STUDIA MATHEMATICA 218 (1) (2013)

Spaces of Whitney jets on self-similar sets

by

Dietmar Vogt (Wuppertal)

Abstract. It is shown that complemented subspaces of s, that is, nuclear Fréchet spaces with properties (DN) and (Ω), which are 'almost normwise isomorphic' to a multiple direct sum of copies of themselves are isomorphic to s. This is applied, for instance, to spaces of Whitney jets on the Cantor set or the Sierpiński triangle and gives new results and also sheds new light on known results.

1. Introduction. In [8] it was shown that complemented subspaces of s which are normwise stable are isomorphic to s. This was applied to the space of Whitney jets on the Cantor set. The present note extends this result to a more general situation, such that it can be applied to the space of Whitney jets on self-similar but connected sets like the Sierpiński triangle, which gives a new result, and also to spaces of C^∞-functions on intervals or on $\mathbb{R}$, which sheds a new light on well-known results.

In this note s will denote the space of rapidly decreasing sequences, that is,

$$s = \Big\{x = (x_0, x_1, \dots) : |x|_k := \sum_n |x_n|(n+1)^k < \infty \text{ for all } k \in \mathbb{N}\Big\}.$$

Equipped with the norms $|x|_k$, it is a nuclear Fréchet space. It is isomorphic to many of the Fréchet spaces which occur in analysis, in particular, spaces of C^∞-functions.

s is an example of a power series space of infinite type. We recall the general definition: for $\alpha : 0 \le \alpha_0 \le \alpha_1 \le\nearrow +\infty$ the set

$$\Lambda_\infty(\alpha) := \Big\{x = (x_0, x_1, \dots) : |x|_t = \sum_{n=0}^{\infty} |x_n| e^{t\alpha_n} < \infty \text{ for all } t > 0\Big\},$$

equipped with the norms $|\cdot|_k$, $k \in \mathbb{N}_0$, is a Fréchet space. It is nuclear if, and only if, $\limsup_n (\log n)/\alpha_n < \infty$. We have $s = \Lambda_\infty(\alpha)$ with $\alpha_n = \log(n+1)$.

2010 *Mathematics Subject Classification*: 46E10, 46A04, 46A63.
Key words and phrases: Whitney jets, space s, Cantor set, Sierpiński triangle.

DOI: 10.4064/sm218-1-5

A Fréchet space E with the fundamental system of seminorms $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \cdots$ has *property* (DN) if

$$\exists p\ \forall k\ \exists K, C > 0 :\ \|\cdot\|_k^2 \leq C\|\cdot\|_p\,\|\cdot\|_K.$$

In this case $\|\cdot\|_p$ is called a *dominating norm.*

E has *property* (Ω) if

$$\forall p\ \exists k\ \forall m\ \exists 0 < \theta < 1, C > 0 :\ \|\cdot\|_k^* \leq C\|\cdot\|_p^{*\theta}\,\|\cdot\|_m^{*\,1-\theta}.$$

Here for any continuous seminorm $\|\cdot\|$ and $y \in E'$ we consider the dual, extended real valued, norm $\|y\|^* = \sup\{|y(x)| : x \in E,\ \|x\| \leq 1\}$.

By Vogt–Wagner [9], a Fréchet space E is isomorphic to a complemented subspace of s if, and only if, it is nuclear and has properties (DN) and (Ω).

It is a longstanding unsolved problem of the structure theory of nuclear Fréchet spaces, going back to Mityagin, whether every complemented subspace of s has a basis. If it does then it is isomorphic to some power series space $\Lambda_\infty(\alpha)$; the latter can be calculated in advance by a method going back to Terzioğlu [5] which we describe now.

Let X be a vector space and let $A \subset B$ be absolutely convex subsets of X. We set

$$\delta_n(A,B) = \inf\{\delta > 0 : \text{there exists a subspace } F \subset X \text{ with } \dim F \leq n \text{ and } A \subset \delta B + F\},$$

and call it the nth *Kolmogorov diameter* of A with respect to B.

If now E is a complemented subspace of s, that is, E is nuclear and has properties (DN) and (Ω), then we choose p such that $\|\cdot\|_p$ is a dominating norm and for p we choose $k > p$ according to property (Ω). We set

$$\alpha_n = -\log \delta_n(U_k, U_p),$$

where $U_k = \{x \in E : \|x\|_k \leq 1\}$. The space $\Lambda_\infty(\alpha)$ is called the *associated power series space*, and $E \cong \Lambda_\infty(\alpha)$ if E has a basis. If $\limsup_n \alpha_{2n}/\alpha_n < \infty$ then, by Aytuna–Krone–Terzioğlu [2, Theorem 2.2], $E \cong \Lambda_\infty(\alpha)$.

Instead of the Kolmogorov diameters we will use the approximation numbers of connecting maps between the respective local Banach spaces and estimate them against the Kolmogorov diameters.

We will consider Fréchet spaces with fixed fundamental systems of seminorms. An exact sequence $0 \to E \to H \to G \to 0$ of such spaces is called *normwise exact* if for every k it induces an exact sequence $0 \to E_k \to H_k \to G_k \to 0$ of local Banach spaces. We set $\omega := \mathbb{C}^{\mathbb{N}}$. A fundamental system of seminorms on ω will be assumed to have been suitably chosen, depending on the situation.

For all these concepts and further results of the structure theory of infinite type power series spaces see [7], and for results and unexplained notation of general functional analysis see [4].

2. Calculation of approximation numbers. Let

$$0 \to E \to H \to G \to 0$$

be a normwise exact sequence of Fréchet spaces with $G = \{0\}$ or $G = \omega$. Then for every $k > p$ we have a commutative diagram of Banach spaces with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_k & \xrightarrow{i_k} & H_k & \xrightarrow{q_k} & G_k & \longrightarrow & 0 \\ & & \Big\downarrow j_E & & \Big\downarrow J & & \Big\downarrow j_G & & \\ 0 & \longrightarrow & E_p & \xrightarrow{i_p} & H_p & \xrightarrow{q_p} & G_p & \longrightarrow & 0 \end{array}$$

Here the spaces are the respective local Banach spaces, and the vertical arrows denote the canonical connecting maps. By assumption, the spaces G_k and G_p are finite-dimensional. Therefore the rows split. We denote by P a projection in H_k on the (finite-dimensional) range of some right inverse of q_k, and we set $Q = \mathrm{id} - P$. Then Q is a projection on the range of i_k. The inverse $i_k^{-1} : R(i_k) \to E_k$ is continuous. We obtain

$$J = i_p \circ j_E \circ i_k^{-1} \circ Q + J \circ P.$$

We set $m := \dim R(J \circ P)$. Then for the approximation numbers we obtain

$$a_{n+m}(J) \le a_n(J - J \circ P) = a_n(i_p \circ j_E \circ i_k^{-1} \circ Q) \le C a_n(j_E)$$

for all n with a suitable constant $C > 0$.

We now assume that H is the direct sum of d copies of E, $d \ge 2$, with $\|x_1 \oplus \cdots \oplus x_d\|_k = \sum_{j=1}^d \|x_j\|_k$ for all k. Then $J = j_E \oplus \cdots \oplus j_E$ and therefore $a_{dn}(J) = d a_n(j_E)$. This implies

$$a_{n+m}(j_E) \le a_{d(n+m)}(J) \le a_{dn+m}(J) \le C a_{dn}(j_E) \le C a_{2n}(j_E)$$

for all n.

We set $a_n := a_n(j_E)$ and extend it, by linear interpolation, to a decreasing function a_t on $[0, \infty)$. For $n \le t \le n+1$ we obtain

$$a_{t+m+1} \le a_{n+m+1} \le C a_{2n+2} \le C a_{2t}.$$

For $t \ge 2(m+1)$ we have $a_{3t/2} \le C a_{2t}$ and, therefore, for $t \ge 3(m+1)$ we get $a_t \le C a_{4t/3}$. With another constant $D > 0$ we have

$$a_t \le D a_{4t/3}, \qquad t \ge 1.$$

For $t = (4/3)^{n-1}$ we obtain $a_{(4/3)^{n-1}} \le D a_{(4/3)^n}$ and, by induction, $a_1 \le D^n a_{(4/3)^n}$. This implies $-\log a_{(4/3)^n} \le n \log D - \log a_1$. For $(4/3)^n \le t \le (4/3)^{n+1}$ we have

$$-\log a_t \le -\log a_{(4/3)^{n+1}} \le (n+1)\log D - \log a_1 \le \frac{\log D}{\log \frac{4}{3}} \log t + \log D - \log a_1.$$

We have shown

Lemma 2.1. *Under the above assumptions there are constants C_1 and C_2 such that*

$$-\log a_n(j_E) \le C_1 \log n + C_2 \quad \text{for all } n \in \mathbb{N}.$$

If X and Y are Banach spaces and $X \subset Y$ with continuous imbedding j, then for the unit balls U_X and U_Y, and all n, we have

$$\delta_n(U_X, U_Y) \le a_n(j) \le (n+1)\delta_n(U_X, U_Y).$$

Thus we have shown

Corollary 2.2. *Under the above assumptions there are constants C_1 and C_2 such that $-\log \delta_n(U_k, U_p) \le C_1 \log(n+1) + C_2$ for all $n \in \mathbb{N}_0$.*

3. Main technical result. Let E fulfill the assumptions of the previous section. Moreover, we assume that E is isomorphic to a complemented subspace of s, that is, E has properties (DN) and (Ω). Let $\| \,\|_p$ be a dominating norm, and $\| \,\|_k$ a norm chosen for $\| \,\|_p$ according to (Ω). Then there are constants C_1 and C_2 such that

$$\alpha_n := -\log \delta_n(U_k, U_p) \le C_1 \log(n+1) + C_2$$

for all $n \in \mathbb{N}_0$. Since E is isomorphic to a subspace of s, which implies the left inequality below, we have shown that there are (possibly other) constants $C_1 > 0$ and C_2 such that

$$\frac{1}{C_1} \log(n+1) - C_2 \le \alpha_n \le C_1 \log(n+1) + C_2$$

for all $n \in \mathbb{N}_0$.

The space $\Lambda_\infty(\alpha)$ with $\alpha = (\alpha_0, \alpha_1, \ldots)$ defined as above is the associated power series space of the space E and we have shown that $\Lambda_\infty(\alpha) = s$. Since $\log(2n+1) \le \log 2 + \log(n+1)$, hence $\limsup_n \log(2n+1)/\log(n+1) = 1$, by the theorem of Aytuna–Krone–Terzioğlu (see introduction) we obtain

Theorem 3.1. *If E is isomorphic to a complemented subspace of s and if there exists a normwise exact sequence*

$$0 \to E \to E \oplus \cdots \oplus E \to G \to 0,$$

where the middle space has $d \ge 2$ direct summands and $G = \{0\}$ or $G \cong \omega$, then $E \cong s$.

4. Application. In a first remark we want to point out that Theorem 3.1 gives a structural reason why spaces like $C^\infty(I)$, I a compact interval in $\mathbb{R}$, are necessarily isomorphic to s. This follows alone from properties (DN), (Ω) and nuclearity. All $C^\infty(I)$ are normwise isomorphic. For the proof we may assume that $I = [-1, 1]$ and define $q : C^\infty([-1, 0]) \oplus C^\infty([0, 1])) \to \omega$ by $q(f \oplus g) = (f^{(p)}(0) - g^{(p)}(0))_{p \in \mathbb{N}_0}$.

Now, we will apply Theorem 3.1 in two cases. First, let K be the classical ternary Cantor set and consider the space $\mathcal{E}(K)$ of Whitney jets on K. We set $J(K) := \{f \in C^\infty(\mathbb{R}) : f|_K = 0\} = \{f \in C^\infty(\mathbb{R}) : f^{(p)}|_K = 0 \text{ for all } p\}$. The second equality holds, since K is perfect. Then $\mathcal{E}(K) = C^\infty(\mathbb{R})/J(K)$, and this implies that $\mathcal{E}(K)$ is a nuclear Fréchet space with property (Ω). By a theorem of Tidten [6] it also has property (DN). Therefore it is isomorphic to a complemented subspace of s (see [9]).

By obvious identifications we have

$$\mathcal{E}(K) \cong \mathcal{E}(K \cap [0, 1/3]) \oplus \mathcal{E}(K \cap [2/3, 1]) \cong \mathcal{E}(K) \oplus \mathcal{E}(K)$$

with normwise isomorphy, that is, the assumptions of Theorem 3.1 are fulfilled with $G = \{0\}$. Thus we have shown

THEOREM 4.1. *If K is the classical Cantor set, then $\mathcal{E}(K) \cong s$.*

This result has also been shown in [8]. In [1] it has been shown that the diametral dimensions of $\mathcal{E}(K)$ and s coincide, from which, by means of the Aytuna–Krone–Terzioğlu theorem, one can derive the same result.

The second case will be the Sierpiński triangle. For its construction we start with a closed equilateral triangle, for instance, given by the points $P_1 = (0, 0)$, $P_2 = (2, 0)$, $P_3 = (1, \sqrt{3})$ in $\mathbb{R}^2$. In a first step we remove from it the open equilateral triangle with vertices $P_4 = (1/2, \sqrt{3}/2)$, $P_5 = (3/2, \sqrt{3}/2)$, $P_6 = (1, 0)$. For the remaining three triangles we repeat the procedure, etc. We obtain a compact set S, called the Sierpiński triangle. The subsets S_1, S_2, S_3 of S given as S intersected with the triangle with vertices P_1, P_4, P_6, or P_3, P_4, P_5, or P_2, P_5, P_6, respectively, are copies of S scaled by the factor $1/2$. We obtain a normwise exact sequence

$$0 \to \mathcal{E}(S) \to \mathcal{E}(S_1) \oplus \mathcal{E}(S_2) \oplus \mathcal{E}(S_3) \xrightarrow{q} G \to 0,$$

where $G = (\mathbb{R}^3)^{\mathbb{N}_0^2} \cong \omega$ and

$$q(f_1 \oplus f_2 \oplus f_3) = \big(f_1^{(\alpha)}(P_4) - f_2^{(\alpha)}(P_4), f_2^{(\alpha)}(P_5) - f_3^{(\alpha)}(P_5), f_3^{(\alpha)}(P_6) - f_1^{(\alpha)}(P_6)\big)_{\alpha \in \mathbb{N}_0^2}.$$

Since $\mathcal{E}(S_j) \cong \mathcal{E}(S)$ for $j = 1, 2, 3$ with normwise isomorphy, one of the assumptions of Theorem 3.1 is fulfilled. In Frerick–Jordá–Wengenroth [3] it is shown that $\mathcal{E}(S)$ admits a continuous linear extension operator (even without loss of derivatives) $\mathcal{E}(S) \to C^\infty(L)$, where L denotes a large rectangle in $\mathbb{R}^2$. This follows from the Main Theorem there, together with direct verification of the conditions (see also [3, Introduction, p. 4]). Therefore $\mathcal{E}(S)$ is isomorphic to a complemented subspace of s. We have shown:

THEOREM 4.2. *If S denotes the Sierpiński triangle, then $\mathcal{E}(S) \cong s$.*

References

[1] B. Arslan, A. P. Goncharov and M. Kocatepe, *Spaces of Whitney functions on Cantor-type sets*, Canad. J. Math. 54 (2002), 225–238.

[2] A. Aytuna, J. Krone and T. Terzioğlu, *Complemented infinite type power series subspaces of nuclear Fréchet spaces*, Math. Ann. 283 (1989), 193–202.

[3] L. Frerick, E. Jordá and J. Wengenroth, *Whitney extension operators without loss of derivatives*, arXiv:1304.2928v1, 2013.

[4] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.

[5] T. Terzioğlu, *On the diametral dimension of some classes of F-spaces*, J. Karadeniz Uni. Ser. Math.-Phys. 8 (1985), 1–13.

[6] M. Tidten, *Fortsetzungen von C^∞-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind*, Manuscripta Math. 27 (1979), 291–312.

[7] D. Vogt, *Structure theory of power series spaces of infinite type*, Rev. R. Acad. Cien. Ser. A. Mat. 97 (2003), no. 2, 1–25.

[8] D. Vogt, *A characteristic property of the space s*, Proc. Amer. Math. Soc., to appear.

[9] D. Vogt und M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. 68 (1980), 225–240.

Dietmar Vogt
Department of Mathematics
Bergische Universität Wuppertal
Gauss-Str. 20
42119 Wuppertal, Germany
E-mail: dvogt@math.uni-wuppertal.de

Received June 19, 2013
Revised version July 24, 2013 (7801)