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Poincaré inequality and Hajłasz–Sobolev spaces on nested fractals

by

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Abstract. Given a nondegenerate harmonic structure, we prove a Poincaré-type inequality for functions in the domain of the Dirichlet form on nested fractals. We then study the Hajłasz–Sobolev spaces on nested fractals. In particular, we describe how the "weak"-type gradient on nested fractals relates to the upper gradient defined in the context of general metric spaces.

1. Introduction. The interest in analysis on fractals arose from mathematical physics, and dates back to the 80's of the past century. The first object to be meticulously defined was the Kigami Laplacian on the Sierpiński gasket [K2], and, somehow in parallel, the Brownian motion on the gasket [BP]. Since then, we have seen an outburst of papers focusing both on analytic and probabilistic aspects of stochastic processes with fractal state space. The analytic approach, concerned mostly with Dirichlet forms, their domains and generators, proved particularly useful while constructing processes on fractals. On the other hand, derivatives on fractals have been defined [K1, Ku, St2, T] and their properties studied. For an account of results from that time, as well as an extended list of references, we refer to [K3] (analytic) and [B] (probabilistic).

In the present paper, starting from the definition of the gradient on nested fractals from [Ku, T], we prove certain Poincaré-type inequalities on nested fractals, for functions belonging to the domain of the Brownian Dirichlet form (which can be seen as a fractal counterpart of the Sobolev space $W^{1,2}(\mathbb{R}^d)$). We will then be concerned with Poincaré–Sobolev spaces and spaces of Korevaar–Schoen type, and our analysis will be much in the spirit of [KM] and [KST].

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In the last paper mentioned, the authors consider general metric measure spaces equipped with a Dirichlet structure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, much like the nested fractals we consider. However, in order to proceed, they make a standing assumption that the intrinsic metric related to the Dirichlet structure,

$$d_E(x,y) = \sup\{\phi(x) - \phi(y) : \phi \in \Gamma,\ d\eta_{\mathbb{R}}(\phi,\phi) \le d\mu\},$$

where Γ is a μ-separating core of $\mathcal{E}$, induces the topology equivalent to the initial one. This assumption fails for fractals: the metric d_E is degenerate there (see [BBK, p. 6]). So, in order to extend the results of [KST], one should either modify the definition of d_E or choose a different approach.

A discussion of gradients with connection to the Poincaré inequality (P.I., for short) and relations between various function spaces can be found in the recent paper [GKZ]. While three types of gradients are considered, the one used in P.I. is the so called *upper gradient*, a notion that depends on rectifiable curves. In the context of nested fractals there may be no such curves at all. Again, for a meaningful theory a different notion of gradient should be considered.

We propose a hands-on approach based on discrete approximations of nested fractals and Kusuoka gradients. By a limiting procedure, the gradient can be reasonably defined for functions belonging to the domain of the Dirichlet form, although it is usually hard to decide whether the limit exists at a given point for functions other than m-harmonic. This gradient can be used in Poincaré-type inequalities and in defining variants of Sobolev spaces on fractals.

We start with a local version of P.I., which then yields a global P.I. on nested fractals. We obtain inequalities of the form

$$\fint_B |f - f_B|\, d\mu \le C r^{d_w/2} \left(\frac{1}{r^d} \int_{B(x_0,Ar)} \langle \nabla f, Z \nabla f \rangle\, d\nu \right)^{1/2}, \tag{1.1}$$

where μ is the d-dimensional Hausdorff measure on the fractal, ν is the Kusuoka energy measure on the fractal (see Section 2.2.3 for a precise definition), d_w is the walk dimension of the fractal we are considering, d its Hausdorff dimension, and $\langle \nabla f, Z \nabla f \rangle$ replaces the square of the norm of the gradient. The measure ν is typically singular with respect to the Hausdorff measure, but does not charge points. Observe that in the Euclidean case we have $d_w = 2$, and so the scale function in P.I. will be linear as it should. Poincaré inequalities involving the Dirichlet energy measure in a general setting have been investigated [BBK], but that paper did not involve the definition of gradients on fractal sets. A choice, or even the existence, of a gradient is not obvious on fractals. We propose to use a weak-type gradient with energy measure (cf. Section 2.2).

As an application, in the second part of our paper, we compare several possible definitions of Sobolev-type functions on fractals. On metric spaces, several definitions of Sobolev-type spaces have been considered (see e.g. [FHK], [H], [KS]), and nested fractals are of particular interest in this context. In the present paper, we introduce Poincaré-inequality based Sobolev spaces on fractals and examine their relation to Korevaar–Schoen spaces and Hajłasz–Sobolev spaces. While in a typical situation on metric spaces the scaling factor in a Poincaré inequality is r, the radius of a given ball, it turns out that on nested fractals this does not yield an interesting inequality. To deal with relevant Sobolev spaces, one should take into account the specific geometry of the fractal and use a scaling factor $r^{d_w/2}$, as in (1.1). For some preliminary relations between Hajłasz–Sobolev and Korevaar–Schoen Sobolev spaces on fractals we refer to a paper by Hu [Hu].

2. Preliminaries. We use C or c to denote a positive constant depending possibly on the fractal set, whose exact value is not important for our purposes and which may change from line to line. We will write $f \asymp g$ (on a set D) if there exists a constant $C > 0$ such that for every $x \in D$ one has $C^{-1}g(x) \leq f(x) \leq Cg(x)$. For an m-integrable function f and a set A of finite measure we adopt the notation

$$f_A = \fint_A f\,dm = \frac{1}{m(A)} \int f\,dm.$$

2.1. Nested fractals. The framework of nested fractals is that of Lindström [L]. Suppose that $\phi_1, \ldots, \phi_M$, $M \geq 2$, are similitudes of $\mathbb{R}^N$ with a common scaling factor $L > 1$. When $A \subset \mathbb{R}^n$, then we write $\Phi(A)$ for $\bigcup_{i=1}^M \phi_i(A)$, and Φ^m for Φ composed m times. There exists a unique nonempty compact set (see [F], [L]) $\mathcal{K} \subset \mathbb{R}^N$ such that

$$\mathcal{K} = \bigcup_{i=1}^{M} \phi_i(\mathcal{K}) = \Phi(\mathcal{K}). \tag{2.1}$$

It is called the *self-similar fractal* generated by the family of similitudes $\phi_1, \ldots, \phi_M$. Since the set $\mathcal{K}$ has a finite nonzero diameter, for simplicity we can and will assume that $\operatorname{diam} \mathcal{K} = 1$.

Each of the mappings ϕ_i has a unique fixed point v_i. Such a point is called an *essential fixed point* if there exists another fixed point v_j such that for some transformations ϕ_k, ϕ_l one has $\phi_k(v_i) = \phi_l(v_j)$. The set of all essential fixed points will be denoted by $V^{(0)} = \{v_1, \ldots, v_r\}$. For $m = 1, 2, \ldots$ we set $V^{(m)} = \Phi^m(V^{(0)})$ and $V^{(\infty)} = \bigcup_{m\geq 0} V^{(m)}$. For nondegeneracy, we assume that $r = \#V^{(0)} \geq 2$.

The system $\{\phi_1, \ldots, \phi_M\}$ is said to satisfy the *open set condition* if there exists an open, nonempty set U such that $\Phi(U) \subset U$ and for all $i \neq j$ one has

$\phi_i(U)\cap\phi_j(U)=\emptyset$. If the open set condition is satisfied, then the Hausdorff dimension of the self-similar fractal $\mathcal{K}$ is equal to $d=d(\mathcal{K})=\frac{\log M}{\log L}$. By μ we denote the d-dimensional Hausdorff measure on $\mathcal{K}$ normalized so that $\mu(\mathcal{K})=1$.

For $m\geq 1$, by a *word* of length m we mean a sequence $w=(w_1,\ldots,w_m)\subset\{1,\ldots,M\}^m$. The collection of all words of length m is denoted by $\mathcal{W}_m$; $\mathcal{W}_*=\bigcup_{m\geq 1}\mathcal{W}_m$ consists of all words of finite length; and $\mathcal{W}$ is the collection of all infinite words. When $w\in\mathcal{W}_*$, then $|w|$ denotes its length. If $w\in\mathcal{W}$ is an infinite word, then $[w]_m$ denotes its restriction to the first m coordinates, i.e. for $w=(w_1,w_2,\ldots)$, $[w]_m=(w_1,\ldots,w_m)$. When $w=(w_1,\ldots,w_m)$ is given, we will write $\phi_w=\phi_{w_1}\circ\cdots\circ\phi_{w_m}$, and for a set A, $A_w=\phi_w(A)$.

Definition 2.1. Let $m\geq 1$.

(1) An *m-simplex* is any set of the form $\phi_w(\mathcal{K})$ with $w\in\mathcal{W}_m$ (m-simplices are just scaled down copies of $\mathcal{K}$). The collection of all m-simplices will be denoted by $\mathcal{T}_m$. The 0-simplex is just $\mathcal{K}$.
(2) For an m-simplex $S=\phi_w(\mathcal{K})$, $w\in\mathcal{W}_m$, let $V(S)=\phi_w(V^{(0)})$ be the set of its vertices. An *m-cell* is any of the sets $\phi_w(V^{(0)})$. Two points $x,y\in V^{(m)}$ are called *m-neighbors*, denoted $x\overset{m}{\sim}y$, if they belong to a common m-cell.
(3) If $\Delta\in\mathcal{T}_m$, $m\geq 1$, we denote by Δ^* the union of Δ and all the adjacent m-simplices, and by Δ^{**} the union of Δ^* and all m-simplices adjacent fo Δ^*.
(4) For any $x\in\mathcal{K}\setminus V^{(\infty)}$ and $m\geq 1$, let $\Delta_m(x)$ be the unique m-simplex that contains x.
(5) For any $x,y\in\mathcal{K}\setminus V^\infty$, we define $\operatorname{ind}(x,y)=\min\{m\geq 1:\Delta_m(x)\cap\Delta_m(y)=\emptyset\}$. When $\operatorname{ind}(x,y)=n$, we set $S(x,y)=\Delta_{n-1}(x)\cup\Delta_{n-1}(y)$.
(6) When an m-simplex $\Delta=\mathcal{K}_w=\phi_w(\mathcal{K})$ with $w\in\mathcal{W}_m$ is given and $\tilde{w}\in\mathcal{W}_n$ is another finite word, then $\Delta_{\tilde{w}}$ denotes the $(m+n)$-simplex $\phi_{w\tilde{w}}(\mathcal{K})$.

From now on we will assume that for every $S,T\in\mathcal{T}_m$, $m\geq 1$, with $S\neq T$, one has $S\cap T=V(S)\cap V(T)$ (nesting). Define the graph structure $E_{(1)}$ on $V^{(1)}$ as follows: we write $(x,y)\in E_{(1)}$ if x and y are 1-neighbors. Then we require the graph $(V^{(1)},E_{(1)})$ to be connected. For $x,y\in V^{(0)}$, let $R_{x,y}$ be the reflection in the hyperplane bisecting the segment $[x,y]$. Then we stipulate that

$$\forall_{i\in\{1,\ldots,M\}}\,\forall_{x,y\in V^{(0)},\,x\neq y}\,\exists_{j\in\{1,\ldots,M\}}\,R_{x,y}(\phi_i(V^{(0)}))=\phi_j(V^{(0)})$$

(natural reflections map 1-cells onto 1-cells).

The self-similar fractal $\mathcal{K}$ is called a *nested fractal* if it satisfies the above open set condition, nesting, invariance under local isometries, and the connectivity assumption.

Part of our results will require the following property **(P)** of the fractal:

PROPERTY **(P)**. There exists $\alpha > 0$ such that for all $n = 1, 2, \ldots$ and x, y nonvertex points such that $y \in \Delta_n^*(x) \setminus \Delta_{n+1}^*(x)$ one has

$$\rho(x,y) \geq \alpha / L^n. \tag{2.2}$$

REMARK 2.2. Property **(P)** holds true for nested fractals such that the similitudes $(\phi_i)_{i=1,\ldots,M}$ have the same unitary part. This class of fractals contains the well-known examples such as Sierpiński gaskets, snowflakes, the Vicsek set etc. A proof of this statement is given in the Appendix.

Clearly, if $\mathrm{ind}(x,y) = n$, then $\Delta_{n-1}(x) \cap \Delta_{n-1}(y) \neq \emptyset$. These sets either coincide or are adjacent (i.e. they meet in exactly one point). Moreover, under Property **(P)**, the index $\mathrm{ind}\,(x,y)$ is closely related to the Euclidean distance of x, y.

LEMMA 2.3. (i) *For any fixed* $x \in \mathcal{K} \setminus V^{(\infty)}$ *and* $n \geq 2$, *one has*

$$\{y : \mathrm{ind}(x,y) = n\} = \Delta_{n-1}^*(x) \setminus \Delta_n^*(x). \tag{2.3}$$

(ii) *Assume additionally that the fractal* $\mathcal{K}$ *has property* **(P)**. *If* $\mathrm{ind}(x,y) = n$ *then*

$$\rho(x,y) \asymp L^{-n}, \tag{2.4}$$

$\rho(x,y)$ *being the Euclidean distance.*

Proof. Fix $x \in \mathcal{K} \setminus V^{(\infty)}$ and $n \geq 2$. Observe that $y \in \Delta_n^*(x)$ if and only if $\Delta_n(x) \cap \Delta_n(y) \neq \emptyset$, which is equivalent to $\mathrm{ind}(x,y) \geq n+1$. Since $\{\Delta_n(x)\}_n$ is a decreasing sequence of sets, (2.3) follows. Relation (2.4) follows from (2.3) and property **(P)**. ∎

2.2. Gradients of nested fractals. To proceed, we need to define the gradient. The material in this section is classical and follows mainly [K3] and [T]. For other results concerning gradients on fractals we refer to [Ku, K1, St2].

2.2.1. *Nondegenerate harmonic structure on* $\mathcal{K}$. Suppose that $\mathcal{K}$ is the nested fractal associated with the system $\{\phi_1, \ldots, \phi_M\}$. Let $A = [a_{x,y}]_{x,y \in V^{(0)}}$ be a *conductivity matrix* on $V^{(0)}$, i.e. a symmetric real matrix with nonnegative off-diagonal entries and such that for any $x \in V^{(0)}$, $\sum_{y \in V^{(0)}} a_{x,y} = 0$. For $f : V^{(0)} \to \mathbb{R}$, set $\mathcal{E}_A^{(0)}(f,f) = \frac{1}{2} \sum_{x,y \in V^{(0)}} a_{x,y}(f(x) - f(y))^2$. Then we define two operations:

(1) *Reproduction.* For $f \in C(V^{(1)})$ we let

$$\widetilde{\mathcal{E}}_A^{(1)}(f,f) = \sum_{i=1}^{M} \mathcal{E}_A^{(0)}(f \circ \phi_i, f \circ \phi_i).$$

The mapping $\mathcal{E}_A^{(0)} \mapsto \widetilde{\mathcal{E}}_A^{(1)}$ is called the *reproduction map* and is denoted by $\mathcal{R}$.

(2) *Decimation.* Given a symmetric form $\mathcal{E}$ on $C(V^{(1)})$, define its restriction to $C(V^{(0)})$, $\mathcal{E}_{V^{(0)}}$, as follows. Take $f : V^{(0)} \to \mathbb{R}$, then set

$$\mathcal{E}|_{V^{(0)}}(f,f) = \inf\{\mathcal{E}(g,g) : g : V^{(1)} \to \mathbb{R} \text{ and } g|_{V^{(0)}} = f\}.$$

This mapping is called the *decimation map* and will be denoted by $\mathcal{D}e$.

Let $\mathbf{G}$ be the symmetry group of $V^{(0)}$, i.e. the group of transformations generated by the symmetries $R_{x,y}$, $x, y \in V^{(0)}$. Then we have ([L], [S]):

THEOREM 2.4. *Suppose $\mathcal{K}$ is a nested fractal. Then there exists a unique number $\rho = \rho(\mathcal{K}) > 1$ and a unique, up to a multiplicative constant, irreducible conductivity matrix A on $V^{(0)}$, invariant under the action of $\mathbf{G}$, and such that*

$$(\mathcal{D}e \circ \mathcal{R})(\mathcal{E}_A^{(0)}) = \frac{1}{\rho}\,\mathcal{E}_A^{(0)}. \tag{2.5}$$

A is called the *symmetric nondegenerate harmonic structure on $\mathcal{K}$*. By analogy with the electrical circuit theory, ρ is called the *resistance scaling factor* of $\mathcal{K}$. The number $d_w = d_w(\mathcal{K}) := \frac{\log(M\rho)}{\log L} > 1$ is called the *walk dimension* of $\mathcal{K}$. For further use, note that $\rho = L^{d_w - d}$.

2.2.2. *The canonical Dirichlet form on $\mathcal{K}$.* Suppose A is the nondegenerate harmonic structure on $\mathcal{K}$. Define $\mathcal{E}^{(0)} = \mathcal{E}_A^{(0)}$ and then let

$$\widetilde{\mathcal{E}}^{(m)}(f,f) = \rho^m \sum_{|w|=m} \mathcal{E}^{(0)}(f \circ \phi_w, f \circ \phi_w), \qquad f \in C(V^{(m)}).$$

The sequence $\widetilde{\mathcal{E}}^{(m)}$ is nondecreasing, i.e. for every $f : V^{(\infty)} \to \mathbb{R}$, one has

$$\widetilde{\mathcal{E}}^{(m)}(f,f) \le \widetilde{\mathcal{E}}^{(m+1)}(f,f), \qquad m = 0, 1, 2, \ldots.$$

Set $\widetilde{\mathcal{D}} = \{f : V^{(\infty)} \to \mathbb{R} : \sup_m \widetilde{\mathcal{E}}^{(m)}(f,f) < \infty\}$ and for $f \in \widetilde{\mathcal{D}}$,

$$\widetilde{\mathcal{E}}(f,f) = \lim_{m\to\infty} \widetilde{\mathcal{E}}^{(m)}(f,f). \tag{2.6}$$

Further, $\mathcal{D} = \mathcal{D}(\mathcal{E}) = \{f \in C(\mathcal{K}) : f|_{V^{(\infty)}} \in \widetilde{\mathcal{D}}\}$, $\mathcal{E}(f,f) = \widetilde{\mathcal{E}}(f|_{V^{(\infty)}}, f|_{V^{(\infty)}})$ for $f \in \mathcal{D}$.

Then $(\mathcal{E}, \mathcal{D})$ is a regular local Dirichlet form on $L^2(\mathcal{K}, \mu)$, which is invariant with respect to the group of local symmetries of $\mathcal{K}$. This Dirichlet form is also called the *Brownian Dirichlet form* on $\mathcal{K}$, and will be essential in defining the gradient. It satisfies the following scaling relation: for any $f \in \mathcal{D}$,

$$\mathcal{E}(f,f) = \rho^m \sum_{w \in \mathcal{W}_m} \mathcal{E}(f \circ \phi_w, f \circ \phi_w). \tag{2.7}$$

2.2.3. *Harmonic functions on $\mathcal{K}$ and energy measure*

DEFINITION 2.5. Suppose $f : V^{(0)} \to \mathbb{R}$ is given. Then $h \in \mathcal{D}(\mathcal{E})$ is called *harmonic* on $\mathcal{K}$ with boundary values f if $\mathcal{E}(h,h)$ minimizes the expression $\mathcal{E}(g,g)$ among all $g \in \mathcal{D}(\mathcal{E})$ such that $g|_{V^{(0)}} = f$. The unique harmonic function that agrees with f on $V^{(0)}$ will be denoted by Hf.

Denote by $\mathcal{H}$ the space of all harmonic functions on $\mathcal{K}$. It is an r-dimensional linear space, which can be equipped with the norm

$$\|h\|_{\mathcal{H}}^2 = \mathcal{E}(h,h) + \Big(\sum_{x\in V^{(0)}} h(x)\Big)^2.$$

Further, $\widetilde{\mathcal{H}}$ denotes the orthogonal complement in $\mathcal{H}$ of the (one-dimensional) subspace of constant functions, and let $\widetilde{P} : \mathcal{H} \to \widetilde{\mathcal{H}}$ be the orthogonal projection onto $\widetilde{\mathcal{H}}$. The norm on $\widetilde{\mathcal{H}}$ is given by $\|h\|^2 = \mathcal{E}(h,h)$ (note that $\|\cdot\|$ is a seminorm on $\mathcal{H}$, vanishing on constant functions), and the corresponding scalar product on $\widetilde{\mathcal{H}}$ will be denoted by $\langle\cdot,\cdot\rangle$.

Next, for $i = 1,\dots,M$, we define the map $M_i : \mathcal{H} \to \mathcal{H}$ by $M_i h = h \circ \phi_i$, and $\widetilde{M_i} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ by $\widetilde{M_i} = \widetilde{P} \circ M_i$. From the scaling relation (2.7) we deduce that for $h \in \widetilde{\mathcal{H}}$ and $m \geq 0$,

$$\|h\|^2 = \rho^m \sum_{|w|=m} \|\widetilde{M_w} h\|^2, \tag{2.8}$$

where by $\widetilde{M_w} h$ we have denoted $\widetilde{M}_{w_m} \circ \cdots \circ \widetilde{M}_{w_1} h = \widetilde{P}(h \circ \phi_w)$.

For $f \in \mathcal{D}$, we define the *energy measure associated with* f as the measure whose value on any given m-simplex $\mathcal{K}_w = \mathcal{K}_{w_1\dots w_m}$ is equal to

$$\nu_f(\mathcal{K}_w) = \rho^m \mathcal{E}(f\circ\phi_w, f\circ\phi_w). \tag{2.9}$$

When $h \in \mathcal{H}$ is a harmonic function and $w \in \mathcal{W}_m$, then $\nu_h(\mathcal{K}_w) = \rho^m \|M_w h\|^2$. Let $h_1,\dots,h_{r-1}$ be an orthonormal basis in $\widetilde{\mathcal{H}}$. Then the expression

$$\nu := \sum_{i=1}^{r-1} \nu_{h_i} \tag{2.10}$$

does not depend on the choice of the orthonormal basis and its value on an m-simplex $\mathcal{K}_w$ is equal to

$$\nu(\mathcal{K}_w) = \rho^m \operatorname{Tr} \widetilde{M}_w^* \widetilde{M}_w.$$

The measure given by (2.10) is called the *Kusuoka measure*, or the *energy measure* on $\mathcal{K}$. This measure has no atoms, and typically is singular with respect to the measure μ.

2.2.4. *Gradients.* When $x \in \mathcal{K}$ is a nonlattice point, then x has a unique *address*, an (infinite) sequence $w = w_1 w_2 \dots$ such that $x = \bigcap_{m=1}^{\infty} \mathcal{K}_{[w]_m}$

(recall that we have denoted $[w]_m = (w_1, \dots, w_m)$). For such a nonlattice point, let

$$Z_m(x) = \begin{cases} \dfrac{\widetilde{M}^*_{[w]_m} \widetilde{M}_{[w]_m}}{\mathrm{Tr}\, \widetilde{M}^*_{[w]_m} \widetilde{M}_{[w]_m}} & \text{if } \mathrm{rank}\, \widetilde{M}_{[w]_m} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

It can be shown that $Z_m(\cdot)$ is a bounded, matrix-valued martingale with respect to ν, and as such it is convergent ν-a.s. to an integrable function $Z(\cdot)$.

For a nonlattice point x with address w, set

$$\nabla_m f(x) = \widetilde{M}^{-1}_{[w]_m} (\widetilde{P}H)(f \circ \phi_{[w]_m}), \qquad m = 1, 2, \dots.$$

Then the *gradient* of f at x is the element of $\widetilde{\mathcal{H}}$ given by

$$\nabla f(x) = \lim_{m\to\infty} \nabla_m f(x),$$

provided the limit exists. For the discussion of 'pointwise gradients' and their properties we refer to [T], [PT] and [Hi]. But even if the pointwise limits of ∇_m are not known to exist, we do know (see [Ku, Lemmas 3.5 and 5.1], and also the discussion in [T, p. 137]) that when $f \in \mathcal{D}$, then there exists a measurable mapping $Y(\cdot, f)$ such that

$$\mathcal{E}(f, f) = \int_{\mathcal{K}} \langle Y(\cdot, f), Z(\cdot) Y(\cdot, f)\rangle \, d\nu(\cdot). \tag{2.12}$$

With abuse of notation, we will write ∇f for $Y(\cdot, f)$, which is defined ν-a.e. When we will use the pointwise value, it will be clearly indicated.

Definition 2.6.

(1) A continuous function $f : \mathcal{K} \to \mathbb{R}$ is called *m-harmonic* if $f \circ \phi_w$ is harmonic for any $w = (w_1, \dots, w_m) \in \mathcal{W}_m$.
(2) There exists a unique m-harmonic function with given values at points from $V^{(m)}$. For a continuous function f on $\mathcal{K}$, we denote by $H_m f$ the unique m-harmonic function that agrees with f on $V^{(m)}$.

Remark 2.7. When f is m-harmonic, then for any nonlattice point $x \in \mathcal{K}$ with address $w \in \mathcal{W}_\infty$ one has

$$\nabla_m f(x) = \nabla_{m+n} f(x)$$

for any $n \geq 0$, and so $\nabla f(x)$ exists at nonlattice points (which are of full ν-measure); note also that $\nabla_m f - f$ (and thus also $\nabla f - f$) is constant inside each $\mathcal{K}_w$ with $|w| = m$.

3. Poincaré inequality on nested fractals. The Poincaré inequalities on nested fractals that one can find in the literature (see e.g. [BBK] and its references) are usually written in the form

$$\int_B |f - f_B|^2 \, d\mu \le c\Psi(R) \int_B d\Gamma(f, f), \tag{3.1}$$

where B is a ball of radius R, $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a scale function (most commonly, $\Psi(R) = R^\sigma$), and $\Gamma(f, f)$ is the energy measure associated with the Brownian Dirichlet form on fractals.

Poincaré inequalities $P(q, p)$, on a metric measure space (X, ρ, μ), with a doubling measure μ and another Radon measure ν, are similar in spirit, but usually involve *two* functions. One says that a pair of measurable functions (f, g) satisfies the (q, p)-*Poincaré inequality* when

$$\Big(\fint_B |f - f_B|^q \, d\mu \Big)^{1/q} \le CR \Big(\fint_{\sigma B} |g|^p \, d\nu \Big)^{1/p}, \tag{3.2}$$

where $\sigma \ge 1$ is a given number, and σB denotes the ball concentric with B, but with radius σ times the radius of B. For an account of Poincaré inequalities in metric spaces, we refer mainly to [HK], and also to [H].

The Poincaré inequalities on nested fractals that we will be concerned with will be variants of two-weight inequalities. The measure μ appearing on the left-hand side will be the Hausdorff measure on $\mathcal{K}$, while the measure ν on the right-hand side will be the Kusuoka energy measure. Recall that in most cases the measure ν is not absolutely continuous with respect to μ. The difference from the classical case is that the integral on the right-hand side will not be a barred integral with respect to ν, but it will be divided by the μ-measure of the underlying set.

We start with a fractal version of Poincaré inequality—where balls are replaced with simplices. This version does not require property **(P)** of the underlying fractal. If L is the scaling factor as at the beginning of Section 2.1, the precise statement reads as follows.

THEOREM 3.1. *Let $f \in \mathcal{D}(\mathcal{E})$, and let Δ be any m-simplex, $m \ge 0$. Then*

$$\begin{aligned} \fint_\Delta |f(x) - f_\Delta| \, d\mu(x) &\le C(\operatorname{diam} \Delta)^{d_w/2} \Big(\frac{1}{\mu(\Delta^*)} \int_{\Delta^*} \langle \nabla f, Z \nabla f \rangle \, d\nu \Big)^{1/2} \\ &\le CL^{-md_w/2} \Big(L^{-md} \int_{\Delta^*} \langle \nabla f, Z \nabla f \rangle \, d\nu \Big)^{1/2}, \end{aligned} \tag{3.3}$$

where Δ^* denotes the union of Δ and all m-simplices adjacent to Δ.

The proof will be given later on. Now, we start with a local version of Poincaré inequality for adjacent lattice points.

Proposition 3.2. *Suppose $f \in \mathcal{D}(\mathcal{E})$, and let $x \overset{m}{\sim} y$. Let $\mathcal{K}_w$ be the m-simplex that contains both points x, y, with address $w \in \mathcal{W}_m$. Then*

$$|f(x) - f(y)|^2 \le C(\operatorname{diam} \mathcal{K}_w)^{d_w - d} \int_{\mathcal{K}_w} \langle \nabla f, Z\nabla f\rangle \, d\nu. \tag{3.4}$$

Proof. Set $c(x,y) = a^{-1}_{x'y'}$ where $x', y' \in V^{(0)}$ are such that $x = \phi_w(x')$ and $y = \phi_w(y')$ (the matrix $A = [a_{x,y}]$ was introduced in Section 2.2.1). Then

$$\begin{aligned}|f(x) - f(y)|^2 &\le c(x,y)) \sum_{u,v\in V^{(0)}} a_{uv}|f\circ\phi_w(u) - f\circ\phi_w(v)|^2 \\ &= c(x,y))\mathcal{E}^{(0)}(f\circ\phi_w, f\circ\phi_w) \le \mathcal{E}(f\circ\phi_w, f\circ\phi_w) \\ &= c(x,y)\int_{\mathcal{K}} \langle \nabla(f\circ\phi_w), Z\,\nabla(f\circ\phi_w)\rangle\, d\nu \\ &\le c_1 \int_{\mathcal{K}} \langle \nabla(f\circ\phi_w), Z\,\nabla(f\circ\phi_w)\rangle\, d\nu,\end{aligned}$$

where $c_1 = \sup\{a_{x,y} : x, y \in V^{(0)}\}$.

Since $\operatorname{diam} \mathcal{K}_w = L^{-m}$, the scaling relation from Lemma 3.3 below gives the desired statement. ∎

Lemma 3.3. *Let $f \in \mathcal{D}(\mathcal{E})$, and let $\mathcal{K}_w$ be an m-simplex. Then*

$$\int_{\mathcal{K}} \langle \nabla(f\circ\phi_w), Z\nabla(f\circ\phi_w)\rangle\, d\nu = L^{-m(d_w-d)} \int_{\mathcal{K}_w} \langle \nabla f, Z\nabla f\rangle\, d\nu. \tag{3.5}$$

Remark 3.4. The right hand side of (3.5) is well-defined since $\langle \nabla f, Z\nabla f\rangle$ exists ν-a.e. and $\int_{\mathcal{K}} \langle \nabla f, Z\nabla f\rangle\, d\nu < \infty$ (see [T, Theorem 4]).

Remark 3.5. While $\nu(\mathcal{K}_w)$ in general depends on w, the scaling factor on the right hand side of (3.5) depends only on $m = |w|$. Thus, the lemma is not tantamount to a simple change of variables but reflects an interplay between ∇f and Z.

Proof of Lemma 3.3. Step 1. Assume that f is m-harmonic. Then $\nabla f(y)$ exists at all nonlattice points y and $\nabla f(y) = \nabla_m f(y)$. Observe that $\nabla_m f(\cdot)$ is constant (ν-a.e.) inside each m-simplex $\mathcal{K}_w$ and that it differs there from $M_w^{-1} f(\cdot)$ by a constant only. It follows that

$$\int_{\mathcal{K}_w} \langle \nabla f, Z\nabla f\rangle\, d\nu = \int_{\mathcal{K}_w} \langle \nabla_m f, Z\nabla_m f\rangle\, d\nu = \lim_{n\to\infty} \int_{\mathcal{K}_w} \langle \nabla_m f, Z_n\nabla_m f\rangle\, d\nu.$$

To justify the last statement, observe that the random variables $X_n = \langle \nabla_m f, Z_n \nabla_m f\rangle$ converge to $X = \langle \nabla_m f, Z\nabla_m f\rangle$ in $L^1(\mathcal{K}, d\nu)$. This is so because $X_n \ge 0$, $X_n \to X$ in measure ν and

$$\int_{\mathcal{K}} X_n\, d\nu = \int_{\mathcal{K}} \langle \nabla_m f, Z_n\nabla_m f\rangle\, d\nu = \mathcal{E}(H_n f, H_n f) \to \mathcal{E}(f,f) = \int_{K} X\, d\nu.$$

The convergence in $L^1(\mathcal{K}, d\nu)$ then follows from Scheffé's theorem.

For short, let us write $F = \nabla_m f \in \widetilde{\mathcal{H}}$. Let $n > m$ be fixed, and let $\underline{i} = (i_{m+1}, \ldots, i_n) \in \mathcal{W}_{n-m}$ so that $w\underline{i} \in \mathcal{W}_n$. Since Z_n is constant on n-simplices, once $n > m$ we have

$$\begin{aligned}\int_{K_w} \langle F, Z_n F\rangle \, d\nu &= \sum_{|\underline{i}|=n-m} \int_{\mathcal{K}_{w\underline{i}}} \langle F, Z_n F\rangle \, d\nu \\ &= \sum_{|\underline{i}|=n-m} \frac{\|\widetilde{M}_{w\underline{i}} F\|^2}{\mathrm{Tr}(\widetilde{M}^*_{w\underline{i}} \widetilde{M}_{w\underline{i}})} \cdot L^{n(d_w-d)} \, \mathrm{Tr}(\widetilde{M}^*_{w\underline{i}} \widetilde{M}_{w\underline{i}}) \\ &= L^{n(d_w-d)} \sum_{|\underline{i}|=n-m} \|\widetilde{M}_{w\underline{i}} F\|^2 \\ &= L^{n(d_w-d)} \sum_{|\underline{i}|=n-m} \mathcal{E}(F \circ \phi_{w\underline{i}}, F \circ \phi_{w\underline{i}}).\end{aligned}$$

From the scaling property of $\mathcal{E}$,

$$\begin{aligned}\sum_{|\underline{i}|=n-m} \mathcal{E}(F \circ \phi_{w\underline{i}}, F \circ \phi_{w\underline{i}}) &= \sum_{|\underline{i}|=n-m} \mathcal{E}((F \circ \phi_w) \circ \phi_{\underline{i}}, (F \circ \phi_w) \circ \phi_{\underline{i}}) \\ &= L^{-(n-m)(d_w-d)} \mathcal{E}(F \circ \phi_w, F \circ \phi_w).\end{aligned}$$

We know that $F \circ \phi_w$ and $f \circ \phi_w$ differ by a constant only, so that

$$\mathcal{E}(F \circ \phi_w, F \circ \phi_w) = \mathcal{E}(f \circ \phi_w, f \circ \phi_w).$$

Piecing everything together, we obtain

$$\begin{aligned}\int_{\mathcal{K}_w} \langle F, Z_n F\rangle \, d\nu &= L^{m(d_w-d)} \mathcal{E}(f \circ \phi_w, f \circ \phi_w) \\ &= L^{m(d_w-d)} \int_{\mathcal{K}} \langle \nabla(f \circ \phi_w), Z \nabla(f \circ \phi_w)\rangle \, d\nu.\end{aligned}$$

The right-hand side does not depend on n, thus we can let $n \to \infty$, obtaining (3.5).

STEP 2. Let now f be n-harmonic, with $n > m$. Then $\mathcal{K}_w = \bigcup_{|\underline{i}|=n-m} \mathcal{K}_{w\underline{i}}$ and

$$\int_{\mathcal{K}_w} \langle \nabla f, Z \nabla f\rangle \, d\nu = \sum_{|\underline{i}|=n-m} \int_{\mathcal{K}_{w\underline{i}}} \langle \nabla f, Z \nabla f\rangle \, d\nu. \tag{3.6}$$

To each of the integrals on the right-hand side of (3.6) we apply Step 1, obtaining

$$\begin{aligned}(3.6) &= L^{n(d_w-d)} \sum_{|\underline{i}|=n-m} \int_{\mathcal{K}} \langle \nabla(f \circ \phi_{w\underline{i}}), Z \nabla(f \circ \phi_{w\underline{i}})\rangle \, d\nu \\ &= L^{m(d_w-d)} \sum_{|\underline{i}|=n-m} L^{(n-m)(d_w-d)} \mathcal{E}((f \circ \phi_w) \circ \phi_{\underline{i}}, (f \circ \phi_w) \circ \phi_{\underline{i}}),\end{aligned}$$

which, from the scaling property, is equal to

$$L^{m(d_w-d)}\mathcal{E}(f\circ\phi_w.f\circ\phi_w)=L^{m(d_w-d)}\int_{\mathcal{K}}\langle\nabla(f\circ\phi_w),Z\nabla(f\circ\phi_w)\rangle\,d\nu.$$

Step 3. Let now f be any function from $\mathcal{D}(\mathcal{E})$. Then

$$\mathcal{E}(f,f)=\lim_{n\to\infty}\mathcal{E}(H_nf,H_nf)$$

and

$$\mathcal{E}(f\circ\phi_w,f\circ\phi_w)=\lim_{n\to\infty}\mathcal{E}(H_n(f\circ\phi_w),H_n(f\circ\phi_w)). \tag{3.7}$$

From Step 2 we have, for $n\ge m$,

$$\begin{aligned}\int_{\mathcal{K}}\langle\nabla_n(H_nf\circ\phi_w),Z\nabla_n(H_nf\circ\phi_w)\rangle\,d\nu\\ =L^{-m(d_w-d)}\int_{\mathcal{K}_w}\langle\nabla(H_nf),Z\nabla(H_nf)\rangle\,d\nu,\end{aligned} \tag{3.8}$$

and the assertion follows from the limiting procedure: the left-hand side of (3.8) is equal to $\mathcal{E}(H_nf\circ\phi_w,H_nf\circ\phi_w)\xrightarrow{n\to\infty}\mathcal{E}(f\circ\phi_w,f\circ\phi_w)$. As to the right-hand side, since $\nabla(H_nf)=\nabla_nf$, and ∇_nf converges to ∇f in the seminorm $(\int_{\mathcal{K}}\langle\cdot,Z\cdot\rangle\,d\nu)^{1/2}$, we also have convergence in the restricted seminorm $(\int_{\mathcal{K}_w}\langle\cdot,Z\cdot\rangle\,d\nu)^{1/2}$, which gives the desired convergence. ∎

From Proposition 3.2 we derive the local Poincaré inequality for nonlattice points.

Theorem 3.6. *Suppose that $\mathcal{K}$ has property* **(P)**. *Let $f\in\mathcal{D}(\mathcal{E})$ and $x,y\in\mathcal{K}\setminus V^{(\infty)}$. Then*

$$|f(x)-f(y)|^2\le C\rho(x,y)^{d_w}\frac{1}{\mu(S(x,y))}\int_{S(x,y)}\langle\nabla f,Z\nabla f\rangle\,d\nu,$$

where $S(x,y)$ was introduced in Definition 2.1(6).

Proof. Step 1. Suppose $z\in V^{(m)}$ is a vertex of $\Delta\in\mathcal{T}_m$ and let $y\in\operatorname{Int}\Delta$. Then one finds a chain $z=z_0,z_1,\ldots,z_k\to y$ such that for all $k=1,2,\ldots$ the points z_{k-1} and z_k are $(m+k)$-neighbors. Denote by $\Delta(z_{k-1},z_k)$ the $(m+k)$-simplex they belong to. From Proposition 3.2 we have, since $\Delta(z_{k+1},z_k)\subset\Delta$,

$$\begin{aligned}|f(z_{k-1})-f(z_k)|^2&\le C(\operatorname{diam}\Delta(z_{k-1},z_k))^{d_w-d}\int_{\Delta(z_{k-1},z_k)}\langle\nabla f,Z\nabla f\rangle\,d\nu\\&\le C(\operatorname{diam}\Delta(z_{k-1},z_k))^{d_w-d}\int_{\Delta}\langle\nabla f,Z\nabla f\rangle\,d\nu.\end{aligned}$$

Since f is continuous, summing over k we obtain

$$\begin{aligned}|f(z)-f(y)| &\le \sum_{k=1}^{\infty} |f(z_{k-1})-f(z_k)| \\ &\le \sum_{k=1}^{\infty} (\operatorname{diam} \varDelta(z_{k-1},z_k))^{(d_w-d)/2}\Big(\int_{\varDelta}\langle\nabla f, Z\nabla f\rangle\, d\nu\Big)^{1/2} \\ &\le \sum_{k=1}^{\infty} L^{-(m+k)(d_w-d)/2}\Big(\int_{\varDelta}\langle\nabla f, Z\nabla f\rangle\, d\nu\Big)^{1/2} \\ &= CL^{-m(d_w-d)/2}\Big(\int_{\varDelta}\langle\nabla f, Z\nabla f\rangle\, d\nu\Big)^{1/2},\end{aligned}$$

and consequently

$$|f(z)-f(y)|^2 \le CL^{m(d-d_w)} \int_{\varDelta}\langle\nabla f, Z\nabla f\rangle\, d\nu. \tag{3.9}$$

Step 2. Suppose x, y belong to a common m-simplex $\varDelta$. Then choose a vertex $v \in V(\varDelta)$, write $|f(x)-f(y)|^2 \le 2(|f(x)-f(v)|^2 + |f(v)-f(y)|^2)$, and apply Step 1 in order to get (3.9) for x and y.

Step 3. The result of Step 2 extends immediately to the case when x, y belong to two adjacent m-simplices: when $x \in \varDelta_1 \in \mathcal{T}_m$, $y \in \varDelta_2 \in \mathcal{T}_m$ and $\varDelta_1, \varDelta_2$ are adjacent, then $\varDelta_1$ and $\varDelta_2$ share a vertex $z \in V^{(m)}$. One applies Step 1 to the pair (x, z) and then to (y, z), getting

$$|f(x)-f(y)|^2 \le CL^{m(d-d_w)} \int_{\varDelta_1\cup\varDelta_2}\langle\nabla f, Z\nabla f\rangle\, d\nu. \tag{3.10}$$

Step 4. Now take any $x, y \in \mathcal{K} \setminus V^{(\infty)}$. Let $\operatorname{ind}(x, y) = m$. Then $S(x, y) = \varDelta_{m-1}(x) \cup \varDelta_{m-1}(y)$ is composed either of a common $(m-1)$-simplex or two adjacent $(m-1)$-simplices. In the former case, apply Step 2, in the latter case, Step 3. In either case, $\mu(S(x, y)) \asymp L^{-(m-1)d}$ and $\rho(x, y) \asymp L^{-m}$, so the theorem is proven. ∎

Proof of Theorem 3.1. Choose $\varDelta \in \mathcal{T}_m$. By Jensen's inequality we have

$$⨍_{\varDelta} |f(x)-f_{\varDelta}|\, d\mu(x) \le \Big(⨍_{\varDelta} |f(x)-f_{\varDelta}|^2\, d\mu(x)\Big)^{1/2},$$

and further

$$\begin{aligned}⨍_{\varDelta} |f(x)-f_{\varDelta}|^2\, d\mu(x) &= ⨍_{\varDelta} \Big|f(x) - ⨍_{\varDelta} f(y)\, d\mu(y)\Big|^2 d\mu(x) \\ &= ⨍_{\varDelta} \Big|⨍_{\varDelta} (f(x)-f(y))\, d\mu(y)\Big|^2 d\mu(x)\end{aligned}$$

$$\le \fint_\Delta \fint_\Delta |f(x)-f(y)|^2\,d\mu(y)\,d\mu(x)$$
$$= \frac{1}{\mu(\Delta)^2}\int_\Delta\int_\Delta |f(x)-f(y)|^2\,d\mu(y)\,d\mu(x).$$

Points x and y under the integral belong to a common m-simplex Δ, and so $\operatorname{ind}(x,y) > m$ (without loss of generality we can and do assume that x,y are nonvertex points). Using Lemma 2.3, we split the inner integral as follows:

$$\tag{3.11} \int_\Delta |f(x)-f(y)|^2\,d\mu(y) = \sum_{n=m+1}^{\infty} \int_{\{y\in\Delta:\,\operatorname{ind}(x,y)=n\}} |f(x)-f(y)|^2\,d\mu(y) = \sum_{n=m+1}^{\infty} \int_{(\Delta^*_{n-1}(x)\setminus\Delta^*_n(x))\cap\Delta} |f(x)-f(y)|^2\,d\mu(y).$$

If $\operatorname{ind}(x,y)=n$, then $\rho(x,y)\asymp L^{-n}$ and moreover there exist two adjacent $(n-1)$-simplices, say S and T, such that $x\in S$, $y\in T$ ($S=T$ is permitted).

Let $v\in V^{(n-1)}$ be a common vertex of S and T. Then, according to (3.10) (which is true without property **(P)** as well),

$$|f(x)-f(y)|^2 \le CL^{-n(d_w-d)}\int_{S\cup T}\langle\nabla f, Z\nabla f\rangle\,d\nu \le CL^{-n(d_w-d)}\int_{\Delta^*_{n-1}(x)}\langle\nabla f, Z\nabla f\rangle\,d\nu.$$

As $\mu(\Delta\cap(\Delta^*_{n-1}(x)\setminus\Delta^*_n(x)))\le\mu(\Delta^*_{n-1}(x))\asymp L^{-nd}$, each of the integrals in (3.11) is bounded by $CL^{-nd_w}\int_{\Delta^*_{n-1}(x)}\langle\nabla f, Z\nabla f\rangle\,d\nu$. Consequently,

$$\tag{3.12} \int_\Delta\int_\Delta |f(x)-f(y)|^2\,d\mu(y)\,d\mu(x) \le C\sum_{n=m+1}^{\infty} L^{-nd_w}\int_\Delta\int_{\Delta^*_{n-1}(x)}\langle\nabla f, Z\nabla f\rangle\,d\nu\,d\mu(x).$$

Let $w\in\mathcal{W}_m$ be such that $\Delta=\phi_w(\mathcal{K})$ and for $\underline{i}\in W_{n-1-m}$ set $\Delta_{\underline{i}}=\phi_{w\underline{i}}(\mathcal{K})\subset\Delta$. Observe that on each $\Delta_{\underline{i}}$ the mapping $x\mapsto\Delta^*_{n-1}(x)$ is constant and equal to $\Delta^*_{\underline{i}}$. It follows that

$$\tag{3.13} \int_\Delta\int_{\Delta^*_{n-1}(x)}\langle\nabla f, Z\nabla f\rangle\,d\nu\,d\mu(x) = \sum_{\underline{i}\in\mathcal{W}_{n-1-m}}\int_{\Delta_{\underline{i}}}\int_{\Delta^*_{\underline{i}}}\langle\nabla f, Z\nabla f\rangle\,d\nu\,d\mu(x)$$
$$= \sum_{\underline{i}\in\mathcal{W}_{n-1-m}}\int_{\Delta^*_{\underline{i}}}\langle\nabla f, Z\nabla f\rangle\,d\nu\,\mu(\Delta_{\underline{i}}) \le C\sum_{\underline{i}\in\mathcal{W}_{n-1-m}} L^{-nd}\int_{\Delta^*_{\underline{i}}}\langle\nabla f, Z\nabla f\rangle\,d\nu.$$

The sets $\Delta^*_{\underline{i}}$ are not pairwise disjoint, but each of them consists of at most $M+1$ simplices from $\mathcal{T}_{n-1}$. Therefore, if in (3.13) we decompose each of the integrals over $\Delta^*_{\underline{i}}$ into a number of integrals over corresponding $(n-1)$-simplices, then each of these $(n-1)$-simplices will appear at most $M+1$ times in the sum. Furthermore, since for any $\underline{i} \in \mathcal{W}_{n-1-m}$ one has $\Delta^*_{\underline{i}} \subset \Delta^*$, and $\bigcup_{\underline{i}\in\mathcal{W}_{n-1-m}} \Delta_{\underline{i}} = \Delta$, it follows that

$$\sum_{i\in\mathcal{W}_{n-1-m}} \int_{\Delta^*_i} \langle \nabla f, Z\nabla f\rangle \, d\nu \le C \int_{\Delta^*} \langle \nabla f, Z\nabla f\rangle \, d\nu. \tag{3.14}$$

Collecting (3.12)–(3.14) we obtain

$$\begin{aligned}\int_\Delta \int_\Delta |f(x)-f(y)|^2 \, d\mu(y)\, d\mu(x) &\le C \sum_{n=m+1}^{\infty} L^{-nd_w} L^{-nd} \int_{\Delta^*} \langle \nabla f, Z\nabla f\rangle \, d\nu \\ &= C L^{-md_w} L^{-md} \int_{\Delta^*} \langle \nabla f, Z\nabla f\rangle \, d\nu.\end{aligned}$$

To complete the proof, observe again that $L^{-md} = c\mu(\Delta)$. ∎

Below we derive a Poincaré inequality that uses balls instead of simplices. This statement requires property **(P)** and will be used throughout the next section.

THEOREM 3.7. *Suppose that $\mathcal{K}$ satisfies* **(P)**. *Suppose $f \in \mathcal{D}(\mathcal{E})$. Let $x_0 \in \mathcal{K} \setminus V^{(\infty)}$ be a nonvertex point and let $r>0$ be given. Denote*

$$B = B(x_0, r) = \{y \in \mathcal{K} : \rho(x_0, y) \le r\}.$$

Then there exist $C>0$ and $A \ge 1$ (independent of x_0 and r) such that

$$⨍_B |f - f_B| \, d\mu \le C r^{d_w/2} \biggl(\frac{1}{r^d} \int_{B(x_0, Ar)} \langle \nabla f, Z\nabla f\rangle \, d\nu \biggr)^{1/2}. \tag{3.15}$$

Proof. Only minor changes need to be introduced in the proof of Theorem 3.1. From property **(P)** there exists $\alpha \in (0,1)$ such that for every nonlattice $x \in \mathcal{K}$, and any $m \ge 1$,

$$B(x, \alpha/L^m) \subseteq \Delta^*_m(x) \subseteq B(x, 2/L^m). \tag{3.16}$$

Let n_0 be the unique integer such that $L^{-(n_0+1)} < r/\alpha \le L^{-n_0}$, so that

$$B(x,r) \subseteq B(x, \alpha L^{-n_0}) \subseteq \Delta^*_{n_0}(x).$$

As before, we get

$$⨍_B |f-f_B|^2 \, d\mu \le \frac{1}{\mu(B)^2} \int_B \int_B |f(x)-f(y)|^2 \, d\mu(x)\, d\mu(y).$$

Since $B \subset \Delta^*_{n_0}(x_0)$ and $\Delta^*_{n_0}(x_0) = S_1 \cup \cdots \cup S_K$ is the sum of a finite number of neighboring n_0-simplices, we estimate the inner integral as

$$\int_{\Delta_0^*} |f(y) - f(x)|^2 \, d\mu(x) \, d\mu(y) = \sum_i \int_{S_i} |f(x) - f(y)|^2 \, d\mu(y). \tag{3.17}$$

Now we work with the integral over each S_i separately. Observe that when x, y are as in the integral in (3.17), then $\Delta_{n_0}(x) \cap \Delta_{n_0}(y) \neq \emptyset$, so that $\operatorname{ind}(x, y) \geq n_0 + 1$. Therefore, for any $i = 1, \ldots, K$, we have

$$\begin{aligned} \int_{S_i} |f(x) - f(y)|^2 \, d\mu(y) &= \sum_{n=n_0+1}^{\infty} \int_{S_i \cap \{y : \operatorname{ind}(x,y) = n\}} |f(x) - f(y)|^2 \, d\mu(y) \\ &= \sum_{n=n_0}^{\infty} \int_{S_i \cap (\Delta_n^*(x) \setminus \Delta_{n+1}^*(x))} |f(x) - f(y)|^2 \, d\mu(y) \\ &\leq c \sum_{n=n_0}^{\infty} L^{-nd_w} \int_{S_i \cap \Delta_n^*(x)} \langle \nabla f, Z \nabla f \rangle \, d\nu. \end{aligned}$$

From now on we proceed as in the proof of (3.3), ending up with

$$\begin{aligned} \int_B \int_B |f - f_B|^2 \, d\mu \, d\mu &\leq \int_B \sum_i \int_{S_i} |f(x) - f(y)|^2 \, d\mu(y) \, d\mu(x) \\ &\leq L^{-n_0 d_w} L^{-n_0 d + f} \sum_i \int_{S_i^*} \langle \nabla f, Z \nabla f \rangle \, d\nu \\ &\leq c L^{-n_0 d} L^{-n_0 d} \int_{B(x_0, 2Lr/\alpha)} \langle \nabla f, Z \nabla f \rangle \, d\nu \\ &\leq c r^{d_w} \frac{1}{\mu(B(x_0, 2Lr/\alpha))} \int_{B(x_0, 2Lr/\alpha)} \langle \nabla f, Z \nabla f \rangle \, d\nu, \end{aligned}$$

where we have used the inclusions $S_i^* \subseteq \Delta_{n_0}^* \subseteq B(x_0, 2L^{-n_0}) \subseteq B(x_0, 2Lr/\alpha)$. Set $A = 2L/\alpha$. The proof is complete. ∎

4. Sobolev spaces on fractals. On metric spaces, several definitions of Sobolev-type spaces are possible (see e.g. [FHK], [H], [KS]). We recall some of them below. Their mutual relations and connections with the Poincaré inequality form now a well established theory ([HK], [H]). Below, we briefly recall the relevant definitions.

Suppose (X, ρ, μ) is a metric measure space, where μ is a doubling Radon measure on a metric space (X, ρ). Any nested fractal $\mathcal{K}$ fits into this definition, with μ not only doubling but even Ahlfors regular. In the following definitions of Sobolev-type spaces we suppose $p \geq 1$.

(1) The *Hajłasz–Sobolev space* $M^{1,p}(X)$ consists of those $f \in L^p(X)$ for which there exists $L^p(X)$, $g \geq 0$, such that

$$|f(x) - f(y)| \leq C\rho(x,y)(g(x) + g(y)) \tag{4.1}$$

for μ-almost all $x, y \in X$.

(2) The space $\mathcal{P}^{1,p}(X)$ consists of those $f \in L^1_{\text{loc}}(X)$ for which there exist $\sigma \geq 1$ and $g \in L^p(X)$ such that for every ball $B = B(x,r)$,

$$\fint_B |f - f_B|\, d\mu \leq r\Big(\fint_{B(x,\sigma r)} g^p\, d\mu\Big)^{1/p}. \tag{4.2}$$

(3) The *Korevaar–Schoen Sobolev space* $\mathrm{KS}^{1,p}(X)$ consists of those functions $f \in L^p(X)$ for which

$$\limsup_{\epsilon \to 0} \int_X \fint_{B(x,\epsilon)} \frac{|f(x) - f(y)|^p}{\epsilon^p}\, d\mu(x)\, d\mu(y) < \infty.$$

One also considers the Newtonian spaces $N^{1,p}(X)$, based on an *upper gradient* which involves integrals over rectifiable curves. On nested fractals, the family of rectifiable curves might be empty or not rich enough to yield a nondegenerate object.

In general, the inclusions $M^{1,p}(X) \subset \mathcal{P}^{1,p}(X) \subset \mathrm{KS}^{1,p}(X)$ hold true, but not always can they be reversed. In some cases however, for example in $\mathbb{R}^d$, all three definitions yield the same function spaces. We refer the reader to [KM] and [H] for more details.

We are now going to adapt the definitions of the spaces $M^{1,p}$, $\mathcal{P}^{1,p}$, and $\mathrm{KS}^{1,p}$ to the fractal setting. As already mentioned in the Introduction, the scale r is not natural here, and it will be replaced by $r^{d_w/2}$. Let us mention that in many cases (Euclidean spaces, some manifolds) the walk dimension d_w, read off from the heat kernel estimates on the underlying space, is equal to 2, so that the scale $r^{d_w/2}$ is just r.

Definition 4.1. Let $\mathcal{K}$ be the nested fractal defined in Section 2.1; let $p \geq 1$ and $\sigma > 0$ be given. Recall that μ denotes the normalized d-dimensional Hausdorff measure on $\mathcal{K}$, and ν the Kusuoka measure. We say that a function $f \in L^p(\mathcal{K}, \mu)$ belongs to:

- the space $M^{1,p}_\sigma(\mathcal{K}, \mu)$ when there exists a nonnegative $g \in L^p(\mathcal{K}, \mu)$ such that for μ-a.e. $x, y \in \mathcal{K}$,

$$|f(x) - f(y)| \leq \rho(x,y)^\sigma (g(x) + g(y)); \tag{4.3}$$

- the space $\mathcal{P}^{1,p}_\sigma(\mathcal{K})$ when there exists a nonnegative $g \in L^p(\mathcal{K}, \nu)$ such that for any $x \in \mathcal{K}$ and $0 < r < \operatorname{diam} \mathcal{K}$,

$$(4.4)\qquad \fint_{B(x,r)} |f - f_{B(x,r)}|\,d\mu \le r^{\sigma}\Big(\frac{1}{\mu(B(x,Ar))}\int_{B(x,Ar)} g^p\,d\nu\Big)^{1/p},$$

with some $A \ge 1$; (4.4) will be called the $(1,p,\sigma)$-*Poincaré inequality*;

- the space $\mathrm{KS}^{1,p}_{\sigma}(\mathcal{K})$ when
$$\limsup_{\epsilon\to 0}\int_{\mathcal{K}}\fint_{B(x,\epsilon)} \frac{|f(x)-f(y)|^p}{\epsilon^{p\sigma}}\,d\mu(x)\,d\mu(y) < \infty;$$
- the *Besov–Lipschitz space* $\mathrm{Lip}(\sigma,p,\infty)$, $\sigma > 0$ (see [J]) if
$$\|f\|_{\mathrm{Lip}} = \sup_{m\ge 0} a_m^{(p)}(f) < \infty,$$
where
$$a_m^{(p)}(f) = L^{m\sigma}\Big(L^{md}\iint_{\rho(x,y)\le c_0/L^m} |f(x)-f(y)|^p\,d\mu(x)\,d\mu(y)\Big)^{1/p}$$
with some $c_0 > 0$. Note that different values of this constant yield the same function space with equivalent norms.

It is immediate to see that the spaces $\mathrm{Lip}(\sigma,p,\infty)(\mathcal{K})$ and $\mathrm{KS}^{1,p}_{\sigma}(\mathcal{K})$ coincide and that their norms are equivalent.

We now turn to relations between the Poincaré–Sobolev and Korevaar–Schoen Sobolev spaces on fractals. The inclusion $\mathcal{P}^{1,p}_{\sigma}(\mathcal{K}) \subset \mathrm{KS}^{1,p}_{\sigma}(\mathcal{K})$ is true under the usual constraints on parameters ($p \ge 1$, $\sigma > d/p$), and it can be reversed for $p = 2$, $\sigma = d_w/2$.

PROPOSITION 4.2. *Suppose that the fractal $\mathcal{K}$ has property* **(P)**. *Let $p \ge 1$ and $\sigma > 0$.*

(1) *If $\sigma > d/p$, then $\mathcal{P}^{1,p}_{\sigma}(\mathcal{K}) \subset \mathrm{KS}^{1,p}_{\sigma}(\mathcal{K})$.*
(2) *If $\sigma = d_w/2$, then $\mathcal{P}^{1,2}_{\sigma}(\mathcal{K}) = \mathrm{KS}^{1,2}_{\sigma}(\mathcal{K})$.*

Proof. Once (1) is proven, the inclusion '$\subset$' in (2) follows from the relation $d_w > d$ (true for any nested fractal). As to the opposite inclusion, Theorem 3.7 implies that the $(1,2,d_w/2)$-Poincaré inequality holds true for any $f \in \mathcal{D}(\mathcal{E})$. As $\mathcal{D}(\mathcal{E}) = \mathrm{Lip}(d_w/2,2,\infty) = \mathrm{KS}^{1,2}_{d_w/2}(\mathcal{K})$ [PP, Theorem 5], the inclusion '$\supset$' in (2) follows.

Therefore we need to prove (1). Our proof is a modification of the proof of [KM, Theorem 4.1]. See also [HK, Theorem 5.3 and its proof].

Assume that $f \in \mathcal{P}^{1,p}_{\sigma}(\mathcal{K})$ and that the pair (f,g) satisfies the $(1,p,\sigma)$-Poincaré inequality. We introduce a fractal version of Riesz potentials:

$$J_p(g,n,x) = \sum_{m=0}^{\infty} L^{-(m+n)\sigma}\Big(\frac{1}{\mu(\Delta^*_{n+m}(x))}\int_{\Delta^*_{n+m}(x)} g^p(z)\,d\nu(z)\Big)^{1/p}.$$

The potentials $J_p(g,n,x)$ are well-defined for all nonlattice points of $\mathcal{K}$ (this is a set of full μ-measure).

We will show that there exists a constant $k_0 \geq 0$ such that for μ-a.a. $x, y \in \mathcal{K}$ with $\operatorname{ind}(x,y) \geq k_0$ one has

$$|f(x) - f(y)| \leq C(J_p(g, \operatorname{ind}(x,y) - k_0, x) + J_p(g, \operatorname{ind}(x,y) - k_0, y)). \tag{4.5}$$

Since by assumption $f \in L^p(\mathcal{K},\mu) \subset L^1(\mathcal{K},\mu)$, μ-almost every point of $\mathcal{K}$ is a μ-Lebesgue point for f (cf. [To]):

$$f(x) = \lim_{r\to 0} \fint_{B(x,r)} f(y)\, d\mu(y) = \lim_{r\to 0} f_{B(x,r)}.$$

Let x, y be two nonlattice Lebesgue points for f and let $n_0 = \operatorname{ind}(x,y)$. We use a classical chaining argument. Denote $r_m = \alpha/(AL^m)$, where $A \geq 1$ is the constant from the Poincaré inequality (4.4), and $\alpha \in (0,1)$ comes from (3.16). Using the Jensen inequality, the doubling property for μ, the Poincaré inequality (4.4) and (3.16), we obtain the following chain of inequalities:

$$\begin{aligned} |f(x) - f_{B(x,r_{n_0})}| &\leq \sum_{m=0}^{\infty} |f_{B(x_0,r_{n_0+m})} - f_{B(x,r_{n_0+m+1})}| \\ &\leq \sum_{m=0}^{\infty} \fint_{B(x,r_{n_0+m+1})} |f(z) - f_{B(x,r_{n_0+m})}|\, d\mu(z) \\ &\leq \sum_{m=0}^{\infty} \fint_{B(x,r_{n_0+m})} |f(z) - f_{B(x,r_{n_0+m})}|\, d\mu(z) \\ &\leq C\sum_{m=0}^{\infty} r_{n_0+m}^{\sigma} \biggl(\frac{1}{\mu(B(x,Ar_{n_0+m}))} \int_{B(x,Ar_{n_0+m})} g^p(z)\, d\nu(z)\biggr)^{1/p} \\ &\leq C\sum_{m=0}^{\infty} L^{-(m+n_0)\sigma} \biggl(\frac{1}{\mu(\Delta^*_{n_0+m}(x))} \int_{\Delta^*_{n_0+m}(x)} g^p(z)\, d\nu(z)\biggr)^{1/p} \\ &= CJ_p(g, \operatorname{ind}(x,y), x). \end{aligned} \tag{4.6}$$

A similar estimate holds for y:

$$|f(y) - f_{B(y,r_{n_0})}| \leq CJ_p(g, \operatorname{ind}(x,y), y). \tag{4.7}$$

From Lemma 2.3, there exists a universal constant $C_1 > 0$ such that when $\operatorname{ind}(x,y) = n_0$, then $\rho(x,y) \leq C_1 L^{-n_0} = (C_1 A/\alpha) r_{n_0}$. For short, denote

$$R = (1 + C_1 A/\alpha) r_{n_0}.$$

Let k_0 be the smallest number such that $B(z, AR) \subset \Delta^*_{n_0-k_0}(z)$ for any $z \in \mathcal{K}$ (cf. (3.16)). Using the Poincaré inequality (4.4) and the Ahlfors regularity of μ we get

$$
\begin{aligned}
(4.8)\qquad |f_{B(x,r_{n_0})} - f_{B(y,r_{n_0})}| &\le |f_{B(x,r_{n_0})} - f_{B(x,R)}| + |f_{B(y,r_{n_0})} - f_{B(x,R)}| \\
&\le \fint_{B(x,r_{n_0})} |f(z) - f_{B(x,R)}|\, d\mu(z) + \fint_{B(y,r_{n_0})} |f(z) - f_{B(x,R)}|\, d\mu(z) \\
&\le \left(\frac{\mu(B(x,R))}{\mu(B(x,r_{n_0}))} + \frac{\mu(B(x,R))}{\mu(B(y,r_{n_0}))} \right) \fint_{B(x,R)} |f(z) - f_{B(x,R)}|\, d\mu(z) \\
&\le CR^\sigma \left(\frac{1}{\mu(B(x,AR)} \int_{B(x,AR)} g^p(z)\, d\nu(z) \right)^{1/p} \\
&\le CJ_p(g, \mathrm{ind}(x,y) - k_0, x).
\end{aligned}
$$

The estimate (4.5) follows when we sum up (4.6)–(4.8).

The proposition will be proven once we show that

$$\sup_{m \ge k_0} (a_m^{(p)}(f))^p < \infty,$$

where

$$(a_m^{(p)}(f))^p = L^{m(\sigma p + d)} \iint_{\rho(x,y) \le \alpha / L^m} |f(x) - f(y)|^p\, d\mu(x)\, d\mu(y).$$

We have

$$
\begin{aligned}
(4.9)\qquad (a_m^{(p)}(f))^p &\le \int_{\mathcal{K}} \int_{\Delta_m^*(x)} |f(x) - f(y)|^p\, d\mu(y)\, d\mu(x) \\
&\le \int_{\mathcal{K}} \Big(\sum_{k=m+1}^{\infty} \int_{\Delta_{k-1}^*(x) \setminus \Delta_k^*(x)} |f(x) - f(y)|^p\, d\mu(y) \Big)\, d\mu(x).
\end{aligned}
$$

Since $y \in \Delta_{k-1}^*(x) \setminus \Delta_k^*(x)$ is tantamount to $\mathrm{ind}(x,y) = k+1$, we can use the estimate (4.5) to get

$$
\begin{aligned}
&\int_{\Delta_{k-1}^*(x) \setminus \Delta_k^*(x)} |f(x) - f(y)|^p\, d\mu(y) \\
&\le C \Big(\int_{\Delta_{k-1}^*(x) \setminus \Delta_k^*(x)} J_p^p(g, k - k_0, x)\, d\mu(y) + \int_{\Delta_k^*(x) \setminus \Delta_{k+1}^*(x)} J_p^p(g, k - k_0, y)\, d\mu(y) \Big) \\
&= C(I_k(x) + II_k(x)).
\end{aligned}
$$

To estimate these two parts we need a lemma, which is similar to [KM, Lemma 4.3(ii), (iii)]:

Lemma 4.3. *Let $N \ge 1$, $p \ge 1$, $\sigma > 0$ be given and let the functions $f \in L^p(\mathcal{K}, \mu)$ and $g \in L^p(\mathcal{K}, \nu)$ satisfy the $(1, p, \sigma)$-Poincaré inequality. Then for μ-almost all $x \in \mathcal{K}$,*

$$(4.10)\qquad \int_{\Delta_N^*(x)} J_p^p(g, N, y)\, d\mu(y) \le CL^{-N\sigma p} \int_{\Delta_N^{**}(x)} g^p\, d\nu$$

and

$$\int_{\mathcal{K}} J_p^p(g,N,y)\,d\mu(y) \le CL^{-N\sigma p}\int_{\mathcal{K}} g^p\,d\nu. \tag{4.11}$$

Proof. For $y \in \Delta_N^*(x)$ and $k \ge N$ one has $\Delta_k^*(y) \subset \Delta_N^*(y) \subset \Delta_N^{**}(x)$ and therefore

$$\begin{aligned} J_p(g,N,y) &= \sum_{m=0}^{\infty} L^{-(m+N)\sigma}\Big(\frac{1}{\mu(\Delta_{N+m}^*(y))}\int_{\Delta_{N+m}^*(y)} g^p(z)\,d\nu(z)\Big)^{1/p} \\ &\le C\sum_{m=0}^{\infty} L^{-(\sigma-d/p)(N+m)}\Big(\int_{\Delta_N^{**}(x)} g^p(z)\,d\nu(z)\Big)^{1/p} \\ &= CL^{-(\sigma-d/p)N}\Big(\int_{\Delta_N^{**}(x)} g^p(z)\,d\nu(z)\Big)^{1/p}. \end{aligned}$$

Since $\mu(\Delta_N^*(x)) \le CL^{-Nd}$, (4.10) follows.

To see (4.11), observe that, using (4.10),

$$\int_{\mathcal{K}} J_p^p(g,N,y)\,d\mu(y) = \sum_{\Delta\in\mathcal{T}_N}\int_{\Delta} J_p^p(g,N,y)\,d\mu(y) \le C\sum_{\Delta\in\mathcal{T}_N} L^{-Np\sigma}\int_{\Delta^{**}} g^p\,d\nu.$$

A covering argument as the one used to conclude the proof of Theorem 3.1 gives (4.11). ■

Conclusion of the proof of Proposition 4.2. Since

$$I_k(x) \le \mu(\Delta_{k-1}^*(x))J_p^p(g,k-k_0,x) \le CL^{-kd}J_p^p(g,k-k_0,x),$$

one has, using (4.11),

$$\begin{aligned} \int_{\mathcal{K}} I_k(x)\,d\mu(x) &\le \int_{\mathcal{K}}\int_{\Delta_k^*(x)} J_p^p(g,k-k_0,x)\,d\mu(y)\,d\mu(x) \\ &\le C\int_{\mathcal{K}} J_p^p(g,k-k_0,x)\mu(\Delta_k^*(x))\,d\mu(x) \\ &\le CL^{-k(d+\sigma p)}\int_{\mathcal{K}} g^p\,d\nu. \end{aligned} \tag{4.12}$$

To estimate the other part, we use (4.10):

(4.13)

$$\begin{aligned} \int_{\mathcal{K}} II_k(x)\,d\mu(x) &\le \int_{\mathcal{K}}\int_{\Delta_k^*(x)} J_p^p(g,k-k_0,y)\,d\mu(y)\,d\mu(x) \\ &\le C\int_{\mathcal{K}}\int_{\Delta_{k-k_0}^*(x)} J_p^p(g,k-k_0,y)\,d\mu(y)\,d\mu(x) \\ &\le C\int_{\mathcal{K}} L^{-k\sigma p}\int_{\Delta_{k-k_0}^{**}(x)} g^p\,d\nu\,d\mu(x) \le CL^{-k(d+\sigma p)}\int_{\mathcal{K}} g^p\,d\nu. \end{aligned}$$

Summing (4.12) and (4.13) over $k \geq m$ we find that the right-hand side of (4.9) is no greater than

$$C \sum_{k=m}^{\infty} L^{-k(d+\sigma p)} \int_{\mathcal{K}} g^p \, d\nu = C L^{-m(d+\sigma p)} \int_{\mathcal{K}} g^p \, d\nu,$$

so that

$$(a_m^{(p)}(f))^p \leq \int_{\mathcal{K}} g^p \, d\nu,$$

once $m \geq k_0$. The proposition follows. ■

We now turn our attention to the relation of Poincaré–Sobolev spaces $\mathcal{P}_\sigma^{1,p}(\mathcal{K})$ to Hajłasz–Sobolev spaces $M_\sigma^{1,p}(\mathcal{K})$. It has been proven by Hu that $M_\sigma^{1,p}(\mathcal{K}) \subset \mathrm{KS}_\sigma^{1,p}(\mathcal{K})$ for all $p \geq 1$ and $\sigma > 0$ ([Hu, Theorem 1.1]). Moreover, Hu's theorem asserts that $\mathrm{KS}_\sigma^{1,p}(\mathcal{K}) \subset M_{\sigma'}^{1,p}(\mathcal{K})$ for all $0 < \sigma' < \sigma$. It is not known whether the inclusion $\mathrm{KS}_\sigma^{1,p}(\mathcal{K}) \subset M_\sigma^{1,p}(\mathcal{K})$ holds true on nested fractals, even if property **(P)** holds.

Recall that for $p \geq 1$, the 'weak' L^p, or the *Marcinkiewicz space* $L_w^p(\mathcal{K}, \mu)$, consists of those measurable functions f for which

$$\sup_{t>0} t^p \mu\{x : |f(x)| > t\} < \infty.$$

We can also consider 'weak' Hajłasz–Sobolev spaces.

Definition 4.4. Let $p \geq 1$ and $\sigma > 0$. One says that $f \in L^p(\mathcal{K}, \mu)$ belongs to the *weak Hajłasz–Sobolev space* $(M_\sigma^{1,p})_w(\mathcal{K})$ if there exists a function $g \in L_w^p(\mathcal{K}, \mu)$ such that (4.1) holds true.

Proposition 4.5. *Suppose that the nested fractal $\mathcal{K}$ has property* **(P)**. *Assume $p \geq 1$, $\sigma > 0$. Then:*

(1) $\mathcal{P}_\sigma^{1,p}(\mathcal{K}) \subset (M_\sigma^{1,p})_w(\mathcal{K}) \subset M_\sigma^{1,p'}(\mathcal{K})$ *with any $1 \leq p' < p$ (the last inclusion requires $p > 1$).*

(2) *If $p = 2$ and $\sigma = d_w/2$, then $M_\sigma^{1,2}(\mathcal{K}) \subset P_\sigma^{1,2}(\mathcal{K})$.*

Proof. (1) Once we prove estimates for fractal Riesz potentials, this result is immediate. Let $f \in \mathcal{P}_\sigma^{1,p}(\mathcal{K})$, and let $(f, \tilde{f})$ satisfy the $(1, p, \sigma)$-Poincaré inequality.

The function g (corresponding to the upper gradient), needed in the definition of Hajłasz–Sobolev spaces, will be a fractal variant of the Hardy–Littlewood maximal function: for $x \in \mathcal{K} \setminus V^{(\infty)}$ we set

$$g(x) = (M\tilde{f})(x) := \sup_{m \geq 1} \left(\frac{1}{\mu(\Delta_m^*(x))} \int_{\Delta_m^*(x)} \tilde{f}^p \, d\nu \right)^{1/p}.$$

It is obvious that for any $n \geq 1$,

$$J_p(\tilde{f}, n, x) \leq C L^{-n\sigma} g(x) \tag{4.14}$$

with some universal constant $C > 0$. Recall the estimate (4.5):

$$|f(x) - f(y)| \le C(J_p(\tilde{f}, \mathrm{ind}(x,y) - k_0, x) + J_p(\tilde{f}, \mathrm{ind}(x,y) - k_0, y))$$

(k_0 was a universal index depending only on the geometry of the fractal), so that further, taking into account (2.4),

$$|f(x) - f(y)| \le CL^{-\sigma\,\mathrm{ind}(x,y)}(g(x) + g(y)) \le C\rho(x,y)^\sigma(g(x) + g(y)).$$

The argument that proves $g \in L^p_w(\mathcal{K}, \nu)$ is also classical. Fix $t > 0$ and suppose that $g(x) > t$ for some $x \in \mathcal{K} \setminus V^{(\infty)}$. By the definition of g, there exists $m = m(x)$ such that

$$\mu(\Delta^*_m(x)) \le \frac{1}{t^p} \int_{\Delta^*_m(x)} \tilde{f}^p \, d\nu. \tag{4.15}$$

Consider the covering of the set $A(t) = \{x \in \mathcal{K} : g(x) > t\}$ by the balls $B(x, 2L^{-m(x)})$, $x \in A(t) \setminus V^{(\infty)}$. By the $5r$-covering lemma there is a countable subcollection of these balls, $B_i = B(x_i, \rho_i)$ with $\rho_i = 2L^{-m(x_i)}$, such that the B_i's are pairwise disjoint, yet $A(t) \subset \bigcup_i B(x_i, 5\rho_i)$. Due to (3.16), the sets $\Delta^*_{m(x_i)}(x_i)$ are disjoint. Then, by the doubling property of μ,

$$\begin{aligned}\mu(\{x : g(x) > t\}) &\le \mu\Big(\bigcup_i B(x_i, 5\rho_i)\Big) \le C \sum_i \mu(B(x_i, \rho_i)) \\ &\le C \sum_i \mu(B(x_i, \alpha L^{-m(x_i)})) \le C \sum_i \mu(\Delta^*_{m(x_i)}) \\ &\overset{(4.15)}{\le} \frac{C}{t^p} \sum_i \int_{\Delta^*_{m(x_i)}} \tilde{f}^p \, d\nu \le \frac{C}{t^p} \int_{\mathcal{K}} \tilde{f}^p \, d\nu.\end{aligned}$$

Since $\mu(\mathcal{K}) < \infty$, we have $L^p_w(\mathcal{K}, \mu) \subset L^{p'}(\mathcal{K}, \mu)$ for $p' < p$. This way (1) is proven. Assertion (2) follows from Hu's inclusion $M^{1,2}_\sigma(\mathcal{K}) \subset \mathrm{KS}^{1,p}_\sigma(\mathcal{K})$ and Proposition 4.2(2) above. ■

5. Appendix. We will now prove the statement from Remark 2.2. Set

$$\alpha_0 = \inf\{\mathrm{dist}(A, B) : A, B \in \mathcal{T}_2,\ A \cap B = \emptyset\} \quad \text{and} \quad \alpha = L\alpha_0.$$

More precisely, we will be proving the following.

PROPOSITION 5.1. *Let $\mathcal{K}$ be the nested fractal associated with similitudes $\{\phi_i\}_{i=1}^r$ with contraction factor L. Suppose that the ϕ_i's share the same unitary part, i.e. there is an isometry $U : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi_i(x) = (1/L)U(x) + t_i$, $t_i \in \mathbb{R}^n$, $i = 1, \ldots, r$. Then* **(P)** *holds.*

The key argument in the proof is provided by the following lemma.

Lemma 5.2. *Let $n \geq 1$. Suppose A, B are two neighbouring n-simplices, and let $A_1 \subset A$, $B_1 \subset B$ be two $(n+1)$-simplices that are disjoint. Then* $\operatorname{dist}(A_1, B_1) \geq \alpha L^{-n}$.

Proof. We proceed by induction on n. Clearly, the statement is true for $n = 1$.

Suppose it is true for $1, \ldots, n-1$. Let $A, B \in \mathcal{T}_n$ and $A_1, B_1 \in \mathcal{T}_{n+1}$ be as in the statement; let $(i_1, \ldots, i_n)$ be the address of A and $(w_1, \ldots, w_n)$ the address of B. Define $k_0 = \min\{l : i_l \neq w_l\}$. One has $1 \leq k_0 \leq n$.

If $k_0 > 1$ then $A, B \subset \mathcal{K}_{i_1 \ldots i_{k_0-1}} \in \mathcal{T}_{k_0-1}$. Set

$$A' = \phi^{-1}_{i_1 \ldots i_{k_0-1}}(A), \qquad B' = \phi^{-1}_{i_1 \ldots i_{k_0-1}}(B) \qquad \text{(we have } A', B' \in \mathcal{T}_{n-k_0+1}),$$
$$A'_1 = \phi^{-1}_{i_1 \ldots i_{k_0-1}}(A_1), \quad B'_1 = \phi^{-1}_{i_1 \ldots i_{k_0-1}}(B_1) \quad \text{(we have } A'_1, B'_1 \in \mathcal{T}_{n-k_0+2}).$$

Those simplices satisfy the assumptions for $n - k_0 + 1 \leq n - 1$ and the statement follows.

Now, suppose that $k_0 = 1$. We have

$$\mathcal{K}_{i_1} \supset \mathcal{K}_{i_1 i_2} \supset \cdots \supset \mathcal{K}_{i_1 \ldots i_n} = A \supset A_1 = \mathcal{K}_{i_1 \ldots i_n i_{n+1}},$$
$$\mathcal{K}_{w_1} \supset \mathcal{K}_{w_1 w_2} \supset \cdots \supset \mathcal{K}_{w_1 \ldots w_n} = B \supset B_1 = \mathcal{K}_{w_1 \ldots w_n w_{n+1}}.$$

Let v be a junction point of A and B. Because of the inclusions above, $v \in \mathcal{K}_{i_1 i_2} \cap \mathcal{K}_{w_1 w_2} \subset \mathcal{K}_{i_1} \cap \mathcal{K}_{w_1}$ as well.

We will now show that $\mathcal{K}_{i_1 i_2} \cup \mathcal{K}_{w_1 w_2}$ is similar to $\mathcal{K}_{i_1} \cup \mathcal{K}_{w_1}$. More precisely, we will see that

$$\mathcal{K}_{i_1 i_2} - v = S(\mathcal{K}_{i_1} - v) \quad \text{and} \quad \mathcal{K}_{w_1 w_2} - v = S(\mathcal{K}_{w_1} - v), \tag{5.1}$$

where $S = (1/L)U$ is the similitude such that $\phi_i = S + t_i$.

Since $v \in \mathcal{K}_{i_1} \cap \mathcal{K}_{w_1} \subset V^{(1)}$, there exist $z_1, z_2 \in V^{(0)}$ and maps ϕ_{j_1}, ϕ_{j_2} such that z_l is the fixed point of ϕ_{j_l}, $l = 1, 2$, and $v = \phi_{i_1}(z_1) = \phi_{w_1}(z_2)$. Further, since $v \in \mathcal{K}_{i_1 i_2}$, there exists another essential fixed point u such that $v = \phi_{i_1 i_2}(u)$. Then $\phi_{i_1}(z_1) = \phi_{i_1 i_2}(u)$ and so $z_1 = \phi_{i_2}(u)$. In particular, $z_1 \in \mathcal{K}_{i_2} \cap \mathcal{K}_{j_1}$. By [L, Proposition IV.13], any element in $V^{(0)}$ belongs to exactly one n-cell for each n. It follows that $j_1 = i_2$. The same argument for $\mathcal{K}_{w_1 w_2}$ gives $j_2 = w_2$.

Since S is linear, we have

$$S(K_{i_1} - v) = S(\phi_{i_1}(K) - \phi_{i_1}(z_1)) = S(SK + t_{i_1} - Sz_1 - t_{i_1}) = S^2(K - z_1).$$

On the other hand, since $z_1 = \phi_{i_2}(z_1)$, we get

$$\begin{aligned} K_{i_1 i_2} - v &= \phi_{i_1}(\phi_{i_2}(K)) - \phi_{i_1}(z_1) = S(\phi_{i_2}(K)) - S(z_1) \\ &= S(\phi_{i_2}(K) - \phi_{i_2}(z_1)) = S(SK - Sz_1) = S^2(K - z_1). \end{aligned}$$

Identical arguments hold for the pair $\mathcal{K}_{w_1}$ and $\mathcal{K}_{w_1 w_2}$, and the proof of (5.1) is complete.

Now,

$$A' = S^{-1}(A - v) + v, \qquad B' = S^{-1}(B - v) + v,$$
$$A_1' = S^{-1}(A_1 - v) + v, \quad B_1' = S^{-1}(B_1 - v) + v$$

are two pairs of $(n-1)$- and $(n-2)$-simplices satisfying the assumptions, hence $\operatorname{dist}(A_1', B_1') \geq \alpha/L^{n-1}$, and thus $\operatorname{dist}(A_1, B_1) \geq \alpha/L^n$. ∎

Proof of Proposition 5.1. We proceed by induction on n.

If $n = 1$ and $y \notin \Delta_1^*(x) \setminus \Delta_2^*(x)$, then the 2-simplices $\Delta_2(x)$ and $\Delta_2(y)$ are disjoint. Thus, $\rho(x, y) \geq \operatorname{dist}(\Delta_2(x), \Delta_2(y)) \geq \alpha_0 = \alpha/L$.

Suppose now that the statement is true for $1, \ldots, n-1$, and take $y \in \Delta_n^*(x) \setminus \Delta_{n+1}^*(x)$. Then the sets $\Delta_{n+1}(x)$ and $\Delta_{n+1}(y)$ are disjoint, whereas $\Delta_n(x)$ and $\Delta_n(y)$ are not. There are two possibilities: either $\Delta_n(x) = \Delta_n(y)$, or they are adjacent n-simplices. Let $(i_1, \ldots, i_n)$ be the address of $\Delta_n(x)$ and $(w_1, \ldots, w_n)$ be the address of $\Delta_n(y)$.

If $\Delta_n(x) = \Delta_n(y)$, we consider the points $x' = \phi_{i_1 \ldots i_{n-1}}^{-1}(x)$ and $y' = \phi_{i_1 \ldots i_{n-1}}^{-1}(y)$. Then $\Delta_2(x')$ and $\Delta_2(y')$ are disjoint 2-simplices, so by assumption we get $\rho(x', y') \geq \alpha/L$, thus $\rho(x, y) \geq \alpha/L^n$.

If $\Delta_n(x)$ and $\Delta_n(y)$ are adjacent n-simplices, we apply Lemma 5.2 to $A = \Delta_n(x)$, $B = \Delta_n(y)$, $A_1 = \Delta_{n+1}(x)$, $B_1 = \Delta_{n+1}(y)$. ∎

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