a sequence $\{b_n\} \subset \mathfrak{A}_0$ such that $\|\lambda_\varphi(y) - \lambda_\varphi(b_n)\| \to 0$. This clearly implies that $\varphi(xa, y) = 0$ for each $y \in \mathfrak{X}$. Hence $xa \in N_\varphi$. Thus, for each $x \in \mathfrak{X}$, the map $\pi^\circ_\varphi(x)$ is a well defined linear operator from $\lambda_\varphi(\mathfrak{A}_0)$ into $\mathcal{H}_\varphi$. We notice that the restriction of π°_φ to $\mathfrak{A}_0$ maps $\lambda_\varphi(\mathfrak{A}_0)$ into itself. This fact and the properties of φ listed in Definition 3.1 easily imply that π°_φ is a *-representation. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then (i) and (ii) follow from the definitions. ∎

Denote by π_φ the closure of π°_φ. The triple $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ is called the *GNS construction* for φ and we refer to π_φ as the *GNS representation* of $\mathfrak{X}$ constructed from φ. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then $\xi_\varphi := \lambda_\varphi(e)$ is *cyclic* for π_φ.

With a proof similar to the usual one in the case of *-algebras one can prove the following

PROPOSITION 3.4. *Let* $(\mathfrak{X}, \mathfrak{A}_0)$ *be a Banach quasi* **-algebra with unit* e *and* $\varphi \in \mathcal{P}(\mathfrak{X})$. *Then the GNS construction* $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ *is unique up to unitary equivalence.*

It is easy to prove

PROPOSITION 3.5. *The* **-representation* π_φ *is bounded if, and only if,* φ *is admissible, i.e, for every* $a \in \mathfrak{A}_0$ *there exists* $\gamma_x > 0$ *such that*

$$\varphi(xa, xa) \leq \gamma_x \varphi(a, a), \qquad \forall a \in \mathfrak{A}_0.$$

Assume that $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e. Then it is clear that, if $\varphi \in \mathcal{P}(\mathfrak{X})$, the linear functional ω_φ defined by

$$\omega_\varphi(x) = \varphi(x, e), \qquad x \in \mathfrak{X},$$

is bounded on $\mathfrak{X}$, i.e. $\omega_\varphi \in \mathfrak{X}^\sharp$. Moreover, it is *positive* on $\mathfrak{X}$, in the sense that $\omega_\varphi(x) \geq 0$ for every $x \in \mathfrak{X}^+$, where $\mathfrak{X}^+$ is the closure in $\mathfrak{X}$ of the set

$$\mathfrak{A}_0^+ = \Big\{ \sum_{k=1}^{n} a_k^* a_k : a_k \in \mathfrak{A}_0,\, k = 1, \ldots, n,\, n \in \mathbb{N} \Big\}.$$

The set of positive elements of $\mathfrak{X}^\sharp$ is denoted by $\mathfrak{X}^\sharp_+$.

Furthermore, the map $\varphi \in \mathcal{P}(\mathfrak{X}) \mapsto \omega_\varphi \in \mathfrak{X}^\sharp_+$ is injective. For, if $\omega_\varphi(x) = 0$ for each $x \in \mathfrak{X}$, then making use of the properties (ii) and (iii) of $\mathcal{P}(\mathfrak{X})$ and of the density of $\mathfrak{A}_0$, it follows that $\varphi(x, y) = 0$ for all $x, y \in \mathfrak{X}$.

Finally, we define

$$\mathcal{S}(\mathfrak{X}) = \{\varphi \in \mathcal{P}(\mathfrak{X}) : \|\varphi\| \leq 1\}.$$

It is easily seen that $\mathcal{S}(\mathfrak{X})$ is a convex subset of $\mathcal{P}(\mathfrak{X})$. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then $\varphi(e, e) \leq \|e\|^2 = 1$ for any $\varphi \in \mathcal{S}(\mathfrak{X})$.

Let $\mathfrak{X}^\sharp_1 = \{\omega \in \mathfrak{X}^\sharp : \|\omega\|^\sharp \leq 1\}$ be the unit ball of $\mathfrak{X}^\sharp$ and

$$\mathfrak{X}^\sharp_{\mathcal{S}} = \{\omega_\varphi : \varphi \in \mathcal{S}(\mathfrak{X})\}.$$

REMARK 3.6. Obviously, it is possible that $\mathcal{S}(\mathfrak{X}) = \{0\}$ (or, equivalently, $\mathfrak{X}^\sharp_{\mathcal{S}} = \{0\}$). It is, however much more interesting to consider Banach quasi *-algebras for which the set $\mathcal{S}(\mathfrak{X})$ is sufficiently rich (Section 3.3).

PROPOSITION 3.7. *Assume that* $(\mathfrak{X}, \mathfrak{A}_0)$ *has a unit and that* $\mathcal{S}(\mathfrak{X}) \neq \{0\}$. *Then the following statements hold*:

(i) $\mathfrak{X}^\sharp_{\mathcal{S}}$ *is a convex, weak***-compact subset of* $\mathfrak{X}^\sharp_1$.

(ii) $\mathfrak{X}^\sharp_{\mathcal{S}}$ *has extreme points. If* ω_φ *is extreme, then* $\|\varphi\| = 1$.

(iii) ω_φ *is extreme in* $\mathfrak{X}^\sharp_{\mathcal{S}}$ *if, and only if,* φ *is extreme in* $\mathcal{S}(\mathfrak{X})$.

The proof is very simple and we omit it.

3.2. *Seminorms.* We will now define some seminorms, closely related to families of sesquilinear forms [22] and to representations. Similar constructions have been considered in the case of *-algebras in [9, 24].

To begin with, we put

$$\mathfrak{p}(x) = \sup_{\varphi \in \mathcal{S}(\mathfrak{X})} \varphi(x,x)^{1/2}.$$

Then p is a seminorm on $\mathfrak{X}$ with $\mathfrak{p}(x) \leq \|x\|$ for every $x \in \mathfrak{X}$.

We also put

$$N(\mathfrak{p}) = \{x \in \mathfrak{X} : \mathfrak{p}(x) = 0\}.$$

REMARK 3.8. Under the assumption of Proposition 3.7, the set $\mathcal{S}(\mathfrak{X})$, which is convex, has extreme elements (of unit norm) whose closed convex hull is exactly $\mathcal{S}(\mathfrak{X})$. Thus, in this case,

$$\mathfrak{p}(x) = \sup_{\|\varphi\|=1} \varphi(x,x)^{1/2}.$$

We also define

$$\mathfrak{q}(x) = \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\, a \in \mathfrak{A}_0,\, \varphi(a,a) = 1\}, \qquad x \in \mathfrak{X}, \tag{3.2}$$

and

$$D(\mathfrak{q}) = \{x \in \mathfrak{X} : \mathfrak{q}(x) < \infty\}.$$

If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then $\mathfrak{q}$ has a simpler form. In fact, if we put

$$\mathfrak{q}'(x) = \sup\{\varphi(x,x)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\, \varphi(e,e) = 1\}, \qquad x \in \mathfrak{X},$$

and

$$D(\mathfrak{q}') = \{x \in \mathfrak{X} : \mathfrak{q}'(x) < \infty\},$$

then $D(\mathfrak{q}) = D(\mathfrak{q}')$ and $\mathfrak{q}(x) = \mathfrak{q}'(x)$ for every $x \in D(\mathfrak{q})$. Indeed, it is clear that

$$\mathfrak{q}'(x) \leq \mathfrak{q}(x), \qquad \forall x \in \mathfrak{X}. \tag{3.3}$$

On the other hand, if $\varphi \in \mathcal{P}(\mathfrak{X})$ and $a \in \mathfrak{A}_0$, then also $\varphi_a \in \mathcal{P}(\mathfrak{X})$, where $\varphi_a(x,y) = \varphi(xa,ya)$ for every $x,y \in \mathfrak{X}$. Clearly, if $a \in \mathfrak{A}_0$ and $\varphi(a,a) = 1$, then $\varphi_a(e,e) = 1$. This implies that

$$\mathfrak{q}(x) \leq \mathfrak{q}'(x), \qquad \forall x \in \mathfrak{X}. \tag{3.4}$$

The inequalities (3.3) and (3.4) also hold when one of their terms is ∞. Thus the statement is proved.

The seminorms $\mathfrak{p}$ and $\mathfrak{q}$ compare as follows.

PROPOSITION 3.9. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. Then*:

(i) $\mathfrak{p}(xa) \leq \mathfrak{q}(x)\mathfrak{p}(a),\ \forall x \in D(\mathfrak{q}),\, a \in \mathfrak{A}_0$.

(ii) *If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit, then*

$$\mathfrak{p}(x) \leq \mathfrak{q}(x), \qquad \forall x \in D(\mathfrak{q}).$$

Proof. (i) Let $x \in D(\mathfrak{q})$. Then from the definition of $\mathfrak{q}(x)$ we get, for each $\varphi \in \mathcal{P}(\mathfrak{X})$,

$$\varphi(xa, xa) \leq \mathfrak{q}(x)^2 \varphi(a, a), \qquad \forall a \in \mathfrak{A}_0. \tag{3.5}$$

The statement then follows by taking the supremum over $\varphi \in \mathcal{S}(\mathfrak{X})$.

(ii) This follows from (i) by choosing $a = e$ and taking into account that $\mathfrak{p}(e) \leq 1$. ∎

PROPOSITION 3.10. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. Then*:

(i) $\mathfrak{A}_0 \subseteq D(\mathfrak{q})$ *and* $q(a) \leq \|a\|_0$, $\forall a \in \mathfrak{A}_0$.

(ii) $D(\mathfrak{q}) = \{x \in \mathfrak{X} : \pi_\varphi(x)$ *bounded*, $\forall \varphi \in \mathcal{P}(\mathfrak{X})$, *and*

$$\sup_{\varphi \in \mathcal{P}(\mathfrak{X})\}} \|\overline{\pi_\varphi(x)}\| < \infty\},$$

$$\mathfrak{q}(x) = \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\|, \qquad \forall x \in D(\mathfrak{q}).$$

(iii) $\mathfrak{q}$ *is an extended C^*-seminorm on* $(\mathfrak{X}, \mathfrak{A}_0)$ (*i.e.* $\mathfrak{q}(x^*) = \mathfrak{q}(x)$, $\forall x \in \mathfrak{X}$; $q(a^*a) = q(a)^2$, $\forall a \in \mathfrak{A}_0$, *see* [22]).

(iv) $\mathfrak{p}(ax) \leq \|a\|_0 \mathfrak{p}(x)$, $\forall x \in \mathfrak{X}$, $a \in \mathfrak{A}_0$.

Proof. (i) Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Then the restriction of φ to $\mathfrak{A}_0 \times \mathfrak{A}_0$ is $\|\cdot\|_0$-bounded. This fact together with a repeated use of the Cauchy–Schwarz inequality gives, for any $a, b \in \mathfrak{A}_0$,

$$\begin{aligned}\varphi(ab, ab) &\leq \varphi(b, b)^{1/2+1/2^2+\cdots+1/2^k} \varphi((a^*a)^{2^{k-1}} b, (a^*a)^{2^{k-1}} b)^{1/2^k} \\ &\leq \varphi(b, b)^{1/2+1/2^2+\cdots+1/2^k} \|\varphi\|^{1/2^k} (\|(a^*a)^{2^{k-1}}\|_0 \|b\|_0)^{1/2^{k-1}}.\end{aligned}$$

For $k \to \infty$, we get

$$\varphi(ab, ab) \leq \|a\|_0^2 \varphi(b, b), \qquad \forall a, b \in \mathfrak{A}_0.$$

This implies that $\mathfrak{q}(a) \leq \|a\|_0$ for every $a \in \mathfrak{A}_0$.

(ii) Let $x \in D(\mathfrak{q})$ and $\varphi \in \mathcal{P}(\mathfrak{X})$. If π_φ denotes the GNS representation constructed from φ, making use of (3.5) we obtain

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) \leq \mathfrak{q}(x)^2 \varphi(a, a) = \mathfrak{q}(x)^2 \|\lambda_\varphi(a)\|^2, \qquad \forall a \in \mathfrak{A}_0.$$

Thus $\pi_\varphi(x)$ is bounded and $\|\pi_\varphi(x)\| \leq \mathfrak{q}(x)$. This implies that

$$M(x) := \sup\{\|\overline{\pi_\varphi(x)}\| : \varphi \in \mathcal{P}(\mathfrak{X})\} \leq \mathfrak{q}(x).$$

Conversely, assume that $x \in \mathfrak{X}$ and $M(x)$ is finite. Then

$$\varphi(xa, xa) = \|\pi_\varphi(x)\lambda_\varphi(a)\|^2 \leq M(x)^2 \|\lambda_\varphi(a)\|^2 = M(x)^2 \varphi(a, a), \qquad \forall a \in \mathfrak{A}_0.$$

Hence, $x \in D(\mathfrak{q})$ and $\mathfrak{q}(x) \leq M(x)$.

(iii) This follows directly from (ii).

(iv) For $x \in \mathfrak{X}$ and $\varphi \in \mathcal{S}(\mathfrak{X})$, define

$$\omega_\varphi^x(a) = \varphi(ax, x), \qquad a \in \mathfrak{A}_0.$$

Then ω_φ^x is positive and $\|\cdot\|_0$-bounded on $\mathfrak{A}_0$. Proceeding as in (i) one gets

$$\varphi(ax, ax) \leq \|a\|_0^2 \varphi(x,x), \qquad \forall a \in \mathfrak{A}_0.$$

Taking the supremum over $\varphi \in \mathcal{S}(\mathfrak{X})$, we obtain the result. ■

So far, we have not proved (or even assumed) anything about the *size* of the families of sesquilinear forms we have considered. There are however examples of Banach quasi *-algebras $(\mathfrak{X}, \mathfrak{A}_0)$ with $\mathcal{P}(\mathfrak{X}) = \{0\}$ (see Example 3.20 below). The previous statements remain of course true, but become mostly trivial. Much more interesting is the case where $\mathcal{P}(\mathfrak{X})$ contains sufficiently many elements, by which we mean that $N(\mathfrak{p}) = \{0\}$.

3.3. *Sufficient families of sesquilinear forms*

DEFINITION 3.11. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. We say that $\mathcal{S}(\mathfrak{X})$ is *sufficient* if the conditions $x \in \mathfrak{X}$ and $\varphi(x,x) = 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$ imply $x = 0$.

REMARK 3.12. We adopted a similar definition for CQ^*-algebras in [6]. Some of the statements that follow generalize results obtained for that situation in [6, 20].

The following lemma allows us to formulate in different ways the notion of sufficiency of $\mathcal{S}(\mathfrak{X})$.

LEMMA 3.13. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit e. For an element $x \in \mathfrak{X}$, the following statements are equivalent.*

(i) $\mathfrak{p}(x) = 0$, *i.e.* $x \in N(\mathfrak{p})$.
(ii) $\varphi(x,x) = 0$ *for every* $\varphi \in \mathcal{S}(\mathfrak{X})$.
(iii) $\varphi(x,y) = 0$ *for every* $\varphi \in \mathcal{S}(\mathfrak{X})$ *and* $y \in \mathfrak{X}$.
(iv) $\omega_\varphi(x) = 0$ *for every* $\varphi \in \mathcal{S}(\mathfrak{X})$.
(v) $\varphi(xa, a) = 0$ *for every* $\varphi \in \mathcal{S}(\mathfrak{X})$ *and* $a \in \mathfrak{A}_0$.
(vi) $\varphi(xa, b) = 0$ *for every* $\varphi \in \mathcal{S}(\mathfrak{X})$ *and* $a, b \in \mathfrak{A}_0$.

PROPOSITION 3.14. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit e. If the set*

$$\mathfrak{X}^\sharp_{\mathcal{P}} := \{\omega_\varphi : \varphi \in \mathcal{P}(\mathfrak{X})\}$$

is weak-dense in $\mathfrak{X}^\sharp_+$, then $\mathcal{S}(\mathfrak{X})$ is sufficient. Conversely, if $(\mathfrak{X}, \|\cdot\|)$ is a reflexive Banach space and $\mathcal{S}(\mathfrak{X})$ is sufficient, then $\mathfrak{X}^\sharp_{\mathcal{P}}$ is weak*-dense in $\mathfrak{X}^\sharp_+$.*

Proof. Assume that $\mathcal{S}(\mathfrak{X})$ is not sufficient. Then there exists $x \in \mathfrak{X}$, $x \neq 0$, such that $\varphi(x,x) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$. This implies that $\omega_\varphi(x) = 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$. Thus the non-zero continuous linear functional f_x on $\mathfrak{X}^\sharp$ defined by $f_x(\omega) = \omega(x)$ is identically zero on $\{\omega_\varphi : \varphi \in \mathcal{P}(\mathfrak{X})\}$. Thus this set is not weak*-dense in $\mathfrak{X}^\sharp_+$.

Conversely, assume that $\mathfrak{X}^\sharp_{\mathcal{P}}$ is not weak*-dense in $\mathfrak{X}^\sharp_+$. Then, by reflexivity, there would exist an $x \in \mathfrak{X}$, $x \neq 0$, such that $\omega_\varphi(x) = \varphi(x, e) = 0$ for each $\varphi \in \mathcal{P}(\mathfrak{X})$. Then, by Lemma 3.13, we get $\varphi(x, x) = 0$ for each $\varphi \in \mathcal{P}(\mathfrak{X})$. This implies that $x = 0$, a contradiction. ∎

PROPOSITION 3.15. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit e and let $\mathcal{S}(\mathfrak{X})$ be sufficient. Let $x \in \mathfrak{X}$. Then*

(i) *$x = x^*$ if, and only if, $\omega_\varphi(x) \in \mathbb{R}$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$.*

Moreover, if $\mathfrak{X}^\sharp_{\mathcal{P}}$ is weak-dense in $\mathfrak{X}^\sharp_+$, then:*

(ii) *If $\omega_\varphi(x) \geq 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$, then x is positive.*
(iii) *$x \in \mathfrak{X}^+ \cap \{-\mathfrak{X}^+\}$ if, and only if, $x = 0$.*

Proof. (i) Assume that $\omega_\varphi(x) \in \mathbb{R}$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$. Then

$$\omega_\varphi(x - x^*) = \omega_\varphi(x) - \omega_\varphi(x^*) = \omega_\varphi(x) - \overline{\omega_\varphi(x)} = 0$$

for every $\varphi \in \mathcal{S}(\mathfrak{X})$. By Lemma 3.13 one has $\varphi(x - x^*, x - x^*) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$. Hence $x = x^*$. The converse implication is obvious.

(ii) This follows immediately from the weak*-denseness of $\mathfrak{X}^\sharp_{\mathcal{P}}$.

(iii) Assume that $x \in \mathfrak{X}^+ \cap \{-\mathfrak{X}^+\}$; then by (ii) it follows that $\omega_\varphi(x) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$. From this we conclude that $x = 0$. ∎

PROPOSITION 3.16. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a BQ*-algebra with unit. If $\mathcal{S}(\mathfrak{X})$ is sufficient, then $\mathfrak{A}_0$ is a *-semisimple Banach *-algebra.*

Proof. It suffices to show that if $a \in \mathfrak{A}_0$ and $\omega(a^* a) = 0$ for each positive linear functional ω on $\mathfrak{A}_0$, then $a = 0$. If this assumption is satisfied, then, in particular, $\omega_\varphi(a^* a) = 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$. This implies that $\varphi(a, a) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$, and so $a = 0$. ∎

If $(\mathfrak{X}, \mathfrak{A}_0)$ has a sufficient $\mathcal{S}(\mathfrak{X})$, then also the multiplications defined in Section 2 behave in a reasonable fashion:

PROPOSITION 3.17. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with sufficient $\mathcal{S}(\mathfrak{X})$. Then $(\mathfrak{X}, \mathfrak{A}_0)$ is normal.*

Proof. Let $x, y \in \mathfrak{X}_{\mathrm{b}}$. For every $\varphi \in \mathcal{S}(\mathfrak{X})$ and $c \in \mathfrak{A}_0$, we have

$$\begin{aligned}\varphi((x \blacktriangleright y)c, c) &= \varphi((\overline{L}_x y)c, c) = \lim_{m\to\infty} \varphi((x b_m)c, c)\\ &= \lim_{m\to\infty} \varphi(x(b_m c), c) = \lim_{m\to\infty} \varphi(b_m c, x^* c)\\ &= \varphi(yc, x^* c),\end{aligned}$$

where $\{b_m\} \subset \mathfrak{A}_0$ converges to y in $\mathfrak{X}$. Analogously, if $\{a_n\} \subset \mathfrak{A}_0$ converges to x in $\mathfrak{X}$, we have

$$\begin{aligned}\varphi((x \blacktriangleleft y)c, c) &= \varphi((\overline{R}_y x)c, c) = \lim_{n\to\infty} \varphi((a_n y)c, c)\\ &= \lim_{n\to\infty} \varphi(a_n(yc), c) = \lim_{n\to\infty} \varphi(yc, a_n^* c) = \varphi(yc, x^* c).\end{aligned}$$

Therefore

$$\varphi((x \blacktriangleright y - x \blacktriangleleft y)c, c) = 0, \qquad \forall \varphi \in \mathcal{S}(\mathfrak{X}),\ c \in \mathfrak{A}_0.$$

By Lemma 3.13 it follows that $x \blacktriangleright y = x \blacktriangleleft y$. This concludes the proof. ∎

If $(\mathfrak{X}, \mathfrak{A}_0)$ has a sufficient $\mathcal{S}(\mathfrak{X})$, then $\mathfrak{p}$ is a norm on $\mathfrak{X}$, weaker in general than the original norm of $\mathfrak{X}$. Thus, it makes sense to consider the case where they coincide. Hence we give the following

Definition 3.18. A Banach quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is called *regular* if

(i) $\mathcal{S}(\mathfrak{X})$ is sufficient;
(ii) $\mathfrak{p}(x) = \|x\|$ for every $x \in \mathfrak{X}$.

A similar definition was given for CQ^*-algebras in [6]. We notice that the equality $\mathfrak{p}(x) = \|x\|$ for every $x \in \mathfrak{X}$ implies that $\mathfrak{p}(x^*) = \mathfrak{p}(x)$ for every $x \in \mathfrak{X}$. This equality fails in general; but it is exactly what is needed to embed $(\mathfrak{X}, \mathfrak{A}_0)$ in a larger *regular* Banach quasi *-algebra.

Proposition 3.19. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with sufficient $\mathcal{S}(\mathfrak{X})$ and $\mathfrak{p}(x^*) = \mathfrak{p}(x)$ for every $x \in \mathfrak{X}$. Then there exists a regular Banach quasi *-algebra, $(\mathfrak{X}_{\mathcal{S}}, \mathfrak{A}_0)$, such that $\mathfrak{X}_{\mathcal{S}}$ contains $\mathfrak{X}$ as a dense subspace.*

Proof. We let $\mathfrak{X}_{\mathcal{S}}$ be the completion of $\mathfrak{A}_0$ with respect to $\mathfrak{p}$; then $(\mathfrak{X}_{\mathcal{S}}, \mathfrak{A}_0)$ is a Banach quasi *-algebra, by Proposition 3.10(vi) and the assumption that $\mathfrak{p}(x^*) = \mathfrak{p}(x)$ for every $x \in \mathfrak{X}$. We now prove that $\mathfrak{X}$ can be identified with a subspace of $\mathfrak{X}_{\mathcal{S}}$. Indeed, if $x \in \mathfrak{X}$ then there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that

$$x = \|\cdot\|\text{-}\lim_{n\to\infty} a_n.$$

It is readily seen that $\{a_n\}$ is also a Cauchy sequence with respect to $\mathfrak{p}$. Thus there exists an element $\overline{x} \in \mathfrak{X}_{\mathcal{S}}$ such that

$$\overline{x} = \mathfrak{p}\text{-}\lim_{n\to\infty} a_n.$$

The element $\overline{x}$ does not depend on the particular sequence $\{a_n\}$ used to approximate x in $\mathfrak{X}$. Indeed, if $\{a'_n\}$ is another such sequence, then

$$\mathfrak{p}(a_n - a'_n) \leq \|a_n - a'_n\| \to 0 \qquad \text{as } n \to \infty.$$

We have defined in this way a map $i : x \in \mathfrak{X} \mapsto \overline{x} \in \mathfrak{X}_{\mathcal{S}}$; we will now prove that i is injective.

Assume that $\overline{x} = 0$ for some $x \in \mathfrak{X}$ and let $\{a_n\}$ be a sequence in $\mathfrak{A}_0$ approximating x in the norm of $\mathfrak{X}$ and such that $\mathfrak{p}(a_n) \to 0$; this implies that $\varphi(a_n, a_n) \to 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$. Therefore

$$\varphi(x, x) = \lim_{n\to\infty} \varphi(a_n, a_n) = 0.$$

From the sufficiency of $\mathcal{S}(\mathfrak{X})$ we get $x = 0$. To conclude the proof, we need to show that $\mathcal{S}(\mathfrak{X}_{\mathcal{S}})$ is sufficient and that $(\mathfrak{X}_{\mathcal{S}}, \mathfrak{A}_0)$ is regular.

First, we prove that the two families of sesquilinear forms, $\mathcal{S}(\mathfrak{X})$ and $\mathcal{S}(\mathfrak{X}_{\mathcal{S}})$, can be identified. Indeed, let $\Phi \in \mathcal{S}(\mathfrak{X}_{\mathcal{S}})$; then its restriction $\Phi_{\mathfrak{X}}$ to $\mathfrak{X}$ belongs, as is easily seen, to $\mathcal{S}(\mathfrak{X})$. Conversely, if $\Phi_0 \in \mathcal{S}(\mathfrak{X})$, then making use of the Cauchy–Schwarz inequality, we get

$$|\Phi_0(x, y)| \leq \mathfrak{p}(x)\mathfrak{p}(y), \quad \forall x, y \in \mathfrak{X}.$$

Therefore Φ_0 has a unique extension Φ to $\mathfrak{X}_{\mathcal{S}}$ and $\Phi \in \mathcal{S}(\mathfrak{X}_{\mathcal{S}})$. Taking this fact into account, the sufficiency of $\mathcal{S}(\mathfrak{X}_{\mathcal{S}})$ follows by the definition of completion. The regularity is a simple consequence of the definition of the norm in the completion. ∎

Example 3.20. The BQ^*-algebra $(L^p(I), C(I))$ is regular [5] if, and only if, $p \geq 2$. For $1 \leq p < 2$, $\mathcal{S}(L^p(I)) = \{0\}$. In the case of the non-commutative L^p of Example 2.5, it has been proved in [8] that, for finite τ, $(L^p(\tau), \mathfrak{M})$ is regular if $p \geq 2$.

Example 3.21. For the Banach quasi *-algebra $(\mathcal{H}, \mathfrak{A}_0)$ of Example 2.6, $\mathcal{S}(\mathfrak{X})$ is sufficient, since it contains the inner product $\langle \cdot \,|\, \cdot \rangle$. For the same reason, $(\mathcal{H}, \mathfrak{A}_0)$ is regular.

We consider again the seminorm $\mathfrak{q}$ defined in (3.2). If $(\mathfrak{X}, \mathfrak{A}_0)$ has a sufficient $\mathcal{S}(\mathfrak{X})$, then $\mathfrak{q}$ is also a norm on $D(\mathfrak{q})$ and has the C^*-property on $\mathfrak{A}_0$. If, in addition, $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit, then (Proposition 3.9)

$$\mathfrak{p}(x) \leq \mathfrak{q}(x), \quad \forall x \in D(\mathfrak{q}). \tag{3.6}$$

The space $D(\mathfrak{q})$ endowed with the topology defined by $\mathfrak{q}$ is denoted by $\mathfrak{X}_{\mathfrak{q}}$. Then we have the following

Proposition 3.22. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit. Assume that $\mathcal{S}(\mathfrak{X})$ is sufficient. Then $\mathfrak{X}_{\mathfrak{q}}$ is a normed space containing $\mathfrak{A}_0$ as a subspace. Moreover if $\mathfrak{X}$ is regular, then $\mathfrak{X}_{\mathfrak{q}}$ is a Banach space.*

Proof. The first part of the statement follows from Proposition 3.10(i). In order to prove that, if $\mathfrak{X}$ is regular, $\mathfrak{X}_{\mathfrak{q}}$ is a Banach space, we only have to show its completeness. Let $\{x_n\}$ be a $\mathfrak{q}$-Cauchy sequence in $\mathfrak{X}_{\mathfrak{q}}$.

Inequality (3.6) in the regular case becomes $\|x\| \leq \mathfrak{q}(x)$ for all $x \in \mathfrak{X}_{\mathfrak{q}}$. Therefore $\{x_n\}$ is also $\|\cdot\|$-Cauchy. Using the $\|\cdot\|$-completeness of $\mathfrak{X}$ we conclude that there exists an element $x \in \mathfrak{X}$ which is the $\|\ \|$-limit of x_n.

Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Then $\varphi(x, x) = \lim_{n\to\infty} \varphi(x_n, x_n)$. The sequence $\mathfrak{q}(x_n)$ is bounded, because $\{x_n\}$ is $\mathfrak{q}$-Cauchy. Let M be its supremum. Then

$$\varphi(x_n a, x_n a)^{1/2} \leq q(x_n) \leq M, \quad \forall a \in \mathfrak{A}_0 \text{ with } \varphi(a, a) = 1.$$

Hence

$$\varphi(xa, xa)^{1/2} = \lim_{n\to\infty} \varphi(x_n a, x_n a)^{1/2} \leq M.$$

Thus, clearly, $\mathfrak{q}(x) \leq M$, i.e. $x \in \mathfrak{X}_{\mathfrak{q}}$. Finally, using the uniqueness of the limit in the completion of $\mathfrak{X}_{\mathfrak{q}}$, we conclude that $x = \mathfrak{q}\text{-}\lim_{n\to\infty} x_n$. Thus $\mathfrak{X}_{\mathfrak{q}}$ is complete. ∎

We observe that in general the inclusion $\mathfrak{A}_0 \subseteq \mathfrak{X}_{\mathfrak{q}}$ is proper. For instance, in $(L^p(I), C(I))$ any step function s defined on $[0,1]$ is in $L^p(I)$ but not in $C(I)$. It is immediate to verify that $s \in (L^p(I))_{\mathfrak{q}}$.

Our next goal is to prove that, for regular Banach quasi *-algebras, $\mathfrak{X}_{\mathfrak{q}}$ is exactly the set of elements having finite spectral radius.

We begin with the following

Proposition 3.23. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with sufficient $\mathcal{S}(\mathfrak{X})$. Then for every $x \in \mathfrak{X}$ the maps*

$$L_x : a \in \mathfrak{A}_0 \mapsto xa \in \mathfrak{X}, \qquad R_x : a \in \mathfrak{A}_0 \mapsto ax \in \mathfrak{X}$$

are closable in $\mathfrak{X}$.

Proof. Let $x \in \mathfrak{X}$ and $\{a_n\} \subset \mathfrak{A}_0$ be a sequence $\|\cdot\|$-converging to zero and such that $xa_n \to y$ with respect to $\|\cdot\|$. Then, if $\varphi \in \mathcal{S}(\mathfrak{X})$ and $b_1, b_2 \in \mathfrak{A}_0$, we get

$$\begin{aligned}|\varphi(yb_1, b_2)| &\leq |\varphi((y - xa_n)b_1, b_2)| + |\varphi(a_n b_1, x^* b_2)| \\ &\leq \|y - xa_n\|\, \|b_1\|_0 \|b_2\|_0 + \|a_n\|\, \|b_1\|_0 \|x^* b_2\| \to 0.\end{aligned}$$

Therefore $\varphi(yb_1, b_2) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$ and $b_1, b_2 \in \mathfrak{A}_0$. By Lemma 3.13, $y = 0$. The proof for R_x is similar. ∎

The previous proposition suggests a handy criterion for the existence of a bounded inverse of an element:

Proposition 3.24. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit e and sufficient $\mathcal{S}(\mathfrak{X})$. Let $x \in \mathfrak{X}$ satisfy the following conditions*:

(i) *there exists $\gamma > 0$ such that*

$$\min\{\|ax\|, \|xa\|\} \geq \gamma \|a\|, \qquad \forall a \in \mathfrak{A}_0;$$

(ii) *the sets $\{ax : a \in \mathfrak{A}_0\}$ and $\{xa : a \in \mathfrak{A}_0\}$ are both dense in $\mathfrak{X}$.*

Then x has a bounded inverse.

Proof. Let $x \in \mathfrak{X}$ satisfy (i) and (ii). Then, making use of standard techniques for closable maps in Banach spaces, one can prove that the range of the closure $\bar{L}_x$ of L_x is the whole space $\mathfrak{X}$. Moreover, it is easy to prove that

$$\|\bar{L}_x y\| \geq \gamma \|y\|, \qquad \forall y \in D(\bar{L}_x),$$

where $D(\bar{L}_x)$ denotes the domain of $\bar{L}_x$. Therefore, there exists a unique $b_1 \in D(\bar{L}_x)$ such that $\bar{L}_x b_1 = e$.

Let $\{z_n\} \subset \mathfrak{A}_0$ with $\|b_1 - z_n\| \to 0$ and $\{xz_n\}$ converging in $\mathfrak{X}$. Then, for every $a \in \mathfrak{A}_0$, $\|b_1 a - z_n a\| \to 0$ and $\{x(z_n a)\}$ converges in $\mathfrak{X}$. Hence $b_1 a \in D(\bar{L}_x)$ and

$$\bar{L}_x(b_1 a) = \lim_{n\to\infty} x(z_n a) = \lim_{n\to\infty} (xz_n)a = (\bar{L}_x b_1)a = ea = a.$$

Therefore $\bar{L}_x(b_1 a) = (\bar{L}_x b_1)a$ for every $a \in \mathfrak{A}_0$. Hence

$$\|\bar{L}_x(b_1 a)\| \ge \gamma \|b_1 a\|, \qquad \forall a \in \mathfrak{A}_0.$$

This implies that

$$\|b_1 a\| \le \frac{1}{\gamma}\,\|a\|, \qquad \forall a \in \mathfrak{A}_0.$$

Hence b_1 is left bounded.

In a similar way one shows the existence of a unique right bounded element $b_2 \in D(\overline{R}_x)$ such that $\overline{R}_x b_2 = e$.

We now prove that $b_1 = b_2$. Let $z_{i,n}$ $(i = 1, 2)$ be a sequence in $\mathfrak{A}_0$ such that $\|z_{i,n} - b_i\| \to 0$ and $\{xz_{i,n}\}$ converges in $\mathfrak{X}$. For every $\varphi \in \mathcal{S}(\mathfrak{X})$ and $c \in \mathfrak{A}_0$, we have

$$\begin{aligned}\varphi(b_2 c, c) &= \varphi(b_2(\bar{L}_x b_1)c, c) = \varphi((\bar{L}_x b_1)c, b_2^* c)\\ &= \lim_{n\to\infty} \varphi((\bar{L}_x z_{1,n})c, z_{2,n}^* c) = \lim_{n\to\infty} \varphi(x(z_{1,n}c), z_{2,n}^* c)\\ &= \lim_{n\to\infty} \varphi(z_{1,n}c, (z_{2,n}x)^* c) = \varphi(b_1 c, c),\end{aligned}$$

since $z_{2,n}x \to \overline{R}_x b_2 = e$. The sufficiency of $\mathcal{S}(\mathfrak{X})$ implies that $b_1 = b_2$. We put $b := b_1 = b_2$. Then $b \in \mathfrak{X}_{\mathrm{b}}$.

We finally prove that $\bar{L}_x b = \overline{R}_b x$. Let $\varphi \in \mathcal{S}(\mathfrak{X})$ and let $\{z_n\} \subset \mathfrak{A}_0$ with $\|b - z_n\| \to 0$ and $\{xz_n\}$ converging in $\mathfrak{X}$. Then, for every $c \in \mathfrak{A}_0$,

$$\varphi((\bar{L}_x b)c, c) = \lim_{n\to\infty} \varphi((xz_n)c, c) = \lim_{n\to\infty} \varphi(z_n c, x^* c) = \varphi(bc, x^* c).$$

On the other hand, if $\{a_n\} \subset \mathfrak{A}_0$ with $\|x - a_n\| \to 0$, then for every $c \in \mathfrak{A}_0$,

$$\begin{aligned}\varphi((\overline{R}_b x)c, c) &= \lim_{n\to\infty} \varphi((R_b a_n)c, c) = \lim_{n\to\infty} \varphi((a_n b)c, c)\\ &= \lim_{n\to\infty} \varphi(bc, a_n^* c) = \varphi(bc, x^* c).\end{aligned}$$

The sufficiency of $\mathcal{S}(\mathfrak{X})$ implies the desired equality.

Analogously, one can prove that $\overline{R}_x b = \bar{L}_b x$. In conclusion, $b \in \mathfrak{X}_{\mathrm{b}}$ and $\bar{L}_b x = \overline{R}_b x = e$, i.e. x has a bounded inverse. ∎

Proposition 3.25. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a regular Banach quasi *-algebra with unit e. Let $x \in \mathfrak{X}_{\mathfrak{q}}$ and $\lambda \in \mathbb{C}$ with $|\lambda| > \mathfrak{q}(x)$. Then $x - \lambda e$ has a bounded inverse $(x - \lambda e)_{\mathrm{b}}^{-1} \in \mathfrak{X}_{\mathrm{b}}$. Thus*

$$\{\lambda \in \mathbb{C} : |\lambda| > \mathfrak{q}(x)\} \subseteq \varrho(x).$$

Proof. Let $\varphi \in \mathcal{S}(\mathfrak{X})$. By definition, if $x \in D(\mathfrak{q})$, then

$$|\lambda| > \mathfrak{q}(x) \ge \varphi(xb, xb), \qquad \forall b \in \mathfrak{A}_0 \text{ with } \varphi(b, b) = 1.$$

Therefore, for every $a \in \mathfrak{A}_0$,

$$\begin{aligned}\varphi((x-\lambda e)a,(x-\lambda e)a)^{1/2} &\geq |\lambda|\varphi(a,a)^{1/2} - \varphi(xa,xa)^{1/2}\\ &\geq (|\lambda| - \mathfrak{q}(x))\,\varphi(a,a)^{1/2}.\end{aligned}$$

Taking the supremum over $\varphi \in \mathcal{S}(\mathfrak{X})$ we get

$$\mathfrak{p}((x-\lambda e)a) \geq \mathfrak{p}(a)(|\lambda| - \mathfrak{q}(x)).$$

From the regularity of $(\mathfrak{X}, \mathfrak{A}_0)$, we finally get

$$\|(x-\lambda e)a\| \geq \|a\|(|\lambda| - \mathfrak{q}(x)), \quad \forall a \in \mathfrak{A}_0.$$

Furthermore, if $q(x) < \infty$ and $|\lambda| > \mathfrak{q}(x)$, then the sets

$$\operatorname{Ran} L_{x-\lambda e} := \{(x-\lambda e)b : b \in \mathfrak{A}_0\}, \quad \operatorname{Ran} R_{x-\lambda e} := \{b(x-\lambda e) : b \in \mathfrak{A}_0\}$$

are $\|\cdot\|$-dense in $\mathfrak{X}$.

Indeed, assume, for instance, that $\operatorname{Ran} L_{x-\lambda e}$ is not dense in $\mathfrak{X}$. Then there exists a non-zero $\| \|$-continuous functional f on $\mathfrak{X}$ such that $f((x-\lambda)b) = 0$ for every $b \in \mathfrak{A}_0$. Therefore $f(xb) = \lambda f(b)$ for every $b \in \mathfrak{A}_0$. From the $\| \|$-continuity of f we get $|f(xb)| \leq \|f\|^\sharp \|xb\|$ for every $b \in \mathfrak{A}_0$.

From the regularity of $(\mathfrak{X}, \mathfrak{A}_0)$ and from Proposition 3.9(i), we get

$$|f(xb)| \leq \|f\|^\sharp\|xb\| = \|f\|^\sharp\mathfrak{p}(xb) \leq \|f\|^\sharp\mathfrak{q}(x)\mathfrak{p}(b) = \|f\|^\sharp\mathfrak{q}(x)\|b\|, \quad \forall b \in \mathfrak{A}_0.$$

The functional f_x defined by $f_x(b) := f(xb)$, $b \in \mathfrak{A}_0$, is $\| \|$-continuous, since

$$|f_x(b)| = |\lambda f(b)| \leq |\lambda|\,\|f\|^\sharp\|b\|, \quad \forall b \in \mathfrak{A}_0.$$

An easy computation shows that $\|f_x\|^\sharp = |\lambda|\,\|f\|^\sharp$. Thus we find the following contradictory inequality: $|\lambda| \leq \mathfrak{q}(x)$. A similar argument shows the corresponding statement for $R_{x-\lambda e}$.

Applying Proposition 3.24 we get the result. ∎

We can now prove the following

THEOREM 3.26. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a regular Banach quasi *-algebra with unit e. Then $D(\mathfrak{q})$ coincides with the set $\mathfrak{X}_{\mathrm{b}}$ of all bounded elements of $\mathfrak{X}$. Moreover*

$$\mathfrak{q}(x) = \|x\|_{\mathrm{b}}, \quad \forall x \in \mathfrak{X}_{\mathrm{b}}.$$

Therefore $(\mathfrak{X}_{\mathrm{b}}, \|\cdot\|_{\mathrm{b}})$ is a C^*-algebra.

Proof. Propositions 2.27 and 3.25 show that $D(\mathfrak{q}) \subseteq \mathfrak{X}_{\mathrm{b}}$. On the other hand, consider, for each $\varphi \in \mathcal{P}(\mathfrak{X})$, the linear functional ω_φ defined by

$$\omega_\varphi(x) = \varphi(x, e), \quad x \in \mathfrak{X}_{\mathrm{b}}.$$

A simple limit argument shows that ω_φ is positive (i.e. $\omega(x^* \bullet x) \geq 0$ for each $x \in \mathfrak{X}_{\mathrm{b}}$), so if π_φ denotes the corresponding GNS representation, then $\pi_\varphi(x)$ is bounded and $\|\overline{\pi_\varphi(x)}\| \leq \|x\|_{\mathrm{b}}$ for every $x \in \mathfrak{X}_{\mathrm{b}}$. Thus, if $x \in \mathfrak{X}_{\mathrm{b}}$, then by Proposition 3.10(ii),

$$\mathfrak{q}(x) = \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| \leq \|x\|_{\mathrm{b}}.$$

From Proposition 3.9(i) it follows that

$$\|xa\| = \mathfrak{p}(xa) \leq \mathfrak{q}(x)\mathfrak{p}(a) = \mathfrak{q}(x)\|a\|, \qquad \forall x \in D(q),\, a \in \mathfrak{A}_0,$$

and, by taking the involution, also

$$\|ax\| \leq \mathfrak{q}(x)\|a\|, \qquad \forall x \in D(q),\, a \in \mathfrak{A}_0.$$

This implies that $\|x\|_{\mathrm{b}} \leq \mathfrak{q}(x)$. Thus, in conclusion, $\|\cdot\|_{\mathrm{b}}$ is a C^*-norm. ∎

A further characterization of the set of bounded elements of $(\mathfrak{X}, \mathfrak{A}_0)$, in the case where $\mathcal{S}(\mathfrak{X})$ is sufficient, can be obtained in terms of representations.

THEOREM 3.27. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit e. Assume that $\mathcal{S}(\mathfrak{X})$ is sufficient. Then $(\mathfrak{X}, \mathfrak{A}_0)$ admits a faithful *-representation π in a Hilbert space $\mathcal{H}$. Moreover*

$$\mathfrak{X}_{\mathrm{b}} = \{x \in \mathfrak{X} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H})\}$$

and

$$\|\overline{\pi(x)}\| = \mathfrak{q}(x), \qquad \forall x \in \mathfrak{X}_{\mathrm{b}}.$$

Proof. For each $\varphi \in \mathcal{P}(\mathfrak{X})$, let π_φ be the corresponding GNS construction with dense domain $\mathcal{D}_\varphi \subseteq \mathcal{H}_\varphi$. Put

$$\mathcal{H} = \bigoplus_{\varphi \in \mathcal{P}(\mathfrak{X})} \mathcal{H}_\varphi = \Big\{(\xi_\varphi)_{\varphi \in \mathcal{P}(\mathfrak{X})} : \xi_\varphi \in \mathcal{H}_\varphi,\, \sum_{\varphi \in \mathcal{P}(\mathfrak{X})} \|\xi_\varphi\|^2 < \infty\Big\},$$

with the usual inner product

$$\langle (\xi_\varphi) \,|\, (\eta_\varphi) \rangle = \sum_{\varphi \in \mathcal{S}(\mathfrak{X})} \langle \xi_\varphi \,|\, \eta_\varphi \rangle, \qquad (\xi_\varphi), (\eta_\varphi) \in \mathcal{H}.$$

Let

$$\mathcal{D} = \Big\{(\xi_\varphi) \in \mathcal{H} : \xi_\varphi \in \mathcal{D}_\varphi,\, \varphi \in \mathcal{P}(\mathfrak{X}),\, \sum_{\varphi \in \mathcal{S}(\mathfrak{X})} \|\pi_\varphi(x)\xi_\varphi\|^2 < \infty,\, \forall x \in \mathfrak{X}\Big\}.$$

Then $\mathcal{D}$ is a dense domain in $\mathcal{H}$ and so we can define, for $x \in \mathfrak{X}$,

$$\pi(x)(\xi_\varphi) = (\pi_\varphi(x)\xi_\varphi), \qquad (\xi_\varphi) \in \mathcal{D}.$$

Then $\pi(x) \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ for each $x \in \mathfrak{X}$ and $\pi : x \in \mathfrak{X} \mapsto \pi(x) \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$. Moreover, π is faithful, since

$$\pi(x) = 0 \;\Leftrightarrow\; \pi_\varphi(x) = 0, \forall \varphi \in \mathcal{P}(\mathfrak{X}) \;\Leftrightarrow\; \varphi(x, x) = 0, \forall \varphi \in \mathcal{P}(\mathfrak{X}).$$

The sufficiency of $\mathcal{S}(\mathfrak{X})$ then implies that $x = 0$.

Finally, $\pi(x)$ is bounded if, and only if, each π_φ for $\varphi \in \mathcal{P}(\mathfrak{X})$ is bounded and

$$\sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| < \infty,$$

and, in this case,

$$\|\overline{\pi(x)}\| = \sup_{\varphi\in\mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\|, \qquad x \in \mathfrak{X}.$$

But, by Proposition 3.10(ii),

$$\sup_{\varphi\in\mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| = \mathfrak{q}(x).$$

This concludes the proof. ∎

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