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An observation on the Turán–Nazarov inequality

by

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Abstract. The main observation of this note is that the Lebesgue measure μ in the Turán–Nazarov inequality for exponential polynomials can be replaced with a certain geometric invariant $\omega \geq \mu$, which can be effectively estimated in terms of the metric entropy of a set, and may be nonzero for discrete and even finite sets. While the frequencies (the imaginary parts of the exponents) do not enter the original Turán–Nazarov inequality, they necessarily enter the definition of ω.

1. Introduction. The classical Turán inequality bounds the maximum of the absolute value of an exponential polynomial $p(t)$ on an interval B through the maximum of its absolute value on any subset Ω of positive measure. Turán [11] assumed Ω to be a subinterval of B, and Nazarov [7] generalized the result to any subset Ω of positive measure. More precisely, we have:

Theorem 1.1 ([7]). *Let $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ be an exponential polynomial, where $c_k, \lambda_k \in \mathbb{C}$. Let $B \subset \mathbb{R}$ be an interval, and let $\Omega \subset B$ be a measurable set. Then*

$$\sup_B |p| \leq e^{\mu_1(B)\cdot \max|\operatorname{Re}\lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\mu_1(\Omega)}\right)^m \cdot \sup_\Omega |p|$$

where μ_1 is the Lebesgue measure on $\mathbb{R}$ and $c > 0$ is an absolute constant.

In this note, we generalize and strengthen the Turán–Nazarov inequality (and its multi-dimensional analogue stated below) by replacing the Lebesgue measure of Ω with a simple geometric invariant $\omega_D(\Omega)$. We call it the metric span of $\Omega \subset \mathbb{R}^n$ with respect to a "diagram" D of p comprising the degree of p and its maximal frequency λ. The metric span always bounds the Lebesgue measure from above, and it is strictly positive for sufficiently dense discrete (in particular, finite) sets Ω. It can be effectively estimated in terms of the metric entropy of Ω. See [13] and Section 2.1 below for some basic properties of $\omega_D(\Omega)$.

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Our approach is as follows: Put $\rho = \sup_\Omega |p|$. Then $\Omega \subset V_\rho$, where $V_\rho = V_\rho(p) = \{t \in B : |p(t)| \le \rho\}$ is the ρ-sublevel set of the exponential polynomial p. Next we use a theorem of Khovanskiĭ [6] to give an upper bound on the number of solutions of $|p(t)| = \rho$ in an interval B in terms of the length of the interval, the degree of p and the maximal frequency of p. This also bounds from above the number of intervals in V_ρ. Next, for V_ρ, consisting of a finite number of closed intervals, it is easy to compare the Lebesgue measure $\mu_1(V_\rho)$ and the metric entropy of $\Omega \subset V_\rho$. We conclude that $\mu_1(V_\rho) \ge \omega_D(\Omega)$. Finally, we apply the original Turán–Nazarov inequality of Theorem 1.1 to the sublevel set V_ρ.

With appropriate modifications this approach also works in higher dimensions. Originally it was applied in [13] in order to produce a Remez-type inequality for algebraic polynomials on discrete sets. The corresponding invariant $\omega_{n,d}(\Omega)$ depends only on the dimension and the degree, and uses Vitushkin's bound (see [12], and [5] for further developments in this direction) for the metric entropy of semialgebraic sets instead of Khovanskiĭ's bound. It replaces the Lebesgue measure of Ω in the classical Remez inequality for algebraic polynomials ([9, 4]).

Now we give an accurate statement of our main results in the one-dimensional case. For a given exponential polynomial $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ with $c_k, \lambda_k \in \mathbb{C}$, and for a given interval $B \subset \mathbb{R}$, we define the diagram $D = D(p, B) = (m, \lambda, l)$. It comprises the degree m of p, the maximal frequency $\lambda = \max_{k=0,\dots,m} |\operatorname{Im} \lambda_k|$, and the length $l = \mu_1(B)$.

Define the constant M_D (which we call a "frequency bound" for p) as $M_D = \lfloor d/2 \rfloor + 1$, where $d = d(m, \lambda, l)$ is the maximal number of solutions of $|p| = \rho$, $\rho \in \mathbb{R}$, on an interval of length l, for a complex exponential polynomial p of degree m and of maximal frequency λ.

For any bounded subset $\Omega \subset \mathbb{R}$ and for $\epsilon > 0$ let $M(\epsilon, \Omega)$ be the minimal number of ϵ-intervals covering Ω. Now the metric span ω_D is defined as follows:

Definition 1.2. The *metric span* $\omega_D(\Omega)$ of $\Omega \subset \mathbb{R}$ is given by

$$\omega_D(\Omega) = \sup_{\varepsilon > 0} \varepsilon[M(\varepsilon, \Omega) - M_D].$$

Now we can state our main result in the one-dimensional case:

Theorem 1.3. *Let $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ be an exponential polynomial, where $c_k, \lambda_k \in \mathbb{C}$. Let $B \subset \mathbb{R}$ be an interval, and let $\Omega \subset B$ be any set. Then*

$$\sup_B |p| \le e^{\mu_1(B) \cdot \max |\operatorname{Re} \lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\omega_D(\Omega)}\right)^m \cdot \sup_\Omega |p|$$

where $c > 0$ is an absolute constant.

Using Khovanskiĭ's [6] bound we can give a more explicit (although somewhat cumbersome) expression for d, and hence for M_D and ω_D. Let us put $\tilde{d} = \tilde{d}(m, \lambda, l) = C(m) l \lambda$. Here $C(m) = n(2n+1)^{2n} 2^{2n^2}$ for $n = (m+1)(m+2)/2 + 1$. Next we define $\tilde{M}_D = \lfloor \tilde{d}/2 \rfloor + 1$ and $\tilde{\omega}_D(\Omega) = \sup_{\varepsilon > 0} \varepsilon [M(\varepsilon, \Omega) - \tilde{M}_D]$. As we shall see below, always $d \le \tilde{d}$, and hence $\tilde{\omega}_D(\Omega) \le \omega_D(\Omega)$.

COROLLARY 1.4. *Under the conditions of Theorem* 1.3*,*

$$\sup_B |p| \le e^{\mu_1(B) \cdot \max |\mathrm{Re}\, \lambda_k|} \cdot \left(\frac{c \mu_1(B)}{\tilde{\omega}_D(\Omega)} \right)^m \cdot \sup_\Omega |p|.$$

REMARK 1.5. The same type of reasoning applies to any class of functions for which a Remez-type inequality and a uniform bound on the number of zeroes hold.

REMARK 1.6. For any *measurable* Ω we always have $\omega_D(\Omega) \ge \mu_1(\Omega)$, with equality if Ω is a sublevel set of p (see Section 2.1.1 below). Thus, Theorem 1.3 provides a true generalization and strengthening of the Turán–Nazarov inequality given in Theorem 1.1.

REMARK 1.7. We insist in Definition 1.2 above that ω_D depends only on the *imaginary parts* of the exponents λ_k, i.e. on the frequencies (and consequently we get a rather complicated bound in Corollary 1.4; compare Theorems 2.5, 2.6 below).

But this separation allows us to preserve and further develop a remarkable feature of the original Turán–Nazarov inequality: The bound does not depend on the frequencies, i.e. on the imaginary parts of λ_k in p. When we allow *discrete* (in particular, *finite*) sets Ω, this feature certainly cannot be completely preserved: Already for a trigonometric polynomial $p(t) = \sin(\lambda t)$, the set Ω of its zeroes (on which the Turán–Nazarov inequality certainly fails) consists of all the points $x_j = j\pi/\lambda$, $j \in \mathbb{N}$, and the number of such points in any interval B is of order $\mu(B)\lambda/\pi$.

However, Theorem 1.3 separates the roles of the real and imaginary parts of the exponents: The first enter the main bound, as in the original Turán–Nazarov inequality, while the second enter the definition of the span $\omega_D(\Omega)$. As the density of Ω grows, the influence of the frequencies decreases: see Section 2.1 below.

REMARK 1.8. Recently, promising applications of Theorem 1.3 have been found in signal processing, specifically, in non-uniform exponential sampling (see [10, 2, 1] and references therein).

There is a version of the Turán–Nazarov inequality for quasipolynomials in one or several variables due to A. Brudnyi [3, Theorem 1.7]. While less accurate than the original one (in particular, the role of real and complex parts of the exponents is not separated), this result gives important information for a wider class of quasipolynomials. In Section 3 we provide a strengthening of Brudnyi's result along the same lines as above: We replace the Lebesgue measure with an appropriate "metric span" which always bounds the Lebesgue measure from above and is strictly positive for sufficiently dense discrete (in particular, finite) sets.

2. Proofs and examples in dimension one. In this section we prove Theorem 1.3 and provide some of its consequences.

Proof of Theorem 1.3. Let $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ be an exponential polynomial with $c_k, \lambda_k \in \mathbb{C}$. Let $B \subset \mathbb{R}$ be an interval. We consider the sublevel set $V_\rho = \{t \in B : |p(t)| \le \rho\}$ of $p(t)$. By definition, $d = d(m, \lambda, \mu_1(B))$ is the maximal number of solutions of $|p| = \rho$, $\rho \in \mathbb{R}$, on the interval B. Hence the boundary of V_ρ consists of at most d points (including the endpoints). Therefore, the set V_ρ consists of at most $M_D = \lfloor d/2 \rfloor + 1$ subintervals Δ_i (i.e. connected components of V_ρ), with M_D defined as in Theorem 1.3. Let us cover each of these subintervals Δ_i by adjacent ε-intervals Q_ε starting from the left endpoint. Since all the adjacent ε-intervals, except possibly one, are inside Δ_i, their number does not exceed $|\Delta_i|/\varepsilon + 1$. Thus, we have

$$M(\varepsilon, V_\rho) \le (\lfloor d/2 \rfloor + 1) + \mu_1(V_\rho)/\varepsilon = M_D + \mu_1(V_\rho)/\varepsilon.$$

Now let a set $\Omega \subset B$ be given.

LEMMA 2.1. *If $\Omega \subset V_\rho$ for a certain $\rho \ge 0$ then $\mu_1(V_\rho) \ge \omega_D(\Omega)$.*

Proof. If $\Omega \subset V_\rho$ then for each $\varepsilon > 0$ we have $M(\varepsilon, \Omega) \le M(\varepsilon, V_\rho) \le M_D + \mu_1(V_\rho)/\varepsilon$, or $\mu_1(V_\rho) \ge \varepsilon[M(\varepsilon, \Omega) - M_D]$. Taking supremum with respect to $\varepsilon > 0$, via Definition 1.2 we conclude that $\mu_1(V_\rho) \ge \omega_D(\Omega)$. ∎

Let us now put $\hat\rho = \sup_\Omega |p|$. Then we have $\Omega \subset V_{\hat\rho}$. Applying Lemma 2.1 we get $\mu_1(V_{\hat\rho}) \ge \omega_D(\Omega)$. Finally, we apply the original Turán–Nazarov inequality (Theorem 1.1) to the subset $V_{\hat\rho} \subset B$, on which $|p|$ by definition does not exceed $\hat\rho$. This completes the proof of Theorem 1.3. ∎

Proof of Corollary 1.4. Let, as above, $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ be an exponential polynomial, where $c_k, \lambda_k \in \mathbb{C}$. Let us write $c_k = \gamma_k e^{i\phi_k}$, $\lambda_k = a_k + ib_k$, $k = 0, 1, \ldots, m$.

LEMMA 2.2.

$$|p(t)|^2 = 2 \sum_{0 \le k \le l \le m} \gamma_k \gamma_l e^{(a_k + a_l)t} \cos(\phi_k - \phi_l + (b_k - b_l)t)$$

is an exponential-trigonometric polynomial of degree $(m+1)(m+2)/2$ with real coefficients.

Proof. We have

$$p(t) = \sum_{k=0}^{m} \gamma_k e^{i\phi_k} e^{(a_k+ib_k)t} = \sum_{k=0}^{m} \gamma_k e^{a_k t + i(\phi_k + b_k t)},$$
$$\bar{p}(t) = \sum_{k=0}^{m} \gamma_k e^{a_k t - i(\phi_k + b_k t)}.$$

Therefore

$$|p(t)|^2 = p(t)\bar{p}(t) = \sum_{k,l=0}^{m} \gamma_k \gamma_l e^{(a_k+a_l)t + i(\phi_k - \phi_l + (b_k - b_l)t)}.$$

Adding the terms in this sum for the indices (k, l) and (l, k) we get

$$|p(t)|^2 = 2 \sum_{k \le l} \gamma_k \gamma_l e^{(a_k+a_l)t} \cos(\phi_k - \phi_l + (b_k - b_l)t). \ \blacksquare$$

The following lemma provides a bound on the number of real solutions of the equation $|p(t)| = \rho$. It is a direct consequence of Khovanskiĭ's bound (Theorem 3.4 and Lemma 3.5 in Section 3.1 below).

Lemma 2.3. *For $p(t)$ as above and for each positive $\eta > 0$, the number of non-degenerate solutions of the equation $|p(t)| = \eta$ in the interval $B \subset \mathbb{R}$ does not exceed*

$$\tilde{d} = C(m)\mu_1(B)\lambda$$

where $\lambda = \max|\mathrm{Im}\,\lambda_k|$ and

$$C(m) = n(2n+1)^{2n}2^{2n^2}, \qquad n = (m+1)(m+2)/2 + 1.$$

So we have $d \le \tilde{d}$, $M_d \le M_{\tilde{d}}$, $\omega_D(\Omega) \ge \tilde{\omega}_D(\Omega)$. This completes the proof of Corollary 1.4. ∎

We expect that the expression for $C(m)$ in Lemma 2.3 provided by the general result of Khovanskiĭ can be strongly improved in our specific case. Let us recall the following result of Nazarov [7, Lemma 4.2], which gives a much more realistic bound on the local distribution of zeroes of an exponential polynomial if the real parts of its exponents are relatively small:

Lemma 2.4. *Let $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ be an exponential polynomial with $c_k, \lambda_k \in \mathbb{C}$. Then the number of zeroes of $p(z)$ inside each disk of radius $r > 0$ does not exceed $4m + 7\hat{\lambda}r$, where $\hat{\lambda} = \max|\lambda_k|$.*

The reason we use the Khovanskiĭ bound in Theorem 1.3 is that it involves only the imaginary parts of the exponents λ_k. In contrast, the bound of Lemma 2.4 is in terms of $\hat{\lambda} = \max|\lambda_k|$ (as opposed to $\max|\mathrm{Im}\,\lambda_k|$). So for the real parts of the exponents of p large, the Khovanskiĭ bound may be better.

In order to apply Lemma 2.4 we notice that

$$|p(t)|^2 = p(t)\bar{p}(t) = \sum_{k,l=0}^{m} c_k \bar{c}_l e^{(\lambda_k + \bar{\lambda}_l)t}$$

is an exponential polynomial of degree at most m^2 with the maximal absolute value of the exponents not exceeding $2\hat{\lambda}$. Adding a constant adds at most one to the degree. We conclude that the number of real solutions of $|p(t)| = \eta$ inside the interval B does not exceed $d_1 = 4m^2 + 14\hat{\lambda}\mu_1(B)$. Now we define ω'_D putting $M'_D = \lfloor d_1/2 \rfloor + 1$ in Definition 1.2. Repeating word for word the proof of Theorem 1.3 above we obtain:

THEOREM 2.5. *For $p(t)$ as above,*

$$\sup_B |p| \le e^{\mu_1(B)\cdot \max |\mathrm{Re}\,\lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\omega'_D(\Omega)} \right)^m \cdot \sup_\Omega |p|.$$

For the case of a real exponential polynomial $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ with $c_k, \lambda_k \in \mathbb{R}$, we get an especially simple and sharp result. Notice that the number of zeroes of a real exponential polynomial is always bounded by its degree m (indeed, the "monomials" $e^{\lambda_k t}$ form a Chebyshev system on each real interval). Applying this fact in the same way as above we get

THEOREM 2.6. *For $p(t)$ a real exponential polynomial of degree m,*

$$\sup_B |p| \le e^{\mu_1(B)\cdot \max |\lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\omega''_D(\Omega)} \right)^m \cdot \sup_\Omega |p|$$

where $\omega''_D(\Omega) = \sup_{\varepsilon>0} \varepsilon[M(\varepsilon,\Omega) - m]$.

Notice that in this case the metric span $\omega''_D(\Omega)$ depends only on the degree m of p and the result is sharp: For any Ω consisting of at least $m+1$ points there is an inequality of the required form, while for any m points there is a real exponential polynomial $p(t)$ of degree m vanishing at exactly those points.

2.1. Some examples. In this section we give just a couple of examples illustrating the properties of the span ω_D, as well as the scope and possible applications of Theorem 1.3.

2.1.1. *$\omega_D(\Omega)$ versus $\mu_1(\Omega)$.* Let us recall that for a given interval B and for an exponential polynomial $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$, $c_k, \lambda_k \in \mathbb{C}$, its diagram $D = D(p,B) = (m,\lambda,l)$ comprises the degree m of p, the maximal frequency $\lambda = \max_{k=0,\dots,m} |\mathrm{Im}\,\lambda_k|$, and the length $l = \mu_1(B)$. Next, $d = d(m,\lambda,l)$ is the maximal number of solutions of $|p| = \rho$, $\rho \in \mathbb{R}$, on an interval of length l, $M_D = \lfloor d/2 \rfloor + 1$, and $\omega_D(\Omega) = \sup_{\varepsilon>0} \varepsilon[M(\varepsilon,\Omega) - M_D]$.

PROPOSITION 2.7. *For any measurable Ω we have $\omega_D(\Omega) \ge \mu_1(\Omega)$, with equality if $\Omega = V_\rho$ is a sublevel set of p.*

Proof. Indeed, for any $\varepsilon > 0$ we have $M(\varepsilon, \Omega) \geq \mu_1(\Omega)/\varepsilon$. Now substitute this into the expression for $\omega_D(\Omega)$ and let $\varepsilon \to 0$. We get $\omega_D(\Omega) \geq \mu_1(\Omega)$. In order to show the equality for $\Omega = V_\rho$ being a sublevel set of p, we shall prove a slightly more general statement: *Let $\Omega \subset B$ consist of s closed intervals. Then for $s \leq M_D$ we have $\omega_D(\Omega) = \mu_1(\Omega)$.* Indeed, let $\varepsilon > 0$ be given. We cover each of these subintervals Δ_i, $i = 1, \ldots, s$, of Ω by adjacent ε-intervals Q_ε starting from the left endpoint. Since all the adjacent ε-intervals, except possibly one, are inside Δ_i, their number does not exceed $|\Delta_i|/\varepsilon + 1$. Thus, $M(\varepsilon, \Omega) \leq s + \mu_1(\Omega)/\varepsilon$, and therefore

$$\varepsilon[M(\varepsilon, \Omega) - M_D] \leq \varepsilon[s + \mu_1(\Omega)/\varepsilon - M_D] \leq \mu_1(\Omega)$$

if $s \leq M_D$. Since this inequality holds for each $\varepsilon > 0$, we conclude that $\omega_D(\Omega) \leq \mu_1(\Omega)$. ∎

Remark 2.8. It looks plausible that the equality in Proposition 2.7 happens if *and only if* $\Omega = V_\rho$ is a sublevel set of p, i.e. it consists of s closed intervals, with $s \leq M_D$. Indeed, for one interval Δ_i if we take ε smaller than, but very close to, $|\Delta_i|/n$, then we have $M(\varepsilon, \Delta_i)$ very close to $|\Delta_i|/\varepsilon + 1$. For two intervals, if their lengths are commensurable, in exactly the same way we can find ε such that $M(\varepsilon, \Delta_i \cup \Delta_j)$ is very close to $(|\Delta_i| + |\Delta_j|)/\varepsilon + 2$. If the lengths are not commensurable, we still can get the same result, using the density of the integer multiples of an irrational angle on the unit circle. Presumably, this reasoning can be extended to any s, providing $\varepsilon > 0$ for which $M(\varepsilon, \Omega)$ is very close to $\mu_1(\Omega)/\varepsilon + s$. So if $s > M_D$, for this specific ε we get $\varepsilon[M(\varepsilon, \Omega) - M_D] \geq \varepsilon[\mu_1(\Omega)/\varepsilon + s - M_D] > \mu_1(\Omega)$. Hence $\omega_D(\Omega) > \mu_1(\Omega)$.

2.1.2. *Subsets Ω dense "in resolution ε".* Here we show that the role of the frequency bound in the results above decreases as the discrete subset $\Omega \subset B$ becomes denser. For $\Omega \subset B$ and for $\varepsilon > 0$ we define the "measure $\mu_1(\varepsilon, \Omega)$ of Ω in resolution ε" as the minimal possible measure of the coverings of Ω with ε-intervals.

Proposition 2.9. *For each diagram D and for any $\varepsilon > 0$ the metric span $\omega_D(\Omega)$ satisfies*

$$\omega_D(\Omega) \geq \mu_1(\varepsilon, \Omega)\left(1 - \frac{\varepsilon M_D}{\mu_1(\varepsilon, \Omega)}\right).$$

Proof. By the definition $\omega_D(\Omega) \geq \varepsilon[M(\varepsilon, \Omega) - M_D]$. Clearly, $M(\varepsilon, \Omega) \geq \varepsilon^{-1}\mu_1(\varepsilon, \Omega)$. Hence $\omega_D(\Omega) \geq \mu_1(\varepsilon, \Omega) - \varepsilon M_D$. ∎

So if in a small resolution ε, the measure $\mu := \mu_1(\varepsilon, \Omega)$ is not 0 then we restore the original Turán–Nazarov inequality for Ω, with a correction factor $1 - \varepsilon M_D/\mu$, where M_D is the frequency bound.

2.1.3. *Combining the discrete and positive measure cases.* Let a diagram D be fixed, and let $\Omega = \Omega_1 \cup \Omega_2 \subset B$, with Ω_1 a set of a positive measure μ, and Ω_2 a discrete set. We assume that the sets Ω_1 and Ω_2 are $2\mu_1(B)/M_D$-separated, where M_D is the frequency bound for D.

PROPOSITION 2.10. $\omega_D(\Omega) \geq \mu + \omega_D(\Omega_2)$.

Proof. By the definition $\omega_D(\Omega) = \sup_\varepsilon \varepsilon[M(\varepsilon, \Omega) - M_D]$, and this supremum is achieved for $\varepsilon \leq \mu_1(B)/M_D$. Indeed, otherwise $M(\varepsilon, \Omega) - M_D$ would be negative. Hence by the separation assumption we have $M(\varepsilon, \Omega) = M(\varepsilon, \Omega_1) + M(\varepsilon, \Omega_2)$ and so $\omega_D(\Omega) = \sup_\varepsilon \varepsilon(M(\varepsilon, \Omega_1) + M(\varepsilon, \Omega_2) - M_D) \geq \mu_1(\Omega_1) + \omega_D(\Omega_2)$. ∎

So in situations as above, Theorem 1.3 improves the original Turán–Nazarov inequality, and the frequency bound applies only to the discrete part of Ω.

2.1.4. *Interpolation with exponential polynomials.* This is a classical topic starting at least with [8] and actively studied today in connection with numerous applications. Theorems 1.3, 2.5, 2.6 connect the Turán–Nazarov inequality on $\Omega \subset B$ with estimates for the robustness of the interpolation from Ω to B. In particular, they provide robustness estimates in solving the "generalized Prony system" for non-uniform samples. See [10, 2, 1] for some initial results in this direction.

3. Multi-dimensional case.

In this section we consider the version of Turán–Nazarov inequality for quasipolynomials in one or several variables due to A. Brudnyi [3, Theorem 1.7]. We provide a strengthening of this result along the same lines as above: the Lebesgue measure is replaced with an appropriate "metric span". First, let us recall some definitions.

DEFINITION 3.1. Let $f_1, \ldots, f_k \in (\mathbb{C}^n)^*$ be a set of pairwise different complex linear functionals f_j which we identify with the scalar products $f_j \cdot z$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We shall write

$$f_j = a_j + ib_j.$$

A *quasipolynomial* is a finite sum

$$p(z) = \sum_{j=1}^{k} p_j(z) e^{f_j \cdot z}$$

where $p_j \in \mathbb{C}[z_1, \ldots, z_n]$ are polynomials in z of degrees d_j. The degree of p is $m = \deg p = \sum_{j=1}^{k}(d_j + 1)$. Following A. Brudnyi [3], we introduce the *exponential type* of p to be

$$t(p) = \max_{1 \leq j \leq k} \max_{z \in B_c(0,1)} |f_j \cdot z|$$

where $B_c(0,1)$ is the complex Euclidean ball of radius 1 centered at 0.

Below we consider $p(x)$ for the real variables $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

THEOREM 3.2 ([3]). *Let p be a quasipolynomial with parameters n, m, k defined on $\mathbb{C}^n$. Let $B \subset \mathbb{R}^n$ be a convex body, and let $\Omega \subset B$ be a measurable set. Then*

$$\sup_B |p| \le \left(\frac{cn\mu_n(B)}{\mu_n(\Omega)}\right)^{\ell} \cdot \sup_{\Omega} |p|$$

where $\ell = (c(m,k) + (m-1)\log(c_1 \max\{1, t(p)\}) + c_2 t(p) \operatorname{diam}(B))$, and c, c_1, c_2 are absolute positive constants, and $c(k,m)$ is a positive number depending only on m and k.

Generalizing this result of Brudnyi, we follow the arguments described in Sections 1 and 2 above, and [13].

3.1. Covering number of sublevel sets. For a relatively compact $A \subset \mathbb{R}^n$, the covering number $M(\varepsilon, A)$ is defined now as the minimal number of ε-cubes Q_ε covering A (they are translations of the standard ε-cubes $Q^n_\varepsilon := [0, \varepsilon]^n$).

LEMMA 3.3. *The function*

$$\begin{aligned} q(x) &:= |p(x)|^2 \\ &= \sum_{0\le i\le j\le k} e^{\langle a_i + a_j, x\rangle}[P_{i,j}(x)\sin\langle b_i - b_j, x\rangle + Q_{i,j}(x)\cos\langle b_i - b_j, x\rangle] \end{aligned}$$

is a real exponential trigonometric quasipolynomial with $P_{i,j}, Q_{i,j}$ real polynomials in x of degree $d_i + d_j$, and at most $\kappa := k(k+1)/2$ exponents, sine and cosine elements.

Proof. Repeat word for word the proof of Lemma 2.2 above. ∎

Clearly, all the partial derivatives $\partial q(x)/\partial x_j$ have exactly the same form. The following bound due to Khovanskiĭ gives an estimate of the number of solutions of a system of real exponential trigonometric quasipolynomials. More precisely, we have

THEOREM 3.4 (Khovanskiĭ bound [6, Section 1.4]). *Let $P_1 = \cdots = P_n = 0$ be a system of n equations with n real unknowns $x = x_1, \ldots, x_n$, where P_i is a polynomial of degree m_i in $n + k + 2p$ real variables x, $y_1, \ldots, y_k$, $u_1, \ldots, u_p$, $v_1, \ldots, v_p$, with $y_i = \exp\langle a_j, x\rangle$, $j = 1, \ldots, k$ and $u_q = \sin\langle b_q, x\rangle$, $v_q = \cos\langle b_q, x\rangle$, $q = 1, \ldots, p$. Then the number of non-degenerate solutions of this system in the region bounded by the inequalities $|\langle b_q, x\rangle| < \pi/2$, $q = 1, \ldots, p$, is finite and less than*

$$m_1 \cdots m_n \Big(\sum m_i + p + 1\Big)^{p+k} 2^{p+(p+k)(p+k-1)/2}.$$

Let us denote the vectors $b_i - b_j \in \mathbb{R}^n$ by $b_{i,j}$ and let $\lambda := \max \|b_{i,j}\|$ be the maximal frequency in q. The next lemma is a simple consequence of the Khovanskiĭ bound:

Lemma 3.5. *Let V be a parallel translation of the coordinate subspace in $\mathbb{R}^n$ generated by $x_{j_1},\dots,x_{j_s}$. Then the number of non-degenerate real solutions in $V\cap Q^n_\rho$ of the system*

$$\frac{\partial q(x)}{\partial x_{j_1}}=\dots=\frac{\partial q(x)}{\partial x_{j_s}}=0$$

is at most $\hat{C}_s\lambda^s$, where

$$\hat{C}_s=\Big(\frac{2}{\pi}\sqrt{s}\,\rho\Big)^s\prod_{r=1}^{s}(d_{j_r}+d_{i_r})\Big(\sum_{r=1}^{s}d_{j_r}+d_{i_r}+2\kappa+1\Big)^{2\kappa}2^{\kappa+(2\kappa)(2\kappa-1)/2}.$$

Proof. The following geometric construction is required by the Khovanskiĭ bound: Let $Q_{i,j}=\{x\in\mathbb{R}^n:|\langle b_{i,j},x\rangle|\le\pi/2\}$ and let $Q=\bigcap_{0\le i\le j\le k}Q_{i,j}$. For any $B\subset\mathbb{R}^n$ we define $M(B)$ as the minimal number of translations of Q covering B. For an affine subspace V of $\mathbb{R}^n$ we define $M(B\cap V)$ as the minimal number of translations of $Q\cap V$ covering $B\cap V$. Notice that for $B=Q^n_r$, a cube of size r, we have $M(Q^n_r)\le((2/\pi)\sqrt{n}\,r\lambda)^n$. Indeed, Q always contains a ball of radius $\pi/(2\lambda)$. Now, applying the Khovanskiĭ bound of Theorem 3.4 to the system

$$\frac{\partial q(x)}{\partial x_{j_1}}=\dots=\frac{\partial q(x)}{\partial x_{j_s}}=0$$

we find that the number of non-degenerate real solutions in $V\cap Q^n_\rho$ is at most

$$\Big(\frac{2}{\pi}\sqrt{s}\,\rho\lambda\Big)^s\prod_{r=1}^{s}(d_{j_r}+d_{i_r})\Big(\sum_{r=1}^{s}d_{j_r}+d_{i_r}+2\kappa+1\Big)^{2\kappa}2^{\kappa+(2\kappa)(2\kappa-1)/2}.\ \blacksquare$$

Let a quasipolynomial p be as above. A sublevel set $A=A_\rho$ of p is defined as $A=\{x\in\mathbb{R}^n:|p(x)|\le\rho\}$. The following lemma extends, to the case of sublevel sets of exponential polynomials, the result of Vitushkin [12] for semialgebraic sets. It can be proved using a general result of Vitushkin in [12] through the use of "multi-dimensional variations". However, in our specific case the proof below is much shorter and it produces explicit ("in one step") constants.

Lemma 3.6. *For any $1\ge\varepsilon>0$ we have*

$$M(\varepsilon,A\cap Q^n_1)\le C_0+C_1\Big(\frac{1}{\varepsilon}\Big)+\dots+C_{n-1}\Big(\frac{1}{\varepsilon}\Big)^{n-1}+\mu_n(A)\Big(\frac{1}{\varepsilon}\Big)^n$$

where $C_0,\dots,C_{n-1}$ are positive constants, which depend only on k, d_i and the maximal frequency λ of the quasipolynomial p.

Proof. The sublevel set A_ρ is defined via the real exponential trigonometric quasipolynomial $q(x)=|p(x)|^2$, i.e. $A=A_\rho(p)=\{x\in Q^n_1:q(x)\le\rho^2\}$.

Let us subdivide Q_1^n into adjacent ε-cubes Q_ε with respect to the standard Cartesian coordinate system. Each Q_ε having non-empty intersection with A is either entirely contained in A, or intersects the boundary ∂A of A. Certainly, the number of those boxes Q_ε which are entirely contained in A is bounded by $\mu_n(A)/\mu_n(Q_\varepsilon) = \mu_n(A)/\varepsilon^n$. In the other case, where Q_ε intersects ∂A, it means that there exist faces of Q_ε that have non-empty intersection with ∂A.

Among all these faces, let us take the one with the smallest dimension s. In other words, there exists an s-face F of the smallest dimension s that intersects ∂A, for some $s = 0, 1, \ldots, n$. Let us fix an s-dimensional affine subspace V which corresponds to F. Then F contains completely some of the connected components of $A \cap V$, otherwise ∂A would intersect a face of Q_ε of dimension strictly less than s. Clearly, inside each compact connected component of $A \cap V$ there is a critical point of q, which is defined by the system of equations

$$\frac{\partial q(x)}{\partial x_{j_1}} = \cdots = \frac{\partial q(x)}{\partial x_{j_s}} = 0$$

(assuming that V is a translation of the coordinate subspace in $\mathbb{R}^n$ generated by $x_{j_1}, \ldots, x_{j_s}$). After a small perturbation of q we can always assume that all such critical points are non-degenerate. Hence by Lemma 3.5 the number of these points, and therefore of the boxes Q_ε of the type considered, is bounded by $\hat{C}_s \lambda^s$.

According to the partitioning construction of Q_1^n, we have at most $(1/\varepsilon + 1)^{n-s}$ s-dimensional affine subspaces with respect to the same s coordinates. On the other hand, the number of different choices of s coordinates is $\binom{n}{s}$. This means that the number of boxes that have an s-face F which contains completely some connected component of $A \cap V$ is at most $\binom{n}{s}(1/\varepsilon + 1)^{n-s}\hat{C}_s\lambda^s$, which does not exceed, assuming $\varepsilon \le 1$, the constant $C_{n-s} := \binom{n}{s}2^{n-s}\hat{C}_s\lambda^s(1/\varepsilon)^{n-s}$. Note that C_0 is the bound on the number of boxes that contain completely some of the connected components of A. Thus, we have

$$M(\varepsilon, A) \le C_0 + C_1\left(\frac{1}{\varepsilon}\right) + \cdots + C_{n-1}\left(\frac{1}{\varepsilon}\right)^{n-1} + \mu_n(A)\left(\frac{1}{\varepsilon}\right)^n. \ \blacksquare$$

4. Metric span and generalized Brudnyi inequality. Let p be a quasipolynomial as above, with parameters n, k, d_j. These parameters, together with the maximal frequency λ of p, form the multi-dimensional diagram D of p. Notice that in contrast to the one-dimensional case (and to Theorem 3.2) we restrict ourselves to the unit box Q_1^n. So B does not appear in the diagram.

For a given $0 < \varepsilon \le 1$ let us denote by $M_D(\varepsilon)$ the quantity $M_D(\varepsilon) = \sum_{j=0}^{n-1} C_j(1/\varepsilon)^j$, where $C_0, \dots, C_{n-1}$ are the constants from Lemma 3.6. Extending the terminology from the one-dimensional case above, we call $M_D(\varepsilon)$ the "frequency bound" for D. Note that the constants C_j depend only on the parameters n, k, d_i and on the maximal frequency λ of the quasipolynomial p. By Lemma 3.6 for any sublevel set A_ρ of p we have

$$M(\varepsilon, A) \le M_D(\varepsilon) + \mu_n(A)(1/\varepsilon)^n.$$

Now for any subset $\Omega \subset Q_1^n$ we introduce the metric span ω_D of Ω with respect to a given diagram D as follows:

Definition 4.1. For a subset $\Omega \subset \mathbb{R}^n$ the *metric span* ω_D is defined as

$$\omega_D(\Omega) = \sup_{\varepsilon > 0} \varepsilon^n [M(\varepsilon, \Omega) - M_D(\varepsilon)].$$

Lemma 4.2. *Let $A \subset Q_1^n$ be a sublevel set of a real quasipolynomial with diagram D. Then for any $\Omega \subset A$ we have*

$$\mu_n(A) \ge \omega_D(\Omega).$$

Proof. This fact follows directly from Lemma 3.6. Indeed, for any $\varepsilon > 0$ we have

$$M(\varepsilon, \Omega) \le M(\varepsilon, A) \le M_D(\varepsilon) + \mu_n(A)\left(\frac{1}{\varepsilon}\right)^n.$$

Consequently, for any $\varepsilon > 0$ we see that $\mu_n(A) \ge \varepsilon^n[M(\varepsilon, \Omega) - M_D(\varepsilon)]$. Now, we can take the supremum with respect to ε. ∎

For some examples and properties of sets in $\mathbb{R}^n$ with positive metric span, see [13, Section 5]. Here we only mention that for a measurable $\Omega \subset \mathbb{R}^n$ we always have $\omega_D(\Omega) \ge \mu_n(\Omega)$. The proof is exactly the same as in the remark after Theorem 1.3. Now we can prove our generalization of Brudnyi's Theorem 3.2 above.

Theorem 4.3. *Let p be as above and let $\Omega \subset Q_1^n$. Then*

$$\sup_{Q_1^n} |p| \le \left(\frac{cn\mu_n(B)}{\omega_D(\Omega)}\right)^\ell \cdot \sup_{\Omega} |p|.$$

Proof. Let $\hat\rho := \sup_\Omega |p|$. For the sublevel set $A_{\hat\rho}$ of the quasipolynomial p we find that $\Omega \subset A_{\hat\rho}$. By Lemma 4.2 we have $\mu_n(A_{\hat\rho}) \ge \omega_D(\Omega)$. Now since p is bounded in absolute value by $\hat\rho$ on $A_{\hat\rho}$ by definition, we can apply Theorem 3.2 with $B = Q_1^n$ and $A_{\hat\rho}$. ∎

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