Bounded elements and spectrum in Banach quasi \(*\)-algebras

by

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Abstract. A normal Banach quasi \(*\)-algebra \((\mathcal{X}, \mathcal{A}_0)\) has a distinguished Banach \(*\)-algebra \(\mathcal{X}_b\) consisting of bounded elements of \(\mathcal{X}\). The latter \(*\)-algebra is shown to coincide with the set of elements of \(\mathcal{X}\) having finite spectral radius. If the family \(P(\mathcal{X})\) of bounded invariant positive sesquilinear forms on \(\mathcal{X}\) contains sufficiently many elements then the Banach \(*\)-algebra of bounded elements can be characterized via a \(C^*\)-seminorm defined by the elements of \(P(\mathcal{X})\).

1. Introduction. A quasi \(*\)-algebra [15] is a couple \((\mathcal{X}, \mathcal{A}_0)\), where \(\mathcal{X}\) is a vector space with involution \(*\), \(\mathcal{A}_0\) is a \(*\)-algebra and a vector subspace of \(\mathcal{X}\), and \(\mathcal{X}\) is an \(\mathcal{A}_0\)-bimodule whose module operations and involution extend those of \(\mathcal{A}_0\).

Quasi \(*\)-algebras were introduced by Lassner [11, 12] with the purpose of providing a reasonable mathematical environment for properly dealing with the thermodynamical limit of local observables of certain quantum statistical models that did not fit into the set-up developed by Haag and Kastler [10]. For this purpose, of course, a topological structure with sufficiently many reasonable properties is needed; in other terms, locally convex quasi \(*\)-algebras have to be considered [1, 18]. The simplest way to construct such an object consists in taking the completion of a locally convex \(*\)-algebra \((\mathcal{A}_0, \tau)\) where the multiplication is separately but not jointly continuous. Of particular interest is, of course, the case where \(\tau\) is a norm topology. This situation has however received so far a rather limited attention, in spite of the fact that it covers very familiar examples such as \(L^p\)-spaces (both commutative and non-commutative). Some results in this direction have been obtained for the so called \(CQ^*\)-algebras in a series of papers [3]–[7], [19]–[21].

In this paper we consider the more general case where \((\mathcal{X}, \mathcal{A}_0)\) is a Banach quasi \(*\)-algebra. This means, roughly speaking, that \(\mathcal{X}\) is a Banach space whose norm \(\| \cdot \|\) has certain coupling properties related to the partial multiplication of \((\mathcal{X}, \mathcal{A}_0)\). In Section 2 we study the set \(\mathcal{X}_b\) of bounded

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elements of $\mathcal{X}$, i.e. elements whose associated multiplication operators are bounded linear maps in $\mathcal{X}$. Then we focus our attention on the class of normal Banach quasi $^*$-algebras: they are characterized by the fact that $\mathcal{X}_b$ is a Banach $^*$-algebra. If $(\mathcal{X}, \mathcal{A}_0)$ is normal, the Banach $^*$-algebra $\mathcal{X}_b$ turns out to be useful for defining a notion of spectrum of an element $x \in \mathcal{X}$, which enjoys properties analogous to the spectrum of an element of a Banach $^*$-algebra.

In Section 3 we discuss some properties of the family of bounded positive sesquilinear forms on $\mathcal{X}$ with certain invariance properties and, starting from them, we construct two seminorms $p, q$ that emulate the Gel’fand–Naĭmark seminorm on a Banach $^*$-algebra (but $q$ is only defined on a domain $D(q) \subseteq \mathcal{X}$; it is actually an unbounded $C^*$-seminorm in the sense of [2]). These seminorms are then used to derive some properties of the spectrum of an element $x \in \mathcal{X}$, under the assumption that the class $\mathcal{P}(\mathcal{X})$ of bounded invariant positive sesquilinear forms is rich enough. The outcome is that, in this case, $D(q)$ exactly equals the $^*$-algebra of bounded elements of $\mathcal{X}$ (or, equivalently, the set of elements of $\mathcal{X}$ that have finite spectral radius). Furthermore, it is shown that $(\mathcal{X}, \mathcal{A}_0)$ admits a faithful $^*$-representation $\pi$ and that $D(q)$ also coincides with the set of elements whose image under $\pi$ is a bounded operator.

2. Banach quasi $^*$-algebras

2.1. Basic definitions

**Definition 2.1.** Let $(\mathcal{X}, \mathcal{A}_0)$ be a quasi $^*$-algebra. $(\mathcal{X}, \mathcal{A}_0)$ is called a *Banach quasi $^*$-algebra* if a norm $\| \cdot \|$ is defined on $\mathcal{X}$ with the properties:

(i) $(\mathcal{X}, \| \cdot \|)$ is a Banach space;
(ii) $\|x^*\| = \|x\|$, $\forall x \in \mathcal{X}$;
(iii) $\mathcal{A}_0$ is dense in $\mathcal{X}$;
(iv) for each $a \in \mathcal{A}_0$, the map $R_a : x \in \mathcal{X} \mapsto xa \in \mathcal{X}$ is continuous in $\mathcal{X}$.

The continuity of the involution implies that

(iv') for each $a \in \mathcal{A}_0$, the map $L_a : x \in \mathcal{X} \mapsto ax \in \mathcal{X}$ is continuous in $\mathcal{X}$.

The *unit* of $(\mathcal{X}, \mathcal{A}_0)$ is an element $e \in \mathcal{A}_0$ such that $xe = ex = x$ for every $x \in \mathcal{X}$. If $(\mathcal{X}, \mathcal{A}_0)$ is a Banach quasi $^*$-algebra with unit $e$, we will assume (without loss of generality) that $\|e\| = 1$. If $(\mathcal{X}, \mathcal{A}_0)$ has no unit, it can always be embedded in a Banach quasi $^*$-algebra with unit $e$ in a standard fashion.

In what follows, we will always assume that if $xa = 0$ for every $a \in \mathcal{A}_0$, then $x = 0$ (of course, this is automatically true if $(\mathcal{X}, \mathcal{A}_0)$ has a unit).
If \((X, \mathcal{A}_0)\) is a Banach quasi*-algebra a norm topology can be defined on \(\mathcal{A}_0\) in the following way. Define
\[
\|a\|_L = \sup_{\|x\| \leq 1} \|Ra\| = \sup_{\|x\| \leq 1} \|xa\|
\]
and
\[
\|a\|_R = \sup_{\|x\| \leq 1} \|La\| = \sup_{\|x\| \leq 1} \|ax\|
\]
and finally
\[
\|a\|_0 = \max\{\|a\|, \|a\|_L, \|a\|_R\}.
\]

Then

**Proposition 2.2.** \((\mathcal{A}_0, \| \cdot \|_0)\) is a normed *-algebra. Moreover
\[
\|ab\| \leq \|a\| \|b\|_0, \quad \|ba\| \leq \|a\| \|b\|_0, \quad \forall a, b \in \mathcal{A}_0.
\]

The above statements follow immediately from the corresponding properties of algebras of bounded operators on a normed space. The two inequalities come directly from the definitions.

Clearly, \(\|b\| \leq \|b\|_0\) for each \(b \in \mathcal{A}_0\).

**Definition 2.3.** A Banach quasi *-algebra \((X, \mathcal{A}_0)\) is called a BQ*-algebra if \((\mathcal{A}_0, \| \cdot \|_0)\) is a Banach *-algebra, and a proper CQ*-algebra if \((\mathcal{A}_0, \| \cdot \|_0)\) is a C*-algebra.

**2.1.1. Examples**

**Example 2.4 (Banach function spaces).** Many Banach function spaces provide examples of Banach quasi *-algebras since they often contain a dense *-algebra of functions. For instance, if \(I = [0, 1]\) then \((L^p(I), C(I))\), where \(C(I)\) denotes the C*-algebra of all continuous functions on \(I\) and \(p \geq 1\), is a Banach quasi *-algebra (more precisely a proper CQ*-algebra). Similarly \((L^p(\mathbb{R}), C^0_0(\mathbb{R}))\) is a Banach quasi *-algebra without unit (here \(C^0_0(\mathbb{R})\) is the *-algebra of continuous functions in \(\mathbb{R}\) with compact support). Other examples are easily found among Sobolev spaces, Besov spaces etc.

**Example 2.5 (Non-commutative \(L^p\)-spaces).** Let \(\mathcal{M}\) be a von Neumann algebra and \(\tau\) a normal semifinite faithful trace \([17]\) on \(\mathcal{M}\). Then the completion of the *-ideal
\[
\mathcal{J}_p = \{X \in \mathcal{M} : \tau(|X|^p) < \infty\}
\]
with respect to the norm
\[
\|X\|_p = \tau(|X|^p)^{1/p}, \quad X \in \mathcal{M},
\]
is usually called \(L^p(\tau)\) \([13, 16]\) and is a Banach space consisting of operators affiliated with \(\mathcal{M}\). Then \((L^p(\tau), \mathcal{J}_p)\) is a Banach quasi *-algebra (without unit). If \(\tau\) is a finite trace then \((L^p(\tau), \mathcal{M})\) is a BQ*-algebra.
Example 2.6 (Hilbert algebras). A Hilbert algebra [14, Section 11.7] is a $\ast$-algebra $\mathfrak{A}_0$ which is also a pre-Hilbert space with inner product $\langle \cdot, \cdot \rangle$ such that

(i) the map $b \mapsto ab$ is continuous with respect to the norm defined by the inner product;
(ii) $\langle ab, c \rangle = \langle b, a^*c \rangle$ for all $a, b, c \in \mathfrak{A}_0$;
(iii) $\langle a, b \rangle = \langle b^*, a^* \rangle$ for all $a, b \in \mathfrak{A}_0$;
(iv) $\mathfrak{A}_0^2$ is total in $\mathfrak{A}_0$.

Let $\mathcal{H}$ denote the Hilbert space which is the completion of $\mathfrak{A}_0$ with respect to the norm defined by the inner product. The involution of $\mathfrak{A}_0$ extends to the whole of $\mathcal{H}$, since (iii) implies that $\ast$ is isometric. Then $(\mathcal{H}, \mathfrak{A}_0)$ is a Banach quasi $\ast$-algebra.

2.2. Bounded elements

Definition 2.7. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi $\ast$-algebra and $x \in \mathfrak{X}$. We say that $x$ is left bounded if there exists $\gamma_x > 0$ such that

$$\|xa\| \leq \gamma_x \|a\|, \quad \forall a \in \mathfrak{A}_0.$$ 

The set of all left bounded elements of $\mathfrak{X}$ is denoted by $\mathfrak{X}⧷$. Analogously, we say that $x$ is right bounded if there exists $\gamma'_x > 0$ such that

$$\|ax\| \leq \gamma'_x \|a\|, \quad \forall a \in \mathfrak{A}_0.$$ 

The set of all right bounded elements of $\mathfrak{X}$ is denoted by $\mathfrak{X}♠$.

The terminology is motivated by the fact that, if $x$ is left bounded, the map

$$a \in \mathfrak{A}_0 \mapsto L_x a = xa$$

is bounded on $\mathfrak{A}_0$ and so it has a bounded extension $\tilde{L}_x$ to $\mathfrak{X}$. We put

$$\|x\|⧷ = \max\{\|x\|, \|\tilde{L}_x\|\}.$$ 

Analogously, we define a norm on $\mathfrak{X}♠$ by

$$\|x\|♠ = \max\{\|x\|, \|\tilde{R}_x\|\}.$$ 

We put $\mathfrak{X}_b = \mathfrak{X}⧷ \cap \mathfrak{X}♠$. Clearly, $\mathfrak{A}_0 \subseteq \mathfrak{X}_b$. On $\mathfrak{X}_b$ we define the norm

$$\|x\|_b = \max\{\|x\|, \|\tilde{L}_x\|, \|\tilde{R}_x\|\}.$$ 

Remark 2.8. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit $e$, then since $\|e\| = 1$, we have $\|\tilde{L}_x\| \geq \|x\|$ for every $x \in \mathfrak{X}⧷$, and therefore $\|x\|⧷ = \|\tilde{L}_x\|$. Analogous statements hold for $\| \cdot \|_\clubsuit$ and $\| \cdot \|_b$.

As usual, we denote by $\mathcal{B}(\mathfrak{X})$ the Banach algebra of bounded operators in the Banach space $\mathfrak{X}$. From the definition it follows that $\mathfrak{X}⧷$, as well as $\mathfrak{X}♠$, can be identified with a subspace of $\mathcal{B}(\mathfrak{X})$. 


Let \( x \in \mathcal{X}_\triangleright \) and \( y \in \mathcal{X} \). Then we put
\[
(2.1) \quad x \triangleright y = \bar{L}_x y.
\]
Similarly, if \( y \in \mathcal{X}_\blacktriangleleft \) and \( x \in \mathcal{X} \), we put
\[
(2.2) \quad x \blacktriangleleft y = \bar{R}_y x.
\]

**Remark 2.9.** We notice that an element \( x \in \mathcal{X}_\triangleright \) is not necessarily right bounded.

If \( x, y \in \mathcal{X}_b \) then both \( x \triangleright y \) and \( x \blacktriangleleft y \) are well defined, but, in general, \( x \triangleright y \neq x \blacktriangleleft y \). Conditions for the equality to hold will be given later.

It is easy to show that if \( x, y \in \mathcal{X}_\triangleright \) and \( \mu \in \mathbb{C} \) then both \( x + y \) and \( \mu x \) belong to \( \mathcal{X}_\triangleright \).

**Proposition 2.10.** If \( (\mathcal{X}, \mathfrak{A}_0) \) is a Banach quasi \(*\)-algebra, then the set \( \mathcal{X}_\triangleright \) of all left bounded elements is a Banach algebra with respect to the multiplication \( \triangleright \) and the norm \( \| \cdot \|_\triangleright \).

**Proof.** (i) We prove that if \( x, y \in \mathcal{X}_\triangleright \) then \( x \triangleright y \in \mathcal{X}_\triangleright \) and
\[
\| x \triangleright y \|_\triangleright \leq \| x \|_\triangleright \| y \|_\triangleright .
\]
Indeed, for each \( a \in \mathfrak{A}_0 \) one has, using the associativity properties of the multiplication in \( \mathcal{X} \),
\[
(L_x y)a = \lim_{m \to \infty} (x b_m) a = \lim_{m \to \infty} x (b_m a) = L_x (ya) = L_x (L_y a),
\]
where \( \{b_m\} \) is a sequence in \( \mathfrak{A}_0 \), \( \| \cdot \| \)-converging to \( y \). Therefore,
\[
\| (L_x y)a \| \leq \| L_x \| \| L_y \| \| a \|, \quad \forall a \in \mathfrak{A}_0.
\]
Hence \( x \triangleright y \in \mathcal{X}_\triangleright \), \( L_x \triangleright y = L_x L_y \) and
\[
\| L_x \triangleright y \| \leq \| L_x \| \| L_y \| \leq \| x \|_\triangleright \| y \|_\triangleright .
\]
Since \( \| x \triangleright y \| \leq \| x \|_\triangleright \| y \|_\triangleright \), we finally get
\[
\| x \triangleright y \|_\triangleright \leq \| x \|_\triangleright \| y \|_\triangleright .
\]
Thus, \( \mathcal{X}_\triangleright \) endowed with \( \| \cdot \|_\triangleright \) is a normed algebra. We will now show that \( (\mathcal{X}_\triangleright , \| \cdot \|_\triangleright ) \) is complete. Let \( \{x_n\} \) be a Cauchy sequence in \( (\mathcal{X}_\triangleright , \| \cdot \|_\triangleright ) \). Then \( \{L_{x_n}\} \) is a Cauchy sequence in \( \mathcal{B}(\mathcal{X}) \). Thus there exists \( L \in \mathcal{B}(\mathcal{X}) \) such that \( L_{x_n} \to L \) with respect to the natural norm of \( \mathcal{B}(\mathcal{X}) \). Since \( \| x_n - x_m \| \to 0 \), there exists \( x \in \mathcal{X} \) such that \( \| x_n - x \| \to 0 \). Since the right multiplication by \( a \) is continuous in \( \mathcal{X} \), it follows that \( x_n a \to xa = L_x a \) in the norm of \( \mathcal{X} \). This implies that \( L_x = L \). From these facts it follows easily that \( x \) is left bounded and \( x_n \to x \) with respect to \( \| \cdot \|_\triangleright \).

A similar result can be proved for \( \mathcal{X}_\blacktriangleleft \) taking into account the following facts concerning the involution \( * \) of \( \mathcal{X} \):
\[
(1^*) \quad x \in \mathcal{X}_\triangleright \Leftrightarrow x^* \in \mathcal{X}_\triangleright ;
\]
(2*) \(\|x^*\|_\|\| = \|x\|_\|\| \) for every \(x \in \mathcal{X}_\|\|\); 
(3*) \((x \triangledown y)^* = y^* \triangledown x^*\) for every \(x, y \in \mathcal{X}_\|\|\).

Definition 2.7 easily yields

**Lemma 2.11.**

(i) If \(x \in \mathcal{X}_\|\|\) and \(y \in \mathcal{X}\), then \(\|x \triangledown y\| \leq \|x\|_\|\| \cdot \|y\|_\|\|\).

(ii) If \(y \in \mathcal{X}_\|\|\) and \(x \in \mathcal{X}\), then \(\|x \triangledown y\| \leq \|x\|_\|\| \cdot \|y\|_\|\|\).

If \(x, y \in \mathcal{X}_b\) then, as noticed before, both \(x \triangledown y\) and \(x \triangledown y\) are well defined, but, in general, \(x \triangledown y \neq x \triangledown y\). We want to analyze this situation more carefully. First of all, if \(x, y \in \mathcal{X}_b\), then \(L_x, L_y \in \mathcal{B}(\mathcal{X})\). As shown in the proof of Proposition 2.10, \(L_x L_y = L_{x \triangledown y}\). Similarly, if \(x, y \in \mathcal{X}_b\), then \(R_y R_x = R_{x \triangledown y}\).

In what follows, we denote by \(\mathcal{X}^\|\|\) the Banach dual space of \((\mathcal{X}, \|\|\|\)). The norm in \(\mathcal{X}^\|\|\) is defined, as usual, by \(\|f\|^2 = \sup_{\|x\|_\|\| \leq 1} |f(x)|\) for \(f \in \mathcal{X}^\|\|\).

**Proposition 2.12.** The following statements are equivalent.

(i) \(x \triangledown y = x \triangleleft y\) for every \(x, y \in \mathcal{X}_b\).

(ii) \(x \triangledown y\) is right bounded and \(\|x \triangledown y\| \leq \|x\|_\|\| \cdot \|y\|_\|\|\) for every \(x, y \in \mathcal{X}_b\).

(iii) \(x \triangleleft y\) is left bounded and \(\|x \triangleleft y\| \leq \|x\|_\|\| \cdot \|y\|_\|\|\) for every \(x, y \in \mathcal{X}_b\).

(iv) For any pair \(\{a_n\}, \{b_n\}\) of sequences of elements of \(\mathcal{A}_0\), \(\|\cdot\|\|\)-converging to elements of \(\mathcal{X}_b\), one has

\[
\lim_{n \to \infty} \lim_{m \to \infty} a_n b_m = \lim_{m \to \infty} \lim_{n \to \infty} a_n b_m.
\]

(v) There exists a weak \(^*\)-dense subspace \(\mathcal{M}\) of \(\mathcal{X}^\|\|\) such that for any pair \(\{a_n\}, \{b_n\}\) of sequences of elements of \(\mathcal{A}_0\), \(\|\cdot\|\|\)-converging to elements of \(\mathcal{X}_b\), one has

\[
\lim_{n \to \infty} \lim_{m \to \infty} f(a_n b_m) = \lim_{m \to \infty} \lim_{n \to \infty} f(a_n b_m), \quad \forall f \in \mathcal{M}.
\]

**Proof.** (i)\(\Rightarrow\)(ii): Clearly, the equality \(x \triangledown y = x \triangleleft y\) implies that \(x \triangledown y\) is right bounded and for \(x \triangledown y\) the inequality in Lemma 2.11(ii) holds.

(ii)\(\Leftrightarrow\)(iii) follows easily by taking \(^*\).

(iii)\(\Rightarrow\)(i): Assume that, for every \(x, y \in \mathcal{X}_b\), \(x \triangleleft y\) is left bounded and \(\|x \triangleleft y\| \leq \|x\|_\|\| \cdot \|y\|_\|\|\). Let \(\{b_n\} \subset \mathcal{A}_0\) be such that \(\|y - b_n\| \to 0\) as \(n \to \infty\). Then, since \(\mathcal{A}_0 \subseteq \mathcal{X}_b\) and \(\mathcal{X}_b\) is a vector space, we get

\[
\|x \triangleleft y - x b_n\| = \|x \triangleleft y - x \triangleleft b_n\| = \|x \triangleleft (y - b_n)\| \leq \|x\|_\|\| \cdot \|y - b_n\| \to 0.
\]

Hence

\[
x \triangleleft y = \lim_{n \to \infty} x b_n = L_x y = x \triangledown y.
\]

(i)\(\Rightarrow\)(iv): Let \(\{a_n\}, \{b_n\} \subset \mathcal{A}_0\) with \(\|x - a_n\| \to 0, \|y - b_n\| \to 0\) and \(x, y \in \mathcal{X}_b\). Then

\[
x \triangledown y = L_x y = \lim_{m \to \infty} x b_m = \lim_{m \to \infty} \lim_{n \to \infty} a_n b_m.
\]
On the other hand, 
\[ x \triangleright y = \overline{R_y} x = \lim_{n \to \infty} a_n y = \lim_{n \to \infty} \lim_{m \to \infty} a_n b_m. \]

The equality \( x \triangleright y = x \blacktriangleleft y \) then implies that the two iterated limits coincide.

(iv) \( \Rightarrow \) (v): This is clear.

(v) \( \Rightarrow \) (i): Assume that (i) fails. Then there exists \( f \in X^\flat \) such that \( f(x \triangleright y) \neq f(x \blacktriangleleft y) \). Since \( M \) is weak*-dense in \( X^\flat \), we may suppose that \( f \in M \).

Then, if \( \{a_n\}, \{b_n\} \subset A_0 \) \( \parallel \cdot \parallel \)-converge, respectively, to \( x \) and \( y \), we have
\[ \lim_{m \to \infty} \lim_{n \to \infty} f(a_n b_m) = f(x \triangleright y) \neq f(x \blacktriangleleft y) = \lim_{n \to \infty} \lim_{m \to \infty} f(a_n b_m). \]
This completes the proof.

If any of the equivalent conditions of Proposition 2.12 hold, we put \( x \bullet y := x \triangleright y = x \blacktriangleleft y \), \( x, y \in X_\flat \).

**Definition 2.13.** A Banach quasi *-algebra \((X, A_0)\) such that \( x \triangleright y = x \blacktriangleleft y \) for every \( x, y \in X_\flat \) is called normal.

**Corollary 2.14.**

(i) \((X, A_0)\) is normal if, and only if, \( X_\flat \) is a *-algebra with respect to \( \triangleright \) (or, equivalently, with respect to \( \blacktriangleleft \)).

(ii) If \((X, A_0)\) is a normal Banach quasi *-algebra, then \((X_\flat, \parallel \cdot \parallel_b)\) is a Banach *-algebra with respect to the multiplication \( \bullet \).

**Proof.** (i) The fact that if \((X, A_0)\) is normal, then \( X_\flat \) is a *-algebra with respect to \( \triangleright \) follows from the previous discussion. On the other hand, assume that \( X_\flat \) is a *-algebra with respect to \( \triangleright \); then, for every \( x, y \in X_\flat \), \( x \triangleright y \in X_\flat \) and
\[ x \blacktriangleleft y = (y^* \triangleright x^*)^* = x \triangleright y. \]

(ii) follows easily from Proposition 2.10 and from the properties of the involution.

**Example 2.15.** Assume that for each \( x \in X_\flat \) there exists a sequence \( \{a_n\} \subset A_0 \) such that 
\[ \sup_n \|a_n\|_0 < \infty \quad \text{and} \quad \lim_{n \to \infty} \|x - a_n\| = 0. \]
Then \((X, A_0)\) is normal. Indeed, in this case, it is easily seen that (ii) or (iii) of Proposition 2.12 holds.

**Remark 2.16.** If \((X, A_0)\) is a commutative Banach quasi *-algebra, i.e. \( xa = ax \) for all \( x \in X \) and \( a \in A_0 \), then it is easily seen that each left bounded element \( x \) is also right bounded and \( x \triangleright y = y \blacktriangleleft x \) for every \( y \in X \). Thus if \( x, y \in X_\flat \) then both \( x \triangleright y \) and \( x \blacktriangleleft y \) are in \( X_\flat \) but they need not be equal. In this case, in general, \( X_\flat \) is an algebra with respect to \( \triangleright \) (and also
with respect to \(\triangleright\). Normality, in the commutative case, is equivalent to \(X_b\) being also commutative.

**Example 2.17.** For the Banach quasi \(*\)-algebra \((L^p(I), C(I))\) considered in Example 2.4, one finds that \((L^p(I))_b = L^\infty(I)\) and the norm \(\| \cdot \|_b\) is exactly the \(L^\infty\)-norm. Since the multiplications \(\triangleright\) and \(\triangleleft\) both coincide with the ordinary multiplication of functions, \((L^p(I), C(I))\) is normal. This example also shows that, in general, \(A_0\) is not dense in \(X_b\) with respect to \(\| \cdot \|_b\) since, as is well known, \(C(I)\) is not dense in \(L^\infty(I)\).

Similarly, \((L^p(\mathbb{R}), C^0(\mathbb{R}))\) is a Banach quasi \(*\)-algebra without unit. In this case \((L^p(\mathbb{R}))_b = L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})\) and \((L^p(\mathbb{R}), C^0(\mathbb{R}))\) is normal. The norm \(\| \cdot \|_b\) is equivalent to \(\| \cdot \|_p + \| \cdot \|_\infty\).

For the non-commutative \(L^p\)-spaces of Example 2.5 one finds that 
\((L^p(\tau))_b = J_p\) if \(\tau\) is semifinite, while \((L^p(\tau))_b = \mathcal{M}\) if \(\tau\) is finite. Normality follows from the fact that the multiplications \(\triangleright\) and \(\triangleleft\) both coincide with the ordinary multiplication of bounded operators.

**Example 2.18.** In the case of the Banach quasi \(*\)-algebra \((H, A_0)\) constructed from a Hilbert algebra \(A_0\) as in Example 2.6, the set \(H_b\) of bounded elements of \(H\) is the so-called *fulfilment* of \(A_0\) (\(A_0\) is called a full Hilbert algebra if \(H_b = A_0\)). \((H, A_0)\) is normal. Indeed, let \(x, y \in H_b\), and let \(\{a_n\}, \{b_n\}\) be sequences in \(A_0\), \(\| \cdot \|\)-converging, respectively, to \(x\) and \(y\). Then
\[
\langle x \triangleright y | a \rangle = \lim_{n \to \infty} \langle xb_n | a \rangle = \lim_{n \to \infty} \langle b_n | x^*a \rangle = \langle y | x^*a \rangle, \quad \forall a \in A_0.
\]
On the other hand,
\[
\langle x \triangleleft y | a \rangle = \lim_{m \to \infty} \langle a_ny | a \rangle = \lim_{m \to \infty} \langle y | a_n^*a \rangle = \langle y | x^*a \rangle, \quad \forall a \in A_0.
\]
This implies that \(x \triangleright y = x \triangleleft y\).

**Lemma 2.19.** If \((X, A_0)\) is a normal Banach quasi \(*\)-algebra, then
\[
(2.3) \quad \bar{L}_x \bar{R}_y = \bar{R}_y \bar{L}_x, \quad \forall x, y \in X_b.
\]

*Proof.* Indeed, let \(x, y \in X_b\), and let \(\{a_n\}, \{b_n\} \subset A_0\) \(\| \cdot \|\)-converge, respectively, to \(x\) and \(y\). Then, for every \(a \in A_0\),
\[
(\bar{L}_x \bar{R}_y)a = \bar{L}_x(\bar{R}_ya) = \lim_{m \to \infty} x(ab_m) = \lim_{m \to \infty} \lim_{n \to \infty} a_n(ab_m).
\]
On the other hand,
\[
(\bar{R}_y \bar{L}_x)a = \bar{R}_y(\bar{L}_xa) = \lim_{n \to \infty} (a_n)a y = \lim_{n \to \infty} \lim_{m \to \infty} (a_n)a b_m.
\]
The statement then follows from Proposition 2.12(iv). \(\square\)

**Remark 2.20.** If \((X, A_0)\) has a unit, then (2.3) implies the normality of \((X, A_0)\).

If \((X, A_0)\) is a normal Banach quasi \(*\)-algebra the products of an element \(x \in X\) and an element \(y \in X_b\) are defined via (2.1) and (2.2).
Proposition 2.21. If \((\mathfrak{X}, \mathfrak{A}_0)\) is a normal Banach quasi \(^*\)-algebra, then 
\((\mathfrak{X}, \mathfrak{X}_b)\) is a BQ\(^*\)-algebra.

Proof. We need only check the module associativity rules. Let \(x \in \mathfrak{X}\) 
and \(y_1, y_2 \in \mathfrak{X}_b\). Then 
\[
x \downarrow (y_1 \bullet y_2) = x \downarrow (y_1 \bullet y_2) = R_{y_1} \downarrow_{y_2} x = (R_{y_2} R_{y_1}) x = R_{y_2} (R_{y_1} x) = (x \downarrow y_1) \bullet y_2.
\]
Using (2.3), we also have 
\[
(y_1 \uparrow x) \bullet y_2 = \overline{R}_{y_2} (y_1 \uparrow x) = \overline{R}_{y_2} (\overline{L}_{y_1} x) = (\overline{R}_{y_2} \overline{L}_{y_1}) x = (\overline{L}_{y_1} \overline{R}_{y_2}) x
\]
\[
= \overline{L}_{y_1} (\overline{R}_{y_2} x) = \overline{L}_{y_1} (x \bullet y_2) = y_1 \uparrow (x \bullet y_2).\]

2.3. The spectrum. Let \((\mathfrak{X}, \mathfrak{A}_0)\) be a normal Banach quasi \(^*\)-algebra with 
unit \(e\) and \(x \in \mathfrak{X}\). We say that \(x\) has a bounded inverse if there exists \(y \in \mathfrak{X}_b\) 
such that \(\overline{R}_y x = \overline{L}_y x = e\). From Proposition 2.21 it follows easily that 
this element \(y\), if any, is unique. If \(x\) has a bounded inverse we denote it 
by \(x_b^{-1}\).

Definition 2.22. The resolvent \(\varrho(x)\) of \(x \in \mathfrak{X}\) is the set 
\[
\varrho(x) = \{ \lambda \in \mathbb{C} : x - \lambda e \text{ has a bounded inverse} \}.
\]
The set \(\sigma(x) = \mathbb{C} \setminus \varrho(x)\) is called the spectrum of \(x\).

Proposition 2.23. Let \(x \in \mathfrak{X}\). Then:

(i) The resolvent \(\varrho(x)\) is an open subset of the complex plane.
(ii) The resolvent function \(R_\lambda(x) : \lambda \in \varrho(x) \mapsto (x - \lambda e)_b^{-1}\) is \(\| \cdot \|_b\)-analytic 
on each connected component of \(\varrho(x)\).
(iii) For any \(\lambda, \mu \in \varrho(x)\), \(R_\lambda(x)\) and \(R_\mu(x)\) commute and 
\[
R_\lambda(x) - R_\mu(x) = (\mu - \lambda) R_\mu(x) \bullet R_\lambda(x).
\]

Proof. (i) Let \(\lambda_0 \in \varrho(x)\) and \(\lambda \in \mathbb{C}\) be such that \(|\lambda - \lambda_0| \leq (\|R_{\lambda_0}(x)\|_b)^{-1}\). 
Then the series 
\[
\sum_{n=1}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}(x)^n
\]
converges in \(\mathfrak{X}_b\) with respect to \(\| \cdot \|_b\) to an element \(S_{\lambda,x}\).

Let now \(T_{\lambda,x} := R_{\lambda_0}(x)(e + S_{\lambda,x})\). It is easily checked, using the \(\| \cdot \|\)-
convergence for the product \(T_{\lambda,x}(x - \lambda e)\), that \(T_{\lambda,x}\) is a bounded inverse of 
\(x - \lambda e\).

(ii) follows immediately from the proof of (i). The proof of (iii) is straightforward. \(\blacksquare\)

The classical argument based on Liouville’s theorem can be applied to prove the following

Proposition 2.24. Let \(x \in \mathfrak{X}\). Then \(\sigma(x)\) is non-empty.
Definition 2.25. Let $x \in \mathfrak{X}$. The non-negative number

$$r(x) = \sup_{\lambda \in \sigma(x)} |\lambda|$$

is called the spectral radius of $x$.

Remark 2.26. Of course, if $x \in \mathfrak{X}_b$ then $\sigma(x)$ coincides with the spectrum of $x$ regarded as an element of the Banach $*$-algebra $\mathfrak{X}_b$. For an arbitrary element $x$, the set $\sigma(x) \subset \mathbb{C}$, which is closed, could be unbounded. The next proposition shows that $\sigma(x)$ is indeed unbounded if $x \in \mathfrak{X} \setminus \mathfrak{X}_b$.

Proposition 2.27. Let $x \in \mathfrak{X}$. Then $r(x) < \infty$ if, and only if, $x \in \mathfrak{X}_b$.

Proof. The “if” part has been discussed in the previous remark. Assume now that $r(x) < \infty$. Then the function $\lambda \mapsto (x - \lambda e)^{-1}$ is $\| \cdot \|^b$-analytic in the region $|\lambda| > r(x)$. Therefore it has there a $\| \cdot \|^b$-convergent Laurent expansion

$$(x - \lambda e)^{-1} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}, \quad |\lambda| > r(x),$$

with $a_k \in \mathfrak{X}_b$ for each $k \in \mathbb{N}$. As usual,

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{(x - \lambda e)^{-1}}{\lambda - k+1} d\lambda, \quad k \in \mathbb{N},$$

where $\gamma$ is a circle centered in 0 and with radius $R > r(x)$. The integral on the r.h.s. converges with respect to $\| \cdot \|^b$. The $\| \cdot \|^b$-continuity of multiplication implies that, as in the ordinary case,

$$xa_k = \frac{1}{2\pi i} \int_{\gamma} \frac{x(x - \lambda e)^{-1}}{\lambda - k+1} d\lambda = \frac{1}{2\pi i} \int_{\gamma} \frac{(x - \lambda e)^{-1}}{\lambda - k} d\lambda = a_{k+1}.$$ 

In particular, using Cauchy’s integral formula, we find $xa_1 = -x$. This implies that $x \in \mathfrak{X}_b$. 

Remark 2.28. If $\lambda \in \rho(x)$ then all powers $(x - \lambda)^{-n}$ exist in $\mathfrak{X}_b$, for every $n \in \mathbb{N}$. This does not imply the existence of $(x - \lambda)^n$ for $n > 1$. As an example, consider the Banach quasi $*$-algebra $(L^2(I), C(I))$ where $I = [0, 1]$ (cf. Example 2.4). The function $v(x) = x^{-1/4}$ is in $L^2(I)$; obviously, $0 \in \rho(v)$ since $v^{-1}(x) = x^{1/4} \in C(I)$. We have $v^{-n}(x) = x^{n/4} \in L^2(I)$ for all $n \in \mathbb{N}$, but $v^2(x) = x^{-1/2} \not\in L^2(I)$.

3. Representations and seminorms. Families of sesquilinear forms have been shown to play a relevant role in the study of the structure of $CQ^*$-algebras [6] or more generally Banach $C^*$-modules [23]. The main reason is that they give rise to representations with operators acting in Hilbert space.
3.1. Representations. Before going on we recall some definitions. Let \( \mathcal{H} \) be a complex Hilbert space and \( D \) a dense subspace of \( \mathcal{H} \). We denote by \( \mathcal{L}^\dagger(D, \mathcal{H}) \) the set of all linear operators \( X \) such that \( D(X) = D \) and \( D(X^*) \supseteq D \). The set \( \mathcal{L}^\dagger(D, \mathcal{H}) \) is a partial *-algebra [1] with respect to the following operations: the usual sum \( X_1 + X_2 \), the scalar multiplication \( \lambda X \), the involution \( X \mapsto X^\dagger = X^*|D \) and the (weak) partial multiplication \( X_1 \diamond X_2 = X_1^\dagger \ast X_2 \), defined whenever \( X_2 \) is a weak right multiplier of \( X_1 \) (equivalently, \( X_1 \) is a weak left multiplier of \( X_2 \)), that is, iff \( X_2 D \subset D(X_1^\dagger) \) and \( X_1^\dagger D \subset D(X_2^\dagger) \) (we write \( X_2 \in R^w(X_1) \) or \( X_1 \in L^w(X_2) \)). Let

\[
\mathcal{L}^\dagger(D) = \{ X \in \mathcal{L}^\dagger(D, \mathcal{H}) : XD \subseteq D, \; X^\dagger D \subseteq D \}.
\]

Then \( \mathcal{L}^\dagger(D) \) is a *-algebra with respect to \( \square \) and \( X_1 \square X_2 \xi = X_1(X_2 \xi) \) for each \( \xi \in D \) (see [15]).

A *-representation of the Banach quasi *-algebra \((\mathfrak{X}, \mathfrak{A}_0)\) is a *-homomorphism of \( \mathfrak{X} \) into \( \mathcal{L}^\dagger(D, \mathcal{H}) \), for some pair \((D, \mathcal{H})\) where \( D \) is a dense subspace of a Hilbert space \( \mathcal{H} \), that is, a linear map \( \pi : \mathfrak{X} \rightarrow \mathcal{L}^\dagger(D, \mathcal{H}) \) such that (i) \( \pi(x^*) = \pi(x)^\dagger \) for every \( x \in \mathfrak{X} \), and (ii) if \( x \in \mathfrak{X} \) and \( a \in \mathfrak{A}_0 \) then \( \pi(x) \in L^w(\pi(a)) \) and \( \pi(x) \square \pi(a) = \pi(xa) \).

A *-representation \( \pi \) of \((\mathfrak{X}, \mathfrak{A}_0)\) is called cyclic if there exists \( \eta \in D \) such that \( \pi(\mathfrak{A}_0) \eta \) is dense in \( \mathcal{H} \), and faithful if \( \pi(x) = 0 \) implies \( x = 0 \).

If \( \pi \) is a *-representation of \((\mathfrak{X}, \mathfrak{A}_0)\) in \( \mathcal{L}^\dagger(D, \mathcal{H}) \), then the closure \( \tilde{\pi} \) of \( \pi \) is defined, for each \( x \in \mathfrak{X} \), as the restriction of \( \overline{\pi(x)} \) to the domain \( \tilde{D} \), which is the completion of \( D \) under the graph topology defined by the seminorms \( \xi \in D \mapsto \| \pi(x) \xi \|, \; x \in \mathfrak{X} \) (see [1]). If \( \pi = \tilde{\pi} \) the representation is said to be closed.

The Gel’fand–Naimark–Segal (GNS) construction for positive linear functionals is one of the most relevant tools when studying the structure of a Banach *-algebra. As customary when a partial multiplication is involved (see [1]), we consider as starting point for the construction a positive sesquilinear form enjoying certain invariance properties.

As usual, a sesquilinear form \( \varphi \) on \( \mathfrak{X} \times \mathfrak{X} \) is said to be bounded if there exists a positive constant \( \gamma \) such that

\[
|\varphi(x, y)| \leq \gamma \| x \| \| y \|, \quad \forall x, y \in \mathfrak{X}.
\]

In this case, we put

\[
\| \varphi \| := \sup_{\| x \| = \| y \| = 1} |\varphi(x, y)| = \sup_{\| x \| = 1} \varphi(x, x).
\]

**Definition 3.1.** Let \( \mathcal{P}(\mathfrak{X}) \) denote the set of all sesquilinear forms on \( \mathfrak{X} \times \mathfrak{X} \) such that

(i) \( \varphi(x, x) \geq 0, \; \forall x \in \mathfrak{X} \);

(ii) \( \varphi(xa, b) = \varphi(a, x^*b), \; \forall x \in \mathfrak{X}, \; a, b \in \mathfrak{A}_0 \);

(iii) \( \varphi \) is bounded.
Remark 3.2. We notice that if \( \varphi \in \mathcal{P}(\mathfrak{X}) \) then an easy limit argument shows that, besides (ii) of Definition 3.1, the following equality holds:

\[
\varphi(ax, y) = \varphi(x, a^* y), \quad \forall x, y \in \mathfrak{X}, a \in \mathfrak{A}_0.
\]

Let \( \varphi \in \mathcal{P}(\mathfrak{X}) \). Then the positivity of \( \varphi \) implies that:

\[
\varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall x, y \in \mathfrak{X};
\]

\[
|\varphi(x, y)|^2 \leq \varphi(x, x)\varphi(y, y), \quad \forall x, y \in \mathfrak{X}.
\]

Hence

\[
N_\varphi := \{ x \in \mathfrak{X} : \varphi(x, x) = 0 \} = \{ x \in \mathfrak{X} : \varphi(x, y) = 0, \forall y \in \mathfrak{X} \},
\]

and so \( N_\varphi \) is a subspace of \( \mathfrak{A} \). For each \( x \in \mathfrak{X} \), we denote by \( \lambda_\varphi(x) \) the coset of \( \mathfrak{X}/N_\varphi \) which contains \( x \), and define an inner product \( \langle \cdot | \cdot \rangle \) on

\[
\lambda_\varphi(\mathfrak{X}) = \mathfrak{X}/N_\varphi
\]

by

\[
(\lambda_\varphi(x) | \lambda_\varphi(y)) = \varphi(x, y), \quad x, y \in \mathfrak{X}.
\]

We denote by \( \mathcal{H}_\varphi \) the Hilbert space obtained by the completion of the pre-Hilbert space \( \lambda_\varphi(\mathfrak{X}) \). The subspace \( \lambda_\varphi(\mathfrak{A}_0) \) is dense in \( \mathcal{H}_\varphi \). Indeed, if \( x \in \mathfrak{X} \), there exists a sequence \( \{a_n\} \subset \mathfrak{A}_0 \) such that \( a_n \to x \) in \( \mathfrak{X} \). Then

\[
\|\lambda_\varphi(x) - \lambda_\varphi(a_n)\|^2 = \varphi(x - a_n, x - a_n) \leq \|\varphi\|^2\|x - a_n\|^2 \to 0.
\]

Proposition 3.3. Let \( \varphi \in \mathcal{P}(\mathfrak{X}) \). Put

\[
(3.1) \quad \pi_\varphi^0(x)\lambda_\varphi(a) = \lambda_\varphi(xa), \quad x \in \mathfrak{X}, a \in \mathfrak{A}_0.
\]

Then \( \pi_\varphi^0 \) is a *-representation of \( \mathfrak{X} \) in \( \mathcal{L}^1(\lambda_\varphi(\mathfrak{A}_0), \mathcal{H}_\varphi) \).

If \( (\mathfrak{X}, \mathfrak{A}_0) \) has a unit \( e \), the following properties also hold:

(i) \( D = \lambda_\varphi(\mathfrak{A}_0) = \pi(\mathfrak{A}_0)\lambda_\varphi(e) \) (i.e. \( \lambda_\varphi(e) \) is ultra-cyclic);

(ii) \( \varphi(x, y) = \langle \pi_\varphi^0(x)\lambda_\varphi(e) | \pi_\varphi^0(y)\lambda_\varphi(e) \rangle \), \( \forall x, y \in \mathfrak{X} \).

Proof. First we prove that, for each \( x \in \mathfrak{X} \), the map \( \pi_\varphi^0(x) \) of (3.1) is well defined. Assume that \( \lambda_\varphi(a) = 0 \) for some \( a \in \mathfrak{A}_0 \). If \( x \in \mathfrak{X} \), we then get \( \varphi(a, x^* b) = 0 \) for every \( b \in \mathfrak{A}_0 \). For each \( y \in \mathfrak{X} \) there exists a sequence \( \{b_n\} \subset \mathfrak{A}_0 \) such that \( \|\lambda_\varphi(y) - \lambda_\varphi(b_n)\| \to 0 \). This clearly implies that \( \varphi(xa, y) = 0 \) for each \( y \in \mathfrak{X} \). Hence \( xa \in N_\varphi \). Thus, for each \( x \in \mathfrak{X} \), the map \( \pi_\varphi^0(x) \) is a well defined linear operator from \( \lambda_\varphi(\mathfrak{A}_0) \) into \( \mathcal{H}_\varphi \). We notice that the restriction of \( \pi_\varphi^0 \) to \( \mathfrak{A}_0 \) maps \( \lambda_\varphi(\mathfrak{A}_0) \) into itself. This fact and the properties of \( \varphi \) listed in Definition 3.1 easily imply that \( \pi_\varphi^0 \) is a *-representation. If \( (\mathfrak{X}, \mathfrak{A}_0) \) has a unit \( e \), then (i) and (ii) follow from the definitions. \( \blacksquare \)

Denote by \( \pi_\varphi \) the closure of \( \pi_\varphi^0 \). The triple \( (\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi) \) is called the GNS construction for \( \varphi \) and we refer to \( \pi_\varphi \) as the GNS representation of \( \mathfrak{X} \) constructed from \( \varphi \). If \( (\mathfrak{X}, \mathfrak{A}_0) \) has a unit \( e \), then \( \xi_\varphi := \lambda_\varphi(e) \) is cyclic for \( \pi_\varphi \).
With a proof similar to the usual one in the case of *-algebras one can prove the following

**Proposition 3.4.** Let \((X, \mathcal{A}_0)\) be a Banach quasi *-algebra with unit \(e\) and \(\varphi \in \mathcal{P}(X)\). Then the GNS construction \((\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)\) is unique up to unitary equivalence.

It is easy to prove

**Proposition 3.5.** The *-representation \(\pi_\varphi\) is bounded if, and only if, \(\varphi\) is admissible, i.e., for every \(a \in \mathcal{A}_0\) there exists \(\gamma_x > 0\) such that
\[
\varphi(xa, xa) \leq \gamma_x \varphi(a, a), \quad \forall a \in \mathcal{A}_0.
\]

Assume that \((X, \mathcal{A}_0)\) has a unit \(e\). Then it is clear that, if \(\varphi \in \mathcal{P}(X)\), the linear functional \(\omega_\varphi\) defined by
\[
\omega_\varphi(x) = \varphi(x, e), \quad x \in X,
\]
is bounded on \(X\), i.e., \(\omega_\varphi \in X^d\). Moreover, it is positive on \(X\), in the sense that \(\omega_\varphi(x) \geq 0\) for every \(x \in X^+\), where \(X^+\) is the closure in \(X\) of the set
\[
\mathcal{A}_0^+ = \{\sum_{k=1}^n a_k^* a_k : a_k \in \mathcal{A}_0, k = 1, \ldots, n, n \in \mathbb{N}\}.
\]

The set of positive elements of \(X^d\) is denoted by \(X^d_S\).

Furthermore, the map \(\varphi \in \mathcal{P}(X) \mapsto \omega_\varphi \in X^d_+\) is injective. For, if \(\omega_\varphi(x) = 0\) for each \(x \in X\), then making use of the properties (ii) and (iii) of \(\mathcal{P}(X)\) and of the density of \(\mathcal{A}_0\), it follows that \(\varphi(x, y) = 0\) for all \(x, y \in X\).

Finally, we define
\[
\mathcal{S}(X) = \{\varphi \in \mathcal{P}(X) : \|\varphi\| \leq 1\}.
\]

It is easily seen that \(\mathcal{S}(X)\) is a convex subset of \(\mathcal{P}(X)\). If \((X, \mathcal{A}_0)\) has a unit \(e\), then \(\varphi(e, e) \leq \|e\|^2 = 1\) for any \(\varphi \in \mathcal{S}(X)\).

Let \(X^d_1 = \{\omega \in X^d : \|\omega\|^2 \leq 1\}\) be the unit ball of \(X^d\) and
\[
X^d_S = \{\omega_\varphi : \varphi \in \mathcal{S}(X)\}.
\]

**Remark 3.6.** Obviously, it is possible that \(\mathcal{S}(X) = \{0\}\) (or, equivalently, \(X^d_S = \{0\}\)). It is, however much more interesting to consider Banach quasi *-algebras for which the set \(\mathcal{S}(X)\) is sufficiently rich (Section 3.3).

**Proposition 3.7.** Assume that \((X, \mathcal{A}_0)\) has a unit and that \(\mathcal{S}(X) \neq \{0\}\). Then the following statements hold:

(i) \(X^d_S\) is a convex, weak*-compact subset of \(X^d_1\).

(ii) \(X^d_S\) has extreme points. If \(\omega_\varphi\) is extreme, then \(\|\varphi\| = 1\).

(iii) \(\omega_\varphi\) is extreme in \(X^d_S\) if, and only if, \(\varphi\) is extreme in \(\mathcal{S}(X)\).

The proof is very simple and we omit it.
3.2. Seminorms. We will now define some seminorms, closely related to families of sesquilinear forms [22] and to representations. Similar constructions have been considered in the case of \( ^* \)-algebras in [9, 24].

To begin with, we put

\[
p(x) = \sup_{\varphi \in \mathcal{S}(X)} \varphi(x, x)^{1/2}.
\]

Then \( p \) is a seminorm on \( X \) with \( p(x) \leq \|x\| \) for every \( x \in X \).

We also put

\[
N(p) = \{ x \in X : p(x) = 0 \}.
\]

Remark 3.8. Under the assumption of Proposition 3.7, the set \( \mathcal{S}(X) \), which is convex, has extreme elements (of unit norm) whose closed convex hull is exactly \( \mathcal{S}(X) \). Thus, in this case,

\[
p(x) = \sup_{\|\varphi\|=1} \varphi(x, x)^{1/2}.
\]

We also define

\[
q(x) = \sup_{\|\varphi\|=1} \{ \varphi(xa, xa)^{1/2} : \varphi \in \mathcal{P}(X), a \in \mathcal{A}_0, \varphi(a, a) = 1 \}, \quad x \in X,
\]

and

\[
D(q) = \{ x \in X : q(x) < \infty \}.
\]

If \( (X, \mathcal{A}_0) \) has a unit \( e \), then \( q \) has a simpler form. In fact, if we put

\[
q'(x) = \sup_{\|\varphi\|=1} \{ \varphi(x, x)^{1/2} : \varphi \in \mathcal{P}(X), \varphi(e, e) = 1 \}, \quad x \in X,
\]

and

\[
D(q') = \{ x \in X : q'(x) < \infty \},
\]

then \( D(q) = D(q') \) and \( q(x) = q'(x) \) for every \( x \in D(q) \). Indeed, it is clear that

\[
q'(x) \leq q(x), \quad \forall x \in X.
\]

On the other hand, if \( \varphi \in \mathcal{P}(X) \) and \( a \in \mathcal{A}_0 \), then also \( \varphi_a \in \mathcal{P}(X) \), where \( \varphi_a(x, y) = \varphi(xa, ya) \) for every \( x, y \in X \). Clearly, if \( a \in \mathcal{A}_0 \) and \( \varphi(a, a) = 1 \), then \( \varphi_a(e, e) = 1 \). This implies that

\[
q(x) \leq q'(x), \quad \forall x \in X.
\]

The inequalities (3.3) and (3.4) also hold when one of their terms is \( \infty \). Thus the statement is proved.

The seminorms \( p \) and \( q \) compare as follows.

**Proposition 3.9.** Let \( (X, \mathcal{A}_0) \) be a Banach quasi \( ^* \)-algebra. Then:

(i) \( p(xa) \leq q(x)p(a), \forall x \in D(q), a \in \mathcal{A}_0 \).

(ii) If \( (X, \mathcal{A}_0) \) has a unit, then

\[
p(x) \leq q(x), \quad \forall x \in D(q).
\]
Conversely, assume that \( \phi \). The statement then follows by taking the supremum over \( \phi \in S(\mathcal{X}) \).

(ii) This follows from (i) by choosing \( a = e \) and taking into account that \( p(e) \leq 1 \).

**Proposition 3.10.** Let \((\mathcal{X}, \mathfrak{A}_0)\) be a Banach quasi *-algebra. Then:

(i) \( \mathfrak{A}_0 \subseteq D(q) \) and \( q(a) \leq ||a||_0, \forall a \in \mathfrak{A}_0 \).

(ii) \( D(q) = \{ x \in \mathcal{X} : \pi_\phi(x) \text{ bounded}, \forall \phi \in \mathcal{P}(\mathcal{X}) \} \), and

\[
\sup_{\phi \in \mathcal{P}(\mathcal{X})} ||\pi_\phi(x)|| < \infty,
\]

\[
q(x) = \sup_{\phi \in \mathcal{P}(\mathcal{X})} ||\pi_\phi(x)||, \quad \forall x \in D(q).
\]

(iii) \( q \) is an extended C*-seminorm on \((\mathcal{X}, \mathfrak{A}_0)\) (i.e. \( q(x^*) = q(x) \), \( \forall x \in \mathcal{X}; q(a^*a) = q(a)^2, \forall a \in \mathfrak{A}_0 \), see [22]).

(iv) \( p(ax) \leq ||a||_0p(x), \forall x \in \mathcal{X}, a \in \mathfrak{A}_0 \).

**Proof.** (i) Let \( \phi \in \mathcal{P}(\mathcal{X}) \). Then the restriction of \( \phi \) to \( \mathfrak{A}_0 \times \mathfrak{A}_0 \) is \( || \cdot ||_0 \)-bounded. This fact together with a repeated use of the Cauchy–Schwarz inequality gives, for any \( a, b \in \mathfrak{A}_0 \),

\[
\phi(ab, ab) \leq \phi(b, b)^{1/2+1/2^2+\cdots+1/2^k} \phi((a^*a)^{2k-1}b, (a^*a)^{2k-1}b)^{1/2^k} \leq \phi(b, b)^{1/2+1/2^2+\cdots+1/2^k} ||\phi||^{1/2^k} \left( ||(a^*a)^{2k-1}||_0 ||b||_0 \right)^{1/2^k-1}.
\]

For \( k \to \infty \), we get

\[
\phi(ab, ab) \leq ||a||_0^2 \phi(b, b), \quad \forall a, b \in \mathfrak{A}_0.
\]

This implies that \( q(a) \leq ||a||_0 \) for every \( a \in \mathfrak{A}_0 \).

(ii) Let \( x \in D(q) \) and \( \phi \in \mathcal{P}(\mathcal{X}) \). If \( \pi_\phi \) denotes the GNS representation constructed from \( \phi \), making use of (3.5) we obtain

\[
||\pi_\phi(x)\lambda_\phi(a)||^2 = \phi(xa, xa) \leq q(x)^2 \phi(a, a) = q(x)^2 ||\lambda_\phi(a)||^2, \quad \forall a \in \mathfrak{A}_0.
\]

Thus \( \pi_\phi(x) \) is bounded and \( ||\pi_\phi(x)|| \leq q(x) \). This implies that

\[
M(x) := \sup\{||\pi_\phi(x)|| : \phi \in \mathcal{P}(\mathcal{X})\} \leq q(x).
\]

Conversely, assume that \( x \in \mathcal{X} \) and \( M(x) \) is finite. Then

\[
\phi(xa, xa) = ||\pi_\phi(x)\lambda_\phi(a)||^2 \leq M(x)^2 ||\lambda_\phi(a)||^2 = M(x)^2 \phi(a, a), \quad \forall a \in \mathfrak{A}_0.
\]

Hence, \( x \in D(q) \) and \( q(x) \leq M(x) \).

(iii) This follows directly from (ii).

(iv) For \( x \in \mathcal{X} \) and \( \phi \in \mathcal{S}(\mathcal{X}) \), define

\[
\omega_\phi^x(a) = \phi(ax, x), \quad a \in \mathfrak{A}_0.
\]
Then $\omega^x_\varphi$ is positive and $\| \cdot \|_0$-bounded on $\mathfrak{A}_0$. Proceeding as in (i) one gets
\[ \varphi(ax, ax) \leq \| a \|^2 \varphi(x, x), \quad \forall a \in \mathfrak{A}_0. \]
Taking the supremum over $\varphi \in \mathcal{S}(X)$, we obtain the result. 

So far, we have not proved (or even assumed) anything about the size of the families of sesquilinear forms we have considered. There are however examples of Banach quasi $^*$-algebras $(\mathfrak{X}, \mathfrak{A}_0)$ with $\mathcal{P}(\mathfrak{X}) = \{0\}$ (see Example 3.20 below). The previous statements remain of course true, but become mostly trivial. Much more interesting is the case where $\mathcal{P}(\mathfrak{X})$ contains sufficiently many elements, by which we mean that $N(p) = \{0\}$.

3.3. Sufficient families of sesquilinear forms

**Definition 3.11.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi $^*$-algebra. We say that $\mathcal{S}(\mathfrak{X})$ is **sufficient** if the conditions $x \in \mathfrak{X}$ and $\varphi(x, x) = 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$ imply $x = 0$.

**Remark 3.12.** We adopted a similar definition for $CQ^*$-algebras in [6]. Some of the statements that follow generalize results obtained for that situation in [6, 20].

The following lemma allows us to formulate in different ways the notion of sufficiency of $\mathcal{S}(\mathfrak{X})$.

**Lemma 3.13.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi $^*$-algebra with unit $e$. For an element $x \in \mathfrak{X}$, the following statements are equivalent.

1. $p(x) = 0$, i.e. $x \in N(p)$.
2. $\varphi(x, x) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$.
3. $\varphi(x, y) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$ and $y \in \mathfrak{X}$.
4. $\omega_\varphi(x) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$.
5. $\varphi(xa, a) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$ and $a \in \mathfrak{A}_0$.
6. $\varphi(xa, b) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$ and $a, b \in \mathfrak{A}_0$.

**Proposition 3.14.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi $^*$-algebra with unit $e$. If the set
\[ \mathfrak{X}_\mathcal{P}^\sharp := \{ \omega_\varphi : \varphi \in \mathcal{P}(\mathfrak{X}) \} \]
is weak*-dense in $\mathfrak{X}_\mathcal{P}^\sharp$, then $\mathcal{S}(\mathfrak{X})$ is sufficient. Conversely, if $(\mathfrak{X}, \| \cdot \|)$ is a reflexive Banach space and $\mathcal{S}(\mathfrak{X})$ is sufficient, then $\mathfrak{X}_\mathcal{P}^\sharp$ is weak*-dense in $\mathfrak{X}_\mathcal{P}^\sharp$.

**Proof.** Assume that $\mathcal{S}(\mathfrak{X})$ is not sufficient. Then there exists $x \in \mathfrak{X}$, $x \neq 0$, such that $\varphi(x, x) = 0$ for every $\varphi \in \mathcal{S}(\mathfrak{X})$. This implies that $\omega_\varphi(x) = 0$ for each $\varphi \in \mathcal{S}(\mathfrak{X})$. Thus the non-zero continuous linear functional $f_x$ on $\mathfrak{X}$ defined by $f_x(\omega) = \omega(x)$ is identically zero on $\{ \omega_\varphi : \varphi \in \mathcal{P}(\mathfrak{X}) \}$. Thus this set is not weak*-dense in $\mathfrak{X}_+^\sharp$.
Conversely, assume that $\mathfrak{X}_P$ is not weak*-dense in $\mathfrak{X}_+$. Then, by reflexivity, there would exist an $x \in \mathfrak{X}$, $x \neq 0$, such that $\omega_\varphi(x) = \varphi(x, e) = 0$ for each $\varphi \in \mathfrak{P}(\mathfrak{X})$. Then, by Lemma 3.13, we get $\varphi(x, x) = 0$ for each $\varphi \in \mathfrak{P}(\mathfrak{X})$. This implies that $x = 0$, a contradiction. ■

**Proposition 3.15.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit $e$ and let $S(\mathfrak{X})$ be sufficient. Let $x \in \mathfrak{X}$. Then

(i) $x = x^*$ if, and only if, $\omega_\varphi(x) \in \mathbb{R}$ for each $\varphi \in S(\mathfrak{X})$.

Moreover, if $\mathfrak{X}_P$ is weak*-dense in $\mathfrak{X}_+$, then:

(ii) If $\omega_\varphi(x) \geq 0$ for each $\varphi \in S(\mathfrak{X})$, then $x$ is positive.

(iii) $x \in \mathfrak{X}^+ \cap \{-\mathfrak{X}^+\}$ if, and only if, $x = 0$.

**Proof.** (i) Assume that $\omega_\varphi(x) \in \mathbb{R}$ for each $\varphi \in S(\mathfrak{X})$. Then

$$\omega_\varphi(x - x^*) = \omega_\varphi(x) - \omega_\varphi(x^*) = \omega_\varphi(x) - \omega_\varphi(x) = 0$$

for every $\varphi \in S(\mathfrak{X})$. By Lemma 3.13 one has $\varphi(x - x^*, x - x^*) = 0$ for every $\varphi \in S(\mathfrak{X})$. Hence $x = x^*$. The converse implication is obvious.

(ii) This follows immediately from the weak*-denseness of $\mathfrak{X}_P$.

(iii) Assume that $x \in \mathfrak{X}^+ \cap \{-\mathfrak{X}^+\}$; then by (ii) it follows that $\omega_\varphi(x) = 0$ for every $\varphi \in S(\mathfrak{X})$. From this we conclude that $x = 0$. ■

**Proposition 3.16.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a BQ*-algebra with unit. If $S(\mathfrak{X})$ is sufficient, then $\mathfrak{A}_0$ is a *-semisimple Banach *-algebra.

**Proof.** It suffices to show that if $a \in \mathfrak{A}_0$ and $\omega(a^*a) = 0$ for each positive linear functional $\omega$ on $\mathfrak{A}_0$, then $a = 0$. If this assumption is satisfied, then, in particular, $\omega_\varphi(a^*a) = 0$ for each $\varphi \in S(\mathfrak{X})$. This implies that $\varphi(a, a) = 0$ for each $\varphi \in S(\mathfrak{X})$, and so $a = 0$. ■

If $(\mathfrak{X}, \mathfrak{A}_0)$ has a sufficient $S(\mathfrak{X})$, then also the multiplications defined in Section 2 behave in a reasonable fashion:

**Proposition 3.17.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with sufficient $S(\mathfrak{X})$. Then $(\mathfrak{X}, \mathfrak{A}_0)$ is normal.

**Proof.** Let $x, y \in \mathfrak{X}_b$. For every $\varphi \in S(\mathfrak{X})$ and $c \in \mathfrak{A}_0$, we have

$$\varphi((x \triangleright y)c, c) = \varphi((L_xy)c, c) = \lim_{m \to \infty} \varphi((xb_m)c, c)$$

$$= \lim_{m \to \infty} \varphi(x(b_mc), c) = \lim_{m \to \infty} \varphi(b_mc, x^*c)$$

$$= \varphi(yc, x^*c),$$

where $\{b_m\} \subset \mathfrak{A}_0$ converges to $y$ in $\mathfrak{X}$. Analogously, if $\{a_n\} \subset \mathfrak{A}_0$ converges to $x$ in $\mathfrak{X}$, we have

$$\varphi((x \triangleleft y)c, c) = \varphi((R_yx)c, c) = \lim_{n \to \infty} \varphi((an_y)c, c)$$

$$= \lim_{n \to \infty} \varphi(a_n(yc), c) = \lim_{n \to \infty} \varphi(yc, a_n^*c) = \varphi(yc, x^*c).$$
Therefore

\[ \varphi((x \triangleright y - x \triangledown y)c, c) = 0, \quad \forall \varphi \in \mathcal{S}(X), \ c \in \mathcal{A}_0. \]

By Lemma 3.13 it follows that \( x \triangleright y = x \triangledown y \). This concludes the proof. \( \blacksquare \)

If \((X, \mathcal{A}_0)\) has a sufficient \( \mathcal{S}(X) \), then \( p \) is a norm on \( X \), weaker in general than the original norm of \( X \). Thus, it makes sense to consider the case where they coincide. Hence we give the following

**Definition 3.18.** A Banach quasi \(^*\)-algebra \((X, \mathcal{A}_0)\) is called **regular** if

(i) \( \mathcal{S}(X) \) is sufficient;

(ii) \( p(x) = \|x\| \) for every \( x \in X \).

A similar definition was given for \( CQ^* \)-algebras in [6]. We notice that the equality \( p(x) = \|x\| \) for every \( x \in X \) implies that \( p(x^*) = p(x) \) for every \( x \in X \). This equality fails in general; but it is exactly what is needed to embed \((X, \mathcal{A}_0)\) in a larger regular Banach quasi \(^*\)-algebra.

**Proposition 3.19.** Let \((X, \mathcal{A}_0)\) be a Banach quasi \(^*\)-algebra with sufficient \( \mathcal{S}(X) \) and \( p(x^*) = p(x) \) for every \( x \in X \). Then there exists a regular Banach quasi \(^*\)-algebra, \((X_S, \mathcal{A}_0)\), such that \( X_S \) contains \( X \) as a dense subspace.

**Proof.** We let \( X_S \) be the completion of \( \mathcal{A}_0 \) with respect to \( p \); then \((X_S, \mathcal{A}_0)\) is a Banach quasi \(^*\)-algebra, by Proposition 3.10(vi) and the assumption that \( p(x^*) = p(x) \) for every \( x \in X \). We now prove that \( X \) can be identified with a subspace of \( X_S \). Indeed, if \( x \in X \) then there exists a sequence \( \{a_n\} \subset \mathcal{A}_0 \) such that

\[ x = \| \cdot \| - \lim_{n \to \infty} a_n. \]

It is readily seen that \( \{a_n\} \) is also a Cauchy sequence with respect to \( p \). Thus there exists an element \( \bar{x} \in X_S \) such that

\[ \bar{x} = p - \lim_{n \to \infty} a_n. \]

The element \( \bar{x} \) does not depend on the particular sequence \( \{a_n\} \) used to approximate \( x \) in \( X \). Indeed, if \( \{a'_n\} \) is another such sequence, then

\[ p(a_n - a'_n) \leq \|a_n - a'_n\| \to 0 \quad \text{as} \ n \to \infty. \]

We have defined in this way a map \( i : x \in X \mapsto \bar{x} \in X_S \); we will now prove that \( i \) is injective.

Assume that \( \bar{x} = 0 \) for some \( x \in X \) and let \( \{a_n\} \) be a sequence in \( \mathcal{A}_0 \) approximating \( x \) in the norm of \( X \) and such that \( p(a_n) \to 0 \); this implies that \( \varphi(a_n, a_n) \to 0 \) for each \( \varphi \in \mathcal{S}(X) \). Therefore

\[ \varphi(x, x) = \lim_{n \to \infty} \varphi(a_n, a_n) = 0. \]
From the sufficiency of $S(\mathfrak{X})$ we get $x = 0$. To conclude the proof, we need to show that $S(\mathfrak{X}_S)$ is sufficient and that $(\mathfrak{X}_S, \mathfrak{A}_0)$ is regular.

First, we prove that the two families of sesquilinear forms, $S(\mathfrak{X})$ and $S(\mathfrak{X}_S)$, can be identified. Indeed, let $\Phi \in S(\mathfrak{X}_S)$; then its restriction $\Phi_{\mathfrak{X}}$ to $\mathfrak{X}$ belongs, as is easily seen, to $S(\mathfrak{X})$. Conversely, if $\Phi_0 \in S(\mathfrak{X})$, then making use of the Cauchy–Schwarz inequality, we get

$$|\Phi_0(x, y)| \leq p(x)p(y), \quad \forall x, y \in \mathfrak{X}.$$ 

Therefore $\Phi_0$ has a unique extension $\Phi$ to $\mathfrak{X}_S$ and $\Phi \in S(\mathfrak{X}_S)$. Taking this fact into account, the sufficiency of $S(\mathfrak{X}_S)$ follows by the definition of completion. The regularity is a simple consequence of the definition of the norm in the completion. ■

**Example 3.20.** The $BQ^*$-algebra $(L^p(I), C(I))$ is regular [5] if, and only if, $p \geq 2$. For $1 \leq p < 2$, $S(L^p(I)) = \{0\}$. In the case of the non-commutative $L^p$ of Example 2.5, it has been proved in [8] that, for finite $\tau$, $(L^p(\tau), \mathcal{M})$ is regular if $p \geq 2$.

**Example 3.21.** For the Banach quasi *-algebra $(\mathcal{H}, \mathfrak{A}_0)$ of Example 2.6, $S(\mathfrak{X})$ is sufficient, since it contains the inner product $\langle \cdot | \cdot \rangle$. For the same reason, $(\mathcal{H}, \mathfrak{A}_0)$ is regular.

We consider again the seminorm $q$ defined in (3.2). If $(\mathfrak{X}, \mathfrak{A}_0)$ has a sufficient $S(\mathfrak{X})$, then $q$ is also a norm on $D(q)$ and has the $C^*$-property on $\mathfrak{A}_0$. If, in addition, $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit, then (Proposition 3.9)

$$p(x) \leq q(x), \quad \forall x \in D(q).$$

The space $D(q)$ endowed with the topology defined by $q$ is denoted by $\mathfrak{X}_q$. Then we have the following

**Proposition 3.22.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit. Assume that $S(\mathfrak{X})$ is sufficient. Then $\mathfrak{X}_q$ is a normed space containing $\mathfrak{A}_0$ as a subspace. Moreover if $\mathfrak{X}$ is regular, then $\mathfrak{X}_q$ is a Banach space.

**Proof.** The first part of the statement follows from Proposition 3.10(i). In order to prove that, if $\mathfrak{X}$ is regular, $\mathfrak{X}_q$ is a Banach space, we only have to show its completeness. Let $\{x_n\}$ be a $q$-Cauchy sequence in $\mathfrak{X}_q$.

Inequality (3.6) in the regular case becomes $\|x\| \leq q(x)$ for all $x \in \mathfrak{X}_q$. Therefore $\{x_n\}$ is also $\| \cdot \|-$Cauchy. Using the $\| \cdot \|-$completeness of $\mathfrak{X}$ we conclude that there exists an element $x \in \mathfrak{X}$ which is the $\| \cdot \|$-limit of $x_n$.

Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Then $\varphi(x, x) = \lim_{n \to \infty} \varphi(x_n, x_n)$. The sequence $q(x_n)$ is bounded, because $\{x_n\}$ is $q$-Cauchy. Let $M$ be its supremum. Then

$$\varphi(x_n a, x_n a)^{1/2} \leq q(x_n) \leq M, \quad \forall a \in \mathfrak{A}_0 \text{ with } \varphi(a, a) = 1.$$ 

Hence

$$\varphi(x a, x a)^{1/2} = \lim_{n \to \infty} \varphi(x_n a, x_n a)^{1/2} \leq M.$$
Thus, clearly, \( q(x) \leq M \), i.e. \( x \in \mathcal{X}_q \). Finally, using the uniqueness of the limit in the completion of \( \mathcal{X}_q \), we conclude that \( x = q\text{-lim}_{n \to \infty} x_n \). Thus \( \mathcal{X}_q \) is complete. ■

We observe that in general the inclusion \( \mathcal{A}_0 \subseteq \mathcal{X}_q \) is proper. For instance, in \( (L^p(I), C(I)) \) any step function \( s \) defined on \([0, 1]\) is in \( L^p(I) \) but not in \( C(I) \). It is immediate to verify that \( s \in (L^p(I))_q \).

Our next goal is to prove that, for regular Banach quasi *-algebras, \( \mathcal{X}_q \) is exactly the set of elements having finite spectral radius.

We begin with the following

**Proposition 3.23.** Let \( (\mathcal{X}, \mathcal{A}_0) \) be a Banach quasi *-algebra with sufficient \( S(\mathcal{X}) \). Then for every \( x \in \mathcal{X} \) the maps

\[
L_x : a \in \mathcal{A}_0 \mapsto ax \in \mathcal{X}, \quad R_x : a \in \mathcal{A}_0 \mapsto xa \in \mathcal{X}
\]

are closable in \( \mathcal{X} \).

**Proof.** Let \( x \in \mathcal{X} \) and \( \{a_n\} \subseteq \mathcal{A}_0 \) be a sequence \( \| \cdot \| \)-converging to zero and such that \( xa_n \to y \) with respect to \( \| \cdot \| \). Then, if \( \varphi \in S(\mathcal{X}) \) and \( b_1, b_2 \in \mathcal{A}_0 \), we get

\[
|\varphi(yb_1, b_2)| \leq |\varphi((y - xa_n)b_1, b_2)| + |\varphi(a_nb_1, x^*b_2)|
\]

\[
\leq \|y - xa_n\| \|b_1\|_0\|b_2\|_0 + \|a_n\| \|b_1\|_0\|x^*b_2\| \to 0.
\]

Therefore \( \varphi(yb_1, b_2) = 0 \) for every \( \varphi \in S(\mathcal{X}) \) and \( b_1, b_2 \in \mathcal{A}_0 \). By Lemma 3.13, \( y = 0 \). The proof for \( R_x \) is similar. ■

The previous proposition suggests a handy criterion for the existence of a bounded inverse of an element:

**Proposition 3.24.** Let \( (\mathcal{X}, \mathcal{A}_0) \) be a Banach quasi *-algebra with unit \( e \) and sufficient \( S(\mathcal{X}) \). Let \( x \in \mathcal{X} \) satisfy the following conditions:

(i) there exists \( \gamma > 0 \) such that

\[
\min\{|\|ax\|, |xa\|\} \geq \gamma \|a\|, \quad \forall a \in \mathcal{A}_0;
\]

(ii) the sets \( \{ax : a \in \mathcal{A}_0\} \) and \( \{xa : a \in \mathcal{A}_0\} \) are both dense in \( \mathcal{X} \).

Then \( x \) has a bounded inverse.

**Proof.** Let \( x \in \mathcal{X} \) satisfy (i) and (ii). Then, making use of standard techniques for closable maps in Banach spaces, one can prove that the range of the closure \( \overline{L}_x \) of \( L_x \) is the whole space \( \mathcal{X} \). Moreover, it is easy to prove that

\[
\|\overline{L}_xy\| \geq \gamma \|y\|, \quad \forall y \in D(\overline{L}_x),
\]

where \( D(\overline{L}_x) \) denotes the domain of \( \overline{L}_x \). Therefore, there exists a unique \( b_1 \in D(\overline{L}_x) \) such that \( \overline{L}_xb_1 = e \).
Let \( \{z_n\} \subset \mathfrak{A}_0 \) with \( \|b_1 - z_n\| \to 0 \) and \( \{x z_n\} \) converging in \( \mathfrak{X} \). Then, for every \( a \in \mathfrak{A}_0 \), \( \|b_1 a - z_n a\| \to 0 \) and \( \{x(z_n a)\} \) converges in \( \mathfrak{X} \). Hence \( b_1 a \in D(L_x) \) and
\[
L_x(b_1 a) = \lim_{n \to \infty} x(z_n a) = \lim_{n \to \infty} (x z_n a) = (L_x b_1) a = e a = a.
\]
Therefore \( L_x(b_1 a) = (L_x b_1) a \) for every \( a \in \mathfrak{A}_0 \). Hence
\[
\|L_x(b_1 a)\| \geq \gamma \|b_1 a\|, \quad \forall a \in \mathfrak{A}_0.
\]
This implies that
\[
\|b_1 a\| \leq \frac{1}{\gamma} \|a\|, \quad \forall a \in \mathfrak{A}_0.
\]
Hence \( b_1 \) is left bounded.

In a similar way one shows the existence of a unique right bounded element \( b_2 \in D(R_x) \) such that \( R_x b_2 = e \).

We now prove that \( b_1 = b_2 \). Let \( z_{i,n} \) \((i = 1, 2)\) be a sequence in \( \mathfrak{A}_0 \) such that \( \|z_{i,n} - b_i\| \to 0 \) and \( \{x z_{i,n}\} \) converges in \( \mathfrak{X} \). For every \( \varphi \in \mathcal{S}(\mathfrak{X}) \) and \( c \in \mathfrak{A}_0 \), we have
\[
\varphi(b_2 c, c) = \varphi(b_2 (L_x b_1) c, c) = \varphi((L_x b_1) c, b_2^* c)
\]
\[
= \lim_{n \to \infty} \varphi((L_x z_{1,n}) c, z_{2,n}^* c) = \lim_{n \to \infty} \varphi(x(z_{1,n} c), z_{2,n}^* c)
\]
\[
= \lim_{n \to \infty} \varphi(z_{1,n} c, (z_{2,n} x)^* c) = \varphi(b_1 c, c),
\]
since \( z_{2,n} x \to R_x b_2 = e \). The sufficiency of \( \mathcal{S}(\mathfrak{X}) \) implies that \( b_1 = b_2 \). We put \( b := b_1 = b_2 \). Then \( b \in \mathfrak{X}_b \).

We finally prove that \( \bar{L}_x b = \bar{R}_b x \). Let \( \varphi \in \mathcal{S}(\mathfrak{X}) \) and let \( \{z_n\} \subset \mathfrak{A}_0 \) with \( \|b - z_n\| \to 0 \) and \( \{x z_n\} \) converging in \( \mathfrak{X} \). Then, for every \( c \in \mathfrak{A}_0 \),
\[
\varphi((\bar{L}_x b) c, c) = \lim_{n \to \infty} \varphi((x z_n) c, c) = \lim_{n \to \infty} \varphi(z_n c, x^* c) = \varphi(b c, x^* c).
\]
On the other hand, if \( \{a_n\} \subset \mathfrak{A}_0 \) with \( \|x - a_n\| \to 0 \), then for every \( c \in \mathfrak{A}_0 \),
\[
\varphi((\bar{R}_b x) c, c) = \lim_{n \to \infty} \varphi((R_b a_n) c, c) = \lim_{n \to \infty} \varphi((a_n b) c, c)
\]
\[
= \lim_{n \to \infty} \varphi(b c, a_n^* c) = \varphi(b c, x^* c).
\]
The sufficiency of \( \mathcal{S}(\mathfrak{X}) \) implies the desired equality.

Analogously, one can prove that \( \bar{R}_x b = \bar{L}_b x \). In conclusion, \( b \in \mathfrak{X}_b \) and \( \bar{L}_b x = \bar{R}_b x = e \), i.e. \( x \) has a bounded inverse. \( \blacksquare \)

**Proposition 3.25.** Let \( \mathfrak{X} \) be a regular Banach quasi *-algebra with unit \( e \). Let \( x \in \mathfrak{X}_q \) and \( \lambda \in \mathbb{C} \) with \( |\lambda| > q(x) \). Then \( x - \lambda e \) has a bounded inverse \( (x - \lambda e)_b^{-1} \in \mathfrak{X}_b \). Thus
\[
\{\lambda \in \mathbb{C} : |\lambda| > q(x)\} \subseteq q(x).
\]

**Proof.** Let \( \varphi \in \mathcal{S}(\mathfrak{X}) \). By definition, if \( x \in D(q) \), then
\[
|\lambda| > q(x) \geq \varphi(xb, xb), \quad \forall b \in \mathfrak{A}_0 \text{ with } \varphi(b, b) = 1.
\]
Therefore, for every $a \in \mathfrak{A}_0$,
\[
\varphi((x-\lambda e)a, (x-\lambda e)a)^{1/2} \geq |\lambda|\varphi(a, a)^{1/2} - \varphi(xa, xa)^{1/2} \\
\geq (|\lambda| - q(x)) \varphi(a, a)^{1/2}.
\]
Taking the supremum over $\varphi \in \mathcal{S}(\mathfrak{X})$ we get
\[
p((x-\lambda e)a) \geq p(a)(|\lambda| - q(x)).
\]
From the regularity of $(\mathfrak{X}, \mathfrak{A}_0)$, we finally get
\[
\|q(x)\| \leq \|\varphi((x-\lambda e)a, (x-\lambda e)a)\|, \quad \forall a \in \mathfrak{A}_0.
\]
Furthermore, if $q(x) < \infty$ and $|\lambda| > q(x)$, then the sets
\[
\text{Ran} L_{x-\lambda e} := \{((x-\lambda e)b : b \in \mathfrak{A}_0), \quad \text{Ran} R_{x-\lambda e} := \{b(x-\lambda e) : b \in \mathfrak{A}_0\}
\]
are $\|\cdot\|$-dense in $\mathfrak{X}$.

Indeed, assume, for instance, that $\text{Ran} L_{x-\lambda e}$ is not dense in $\mathfrak{X}$. Then there exists a non-zero $\|\cdot\|$-continuous functional $f$ on $\mathfrak{X}$ such that $f((x-\lambda)b) = 0$ for every $b \in \mathfrak{A}_0$. Therefore $f(xb) = \lambda f(b)$ for every $b \in \mathfrak{A}_0$. From the $\|\cdot\|$-continuity of $f$ we get $|f(xb)| \leq \|f\|^2\|xb\|$ for every $b \in \mathfrak{A}_0$.

From the regularity of $(\mathfrak{X}, \mathfrak{A}_0)$ and from Proposition 3.9(i), we get
\[
|f(xb)| \leq \|f\|^2\|xb\| = \|f\|^2p(xb) \leq \|f\|^2q(x)p(b) = \|f\|^2q(x)b, \quad \forall b \in \mathfrak{A}_0.
\]

The functional $f_x$ defined by $f_x(b) := f(xb), b \in \mathfrak{A}_0$, is $\|\cdot\|$-continuous, since
\[
|f_x(b)| = |\lambda f(b)| \leq |\lambda|\|f\|^2|b|, \quad \forall b \in \mathfrak{A}_0.
\]
An easy computation shows that $\|f_x\|^2 = |\lambda|\|f\|^2$. Thus we find the following contradictory inequality: $|\lambda| \leq q(x)$. A similar argument shows the corresponding statement for $R_{x-\lambda e}$.

Applying Proposition 3.24 we get the result. 

We can now prove the following

**Theorem 3.26.** Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a regular Banach quasi *-algebra with unit $e$. Then $D(q)$ coincides with the set $\mathfrak{X}_b$ of all bounded elements of $\mathfrak{X}$. Moreover
\[
q(x) = \|x\|_b, \quad \forall x \in \mathfrak{X}_b.
\]
Therefore $(\mathfrak{X}_b, \|\cdot\|_b)$ is a $C^*$-algebra.

**Proof.** Propositions 2.27 and 3.25 show that $D(q) \subseteq \mathfrak{X}_b$. On the other hand, consider, for each $\varphi \in \mathcal{P}(\mathfrak{X})$, the linear functional $\omega_\varphi$ defined by
\[
\omega_\varphi(x) = \varphi(x, e), \quad x \in \mathfrak{X}_b.
\]
A simple limit argument shows that $\omega_\varphi$ is positive (i.e. $\omega(x^* \bullet x) \geq 0$ for each $x \in \mathfrak{X}_b$), so if $\pi_\varphi$ denotes the corresponding GNS representation, then $\pi_\varphi(x)$ is bounded and $\|\pi_\varphi(x)\| \leq \|x\|_b$ for every $x \in \mathfrak{X}_b$. Thus, if $x \in \mathfrak{X}_b$,
then by Proposition 3.10(ii),
\[ q(x) = \sup_{\varphi \in \mathcal{P}(x)} \| \pi_{\varphi}(x) \| \leq \| x \|_b. \]

From Proposition 3.9(i) it follows that
\[ \| xa \| = p(xa) \leq q(x)p(a) = q(x)\| a \|, \quad \forall x \in D(q), a \in \mathfrak{A}_0, \]
and, by taking the involution, also
\[ \| ax \| \leq q(x)\| a \|, \quad \forall x \in D(q), a \in \mathfrak{A}_0. \]
This implies that \( \| x \|_b \leq q(x) \). Thus, in conclusion, \( \| \cdot \|_b \) is a \( C^* \)-norm.

**Theorem 3.27.** Let \((\mathfrak{X}, \mathfrak{A}_0)\) be a Banach quasi \(*\)-algebra with unit \( e \). Assume that \( S(\mathfrak{X}) \) is sufficient. Then \((\mathfrak{X}, \mathfrak{A}_0)\) admits a faithful \(*\)-representation \( \pi \) in a Hilbert space \( \mathcal{H} \). Moreover
\[ \mathfrak{X}_b = \{ x \in \mathfrak{X} : \pi(x) \in \mathcal{B}(\mathcal{H}) \} \]
and
\[ \| \pi(x) \| = q(x), \quad \forall x \in \mathfrak{X}_b. \]

**Proof.** For each \( \varphi \in \mathcal{P}(\mathfrak{X}) \), let \( \pi_{\varphi} \) be the corresponding GNS construction with dense domain \( D_{\varphi} \subseteq \mathcal{H}_{\varphi} \). Put
\[ \mathcal{H} = \bigoplus_{\varphi \in \mathcal{P}(\mathfrak{X})} \mathcal{H}_{\varphi} = \left\{ (\xi_{\varphi})_{\varphi \in \mathcal{P}(\mathfrak{X})} : \xi_{\varphi} \in \mathcal{H}_{\varphi}, \sum_{\varphi \in \mathcal{P}(\mathfrak{X})} \| \xi_{\varphi} \|^2 < \infty \right\}, \]
with the usual inner product
\[ \langle (\xi_{\varphi}), (\eta_{\varphi}) \rangle = \sum_{\varphi \in \mathcal{S}(\mathfrak{X})} \langle \xi_{\varphi}, \eta_{\varphi} \rangle, \quad (\xi_{\varphi}), (\eta_{\varphi}) \in \mathcal{H}. \]

Let
\[ \mathcal{D} = \left\{ (\xi_{\varphi}) \in \mathcal{H} : \xi_{\varphi} \in \mathcal{D}_{\varphi}, \varphi \in \mathcal{P}(\mathfrak{X}), \sum_{\varphi \in \mathcal{S}(\mathfrak{X})} \| \pi_{\varphi}(x)\xi_{\varphi} \|^2 < \infty, \forall x \in \mathfrak{X} \right\}. \]
Then \( \mathcal{D} \) is a dense domain in \( \mathcal{H} \) and so we can define, for \( x \in \mathfrak{X} \),
\[ \pi(x)(\xi_{\varphi}) = (\pi_{\varphi}(x)\xi_{\varphi}), \quad (\xi_{\varphi}) \in \mathcal{D}. \]
Then \( \pi(x) \in \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \) for each \( x \in \mathfrak{X} \) and \( \pi : x \in \mathfrak{X} \mapsto \pi(x) \in \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \) is a \(*\)-representation of \((\mathfrak{X}, \mathfrak{A}_0)\). Moreover, \( \pi \) is faithful, since
\[ \pi(x) = 0 \iff \pi_{\varphi}(x) = 0, \forall \varphi \in \mathcal{P}(\mathfrak{X}) \iff \varphi(x, x) = 0, \forall \varphi \in \mathcal{P}(\mathfrak{X}). \]
The sufficiency of \( S(\mathfrak{X}) \) then implies that \( x = 0 \).

Finally, \( \pi(x) \) is bounded if, and only if, each \( \pi_{\varphi} \) for \( \varphi \in \mathcal{P}(\mathfrak{X}) \) is bounded and
\[ \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \| \pi_{\varphi}(x) \| < \infty, \]
and, in this case,
\[ \| \pi(x) \| = \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \| \pi_\varphi(x) \|, \quad x \in \mathfrak{X}. \]

But, by Proposition 3.10(ii),
\[ \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \| \pi_\varphi(x) \| = q(x). \]

This concludes the proof. ■

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References


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