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Quasi *-algebras of measurable operators

by

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Abstract. Non-commutative L^p-spaces are shown to constitute examples of a class of Banach quasi *-algebras called CQ^*-algebras. For $p \geq 2$ they are also proved to possess a sufficient family of bounded positive sesquilinear forms with certain invariance properties. CQ^*-algebras of measurable operators over a finite von Neumann algebra are also constructed and it is proven that any abstract CQ^*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ with a sufficient family of bounded positive tracial sesquilinear forms can be represented as a CQ^*-algebra of this type.

1. Introduction and preliminaries. A *quasi *-algebra* is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{X}$, and $\mathfrak{X}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. Quasi *-algebras were introduced by Lassner [8, 9, 11] to provide an appropriate mathematical framework for certain quantum physical systems for which the usual algebraic approach in terms of C^*-algebras turned out to be insufficient. In these applications they usually arise by taking the completion of the C^*-algebra of observables in a weaker topology satisfying certain physical requirements. The case where this weaker topology is a norm topology has been considered in a series of previous papers [3]–[2], where CQ^*-algebras were introduced: a CQ^*-algebra is, indeed, a quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ where $\mathfrak{X}$ is a Banach space with respect to a norm $\| \ \|$ possessing an isometric involution and $\mathfrak{A}_0$ is a C^*-algebra with respect to a norm $\| \cdot \|_0$, which is dense in $\mathfrak{X}[\| \cdot \|]$.

Since any C^*-algebra $\mathfrak{A}_0$ has a faithful *-representation π, it is natural to ask if this completion can also be realized as a quasi *-algebra of operators affiliated to $\pi(\mathfrak{A}_0)''$. The Segal–Nelson theory [12, 10] of non-commutative integration provides a number of mathematical tools for dealing with this problem.

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The paper is organized as follows. In Section 2 we consider non-commutative L^p-spaces constructed starting from a von Neumann algebra $\mathfrak{M}$ and a normal, semifinite, faithful trace τ as Banach quasi *-algebras. In particular if φ is finite, then it is shown that $(L^p(\varphi), \mathfrak{M})$ is a CQ^*-algebra. If $p \geq 2$, they even possess a sufficient family of positive sesquilinear forms enjoying certain invariance properties.

In Section 3, starting from a family $\mathfrak{F}$ of normal finite traces on a von Neumann algebra $\mathfrak{M}$, we prove that the completion of $\mathfrak{M}$ with respect to a norm defined in a natural way by $\mathfrak{F}$ is indeed a CQ^*-algebra consisting of measurable operators, in Segal's sense, and therefore affiliated with $\mathfrak{M}$.

Finally, in Section 4, we prove that any CQ^*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ with a sufficient family of bounded positive tracial sesquilinear forms can be continuously embedded into the CQ^*-algebra of measurable operators constructed in Section 3.

To keep the paper sufficiently self-contained, we collect below some preliminary definitions and propositions that will be used in what follows.

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a quasi *-algebra. The *unit* of $(\mathfrak{X}, \mathfrak{A}_0)$ is an element $e \in \mathfrak{A}_0$ such that $xe = ex = x$ for every $x \in \mathfrak{X}$. A quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is said to be *locally convex* if $\mathfrak{X}$ is endowed with a topology τ which makes of $\mathfrak{X}$ a locally convex space and such that the involution $a \mapsto a^*$ and the multiplications $a \mapsto ab$, $a \mapsto ba$, $b \in \mathfrak{A}_0$, are continuous. If τ is a norm topology and the involution is isometric with respect to the norm, we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a *normed quasi *-algebra* and, if it is complete, we say it is a *Banach quasi *-algebra*.

Definition 1.1. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with norm $\|\cdot\|$ and involution *. Assume that a second norm $\|\cdot\|_0$ is defined on $\mathfrak{A}_0$, satisfying the following conditions:

(a.1) $\|a^*a\|_0 = \|a\|_0^2$, $\forall a \in \mathfrak{A}_0$;
(a.2) $\|a\| \leq \|a\|_0$, $\forall a \in \mathfrak{A}_0$;
(a.3) $\|ax\| \leq \|a\|_0\|x\|$, $\forall a \in \mathfrak{A}_0$, $x \in \mathfrak{X}$;
(a.4) $\mathfrak{A}_0[\|\cdot\|_0]$ is complete.

Then we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a *CQ^*-algebra*.

Remark 1.2. (1) If $\mathfrak{A}_0[\|\cdot\|_0]$ is not complete, we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a pre-CQ^*-algebra.

(2) In previous papers the name of CQ^*-algebra was given to a more complicated structure where two different involutions were considered on $\mathfrak{A}_0$. When these involutions coincide, we spoke of a *proper CQ^*-algebra*. In this paper only this case will be considered and so we systematically omit the word *proper*.

The following basic definitions and results on non-commutative measure theory are also needed in what follows.

Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$. Put

$$\mathcal{J} = \{X \in \mathfrak{M} : \varphi(|X|) < \infty\}.$$

Then $\mathcal{J}$ is a *-ideal of $\mathfrak{M}$. We denote by $\mathrm{Proj}(\mathfrak{M})$ the lattice of projections of $\mathfrak{M}$.

Definition 1.3. A vector subspace $\mathcal{D}$ of $\mathcal{H}$ is said to be *strongly dense* (resp., *strongly φ-dense*) if

- $U'\mathcal{D} \subset \mathcal{D}$ for any unitary U' in $\mathfrak{M}'$,
- there exists a sequence $P_n \in \mathrm{Proj}(\mathfrak{M})$ such that $P_n\mathcal{H} \subset \mathcal{D}$, $P_n^\perp \downarrow 0$ and $P_n^\perp$ is a finite projection (resp., $\varphi(P_n^\perp) < \infty$).

Clearly, every strongly φ-dense domain is strongly dense.

Throughout this paper, when we say that an operator T is *affiliated with a von Neumann algebra* $\mathfrak{M}$, written $T\eta\mathfrak{M}$, we always mean that T is closed, densely defined and $TU \supseteq UT$ for every unitary operator $U \in \mathfrak{M}'$.

Definition 1.4. An operator $T\eta\mathfrak{M}$ is called

- *measurable* (with respect to $\mathfrak{M}$) if its domain $D(T)$ is strongly dense;
- *φ-measurable* if its domain $D(T)$ is strongly φ-dense.

From the very definition it follows that, if T is φ-measurable, then there exists $P \in \mathrm{Proj}(\mathfrak{M})$ such that TP is bounded and $\varphi(P^\perp) < \infty$.

We recall that any operator affiliated with a finite von Neumann algebra is measurable [12, Cor. 4.1] but it is not necessarily φ-measurable.

2. Non-commutative L^p-spaces as CQ^*-algebras. In this section we will discuss the structure of non-commutative L^p-spaces as quasi *-algebras. We begin by recalling the basic definitions.

Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathfrak{M}_+$. For each $p \geq 1$, let

$$\mathcal{J}_p = \{X \in \mathfrak{M} : \varphi(|X|^p) < \infty\}.$$

Then $\mathcal{J}_p$ is a *-ideal of $\mathfrak{M}$. Following [10], we denote by $L^p(\varphi)$ the Banach space completion of $\mathcal{J}_p$ with respect to the norm

$$\|X\|_p := \varphi(|X|^p)^{1/p}, \qquad X \in \mathcal{J}_p.$$

One usually defines $L^\infty(\varphi) = \mathfrak{M}$. Thus, if φ is a finite trace, then $L^\infty(\varphi) \subset L^p(\varphi)$ for every $p \geq 1$. As shown in [10], if $X \in L^p(\varphi)$, then X is a measurable operator.

Proposition 2.1. *Let $\mathfrak{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace on $\mathfrak{M}_+$. Then $(L^p(\varphi), L^\infty(\varphi) \cap L^p(\varphi))$ is a Banach quasi *-algebra. If φ is a finite trace and $\varphi(\mathbb{I}) = 1$, then $(L^p(\varphi), L^\infty(\varphi))$ is a CQ^*-algebra.*

Proof. Indeed, it is easily seen that the norms $\|\cdot\|_\infty$ of $L^\infty(\varphi) \cap L^p(\varphi)$ and $\|\cdot\|_p$ on $L^p(\varphi)$ satisfy conditions (a.1)–(a.2) of Definition 1.1. Moreover, if φ is finite, then $L^\infty(\varphi) \subset L^p(\varphi)$ and thus $(L^p(\varphi), L^\infty(\varphi))$ is a CQ^*-algebra. ■

Remark 2.2. Of course the condition $\varphi(\mathbb{I}) = 1$ can be easily removed by rescaling the trace.

Definition 2.3. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. We denote by $\mathcal{S}(\mathfrak{X})$ the set of all sesquilinear forms Ω on $\mathfrak{X} \times \mathfrak{X}$ with the following properties:

(i) $\Omega(x, x) \geq 0, \ \forall x \in \mathfrak{X}$,
(ii) $\Omega(xa, b) = \Omega(a, x^*b), \ \forall x \in \mathfrak{X}, \ a, b \in \mathfrak{A}_0$,
(iii) $|\Omega(x, y)| \leq \|x\| \, \|y\|, \ \forall x, y \in \mathfrak{X}$.

A subfamily $\mathcal{A}$ of $\mathcal{S}(\mathfrak{X})$ is called *sufficient* if the conditions $x \in \mathfrak{X}$ and $\Omega(x, x) = 0$ for every $\Omega \in \mathcal{A}$ imply $x = 0$.

If $(\mathfrak{X}, \mathfrak{A}_0)$ is a Banach quasi *-algebra, then the Banach dual space $\mathfrak{X}^\sharp$ of $\mathfrak{X}$ can be made into a Banach $\mathfrak{A}_0$-bimodule with norm

$$\|f\|^\sharp = \sup_{\|x\| \leq 1} |\langle x, f \rangle|, \qquad f \in \mathfrak{X}^\sharp,$$

by defining, for $f \in \mathfrak{X}^\sharp$, $a \in \mathfrak{A}_0$, the module operations in the following way:

$$\langle x, f \circ a \rangle := \langle ax, f \rangle, \qquad x \in \mathfrak{X},$$
$$\langle x, a \circ f \rangle := \langle xa, f \rangle, \qquad x \in \mathfrak{X}.$$

As usual, an involution $f \mapsto f^*$ can be defined on $\mathfrak{X}^\sharp$ by $\langle x, f^* \rangle = \overline{\langle x^*, f \rangle}$ for $x \in \mathfrak{X}$. With these notations we can easily prove the following (see also [15]):

Proposition 2.4. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra and Ω a positive sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$. The following statements are equivalent*:

(i) $\Omega \in \mathcal{S}(\mathfrak{X})$;
(ii) *there exists a bounded conjugate linear operator $T : \mathfrak{X} \to \mathfrak{X}^\sharp$ with the properties*:

(ii.1) $\langle x, Tx \rangle \geq 0, \ \forall x \in \mathfrak{X}$;
(ii.2) $T(ax) = (Tx) \circ a^*, \ \forall a \in \mathfrak{A}_0, \ x \in \mathfrak{X}$;
(ii.3) $\|T\|_{\mathcal{B}(\mathfrak{X}, \mathfrak{X}^\sharp)} \leq 1$;
(ii.4) $\Omega(x, y) = \langle x, Ty \rangle, \ \forall x, y \in \mathfrak{X}$.

We will now focus on the question whether for the Banach quasi *-algebra $(L^p(\varphi), L^\infty(\varphi)\cap L^p(\varphi))$, the family $\mathcal{S}(L^p(\varphi))$, which we are going to describe in more detail, is or is not sufficient.

Before going forth, we recall that many of the familiar results of the ordinary theory of L^p-spaces hold in the very same form for non-commutative L^p-spaces. This is the case, for instance, of Hölder's inequality and also of the characterization of the dual of L^p: the form defining the duality is an extension of φ (denoted by the same symbol) to products of the type XY with $X \in L^p(\varphi)$, $Y \in L^{p'}(\varphi)$ with $p^{-1}+p'^{-1} = 1$, and one has $(L^p(\varphi))^\sharp \simeq L^{p'}(\varphi)$.

In order to study $\mathcal{S}(L^p(\varphi))$, we introduce, for $p \geq 2$, the following notation:

$$\mathcal{B}^p_+ = \{X \in L^{p/(p-2)}(\varphi) : X \geq 0,\ \|X\|_{p/(p-2)} \leq 1\}$$

where $p/(p-2) = \infty$ if $p = 2$.

For each $W \in \mathcal{B}^p_+$, we consider the right multiplication operator

$$R_W : L^p(\varphi) \to L^{p/(p-1)}(\varphi), \qquad R_W X = XW, \qquad X \in L^p(\varphi).$$

Since $L^\infty(\varphi) \cap L^p(\varphi) = \mathcal{J}_p$, we use, for brevity, the latter notation.

Lemma 2.5. *The following statements hold.*

- (i) *Let $p \geq 2$. For every $W \in \mathcal{B}^p_+$, the sesquilinear form $\Omega(X,Y) = \varphi[X(R_W Y)^*]$ is an element of $\mathcal{S}(L^p(\varphi))$.*
- (ii) *If φ is finite, then for each $\Omega \in \mathcal{S}(L^p(\varphi))$, there exists $W \in \mathcal{B}^p_+$ such that*

$$\Omega(X,Y) = \varphi[X(R_W Y)^*], \qquad \forall X, Y \in L^p(\varphi).$$

Proof. (i) We check that the sesquilinear form $\Omega(X,Y) = \varphi[X(R_W Y)^*]$, $X, Y \in L^p(\varphi)$, satisfies conditions (i)–(iii) of Definition 2.3. For every $X \in L^p(\varphi)$ we have

$$\Omega(X,X) = \varphi[X(R_W X)^*] = \varphi[X(XW)^*] = \varphi[(XW)^*X] = \varphi[W|X|^2] \geq 0.$$

For every $X \in L^p(\varphi)$, $A, B \in \mathcal{J}_p$, we get

$$\begin{aligned}\Omega(XA,B) &= \varphi(XA(BW)^*) = \varphi(WB^*XA) = \varphi(A(X^*BW)^*)\\ &= \Omega(A, X^*B).\end{aligned}$$

Finally, for every $X, Y \in L^p(\varphi)$,

$$|\Omega(X,Y)| \leq \|X\|_p \|Y\|_p \|W\|_{p/(p-2)} \leq \|X\|_p \|Y\|_p.$$

(ii) Let $\Omega \in \mathcal{S}(L^p(\varphi))$. Let $T : L^p(\varphi) \to L^{p'}(\varphi)$ be the operator which represents Ω in the sense of Proposition 2.4. The finiteness of φ implies that $\mathcal{J}_p = \mathfrak{M}$; thus we can put $W = T(\mathbb{I})$. It is easy to check that $R_W = T$. This concludes the proof. ∎

Proposition 2.6. *If $p \geq 2$, then $\mathcal{S}(L^p(\varphi))$ is sufficient.*

Proof. Let $X \in L^p(\varphi)$ be such that $\Omega(X,X)=0$ for every $\Omega \in \mathcal{S}(L^p(\varphi))$. By the previous lemma, since $|X|^{p-2} \in L^{p/(p-2)}(\varphi)$, the right multiplication operator R_W with $W = |X|^{p-2}/\alpha$, $\alpha \in \mathbb{R}$, satisfying $\||X|^{p-2}/\alpha\|_{p/(p-2)} \le 1$ represents a sesquilinear form $\Omega \in \mathcal{S}(L^p(\varphi))$. By assumption, $\Omega(X,X)=0$. We then have

$$\Omega(X,X) = \varphi[X(R_W X)^*] = \frac{\varphi[X(X|X|^{p-2})^*]}{\alpha} = \frac{\varphi[(X|X|^{p-2})^* X]}{\alpha} = \frac{\varphi[|X|^p]}{\alpha} = 0,$$

so $X=0$, by the faithfulness of φ. ∎

3. CQ^*-algebras over finite von Neumann algebras. Let $\mathfrak{M}$ be a von Neumann algebra and $\mathfrak{F} = \{\varphi_\alpha : \alpha \in \mathcal{I}\}$ be a family of normal *finite* traces on $\mathfrak{M}$. As usual, we say that the family $\mathfrak{F}$ is *sufficient* if the conditions $X \in \mathfrak{M}$, $X \ge 0$ and $\varphi_\alpha(X) = 0$ for every $\alpha \in \mathcal{I}$ imply $X = 0$ (clearly, if $\mathfrak{F} = \{\varphi\}$, then $\mathfrak{F}$ is sufficient if, and only if, φ is faithful). In this case, $\mathfrak{M}$ is a finite von Neumann algebra [13, Ch. 7]. We assume in addition that the following condition (P) is satisfied:

$$\text{(P)} \qquad \varphi_\alpha(\mathbb{I}) \le 1, \quad \forall \alpha \in \mathcal{I}.$$

Then we define

$$\|X\|_{p,\mathcal{I}} = \sup_{\alpha\in\mathcal{I}} \|X\|_{p,\varphi_\alpha} = \sup_{\alpha\in\mathcal{I}} \varphi_\alpha(|X|^p)^{1/p}.$$

Since $\mathfrak{F}$ is sufficient, $\|\cdot\|_{p,\mathcal{I}}$ is a norm on $\mathfrak{M}$.

We will need the following lemmas whose simple proofs will be omitted.

LEMMA 3.1. *Let $\mathfrak{M}$ be a von Neumann algebra in a Hilbert space $\mathcal{H}$, and $\{P_\alpha\}_{\alpha\in\mathcal{I}}$ a family of projections of $\mathfrak{M}$ with*

$$\bigvee_{\alpha\in\mathcal{I}} P_\alpha = \overline{P}.$$

If $A \in \mathfrak{M}$ and $AP_\alpha = 0$ for every $\alpha \in \mathcal{I}$, then $A\overline{P} = 0$.

LEMMA 3.2. *Let $\mathfrak{F} = \{\varphi_\alpha\}_{\alpha\in\mathcal{I}}$ be a sufficient family of normal finite traces on the von Neumann algebra $\mathfrak{M}$ and let P_α be the support of φ_α. Then $\bigvee P_\alpha = \mathbb{I}$, where $\mathbb{I}$ denotes the identity of $\mathfrak{M}$.*

It is well known that the support of each φ_α enjoys the following properties:

(i) $P_\alpha \in \mathcal{Z}(\mathfrak{M})$, the center of $\mathfrak{M}$, for each $\alpha \in I$;
(ii) $\varphi_\alpha(X) = \varphi_\alpha(XP_\alpha)$ for each $\alpha \in I$.

From the preceding two lemmas it follows that, if the P_α's are as in Lemma 3.2, then

$$AP_\alpha = 0, \forall \alpha \in \mathcal{I} \;\Rightarrow\; A = 0.$$

If Condition (P) is fulfilled, then

$$\|X\|_{p,\mathcal{I}} = \sup_{\alpha\in\mathcal{I}} \|XP_\alpha\|_{p,\alpha}, \qquad \forall X \in \mathfrak{M}.$$

Clearly, the sufficiency of the family of traces and Condition (P) imply that $\|\cdot\|_{p,\mathcal{I}}$ is a norm on $\mathfrak{M}$.

PROPOSITION 3.3. *Let $\mathfrak{M}(p,\mathcal{I})$ denote the Banach space completion of $\mathfrak{M}$ with respect to the norm $\|\cdot\|_{p,\mathcal{I}}$. Then $(\mathfrak{M}(p,\mathcal{I})[\|\cdot\|_{p,\mathcal{I}}], \mathfrak{M}[\|\cdot\|_{\mathcal{B}(\mathcal{H})}])$ is a CQ^*-algebra.*

Proof. Indeed, we have

$$(1) \qquad \|X^*\|_{p,\mathcal{I}} = \sup_{\alpha\in\mathcal{I}} \|X^*P_\alpha\|_{p,\alpha} = \sup_{\alpha\in\mathcal{I}} \|(XP_\alpha)^*\|_{p,\alpha} = \|X\|_{p,\mathcal{I}}, \qquad \forall X \in \mathfrak{M}.$$

Furthermore, for every $X, Y \in \mathfrak{M}$,

$$(2) \qquad \begin{aligned}\|XY\|_{p,\mathcal{I}} &= \sup_{\alpha\in\mathcal{I}} \|XYP_\alpha\|_{p,\alpha} \le \|X\|_{\mathcal{B}(\mathcal{H})} \sup_{\alpha\in\mathcal{I}} \|YP_\alpha\|_{p,\alpha} \\ &= \|X\|_{\mathcal{B}(\mathcal{H})}\|Y\|_{p,\mathcal{I}}.\end{aligned}$$

Finally, Condition (P) implies that

$$\|X\|_{p,\mathcal{I}} \le \|X\|_{\mathcal{B}(\mathcal{H})}, \qquad \forall X \in \mathfrak{M}.$$

From (1) and (2) it follows that $\mathfrak{M}(p,\mathcal{I})$ is a Banach quasi *-algebra. It is clear that $\|\ \|_{\mathcal{B}(\mathcal{H})}$ satisfies conditions (a.1)–(a.4) of Section 1. Therefore $(\mathfrak{M}(p,\mathcal{I}), \mathfrak{M})$ is a CQ^*-algebra. ∎

The next step consists in investigating the Banach space $\mathfrak{M}(p,\mathcal{I})[\|\cdot\|_{p,\mathcal{I}}]$. In particular we are interested in whether $\mathfrak{M}(p,\mathcal{I})[\|\cdot\|_{p,\mathcal{I}}]$ can be identified with a space of operators affiliated with $\mathfrak{M}$. For brevity, whenever no ambiguity can arise, we write $\mathfrak{M}_p$ instead of $\mathfrak{M}(p,\mathcal{I})$.

Let $\mathfrak{F} = \{\varphi_\alpha\}_{\alpha\in\mathcal{I}}$ be a sufficient family of normal, finite traces on the von Neumann algebra $\mathfrak{M}$ satisfying Condition (P). The traces φ_α are not necessarily faithful. Put $\mathfrak{M}_\alpha = \mathfrak{M}P_\alpha$, where, as before, P_α denotes the support of φ_α. Each $\mathfrak{M}_\alpha$ is a von Neumann algebra and φ_α is faithful in $\mathfrak{M}P_\alpha$ [14, Proposition V. 2.10].

More precisely,

$$\mathfrak{M}_\alpha := \mathfrak{M}P_\alpha = \{Z = XP_\alpha \text{ for some } X \in \mathfrak{M}\}.$$

The positive cone $\mathfrak{M}_\alpha^+$ of $\mathfrak{M}_\alpha$ equals

$$\{Z = XP_\alpha \text{ for some } X \in \mathfrak{M}^+\}.$$

For $Z = XP_\alpha \in \mathfrak{M}_\alpha^+$, we put

$$\sigma_\alpha(Z) := \varphi_\alpha(XP_\alpha).$$

The definition of $\sigma_\alpha(Z)$ does not depend on the particular choice of X. Each σ_α is a normal finite faithful trace on $\mathfrak{M}_\alpha$. It is then possible to consider the spaces $L^p(\mathfrak{M}_\alpha, \sigma_\alpha)$, $p \geq 1$, in the usual way. The norm of $L^p(\mathfrak{M}_\alpha, \sigma_\alpha)$ is indicated as $\|\cdot\|_{p,\alpha}$.

Let now (X_k) be a Cauchy sequence in $\mathfrak{M}[\|\cdot\|_{p,\mathcal{I}}]$. For each $\alpha \in \mathcal{I}$, we put $Z_k^{(\alpha)} = X_k P_\alpha$. Then, for each $\alpha \in \mathcal{I}$, $(Z_k^{(\alpha)})$ is a Cauchy sequence in $\mathfrak{M}_\alpha[\|\cdot\|_{p,\alpha}]$. Indeed, since $|Z_k^{(\alpha)} - Z_h^{(\alpha)}|^p = |X_k - X_h|^p P_\alpha$, we have

$$\begin{aligned}\|Z_k^{(\alpha)} - Z_h^{(\alpha)}\|_{p,\alpha} &= \sigma_\alpha(|Z_k^{(\alpha)} - Z_h^{(\alpha)}|^p)^{1/p} = \varphi_\alpha(|X_k - X_h|^p P_\alpha)^{1/p} \\ &= \varphi_\alpha(|X_k - X_h|^p)^{1/p} \to 0.\end{aligned}$$

Therefore, for each $\alpha \in \mathcal{I}$, there exists an operator $Z^{(\alpha)} \in L^p(\mathfrak{M}_\alpha, \sigma_\alpha)$ such that

$$Z^{(\alpha)} = \|\cdot\|_{p,\alpha}\text{-}\lim_{k\to\infty} Z_k^{(\alpha)}.$$

It is now natural to ask whether there exists an operator X, closed, densely defined, affiliated with $\mathfrak{M}$, which reduces to $Z^{(\alpha)}$ on $\mathfrak{M}_\alpha$. To begin with, we assume that the projections P_α are mutually orthogonal. In this case, putting $\mathcal{H}_\alpha = P_\alpha \mathcal{H}$, we have

$$\mathcal{H} = \bigoplus_{\alpha\in\mathcal{I}} \mathcal{H}_\alpha = \Big\{(f_\alpha) : f_\alpha \in \mathcal{H}_\alpha, \sum_{\alpha\in I} \|f_\alpha\|^2 < \infty\Big\}.$$

We put

$$D(X) = \Big\{(f_\alpha) \in \mathcal{H} : f_\alpha \in D(Z^{(\alpha)}), \sum_{\alpha\in I} \|Z^{(\alpha)} f_\alpha\|^2 < \infty\Big\}$$

and for $f = (f_\alpha) \in D(X)$ we define

$$Xf = (Z^{(\alpha)} f_\alpha).$$

Then

(i) $D(X)$ is dense in $\mathcal{H}$. Indeed, $D(X)$ contains all $f = (f_\alpha)$ with $f_\alpha = 0$ except for a finite subset of indices.

(ii) X is closed in $\mathcal{H}$. Indeed, let $f_n = (f_{n,\alpha})$ be a sequence of elements of $D(X)$ with $f_n \to g = (g_\alpha) \in \mathcal{H}$ and $Xf_n \to h$. Since

$$f_n \to g \;\Leftrightarrow\; f_{n,\alpha} \to g_\alpha \in \mathcal{H}_\alpha,\ \forall \alpha \in \mathcal{I},$$

and

$$Xf_n \to h \;\Leftrightarrow\; (Xf_n)_\alpha \to h_\alpha \in \mathcal{H}_\alpha,\ \forall \alpha \in \mathcal{I},$$

the equalities $(Xf_n)_\alpha = Z^{(\alpha)} f_{n,\alpha}$ and the closedness of each $Z^{(\alpha)}$ in $\mathcal{H}_\alpha$ yield

$$g_\alpha \in D(Z^{(\alpha)}) \quad \text{and} \quad h_\alpha = Z^{(\alpha)} g_\alpha.$$

It remains to check that $\sum_{\alpha\in\mathcal{I}} \|Z^{(\alpha)} g_\alpha\|^2 < \infty$; but this is clear, since both $(Z^{(\alpha)} g_\alpha)$ and $h = (h_\alpha)$ are in $\mathcal{H}$.

(iii) $X\eta\mathfrak{M}$. Let $Y \in \mathfrak{M}'$. Then $Yf = (YP_\alpha f)$ for all $f \in \mathcal{H}$ and $YP_\alpha \in (\mathfrak{M}P_\alpha)' = \mathfrak{M}'P_\alpha$. Therefore

$$XYf = ((XY)P_\alpha f) = (YXP_\alpha f) = YXf.$$

In conclusion, X is a measurable operator.

Thus, we have proved the following

PROPOSITION 3.4. *Let $\mathfrak{F} = \{\varphi_\alpha\}_{\alpha\in\mathcal{I}}$ be a sufficient family of normal finite traces on the von Neumann algebra $\mathfrak{M}$. Assume that Condition* (P) *is fulfilled and that the φ_α's have mutually orthogonal supports. Then $\mathfrak{M}_p$, $p \geq 1$, consists of measurable operators.*

The analysis of the general case would be much simplified if, from a given sufficient family $\mathfrak{F}$ of normal finite traces, one could extract (or construct) a *sufficient* subfamily $\mathcal{G}$ of traces with mutually orthogonal supports. Apart from quite simple situations (for instance when $\mathfrak{F}$ is finite or countable), we do not know if this is possible or not. There is however a relevant case where this can be fairly easily done. This occurs when $\mathfrak{F}$ is a convex and w^*-compact family of traces on $\mathfrak{M}$.

LEMMA 3.5. *Let $\mathfrak{F}$ be a convex w^*-compact family of normal finite traces on a von Neumann algebra $\mathfrak{M}$; assume that for each central operator Z with $0 \leq Z \leq \mathbb{I}$ and each $\eta \in \mathfrak{F}$ the functional $\eta_Z(X) := \eta(XZ)$ belongs to $\mathfrak{F}$. Let $\mathfrak{E}\mathfrak{F}$ be the set of extreme elements of $\mathfrak{F}$. If $\eta_1, \eta_2 \in \mathfrak{E}\mathfrak{F}$, $\eta_1 \neq n_2$, and P_1 and P_2 are their respective supports, then P_1 and P_2 are orthogonal.*

Proof. Let P_1, P_2 be, respectively, the supports of η_1 and η_2. We begin by proving that either $P_1 = P_2$ or $P_1P_2 = 0$. Indeed, assume that $P_1P_2 \neq 0$. We define

$$\eta_{1,2}(X) = \eta_1(XP_2), \qquad X \in \mathfrak{M}.$$

Were $\eta_{1,2} = 0$, then, in particular $\eta_{1,2}(P_2) = 0$, i.e. $\eta_1(P_2) = 0$ and therefore, by definition of support, $P_2 \leq 1 - P_1$. This implies that $P_1P_2 = 0$, contrary to the assumption. We now show that the support of $\eta_{1,2}$ is P_1P_2. Let, in fact, Q be a projection such that $\eta_{1,2}(Q) = 0$. Then

$$\eta_1(QP_2) = 0 \;\Rightarrow\; QP_2 \leq 1 - P_1 \;\Rightarrow\; QP_2(1 - P_1) = QP_2 \;\Rightarrow\; QP_2P_1 = 0.$$

Thus the largest Q for which this happens is $1 - P_2P_1$. We conclude that the support of the trace $\eta_{1,2}$ is P_1P_2. Finally, by definition, one has $\eta_{1,2}(X) = \eta_1(XP_2)$, and, since $XP_2 \leq X$,

$$\eta_{1,2}(X) = \eta_1(XP_2) \leq \eta_1(X), \qquad \forall X \in \mathfrak{M}.$$

Thus η_1 majorizes $\eta_{1,2}$. But η_1 is extreme in $\mathfrak{F}$. Therefore $\eta_{1,2}$ has the form $\lambda\eta_1$ with $\lambda \in \,]0, 1]$. This implies that $\eta_{1,2}$ has the same support as η_1;

therefore $P_1P_2 = P_1$, i.e. $P_1 \le P_2$. Starting from $\eta_{2,1}(X) = \eta_2(XP_1)$, we get, in a similar way, $P_2 \le P_1$. Therefore, $P_1P_2 \ne 0$ implies $P_1 = P_2$. However, two different traces of $\mathfrak{E}\mathfrak{F}$ cannot have the same support. Indeed, assume that there exist $\eta_1, \eta_2 \in \mathfrak{F}$ having the same support P. Since P is central, we can consider the von Neumann algebra $\mathfrak{M}P$. The restrictions of η_1, η_2 to $\mathfrak{M}P$ are normal faithful semifinite traces. By [14, Prop. V.2.31] there exists a central element Z in $\mathfrak{M}P$ with $0 \le Z \le P$ (P is here considered as the unit of $\mathfrak{M}P$) such that

$$\eta_1(X) = (\eta_1 + \eta_2)(ZX), \qquad \forall X \in (\mathfrak{M}P)_+. \tag{3}$$

Then Z also belongs to the center of $\mathfrak{M}$, since for every $V \in \mathfrak{M}$,

$$ZV = Z(VP + VP^{\perp}) = ZVP = VZP = VZ.$$

Therefore the functionals

$$\eta_{1,Z}(X) := \eta_1(XZ), \qquad \eta_{2,Z}(X) := \eta_2(XZ), \qquad X \in \mathfrak{M},$$

belong to the family $\mathfrak{F}$ and are majorized, respectively, by the extreme elements η_1, η_2. Then there exist $\lambda, \mu \in [0, 1]$ such that

$$\eta_1(XZ) = \lambda\eta_1(X), \qquad \eta_2(XZ) = \mu\eta_1(X), \qquad \forall X \in \mathfrak{M}.$$

If $\lambda = 1$ we would have, from (3), $\eta_2(ZX) = 0$ for every $X \in (\mathfrak{M}P)_+$; in particular, $\eta_2(|Z|^2) = 0$; this implies that $Z = 0$. Thus $\lambda \ne 1$. Analogously, $\mu \ne 0$: indeed, if $\mu = 0$, then $\eta_1(X) = \lambda\eta_1(X)$ and thus $\lambda = 1$. Therefore there exist $\lambda, \mu \in (0, 1)$ such that

$$\eta_1(X) = \lambda\eta_1(X) + \mu\eta_2(X), \qquad \forall X \in \mathfrak{M}P,$$

which in turn implies

$$\eta_1(X) = \lambda\eta_1(X) + \mu\eta_2(X), \qquad \forall X \in \mathfrak{M}.$$

Hence,

$$(1 - \lambda)\eta_1(X) = \mu\eta_2(X), \qquad \forall X \in \mathfrak{M}.$$

From the last equality, dividing by $\max\{1 - \lambda, \mu\}$ one finds that one of the two elements is a convex combination of the other and of 0, which is absurd. In conclusion, different supports of extreme traces of $\mathfrak{F}$ are orthogonal. ∎

Since, for every $X \in \mathfrak{M}$, $\|X\|_{p,\mathcal{I}}$ remains the same if computed either with respect to $\mathfrak{F}$ or to $\mathfrak{E}\mathfrak{F}$, we can deduce the following

THEOREM 3.6. *Let $\mathfrak{F}$ be a convex and w^*-compact sufficient family of normal finite traces on the von Neumann algebra $\mathfrak{M}$. Assume that $\mathfrak{F}$ satisfies Condition* (P) *and that for each central operator Z with $0 \le Z \le \mathbb{I}$ and each $\eta \in \mathfrak{F}$ the functional $\eta_Z(X) := \eta(XZ)$ belongs to $\mathfrak{F}$. Then the completion $\mathfrak{M}_p[\|\cdot\|_{p,\mathcal{I}}]$ consists of measurable operators.*

Families of traces satisfying the assumptions of Theorem 3.6 will be constructed in the next section.

4. A representation theorem. Once we have constructed some CQ^*-algebras of operators affiliated to a given von Neumann algebra, it is natural to ask under which conditions an abstract CQ^*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ can be realized as a CQ^*-algebra of this type.

Let $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ be a CQ^*-algebra with unit e and let

$$\mathcal{T}(\mathfrak{X}) = \{\Omega \in \mathcal{S}(\mathfrak{X}) : \Omega(x,x) = \Omega(x^*,x^*),\ \forall x \in \mathfrak{X}\}.$$

We remark that if $\Omega \in \mathcal{T}(\mathfrak{X})$ then, by polarization, $\Omega(y^*,x^*) = \Omega(x,y)$ for all $x, y \in \mathfrak{X}$. It is easy to prove that the set $\mathcal{T}(\mathfrak{X})$ is convex.

For each $\Omega \in \mathcal{T}(\mathfrak{X})$, we define a linear functional ω_Ω on $\mathfrak{A}_0$ by

$$\omega_\Omega(a) := \Omega(a,e), \qquad a \in \mathfrak{A}_0.$$

We have

$$\omega_\Omega(a^*a) = \Omega(a^*a,e) = \Omega(a,a) = \Omega(a^*,a^*) = \omega_\Omega(aa^*) \ge 0.$$

This shows at once that ω_Ω is positive and tracial. We put

$$\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0) = \{\omega_\Omega : \Omega \in \mathcal{T}(\mathfrak{X})\}.$$

From the convexity of $\mathcal{T}(\mathfrak{X})$ it follows easily that $\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$ is also convex. If we denote by $\|f\|^\sharp$ the norm of the bounded functional f on $\mathfrak{A}_0$, we also get

$$\|\omega_\Omega\|^\sharp = \omega_\Omega(e) = \Omega(e,e) \le \|e\|^2.$$

Therefore

$$\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0) \subseteq \{\omega \in \mathfrak{A}_0^\sharp : \|\omega\|^\sharp \le \|e\|^2\},$$

where $\mathfrak{A}_0^\sharp$ denotes the topological dual of $\mathfrak{A}_0[\|\cdot\|_0]$. Setting

$$f_\Omega(a) := \frac{\omega_\Omega(a)}{\|e\|^2}$$

we get

$$f_\Omega \in \{\omega \in \mathfrak{A}_0^\sharp : \|\omega\|^\sharp \le 1\}.$$

By the Banach–Alaglou theorem, the set $\{\omega \in \mathfrak{A}_0^\sharp : \|\omega\|^\sharp \le 1\}$ is w^*-compact in $\mathfrak{A}_0^\sharp$. Then $\{\omega \in \mathfrak{A}_0^\sharp : \|\omega\|^\sharp \le \|e\|^2\}$ is also w^*-compact.

PROPOSITION 4.1. *$\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$ is w^*-closed and, therefore, w^*-compact.*

Proof. Let (ω_{Ω_α}) be a net in $\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$ w^*-converging to a functional $\omega \in \mathfrak{A}_0^\sharp$. We will show that $\omega = \omega_\Omega$ for some $\Omega \in \mathcal{T}(\mathfrak{X})$. Let us begin by defining $\Omega_0(a,b) = \omega(b^*a)$, $a, b \in \mathfrak{A}_0$. By the very definition, $\omega_{\Omega_\alpha}(a) \to \omega(a) = \Omega_0(a,e)$. Moreover, for every $a, b \in \mathfrak{A}_0$,

$$\Omega_0(a,b) = \omega(b^*a) = \lim_\alpha \omega_{\Omega_\alpha}(b^*a) = \lim_\alpha \Omega_\alpha(a,b).$$

Therefore

$$\Omega_0(a,a) = \lim_\alpha \Omega_\alpha(a,a) \ge 0.$$

We also have

$$|\Omega_0(a,b)| = \lim_\alpha |\Omega_\alpha(a,b)| \le \|a\|\,\|b\|.$$

Hence Ω_0 can be extended by continuity to $\mathfrak{X}\times\mathfrak{X}$. Indeed, let

$$x = \|\cdot\|\text{-}\lim_n a_n, \qquad y = \|\cdot\|\text{-}\lim_n b_n, \qquad (a_n),(b_n) \subseteq \mathfrak{A}_0.$$

Then

$$\begin{aligned}|\Omega_0(a_n,b_n) - \Omega_0(a_m,b_m)| \\ &= |\Omega_0(a_n,b_n) - \Omega_0(a_m,b_n) + \Omega_0(a_m,b_n) - \Omega_0(a_m,b_m)| \\ &\le |\Omega_0(a_n - a_m, b_n)| + |\Omega_0(a_m, b_n - b_m)| \\ &\le \|a_n - a_m\|\,\|b_n\| + \|a_m\|\,\|b_n - b_m\| \to 0,\end{aligned}$$

since $(\|a_n\|)$ and $(\|b_n\|)$ are bounded sequences. Therefore we can define

$$\Omega(x,y) = \lim_n \Omega_0(a_n,b_n).$$

Clearly, $\Omega(x,x) \ge 0$ for all $x \in \mathfrak{X}$. It is easily checked that $\Omega \in \mathcal{T}(\mathfrak{X})$. This concludes the proof. ∎

Since $\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$ is convex and w^*-compact, by the Krein–Milman theorem it follows that it has extreme points and it coincides with the w^*-closure of the convex hull of the set $\mathfrak{E}\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$ of its extreme points.

By the Gelfand–Naimark theorem each C^*-algebra is isometrically *-isomorphic to a C^*-algebra of bounded operators in Hilbert space. This isometric *-isomorphism is called the *universal *-representation.*

Thus, let π be the universal *-representation of $\mathfrak{A}_0$ and $\pi(\mathfrak{A}_0)''$ the von Neumann algebra generated by $\pi(\mathfrak{A}_0)$.

For every $\Omega \in \mathcal{T}(\mathfrak{X})$ and $a \in \mathfrak{A}_0$, we put

$$\varphi_\Omega(\pi(a)) = \omega_\Omega(a).$$

Then, for each $\Omega \in \mathcal{T}(\mathfrak{X})$, φ_Ω is a positive bounded linear functional on the operator algebra $\pi(\mathfrak{A}_0)$. Clearly,

$$\varphi_\Omega(\pi(a)) = \omega_\Omega(a) = \Omega(a,e),$$
$$|\varphi_\Omega(\pi(a))| = |\omega_\Omega(a)| = |\Omega(a,e)| \le \|a\|\,\|e\| \le \|a\|_0\|e\|^2 = \|\pi(a)\|\,\|e\|^2.$$

Thus φ_Ω is continuous on $\pi(\mathfrak{A}_0)$.

By [7, Theorem 10.1.2], φ_Ω is weakly continuous and so it extends uniquely to $\pi(\mathfrak{A}_0)''$. Moreover, since φ_Ω is a trace on $\pi(\mathfrak{A}_0)$, the extension $\widetilde{\varphi}_\Omega$ is also a trace on $\mathfrak{M} := \pi(\mathfrak{A}_0)''$. The norm $\|\widetilde{\varphi}_\Omega\|^\sharp$ of $\widetilde{\varphi}_\Omega$ as a linear functional on $\mathfrak{M}$ equals the norm of φ_Ω as a functional on $\pi(\mathfrak{A}_0)$. We have

$$\|\widetilde{\varphi}_\Omega\|^\sharp = \widetilde{\varphi}_\Omega(\pi(e)) = \varphi_\Omega(\pi(e)) = \omega_\Omega(e) \le \|e\|^2.$$

The set

$$\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0) = \{\widetilde{\varphi}_\Omega : \Omega \in \mathcal{T}(\mathfrak{X})\}$$

is convex and w^*-compact in $\mathfrak{M}^\sharp$, as can be easily seen by considering the map

$$\omega_\Omega \in \mathfrak{M}_\mathcal{T}(\mathfrak{A}_0) \mapsto \widetilde{\varphi}_\Omega \in \mathfrak{N}_\mathcal{T}(\mathfrak{A}_0),$$

which is linear and injective, and by taking into account the fact that, if $a_\alpha \to a$ in $\mathfrak{A}_0[\|\cdot\|]$, then $\widetilde{\varphi}_\Omega(\pi(a_\alpha) - \pi(a)) = \omega_\Omega(a_\alpha - a) \to 0$.

Let $\mathfrak{E}\mathfrak{N}_\mathcal{T}(\mathfrak{A}_0)$ be the set of extreme points of $\mathfrak{N}_\mathcal{T}(\mathfrak{A}_0)$; then $\mathfrak{N}_\mathcal{T}(\mathfrak{A}_0)$ coincides with the w^*-closure of the convex hull of $\mathfrak{E}\mathfrak{N}_\mathcal{T}(\mathfrak{A}_0)$. The extreme elements of $\mathfrak{N}_\mathcal{T}(\mathfrak{A}_0)$ are easily characterized by the following

PROPOSITION 4.2. *$\widetilde{\varphi}_\Omega$ is extreme in $\mathfrak{N}_\mathcal{T}(\mathfrak{A}_0)$ if, and only if, ω_Ω is extreme in $\mathfrak{M}_\mathcal{T}(\mathfrak{A}_0)$.*

DEFINITION 4.3. A Banach quasi *-algebra $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ is said to be *strongly regular* if $\mathcal{T}(\mathfrak{X})$ is sufficient and

$$\|x\| = \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \Omega(x,x)^{1/2}, \qquad \forall x \in \mathfrak{X}.$$

EXAMPLE 4.4. If $\mathfrak{M}$ is a von Neumann algebra with a sufficient family $\mathfrak{F}$ of normal finite traces, then the CQ^*-algebra $(\mathfrak{M}_p, \mathfrak{M})$ constructed in Section 3 is strongly regular. This follows from the definition of the norm in the completion.

EXAMPLE 4.5. If φ is a normal faithful finite trace on $\mathfrak{M}$, then $\mathcal{T}(L^p(\varphi))$, for $p \geq 2$, is sufficient. To see this, we first define Ω_0 on $\mathfrak{M} \times \mathfrak{M}$ by

$$\Omega_0(X,Y) = \varphi(Y^*X), \qquad X, Y \in \mathfrak{M}.$$

Then

$$|\Omega_0(X,Y)| = |\varphi(Y^*X)| \leq \|X\|_p \|Y\|_{p'}, \qquad \forall X, Y \in \mathfrak{M}.$$

Since $p \geq 2$, $L^p(\varphi)$ is continuously embedded into $L^{p'}(\varphi)$. Thus, there exists $\gamma > 0$ such that $\|Y\|_{p'} \leq \gamma \|Y\|_p$ for every $Y \in \mathfrak{M}$. Define

$$\widetilde{\Omega}(X,Y) = \frac{1}{\gamma}\,\Omega_0(X,Y), \qquad X, Y \in \mathfrak{M}.$$

Then

$$|\widetilde{\Omega}(X,Y)| \leq \|X\|_p \|Y\|_p, \qquad \forall X, Y \in \mathfrak{M}.$$

Hence, $\widetilde{\Omega}$ has a unique extension, denoted by the same symbol, to $L^p(\varphi) \times L^p(\varphi)$. It is easily seen that $\widetilde{\Omega} \in \mathcal{T}(L^p(\varphi))$.

Were, for some $X \in L^p(\varphi)$, $\Omega(X,X) = 0$ for every $\Omega \in \mathcal{T}(L^p(\varphi))$, we would have $\widetilde{\Omega}(X,X) = \|X\|_2^2 = 0$. This clearly implies $X = 0$. The equality $\widetilde{\Omega}(X,X) = \|X\|_2^2$ also shows that $L^2(\varphi)$ is strongly regular.

Let now $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ be a CQ^*-algebra with unit e and sufficient $\mathcal{T}(\mathfrak{X})$. Let $\pi : \mathfrak{A}_0 \hookrightarrow \mathcal{B}(\mathcal{H})$ be the universal representation of $\mathfrak{A}_0$. Assume that the C^*algebra $\pi(\mathfrak{A}_0) := \mathfrak{M}$ is a von Neumann algebra. In this case,

$\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0) = \mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ and $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ is a family of traces satisfying Condition (P). Therefore, by Proposition 3.3, we can construct, for $p \geq 1$, the CQ^*-algebras $(\mathfrak{M}_p[\|\cdot\|_{p,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)}], \mathfrak{M}[\|\cdot\|])$. Clearly, $\mathfrak{A}_0$ can be identified with $\mathfrak{M}$. It is then natural to ask if $\mathfrak{X}$ can also be identified with some $\mathfrak{M}_p$. The next theorem provides the answer to this question.

THEOREM 4.6. *Let* $(\mathfrak{X}[\|\cdot\|], \mathfrak{A}_0[\|\cdot\|_0])$ *be a CQ^*-algebra with unit e and and sufficient $\mathcal{T}(\mathfrak{X})$. Then there exist a von Neumann algebra $\mathfrak{M}$ and a monomorphism*

$$\Phi : x \in \mathfrak{X} \mapsto \Phi(x) := \widetilde{X} \in \mathfrak{M}_2$$

with the following properties:

(i) *Φ extends the universal *-representation π of $\mathfrak{A}_0$;*
(ii) *$\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathfrak{X}$;*
(iii) *$\Phi(xy) = \Phi(x)\Phi(y)$ for every $x, y \in \mathfrak{X}$ such that $x \in \mathfrak{A}_0$ or $y \in \mathfrak{A}_0$.*

Then $\mathfrak{X}$ can be identified with a space of operators affiliated with $\mathfrak{M}$.

If, *in addition*, *$(\mathfrak{X}, \mathfrak{A}_0)$ is strongly regular*, *then*

(iv) *Φ is an isometry of $\mathfrak{X}$ into $\mathfrak{M}_2$;*
(v) *if $\mathfrak{A}_0$ is a W^*-algebra*, *then Φ is an isometric *-isomorphism of $\mathfrak{X}$ onto $\mathfrak{M}_2$.*

Proof. Let π be the universal representation of $\mathfrak{A}_0$ and assume first that $\pi(\mathfrak{A}_0) =: \mathfrak{M}$ is a von Neumann algebra. By Proposition 4.1, the family $\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$ of traces is convex and w^*-compact. Moreover, for each central positive element Z with $0 \leq Z \leq \mathbb{I}$ and for $\varphi \in \mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$, the trace $\varphi_Z(X) := \varphi(ZX)$ still belongs to $\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)$. Indeed, starting from the form $\Omega \in \mathcal{T}(\mathfrak{X})$ which generates φ, one can define the sesquilinear form

$$\Omega_Z(x, y) := \Omega(x\pi^{-1}(Z^{1/2}), y\pi^{-1}(Z^{1/2})), \qquad x, y \in \mathfrak{X}.$$

We check that $\Omega_Z \in \mathcal{T}(\mathfrak{X})$:

(i) $\Omega_Z(x, x) = \Omega(x\pi^{-1}(Z^{1/2}), x\pi^{-1}(Z^{1/2})) \geq 0$ for all $x \in \mathfrak{X}$.

(ii) For every $x \in \mathfrak{X}$ and every $a, b \in \mathfrak{A}_0$, we have

$$\begin{aligned}\Omega_Z(xa, b) &= \Omega(xa\pi^{-1}(Z^{1/2}), b\pi^{-1}(Z^{1/2})) = \Omega(a\pi^{-1}(Z^{1/2}), x^*b\pi^{-1}(Z^{1/2}))\\ &= \Omega_Z(a, x^*b).\end{aligned}$$

(iii) For every $x, y \in \mathfrak{X}$, we have

$$\begin{aligned}|\Omega_Z(x, y)| &= |\Omega(x\pi^{-1}(Z^{1/2}), y\pi^{-1}(Z^{1/2}))| \leq \|x\pi^{-1}(Z^{1/2})\|\,\|\pi^{-1}(Z^{1/2})y\|\\ &\leq \|x\|\,\|\pi^{-1}(Z^{1/2})\|_0\|y\|\,\|\pi^{-1}(Z^{1/2})\|_0 \leq \|x\|\,\|y\|.\end{aligned}$$

(iv) For every $x \in \mathfrak{X}$,

$$\begin{aligned}\Omega_Z(x^*, x^*) &= \Omega(x^*\pi^{-1}(Z^{1/2}), x^*\pi^{-1}(Z^{1/2})) = \Omega(x\pi^{-1}(Z^{1/2}), x\pi^{-1}(Z^{1/2}))\\ &= \Omega_Z(x, x).\end{aligned}$$

Moreover, Ω_Z defines, for every $A = \pi(a) \in \mathfrak{M} = \pi(\mathfrak{A}_0)$, the following trace:

$$\begin{aligned}\varphi_{\Omega_Z}(A) &= \Omega_Z(a,e) = \Omega(a\pi^{-1}(Z^{1/2}), \pi^{-1}(Z^{1/2}))\\ &= \Omega(a\pi^{-1}(Z), e) = \Omega(\pi^{-1}(AZ), e) = \varphi_\Omega(AZ).\end{aligned}$$

Thus, the family of traces $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)\ (= \mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0))$ satisfies the assumptions of Lemma 3.5; therefore, if $\eta_1, \eta_2 \in \mathfrak{E}\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$, and if P_1 and P_2 denote their respective supports, one has $P_1P_2 = 0$.

By the sufficiency of $\mathcal{T}(\mathfrak{X})$ we get

$$\|X\|_{2,\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)} := \sup_{\varphi \in \mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)} \|X\|_{2,\varphi} = \sup_{\varphi \in \mathfrak{E}\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)} \|X\|_{2,\varphi}, \qquad \forall X \in \pi(\mathfrak{A}_0).$$

By Proposition 3.3, the Banach space $\mathfrak{M}_2$, completion of $\mathfrak{M}$ with respect to the norm $\|\cdot\|_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)}$, is a CQ^*-algebra. Moreover, since the supports of the extreme traces satisfy the assumptions of Theorem 3.6, the CQ^*-algebra $(\mathfrak{M}_2[\|\cdot\|_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)}], \mathfrak{M}[\|\cdot\|])$ consists of operators affiliated with $\mathfrak{M}$.

We now define the map Φ. For every $x \in \mathfrak{X}$, there exists a sequence (a_n) of elements of $\mathfrak{A}_0$ converging to x with respect to the norm of $\mathfrak{X}(\|\cdot\|)$. Put $X_n = \pi(a_n)$, $n \in \mathbb{N}$. Then

$$\begin{aligned}\|X_n - X_m\|_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)} :=& \sup_{\varphi \in \mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)} \|\pi(a_n) - \pi(a_m)\|_{2,\varphi}\\ &= \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} [\Omega((a_n - a_m)^*(a_n - a_m), e)]^{1/2}\\ &= \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} [\Omega(a_n - a_m, a_n - a_m)]^{1/2} \le \|a_n - a_m\| \to 0.\end{aligned}$$

Let $\widetilde{X}$ be the $\|\cdot\|_{2,\mathfrak{M}_{\mathcal{T}}(\mathfrak{A}_0)}$-limit of the sequence (X_n) in $\mathfrak{M}_2$. We define

$$\Phi(x) := \widetilde{X}.$$

For each $x \in \mathfrak{X}$, we put

$$p_{\mathcal{T}(\mathfrak{X})}(x) = \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \Omega(x,x)^{1/2}.$$

Owing to the sufficiency of $\mathcal{T}(\mathfrak{X})$, $p_{\mathcal{T}(\mathfrak{X})}$ is a norm on $\mathfrak{X}$ weaker than $\|\cdot\|$. This implies that

$$\|\widetilde{X}\|^2_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)} = \lim_{n\to\infty} \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \Omega(a_n, a_n) = \lim_{n\to\infty} p_{\mathcal{T}(\mathfrak{X})}(a_n)^2 = p_{\mathcal{T}(\mathfrak{X})}(x)^2.$$

From this equality it follows easily that the linear map Φ is well defined and injective. Condition (iii) can be easily proved. If $(\mathfrak{X}, \mathfrak{A}_0)$ is strongly regular, then $p_{\mathcal{T}(\mathfrak{X})}(x) = \|x\|$ for every $x \in \mathfrak{X}$. Thus Φ is isometric. Moreover, in this case, Φ is surjective: indeed, if $T \in \mathfrak{M}_2$, then there exists a sequence (T_n) of bounded operators on $\pi(\mathfrak{A}_0)$ which converges to T with respect to the norm $\|\cdot\|_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)}$. The corresponding sequence $(t_n) \subset \mathfrak{A}_0$, $T_n = \Phi(t_n)$, converges to t with respect to the norm of $\mathfrak{X}$ and $\Phi(t) = T$ by definition. Therefore Φ is an isometric *-isomorphism.

To complete the proof, it is enough to prove that the given CQ^*-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ can be embedded in a CQ^*-algebra $(\mathfrak{K}, \mathfrak{B}_0)$ where $\mathfrak{B}_0$ is a W^*-algebra. Of course, we may directly work with $\pi(\mathfrak{A}_0)$ with π the universal representation of $\mathfrak{A}_0$. The family of traces $\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)$ defined on $\pi(\mathfrak{A}_0)''$ is not necessarily sufficient. Let P_Ω, $\Omega \in \mathcal{T}(\mathfrak{X})$, denote the support of $\widetilde{\varphi}_\Omega$ and let

$$P = \bigvee_{\Omega \in \mathcal{T}(\mathfrak{X})} P_\Omega.$$

Then $\mathfrak{B}_0 := \pi(\mathfrak{A}_0)''P$ is a von Neumann algebra that we can complete with respect to the norm

$$\|X\|_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)} = \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \widetilde{\varphi}_\Omega(X^*X), \qquad X \in \pi(\mathfrak{A}_0)''P.$$

We obtain in this way a CQ^*-algebra $(\mathfrak{K}, \mathfrak{B}_0)$ with $\mathfrak{B}_0$ a W^*-algebra. The faithfulness of π on $\mathfrak{A}_0$ implies that

$$\pi(a)P = \pi(a), \qquad \forall a \in \mathfrak{A}_0.$$

It remains to prove that $\mathfrak{X}$ can be identified with a subspace of $\mathfrak{K}$. But this can be shown as in the first part: for each $x \in \mathfrak{X}$ there exists a sequence $(a_n) \subset \mathfrak{A}_0$ such that $\|x - a_n\| \to 0$ as $n \to \infty$. We now put $X_n = \pi(a_n)$. Then, proceeding as before, we determine the element $\widehat{X} \in \mathfrak{K}$, where

$$\widehat{X} = \|\cdot\|_{2,\mathfrak{N}_{\mathcal{T}}(\mathfrak{A}_0)}\text{-}\lim \pi(a_n)P.$$

It is easy to see that the map $x \in \mathfrak{X} \mapsto \widehat{X} \in \mathfrak{K}$ is injective. If $(\mathfrak{X}, \mathfrak{A}_0)$ is regular, but $\pi(\mathfrak{A}_0) \subset \pi(\mathfrak{A}_0)''$, then Φ is an isometry of $\mathfrak{X}$ into $\mathfrak{M}_2$, but need not be surjective. ■

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