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An extension of a multiplicity theorem by Ricceri with an application to a class of quasilinear equations

by

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Abstract. A recent multiplicity result by Ricceri, stated for equations in Hilbert spaces, is extended to a wider class of Banach spaces. Applications to nonlinear boundary value problems involving the p-Laplacian are presented.

1. Introduction. In [5, Theorem 1], B. Ricceri has established a multiplicity theorem for the critical points of a continuously Gateaux differentiable functional J over a Hilbert space X, satisfying

$$\limsup_{\|x\|\to\infty} \frac{J(x)}{\|x\|^2} \leq 0. \tag{1}$$

Ricceri's result ensures that, for each real r within the range of J and $x_0 \in J^{-1}(]-\infty, r[)$, either the functional

$$x \mapsto \|x - x_0\|^2/2 - \lambda J(x) \quad \text{(for some } \lambda > 0)$$

admits at least three critical points, or the set $J^{-1}([r,\infty[)$ has a unique point minimizing the distance from x_0. Then he proves that, under very general hypotheses (namely, the nonconvexity of $J^{-1}([r,\infty[))$, for some x_0 in a convex dense subset of X and some positive λ, the first case occurs, thus obtaining a multiplicity result. To this end, he employs a result of Tsar'kov ([6, Corollary 2]) dealing with the problem of best approximation and Chebyshev sets, which sharpens a previous result of Efimov and Stechkin ([2, Theorem 3]).

For our purposes, Tsar'kov's result can be stated as follows: in a Hilbert space, let M be a nonconvex, sequentially weakly closed set and S be a convex, dense set; then there exists a point $x_0 \in S$ admitting at least two points of best approximation in M.

Both in [5] and in [6], an interesting link is established between best approximation theory and critical point theory, as solutions of boundary

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value problems are found as critical points of functionals of the type above. Namely, in [5] it is shown that, whenever g is a continuous, nonconstant and nondecreasing real-valued function on $\mathbb{R}$ (with suitable asymptotic behavior to make (1) true), there exist $w_0 \in C_0^\infty(]0,1[)$ and a positive λ such that the two-point Dirichlet problem

$$\begin{cases} -u'' = \lambda g(u) - w_0(x) & \text{in }]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

has at least three classical solutions.

The aim of the present paper is to prove an analogous result for a more general class of problems, built on partial differential equations involving the p-Laplacian: namely, let g be a continuous function on $\Omega \times \mathbb{R}$ (where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary), satisfying standard growth conditions and a certain asymptotic assumption (see Section 3 for more details); moreover, let $g(x,\cdot)$ be nondecreasing for all $x \in \Omega$. We shall prove that, for an arbitrary convex, dense subset S of $W_0^{1,p}(\Omega)$, there exist $u_0 \in S$ and $\lambda > 0$ such that the Dirichlet problem

$$(P) \quad \begin{cases} -\Delta_p u = \lambda g(x, u + u_0(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

has at least three weak solutions.

The same approach leads to a multiplicity theorem for the Neumann problem as well: namely, under the same assumptions on Ω and g, for an arbitrary convex, dense subset S' of $W^{1,p}(\Omega)$, there exist $u_0 \in S'$ and $\lambda > 0$ such that the Neumann problem

$$(Q) \quad \begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda g(x, u + u_0(x)) & \text{in } \Omega, \\ \partial u/\partial n = 0 & \text{in } \partial\Omega, \end{cases}$$

(where n denotes the outward unit normal to $\partial\Omega$) has at least three weak solutions.

Notice that u_0 and λ are not explicitly determined.

Solutions of (P) (resp. (Q)) are found as critical points of a certain functional on $W_0^{1,p}(\Omega)$ (resp. $W^{1,p}(\Omega)$), so we need to extend Ricceri's theorem to a class of Banach spaces wide enough to embrace this case. Moreover, it will be necessary to refer to a more general form of Tsar'kov's result:

THEOREM A ([6, Theorem 2], [2, Lemma 1]). *Let X be a uniformly convex Banach space with strictly convex topological dual, and M a sequentially weakly closed and nonconvex subset of X. Then, for any convex, dense subset S of X, there exists $x_0 \in S$ such that the set*

$$\{y \in M : \|y - x_0\| = d(x_0, M)\}$$

has at least two points.

Here we denote by $d(x_0, M)$ the distance between x_0 and the set M.

A further condition on the space arises from the need that the functional $x \mapsto \|x\|^p$ ($p > 1$) be continuously Gateaux differentiable (which is trivial in the case of Hilbert spaces with $p = 2$): for this purpose, it is sufficient to require that the norm be Fréchet differentiable on the unit sphere (see [1] and Section 2 for more details). This assumption places our space somewhere between the classes of "very smooth" and of "uniformly smooth" Banach spaces. Notice that such a space, provided it is reflexive, has a strictly convex topological dual ([1, Theorem 1 and Corollary 1, Section 1, Chapter 2]).

In the proof of our abstract result, we will refer to the following minimax theorem:

THEOREM B ([4, Theorem 1 and Remark 1]). *Let X be a topological space, I a real interval, and $f : X \times I \to \mathbb{R}$ a function satisfying the following conditions*:

(c_1) *for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous;*

(c_2) *for every $\lambda \in I$, the function $f(\cdot, \lambda)$ is lower semicontinuous and each of its local minima is a global minimum;*

(c_3) *there exist $\varrho_0 > \sup_I \inf_X f$ and $\lambda_0 \in I$ such that the set*

$$\{x \in X : f(x, \lambda_0) \le \varrho_0\}$$

is compact.

Then

$$\sup_I \inf_X f = \inf_X \sup_I f.$$

2. Abstract results. Before stating our results, we need the following definition.

Let X be a Banach space. After [1], we define a *support mapping* as a function $\varphi : X \setminus \{0\} \to X^* \setminus \{0\}$ satisfying:

(SM_1) $\|\varphi(x)\|_{X^*} = 1 = \langle \varphi(x), x \rangle$ for all $x \in S(X)$, where $S(X)$ is the unit sphere in X;

(SM_2) $\varphi(\varrho x) = \varrho \varphi(x)$ for all $\varrho > 0$ and $x \in X \setminus \{0\}$.

Our main result reads as follows:

THEOREM 1. *Let X be a real uniformly convex Banach space with Fréchet differentiable norm on $S(X)$, $p > 1$, and J be a nonconstant, continuously Gateaux differentiable functional with compact derivative, satisfying*

$$\limsup_{\|x\| \to \infty} \frac{J(x)}{\|x\|^p} \le 0. \tag{2}$$

Then for every $r \in \,]\inf_X J, \sup_X J[$ *and every* $x_0 \in J^{-1}(]-\infty, r[)$ *one of the following conditions is true*:

(a) *there exists* $\lambda > 0$ *such that the functional defined on* X *by*

$$x \mapsto \|x - x_0\|^p/p - \lambda J(x)$$

admits at least three critical points;

(b) *there exists* $y \in J^{-1}(r)$ *such that*

$$\|x - x_0\| > \|y - x_0\|$$

for all $x \in J^{-1}([r, \infty[)$ *with* $x \neq y$.

Proof. First of all, we show that condition (a) is meaningful.

From [1, Theorem 1, Section 2, Chapter 2] it follows that, since the norm of X is Fréchet differentiable on $S(X)$, there exists a support mapping φ which is norm-to-norm continuous from $S(X)$ to $S(X^*)$. It is easily seen that, actually, φ is continuous on $X \setminus \{0\}$ and the functional $x \mapsto \|x\|$ is continuously Gateaux differentiable on $X \setminus \{0\}$ with derivative given by

$$x \mapsto \varphi(x)/\|x\|.$$

Thus, for every $x_0 \in X$, the functional $x \mapsto \|x - x_0\|^p/p$ is continuously Gateaux differentiable in X and its derivative is the operator $A : X \to X^*$ defined by

$$A(x) = \begin{cases} \|x - x_0\|^{p-2}\varphi(x - x_0) & \text{if } x \neq x_0, \\ 0 & \text{if } x = x_0. \end{cases}$$

Now we can prove our result. Fix r, x_0 as in the statement, and assume that (a) does not hold: we shall prove that (b) is true.

Set $I = [0, \infty[$ and define $f : X \times I \to \mathbb{R}$ by putting

$$f(x, \lambda) = \|x - x_0\|^p/p + \lambda(r - J(x)).$$

We are going to apply Theorem B to f, endowing X with the weak topology, so let us check the hypotheses. Condition (c_1) is trivial. Since $r < \sup_X J$, it is immediately seen that $\sup_I \inf_X f < \infty$, so condition (c_3) is obviously fulfilled with $\lambda_0 = 0$.

The discussion of condition (c_2) is more complex. Fix $\lambda \geq 0$. Observe that $f(\cdot, \lambda)$ is sequentially weakly lower semicontinuous (l.s.c.), as it is the sum of $x \mapsto \|x - x_0\|^p/p$, which is weakly l.s.c., and of $x \mapsto \lambda(r - J(x))$, which is sequentially weakly continuous since J' is compact and X is reflexive (see [7, Corollary 41].

Moreover, we prove that $f(\cdot, \lambda)$ is coercive: in fact, for every $\varepsilon \in \,]0, 1/p\lambda[$ it follows from (2) that there exists $\delta > 0$ such that

$$J(x) < \varepsilon\|x\|^p \quad \text{whenever} \quad \|x\| \geq \delta;$$

thus, for all $x \in X$ with $\|x\| \geq \max\{\delta, \|x_0\|\}$,

$$f(x,\lambda) > \|x\|^p\left(\frac{\|x-x_0\|^p}{p\|x\|^p} - \lambda\varepsilon\right) + \lambda r$$
$$\geq \|x\|^p\left(\frac{(\|x\|-\|x_0\|)^p}{p\|x\|^p} - \lambda\varepsilon\right) + \lambda r,$$

and the latter goes to ∞ as $\|x\| \to \infty$. As a consequence of the Eberlein–Shmul'yan Theorem, $f(\cdot,\lambda)$ is weakly l.s.c.

We need to verify that every local minimum of $f(\cdot,\lambda)$ is a global minimum: with this aim in mind, we first observe that $f(\cdot,\lambda)$ is a continuously Gateaux differentiable functional with derivative $A - \lambda J'$. Then we prove that $f(\cdot,\lambda)$ satisfies the Palais–Smale condition.

Let $\{z_n\}$ be a *Palais–Smale sequence*, that is, a sequence in X such that

(PS_1) $\{f(z_n,\lambda)\}$ is bounded;
(PS_2) $\lim_n \|A(z_n) - \lambda J'(z_n)\|_{X^*} = 0$.

From (PS_1), together with the coercivity of $f(\cdot,\lambda)$, it follows that $\{z_n\}$ is bounded, that is, there exists a positive constant M such that $\|z_n\| \leq M$ for all n, hence we find a subsequence, which we still denote $\{z_n\}$, weakly convergent to a point $z_0 \in X$. Then, again by compactness of J', we can assume that $\{J'(z_n)\}$ converges to some $\psi \in X^*$.

Let us prove that $\{z_n\}$ is strongly convergent to z_0.

By (PS_2), for every $\varepsilon > 0$ we can find $\nu \in \mathbb{N}$ such that for all $n > \nu$,

$$\|A(z_n) - \lambda J'(z_n)\|_{X^*} < \frac{\varepsilon}{M + \|z_0\|}$$

so in particular

$$\langle A(z_n) - \lambda J'(z_n), z_n - z_0\rangle < \varepsilon.$$

Due to the convergence of $\{J'(z_n)\}$ to ψ,

$$\lim_n \langle J'(z_n), z_n - z_0\rangle = 0,$$

so we get

$$\lim_n \langle A(z_n), z_n - z_0\rangle = 0;$$

moreover, by the weak convergence of $\{z_n\}$ to z_0,

$$\lim_n \langle A(z_n) - A(z_0), z_n - z_0\rangle = 0. \tag{3}$$

To avoid trivial cases, we assume that $z_n \neq x_0$ for all n and $z_0 \neq x_0$; then we have

$$\begin{aligned}&\langle A(z_n)-A(z_0),z_n-z_0\rangle\\&\qquad=\langle\|z_n-x_0\|^{p-2}\varphi(z_n-x_0)-\|z_0-x_0\|^{p-2}\varphi(z_0-x_0),z_n-z_0\rangle\\&\qquad=\langle\|z_n-x_0\|^{p-2}\varphi(z_n-x_0)-\|z_0-x_0\|^{p-2}\varphi(z_0-x_0),\\&\qquad\qquad\qquad(z_n-x_0)+(x_0-z_0)\rangle\\&\qquad\geq(\|z_n-x_0\|^{p-1}-\|z_0-x_0\|^{p-1})(\|z_n-x_0\|-\|z_0-x_0\|).\end{aligned}$$

It is well known that there exists a constant k_p such that

$$\begin{aligned}&(\|z_n-x_0\|^{p-1}-\|z_0-x_0\|^{p-1})(\|z_n-x_0\|-\|z_0-x_0\|)\\&\qquad\qquad\geq k_p(\|z_n-x_0\|-\|z_0-x_0\|)^{\overline{p}},\end{aligned}$$

where $\overline{p}=\max\{p,2\}$. Then, from (3) it follows that

$$\lim_n\|z_n-x_0\|=\|z_0-x_0\|$$

and hence $\{z_n\}$ is strongly convergent to z_0, that is, the Palais–Smale condition is fulfilled.

We can now check condition (c_2), arguing by contradiction: suppose that $f(\cdot,\lambda)$ admits a local, nonglobal minimum; also being coercive, it also has a global minimum too, that is, it has two strong local minima. Applying the Pucci–Serrin Mountain Pass Theorem, we deduce the existence of a third critical point for $f(\cdot,\lambda)$; thus, the functional of condition (a) would have three critical points, contrary to our assumption.

Now Theorem B ensures that

$$\sup_{\lambda\in I}\inf_{x\in X}f(x,\lambda)=\inf_{x\in X}\sup_{\lambda\in I}f(x,\lambda)=:\alpha. \tag{4}$$

Notice that the function $\lambda\mapsto\inf_{x\in X}f(x,\lambda)$ is upper semicontinuous on I, and tends to $-\infty$ as $\lambda\to\infty$ (since $r<\sup_X J$); thus, it attains its supremum at some $\lambda^*\in I$, that is,

$$\alpha=\inf_{x\in X}\left(\frac{\|x-x_0\|^p}{p}+\lambda^*(r-J(x))\right). \tag{5}$$

The infimum on the right hand side of (4) is easily determined as

$$\alpha=\inf_{x\in J^{-1}([r,\infty[)}\frac{\|x-x_0\|^p}{p}=\frac{\|y-x_0\|^p}{p}$$

for some $y\in J^{-1}([r,\infty[)$.

Actually we have $y\in J^{-1}(r)$: in fact, if $J(y)>r$, there would exist a point z belonging to the segment joining x_0 and y such that $J(z)=r$ and

$$\|x_0-z\|<d(x_0,J^{-1}([r,\infty[)),$$

which is a contradiction. Hence

$$\alpha=\inf_{x\in J^{-1}(r)}\frac{\|x-x_0\|^p}{p}\quad\text{(in particular }\alpha>0). \tag{6}$$

By (5) and (6) it follows that

$$\inf_{x\in X}\left(\frac{\|x-x_0\|^p}{p}-\lambda^* J(x)\right)=\inf_{x\in J^{-1}(r)}\left(\frac{\|x-x_0\|^p}{p}-\lambda^* J(x)\right). \tag{7}$$

We deduce that $\lambda^* > 0$: if $\lambda^* = 0$, then (7) becomes $\alpha = 0$, contrary to (6).

Now we can prove (b). Arguing by contradiction, let $z \in J^{-1}([r,\infty[)\setminus\{y\}$ be such that $\|z - x_0\| = \|y - x_0\|$. As above, we see that $z \in J^{-1}(r)$, and so both z and y are global minima of the functional

$$x \mapsto \|x - x_0\|^p/p - \lambda^* J(x)$$

on $J^{-1}(r)$, and so by (7) on X. Thus, the Pucci–Serrin Mountain Pass Theorem shows that this functional has at least three critical points, contrary to the assumption that (a) does not hold (recall that λ^* is positive).

This concludes the proof. ■

Corollary 1. *Let X, J, p be as in Theorem* 1 *and let S be a convex, dense subset of X. Then for every $r \in\,]\inf_X J, \sup_X J[$ such that $J^{-1}([r,\infty[)$ is not convex there exist $x_0 \in J^{-1}(]-\infty, r[) \cap S$ and $\lambda > 0$ such that the functional on X defined by*

$$x \mapsto \|x - x_0\|^p/p - \lambda J(x)$$

admits at least three critical points.

Proof. First of all, since X has a norm Fréchet differentiable on $S(X)$ and is reflexive (as seen in the Introduction), it follows that X^* is strictly convex. Moreover, $J^{-1}([r,\infty[)$ is sequentially weakly closed.

By Theorem A, for some $x_0 \in J^{-1}(]-\infty, r[) \cap S$ there exist two distinct points $y_1, y_2 \in J^{-1}([r,\infty[)$ satisfying

$$\|y_1 - x_0\| = \|y_2 - x_0\| = d(x_0, J^{-1}([r,\infty[)).$$

Thus, condition (b) of Theorem 1 is false, so there exists $\lambda > 0$ such that $x \mapsto \|x - x_0\|^p/p - \lambda J(x)$ has at least three critical points in X. ■

3. An application to PDE. Throughout the following, Ω will denote a bounded domain in $\mathbb{R}^N$ with C^1 boundary, p a real number with $1 < p < N$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a continuous function satisfying the growth condition

$$|g(x,\xi)| \le a|\xi|^{q-1} + b(x) \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R} \tag{8}$$

for some $a > 0$, $q \in\,]1, p^*[$ (p^* being the critical Sobolev exponent), and $b \in L^{q'}(\Omega)$. We define $G : \Omega \times \mathbb{R} \to \mathbb{R}$ by putting

$$G(x,\xi) = \int_0^{\xi} g(x,\eta)\, d\eta.$$

We denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u(x)|^p \, dx \Big)^{1/p},$$

and recall that for all $q \in \,]1, p^*[$, X is compactly embedded in $L^q(\Omega)$ (we denote by c_q the embedding constant).

By classical results, the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega G(x, u(x)) \, dx$$

is continuously Gateaux differentiable with compact derivative given by

$$\langle J'(u), v\rangle = \int_\Omega g(x, u(x)) v(x) \, dx$$

for all $u, v \in X$.

THEOREM 2. *Let $\Omega \subset \mathbb{R}^N$, g, p be as above and let* (8) *be satisfied. Moreover assume*:

(g$_1$) *$g(x, \cdot)$ is nondecreasing for all $x \in \Omega$*;
(g$_2$) *there exists $x_0 \in \Omega$ such that $g(x_0, \cdot)$ is not constant*;
(g$_3$) $\displaystyle\lim_{|\xi|\to\infty} \sup_{x\in\Omega} \frac{G(x,\xi)}{|\xi|^p} = 0.$

Then, for every $r \in \,]\inf_X J, \sup_X J[$ and every convex, dense subset S of X there exist $u_0 \in S$ with $J(u_0) < r$ and $\lambda > 0$ such that problem (P) has at least three weak solutions.

Proof. We observe that X is a uniformly convex Banach space and its norm is continuously Gateaux differentiable over $X \setminus \{0\}$. *A fortiori*, it is Fréchet differentiable on $S(X)$.

We claim that J satisfies (2). Fix $\varepsilon > 0$. By (g$_3$), there exists $\delta > 0$ such that

$$\sup_{x\in\Omega} \frac{G(x,\xi)}{|\xi|^p} < \frac{\varepsilon}{2c_p^p}$$

whenever $|\xi| \ge \delta$.

Define $k = \sup_{x\in\Omega,\, |\xi|\le\delta} |G(x,\xi)|$ (without loss of generality we may assume $k > 0$). Choose $u \in X$ with

$$\|u\| > \Big(\frac{2k|\Omega|}{\varepsilon}\Big)^{1/p}$$

and put

$$\Omega_1 = \{x \in \Omega : |u(x)| \le \delta\} \quad \text{and} \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Then we have

$$J(u) \le \int_{\Omega_1} |G(x,u(x))|\,dx + \int_{\Omega_2} \frac{\varepsilon}{2c_p^p}\,|u(x)|^p\,dx$$
$$\le k|\Omega_1| + \frac{\varepsilon}{2c_p^p}\,\|u\|^p_{L^p(\Omega)} \le k|\Omega| + \frac{\varepsilon}{2}\|u\|^p.$$

Hence,

$$\frac{J(u)}{\|u\|^p} \le \frac{k|\Omega|}{\|u\|^p} + \frac{\varepsilon}{2} < \varepsilon,$$

which is our claim.

We now prove that J is not constant. By (g_1) and (g_2) there exist α' and α'' with $\alpha' < \alpha''$ such that

$$g(x_0,\alpha') < g(x_0,\alpha'').$$

Assume, for instance, that $g(x_0,\alpha') \ne 0$.

Arguing by contradiction, suppose that J is constant. Then

$$\langle J'(u), v\rangle = \int_{\Omega} g(x,u(x))v(x)\,dx = 0$$

for all $u, v \in X$, and in particular, $g(x,u(x)) = 0$ for all $u \in X$ and a.e. x in Ω. Choose $w \in C_0^1(\Omega)$ such that $w(x_0) = \alpha'$. We deduce that $g(x,w(x)) = 0$ for all $x \in \Omega$, and so in particular $g(x_0,\alpha') = 0$, a contradiction.

Thus, we may choose $r \in \,]\inf_X J, \sup_X J[$ and prove that $J^{-1}([r,\infty[)$ is not convex: first of all, by (g_1), $G(x,\cdot)$ is a convex function over $\mathbb{R}$ for all $x \in \Omega$, which implies that J is a convex functional over X; then $J^{-1}(]-\infty,r])$ is convex, so our claim is equivalent to the nonconvexity of $J^{-1}(r)$. We will assume by contradiction that $J^{-1}(r)$ is convex.

By the continuity of g, we can find $\varepsilon \in \,]0,(\alpha''-\alpha')/2[$ such that

$$g(x_0,\xi') < g(x_0,\xi'') \quad \text{for all } \xi' \in \,]\alpha'-\varepsilon,\alpha'+\varepsilon[,\ \xi'' \in \,]\alpha''-\varepsilon,\alpha''+\varepsilon[. \tag{9}$$

We recall that $C_0^\infty(\Omega)$ is everywhere dense in X so since $J^{-1}(]-\infty,r[)$ and $J^{-1}(]r,\infty[)$ are nonempty open sets in X, there exist $v_1, v_2 \in C_0^\infty(\Omega)$ such that

$$J(v_1) := r_1 < r < r_2 =: J(v_2).$$

Define

$$K = \sup\left\{|G(x,\xi)| : x \in \Omega,\ |\xi| \le \max(|\alpha'|, \|v_1\|_{L^\infty(\Omega)}, \|v_2\|_{L^\infty(\Omega)})\right\}.$$

Notice that $K > 0$. Fix $\sigma \in \mathbb{R}$ such that $r_1 + \sigma < r < r_2 - \sigma$. Since the functions $G(\cdot, v_i(\cdot))$ $(i = 1,2)$ belong to $L^1(\Omega)$, there exists $\delta > 0$ such that

$$\left|\int_A G(x,v_i(x))\,dx\right| < \frac{\sigma}{2} \qquad (i = 1,2)$$

for all subsets A of Ω with $|A| < \delta$.

Denote by B a closed ball centered at x_0 whose measure satisfies

$$|B| < \min\left\{\frac{\sigma}{2K}, \delta\right\}.$$

By classical results, it is possible to construct continuous functions $w_i : \mathbb{R}^N \to \mathbb{R}$ ($i = 1, 2$) such that

$$w_i(x) = \begin{cases} \alpha' & \text{if } x = x_0, \\ v_i(x) & \text{if } x \in \Omega \setminus B, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

and $\|w_i\|_{L^\infty(\mathbb{R}^N)} \le \max\{|\alpha'|, \|v_i\|_{L^\infty(\Omega)}\}$.

The previous conditions yield

$$(10)\quad \int_\Omega G(x, w_1(x))\,dx = \int_B G(x, w_1(x))\,dx + J(v_1) - \int_B G(x, v_1(x))\,dx$$
$$< K|B| + r_1 + \sigma/2 < r_1 + \sigma < r.$$

In a similar way

$$(11)\quad \int_\Omega G(x, w_2(x))\,dx > r.$$

Let $\{\varrho_n\}$ be a mollifier sequence and h_n^i ($i = 1, 2$) the convolution

$$h_n^i = \varrho_n * w_i.$$

By the theory of convolution h_n^i belongs to $C_0^\infty(\mathbb{R}^N)$ and in particular to X (for n large enough). Moreover $\{h_n^i\}$ uniformly converges to w_i on compact sets of $\mathbb{R}^N$. Thus,

$$\lim_n J(h_n^i) = \int_\Omega G(x, w_i(x))\,dx,$$

so by (10) and (11) and the uniform convergence we can choose m so large that $J(h_m^1) < r < J(h_m^2)$ and $|h_m^i(x_0) - \alpha'| < \varepsilon$ for $i = 1, 2$. The set

$$\Gamma = \{u \in C_0^\infty(\Omega) : |u(x_0) - \alpha'| < \varepsilon\}$$

is convex. So the segment joining h_m^1 and h_m^2 is contained in Γ; moreover by the continuity of J, there exists a point u' of the segment such that $J(u') = r$. Analogously there exists $u'' \in C_0^\infty(\Omega)$ such that $|u''(x_0) - \alpha''| < \varepsilon$ and $J(u'') = r$.

Since $J^{-1}(r)$ is assumed to be convex, the function $\mu \mapsto J(\mu u' + (1-\mu)u'')$ is constant on $[0, 1]$. Hence its derivative is zero, that is,

$$\langle J'(\mu u' + (1-\mu)u''), u' - u''\rangle = 0$$

for all $\mu \in [0, 1]$. In particular

$$\int_\Omega [g(x, u'(x)) - g(x, u''(x))](u'(x) - u''(x))\,dx = 0,$$

and by (g_1) we get

$$[g(x,u'(x))-g(x,u''(x))](u'(x)-u''(x))=0$$

for all $x\in\Omega$.

On the other hand,

$$\alpha'-\varepsilon<u'(x_0)<\alpha'+\varepsilon<\alpha''-\varepsilon<u''(x_0)<\alpha''+\varepsilon,$$

and so $g(x_0,u'(x_0))=g(x_0,u''(x_0))$ contrary to (9).

We have thus proved that $J^{-1}([r,\infty[)$ is not convex. By Corollary 1, there exist $u_0\in S$ with $J(u_0)<r$ and $\lambda>0$ such that the functional $u\mapsto\|u-u_0\|^p/p-\lambda J(u)$ has at least three critical points in X. Since for such critical point $u\in X$, $u-u_0$ is a weak solution of (P), the proof is concluded. ■

If our assumptions are fulfilled for $p=2<N$ and we choose $u_0\in C_0^\infty(\Omega)$, then due to the linearity of the Laplacian operator, problem (P) is in a sense equivalent to the following:

$$(P')\quad\begin{cases}-\Delta u=\lambda g(x,u)-\Delta u_0(x) & \text{in } \Omega,\\ u=0 & \text{in } \partial\Omega.\end{cases}$$

In fact, it is immediately seen that u is a weak solution of (P) iff $u+u_0$ is a weak solution of (P'). Thus, the next result follows at once from Theorem 2:

Corollary 2. *Let $\Omega\subset\mathbb{R}^N$ and g be as above and let* (8) *and* (g_1)–(g_3) *be satisfied for $p=2$. Then for every $r\in\,]\inf_X J,\sup_X J[$ and every convex, dense subset S of $C_0^\infty(\Omega)$ there exist $u_0\in S$ with $J(u_0)<r$ and $\lambda>0$ such that problem (P') has at least three weak solutions.*

Remark 1. If, in addition to the hypotheses of Theorem 2, we assume that $g(x,\cdot)$ is positively homogeneous with some exponent $\beta\neq p-1$ for all $x\in\Omega$, we can "hide" the parameter λ. Let V be a dense linear subspace of X, and fix $r\in\,]\inf_X J,\sup_X J[$; then we find $u_0\in V$ and $\lambda>0$ such that problem (P) has at least three solutions. Now put $\gamma=1/(\beta+1-p)$ and $v_0=\lambda^\gamma u_0$ (v_0 still belongs to V); then the problem

$$(P'')\quad\begin{cases}-\Delta_p v=g(x,v+v_0(x)) & \text{in } \Omega,\\ v=0 & \text{in } \partial\Omega,\end{cases}$$

has at least three solutions. In fact, for each solution $u\in X$ of (P), the function $v=\lambda^\gamma u$ is a solution of (P''). Notice that $J(v_0)<\lambda^{\gamma(\beta+1)}r$.

Remark 2. There exist a sequence $\{u_n\}\subset S$ and a sequence $\{\lambda_n\}\subset\,]0,\infty[$ such that the problem

$$(P_n)\quad\begin{cases}-\Delta_p u=\lambda_n g(x,u+u_n(x)) & \text{in } \Omega,\\ u=0 & \text{in } \partial\Omega,\end{cases}$$

has at least three solutions for all $n \in \mathbb{N}$, $n \geq 0$. In fact, the existence of a first pair u_0, λ_0 is ensured by Theorem 2, and the result can be generalized by induction: once u_i is determined for $i = 0, \ldots, n$, there exists a subset S_{n+1} of S, convex and everywhere dense in S (hence in X), not containing the points u_i (see [6, Remark 1]); then it suffices to apply Theorem 2 to S_{n+1} to find a new function $u_{n+1} \in S_{n+1}$ and a positive λ_{n+1} such that problem (P_{n+1}) has at least three solutions.

Example 1. The above results allow us to establish that the uniqueness of solution of boundary value problems can be unstable: that is, a problem admitting exactly one nontrivial solution can be perturbed in infinitely many ways so that each of the perturbed problems has at least three nontrivial solutions.

Namely, let Ω be as above, $q \in \,]0,1[$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ \xi^q & \text{if } \xi > 0. \end{cases}$$

Then it is well known that the problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

has exactly one nontrivial solution.

Notice that g satisfies all hypotheses of Corollary 2 and Remark 1 (with $\beta = q$); thus, recalling Remark 2, we observe that, choosing $S = C_0^\infty(\Omega)$, there exists a sequence $\{u_n\} \subset C_0^\infty(\Omega)$ ($u_n \neq 0$ for all $n \in \mathbb{N}$) such that for all $n \in \mathbb{N}$ the problem

$$\begin{cases} -\Delta u = g(u) - \Delta u_n(x) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

has at least three solutions (each of them is obviously different from zero).

Finally, we observe that a similar result can be proved for the problem (Q), as pointed out in the Introduction. Under the same hypotheses on Ω, p, g, set $Y = W^{1,p}(\Omega)$, endowed with the norm

$$\|u\| = \Big(\int_\Omega (|\nabla u(x)|^p + |u(x)|^p)\, dx \Big)^{1/p}.$$

Notice that Y has all the properties we need, and contains $C^\infty(\overline{\Omega})$ as a dense linear subspace. Let $G, J : Y \to \mathbb{R}$ be defined as above. Then the following result holds:

Theorem 3. *Let $\Omega \subset \mathbb{R}^N$, g, p be as above and let* (8) *and* (g_1)–(g_3) *be satisfied. Then for every $r \in \,]\inf_Y J, \sup_Y J[$ and every convex, dense subset S' of Y there exist $u_0 \in S'$ with $J(u_0) < r$ and $\lambda > 0$ such that problem* (Q) *has at least three weak solutions.*

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