STUDIA MATHEMATICA 163 (2) (2004)

# The diagonal mapping in mixed norm spaces

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**Abstract.** For any holomorphic function $F$ in the unit polydisc $U^n$ of $\mathbb{C}^n$, we consider its restriction to the diagonal, i.e., the function in the unit disc $U$ of $\mathbb{C}$ defined by $\mathcal{D}F(z) = F(z, \ldots, z)$, and prove that the diagonal mapping $\mathcal{D}$ maps the mixed norm space $H^{p,q,\alpha}(U^n)$ of the polydisc onto the mixed norm space $H^{p,q,|\alpha|+(p/q+1)(n-1)}(U)$ of the unit disc for any $0 < p < \infty$ and $0 < q \le \infty$.

**1. Introduction.** Let $U^n$ be the polydisc in $\mathbb{C}^n$ and $T^n$ be its Shilov boundary (see [Ru1]). Denote by $dm_n$ the normalized volume measure in $U^n$, and by $d\sigma_n$ the normalized surface measure on $T^n$. For any Lebesgue measurable function $f$ in $U^n$, we define

$$M_q(r, f) = \Big( \int_{T^n} |f(r\zeta)|^q \, d\sigma_n(\zeta) \Big)^{1/q}, \tag{1.1}$$

where $0 < q < \infty$ and $r\zeta = (r_1\zeta_1, \ldots, r_n\zeta_n)$. When $q = \infty$, as usual, we define $M_\infty(r, f)$ to be the essential supremum of $|f(r\zeta)|$ over $\zeta \in T^n$. If $0 < p < \infty$, $0 < q \le \infty$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > -1$, $\forall j = 1, \ldots, n$, let

$$\|f\|^p_{p,q,\alpha} = \int_{I^n} \prod_{j=1}^n (1 - r_j^2)^{\alpha_j} M_q^p(r, f) \, dr, \tag{1.2}$$

where $I^n = [0, 1)^n$ and $dr = dr_1 \cdots dr_n$. The *mixed norm space* $L^{p,q,\alpha}(U^n)$ is then defined to be the space of functions $f$ in $U^n$ such that $\|f\|_{p,q,\alpha} < \infty$, and the *holomorphic mixed norm space* $H^{p,q,\alpha}(U^n)$ is its subspace consisting of holomorphic functions. The mixed norm spaces have been studied extensively; see, for example, [BP], [AJ], [J], [L], [Sh2], [Pa] and [SR].

The main purpose of this article is to consider the action of the diagonal mapping on mixed norm spaces on $U^n$.

2000 *Mathematics Subject Classification*: 32A37, 47B38.

*Key words and phrases*: diagonal mapping, mixed norm spaces, weighted Bergman projection.

The first author was supported in part by the NNSF of China (#10001030) and a Post-doctoral Fellowship of the University of Aveiro, UI&D "Matemática e Aplicações".

To each holomorphic function $F$ in the unit polydisc $U^n$ of $\mathbb{C}^n$, we associate a function $\mathcal{D}F$, defined on the unit disc $U$ of $\mathbb{C}$ by

$$\mathcal{D}F(z) = F(z, \ldots, z). \tag{1.3}$$

The operator $\mathcal{D}$ is called the *diagonal mapping*. In his book [Ru1], Rudin suggested the study of this mapping. Afterwards, the diagonal mapping has been completely investigated in the Hardy spaces and Bergman spaces; see [Ru1], [HO], [DS], [Sha], [MR], [Sh], [Djs], and [RL]. For instance, Shapiro [Sha] and Shamoian [Sh] proved that

$$\mathcal{D}H^{p,p,\alpha}(U^n) = H^{p,p,|\alpha|+2n-2}(U) \tag{1.4}$$

for any $0 < p < \infty$ and $\alpha_j > -1$, $\forall j = 1, \ldots, n$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

In view of (1.4), the interesting phenomenon in weighted Bergman spaces is that for any given weight $\alpha$, the resulting weight $|\alpha|+2n-2$ is independent of $p$. But this fails in mixed norm spaces, i.e.,

$$\mathcal{D}H^{p,q,\alpha}(U^n) \neq H^{p,q,|\alpha|+2n-2}(U).$$

In fact, taking $f(z_1, z_2) = (1-z_1)^{-\beta_1}(1-z_2)^{-\beta_2}$ with $\beta_i = (1+\alpha_i)/2+4/5$ for $i = 1, 2$, one can easily verify that $f \in H^{2,1,\alpha}(U^2)$ but $\mathcal{D}f \notin H^{2,1,|\alpha|+2}(U)$.

Our main result is the following theorem.

THEOREM 1.1. *Let $0 < p < \infty$, $0 < q \leq \infty$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > -1$, $\forall j = 1, \ldots, n$. Then*

$$\mathcal{D}H^{p,q,\alpha}(U^n) = H^{p,q,|\alpha|+(p/q+1)(n-1)}(U). \tag{1.5}$$

Theorem 1.1 also shows that, by the closed graph theorem, the composition operator $C_\Phi$ defined by

$$C_\Phi F = F \circ \Phi,$$

where $\Phi(z) = (z, \ldots, z)$ for any $z \in U$, is bounded from $H^{p,q,\alpha}(U^n)$ onto $H^{p,q,|\alpha|+(p/q+1)(n-1)}(U)$. For the theory of composition operators, we refer to [CM].

The paper is organized as follows. In the next section, we provide an integral representation for the diagonal mapping, given by the diagonalization of weighted Bergman operators of $U^n$. Similarly, we provide an integral representation for a right inverse operator of the diagonal mapping, given by the polarization of weighted Bergman operators of $U$. In the third section, we extend Hardy's inequalities ([HL], [Fl], [AB]) to higher dimensions, which is a key tool to proving the boundedness of integral operators in mixed norm spaces. In the fourth section, we show that the weighted Bergman projection $\mathcal{T}_\beta$, which is the orthogonal projection from $L^2(U^n, \prod_{j=1}^n (1-|u_j|^2)^{\beta_j} dm_n(u))$ onto $H^{2,2,\beta}(U^n)$, induces a bounded operator from $L^{p,q,\alpha}(U^n)$ to $H^{p,q,\alpha}(U^n)$.

**2. Diagonalization and polarization.** Set $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta, \dots, \beta_n)$.

HYPOTHESIS 2.1. *Throughout the paper, we assume that*

(i) $0 < p < \infty$ *and* $0 < q \le \infty$;
(ii) $\alpha_j > -1, \quad \forall j = 1, \dots, n$;
(iii) $\beta_j > \dfrac{\alpha_j + 1}{p} + \dfrac{1}{\min(q,1)} - 2, \quad \forall j = 1, \dots, n.$

By Hypothesis 2.1, it is clear that always $\beta_j > -1$. We remark that Hypothesis 2.1(iii) is required only to assure the validity of technical lemmas in our applications, i.e., Lemmas 3.2, 3.3 and 4.2 below.

For any $u \in U^n$ and $r \in I^n$, we write $u = (u_1, \dots, u_n)$ and $r = (r_1, \dots, r_n)$. We shall often use the *polar coordinates formula*:

$$\int_{U^n} \prod_{j=1}^n (1 - |u_j|^2)^{\beta_j} |F(u)|^p \, dm_n(u) = 2^n \int_{I^n} \prod_{j=1}^n r_j (1 - r_j^2)^{\beta_j} M_p^p(r, F) \, dr$$

for any measurable function $F$ in $U^n$. Further, if $F$ is holomorphic, then it is well known (see [Sh1]) that

$$\int_{U^n} \prod_{j=1}^n (1 - |u_j|^2)^{\beta_j} |F(u)|^p \, dm_n(u) \simeq \int_{I^n} \prod_{j=1}^n (1 - r_j^2)^{\beta_j} M_p^p(r, F) \, dr,$$

where $A \simeq B$ means $K^{-1}A \le B \le KA$. Here and afterwards, $K$ always denotes some positive absolute constant which may vary from line to line.

Our starting point is the weighted Bergman projection operator. For any function $F$ in $U^n$ and $v \in U^n$, we let

$$\mathcal{T}_\beta F(v) = \prod_{j=1}^n (\beta_j + 1) \int_{U^n} \frac{\prod_{j=1}^n (1 - |u_j|^2)^{\beta_j}}{\prod_{j=1}^n (1 - v_j \overline{u}_j)^{\beta_j + 2}} F(u) \, dm_n(u). \tag{2.1}$$

It is well known that $\mathcal{T}_\beta$ is the orthogonal projection from the Hilbert space $L^2(U^n, \prod_{j=1}^n (1 - |u_j|^2)^{\beta_j} dm_n(u))$ onto its holomorphic Hilbert subspace, i.e. the weighted Bergman space $H^{2,2,\beta}(U^n)$.

We also consider the diagonalization and polarization of the Bergman projection. For any functions $F$ in $U^n$ and $f$ in $U$, we define functions $\mathcal{D}_\beta F$ in $U$ and $\mathcal{E}_\beta f$ in $U^n$:

$$\mathcal{D}_\beta F(z) = \prod_{j=1}^n (\beta_j + 1) \int_{U^n} \frac{\prod_{j=1}^n (1 - |u_j|^2)^{\beta_j}}{\prod_{j=1}^n (1 - z\overline{u}_j)^{\beta_j + 2}} F(u) \, dm_n(u), \tag{2.2}$$

$$\mathcal{E}_\beta f(u) = (|\beta| + 2n - 1) \int_{U} \frac{(1 - |z|^2)^{|\beta| + 2n - 2}}{\prod_{j=1}^n (1 - u_j \overline{z})^{\beta_j + 2}} f(z) \, dm_1(z). \tag{2.3}$$

It is clear that $\mathcal{D}_\beta$ is the diagonalization of the weighted Bergman operator $\mathcal{T}_\beta$ of $U^n$, and $\mathcal{E}_\beta$ is the polarization of the weighted Bergman projection $\mathcal{T}_{|\beta|+2n-2}$ of $U$. We shall see that, when restricted to holomorphic mixed norm spaces, $\mathcal{D}_\beta$ coincides with the diagonal mapping $\mathcal{D}$, while $\mathcal{E}_\beta$ plays the role of a right inverse of $\mathcal{D}$.

THEOREM 2.2. *Under Hypothesis* 2.1,

(i) $\mathcal{T}_\beta f = f$ *for every* $f \in H^{p,q,\alpha}(U^n)$;
(ii) $\mathcal{D}_\beta : H^{p,q,\alpha}(U^n) \to H^{p,q,|\alpha|+(p/q+1)(n-1)}(U)$ *is bounded*;
(iii) $\mathcal{E}_\beta : H^{p,q,|\alpha|+(p/q+1)(n-1)}(U) \to H^{p,q,\alpha}(U^n)$ *is bounded.*

COROLLARY 2.3. *Under Hypothesis* 2.1, *for any* $F \in H^{p,q,\alpha}(U^n)$, *we have* $\mathcal{D}F \in H^{p,q,|\alpha|+(p/q+1)(n-1)}(U)$ *and*

$$\mathcal{D} = \mathcal{D}_\beta|_{H^{p,q,\alpha}(U^n)}\,. \tag{2.4}$$

*Proof.* From (1.3), Theorem 2.2(i), (2.1) and (2.2), we have

$$\mathcal{D}F(z) = F(z,\dots,z) = \mathcal{T}_\beta F(z,\dots,z) = \mathcal{D}_\beta F(z). \tag{2.5}$$

Therefore Theorem 2.2(ii) shows that $\mathcal{D}F \in H^{p,q,|\alpha|+(p/q+1)(n-1)}(U))$. ∎

COROLLARY 2.4. *Under Hypothesis* 2.1, *for* $f \in H^{p,q,|\alpha|+(p/q+1)(n-1)}(U)$, *we have* $\mathcal{E}_\beta f \in H^{p,q,\alpha}(U^n)$ *and*

$$\mathcal{D}(\mathcal{E}_\beta f) = f. \tag{2.6}$$

*Proof.* Let $\beta$ be a multi-index satisfying Hypothesis 2.1. For any $f \in H^{p,q,|\alpha|+(p/q+1)(n-1)}(U)$, Theorem 2.2(iii) shows that $\mathcal{E}_\beta f \in H^{p,q,\alpha}(U^n)$.

Let $A(U^n)$ be the space of functions holomorphic in $U^n$ and continuous in the closure of $U^n$. When $n = 1$, it is known [Sh2 (I), Proposition 2.3] that $A(U)$ is dense in $H^{p,q,\gamma}(U)$ for any $\gamma > -1$. From (2.4) and the boundedness of $\mathcal{D}_\beta$ and $\mathcal{E}_\beta$, as shown in Theorem 2.2(ii), (iii), we need only show that

$$\mathcal{D}_\beta(\mathcal{E}_\beta f) = f, \quad \forall f \in A(U).$$

Fix $w \in U$ and define

$$h_w(u) = \prod_{j=1}^{n} (1 - u_j\overline{w})^{-(\beta_j+2)}, \quad u \in U^n.$$

Then $h_w$ is a bounded holomorphic function in $U^n$, so that $\mathcal{T}_\beta h_w = h_w$ by Theorem 2.2(i). Recalling $\Phi(z) = (z,\dots,z)$, we thus have

$$\mathcal{T}_\beta h_w(\Phi(z)) = h_w(\Phi(z)) = (1 - z\overline{w})^{-(|\beta|+2n)}.$$

From (2.2) and (2.3), Fubini's theorem shows that for any $f \in A(U)$ and $z \in U$,

$$\begin{aligned}\mathcal{D}_\beta(\mathcal{E}_\beta f)(z) &= (|\beta|+2n-1)\int_U (1-|w|^2)^{|\beta|+2n-2}\mathcal{T}_\beta h_w(\Phi(z))f(w)\,dm_1(w)\\ &= (|\beta|+2n-1)\int_U \frac{(1-|w|^2)^{|\beta|+2n-2}}{(1-z\overline{w})^{|\beta|+2n}}\,f(w)\,dm_1(w)\\ &= \mathcal{T}_{|\beta|+2n-2}f(z) = f(z). \ \blacksquare\end{aligned}$$

Theorem 1.1 is a direct consequence of Corollaries 2.3 and 2.4.

**3. Extended Hardy inequalities.** In order to prove the boundedness of integral operators in mixed norm spaces, we need to establish some useful inequalities concerning mixed integrals over $I^n$ or $I$, which are closely related to Hardy's inequalities when $n=1$ (see [HL], [Fl], [AB]).

PROPOSITION 3.1. *Let $b_j > a_j > 0$, $c_j > 0$, $\forall j = 1,\ldots,n$, $\delta > 0$, and let $g : I^n \to [0,\infty)$ be measurable. Assume either $0 < k < 1$ and $g$ is increasing in each variable, or $1 \le k < \infty$. Then*

(i)
$$\int_I (1-\varrho)^{k|a|-1}\bigg(\int_{I^n}\frac{\prod_{j=1}^n(1-r_j)^{c_j-1}}{\prod_{j=1}^n(1-r_j\varrho)^{b_j}}\,g(r)\,dr\bigg)^k d\varrho \le K(a,b,c,k)\int_{I^n}\prod_{j=1}^n(1-r_j)^{k(a_j-b_j+c_j)-1}g^k(r)\,dr;$$

(ii)
$$\int_{I^n}\prod_{j=1}^n(1-r_j)^{ka_j-1}\bigg(\int_{I^n}\frac{\prod_{j=1}^n(1-t_j)^{c_j-1}}{\prod_{j=1}^n(1-r_jt_j)^{b_j}}\,g(t)\,dt\bigg)^k dr \le K(a,b,c,k)\int_{I^n}\prod_{j=1}^n(1-r_j)^{k(a_j-b_j+c_j)-1}g^k(r)\,dr;$$

(iii)
$$\int_{I^n}\prod_{j=1}^n(1-r_j)^{ka_j-1}\bigg(\int_I\frac{(1-\varrho)^{\delta-1}}{\prod_{j=1}^n(1-r_j\varrho)^{b_j}}\,g(\varrho)\,d\varrho\bigg)^k dr \le K(a,b,\delta,k)\int_I(1-\varrho)^{k(|a|-|b|+\delta)-1}g^k(\varrho)\,d\varrho.$$

For the proof we need some technical lemmas.

LEMMA 3.2. *Let $b_j > a_j > 0$, $j=1,\ldots,n$, and $r\in I^n$. Then*

(i)
$$\int_I\frac{(1-\varrho)^{|a|-1}}{\prod_{j=1}^n(1-r_j\varrho)^{b_j}}\,d\varrho \le K(a,b)\,\frac{1}{\prod_{j=1}^n(1-r_j)^{b_j-a_j}};$$

(ii)
$$\int_{I^n}\frac{\prod_{j=1}^n(1-t_j)^{a_j-1}}{\prod_{j=1}^n(1-r_jt_j)^{b_j}}\,dt \le K(a,b)\,\frac{1}{\prod_{j=1}^n(1-r_j)^{b_j-a_j}}.$$

*Proof.* (i) When $n = 1$, the inequality is well known (see for example [SW]). We now apply induction on $n$ to deal with the general case. Assume that (i) holds for $n-1$. For any given $r = (r_1, \ldots, r_n) \in I^n$, let $r_0 = \min\{r_j : j = 1, \ldots, n\}$. Then

$$\int_{r_0}^{1} \frac{(1-\varrho)^{|a|-1}}{\prod_{j=1}^n (1-r_j\varrho)^{b_j}}\, d\varrho \le \frac{(1-r_0)^{a_1}}{(1-r_0^2)^{b_1}} \int_{r_0}^{1} \frac{(1-\varrho)^{a_2+\cdots+a_n-1}}{\prod_{j=2}^n (1-r_j\varrho)^{b_j}}\, d\varrho \le \frac{1}{(1-r_1)^{b_1-a_1}} \frac{K}{\prod_{j=2}^n (1-r_j)^{b_j-a_j}}.$$

Since $\prod_{j=1}^n (1-r_j\varrho)^{-b_j} \le \prod_{j=1}^n (1-\varrho)^{-b_j}$ for any $\varrho \in [0,1)$, we also have

$$\int_{0}^{r_0} \frac{(1-\varrho)^{|a|-1}}{\prod_{j=1}^n (1-r_j\varrho)^{b_j}}\, d\varrho \le K \frac{1}{(1-r_0)^{|b|-|a|}} \le K \frac{1}{\prod_{j=1}^n (1-r_j)^{b_j-a_j}}.$$

This proves (i).

(ii) This is obvious since the integral can be decomposed as the product of integrals over $I$. ■

Applying the standard technique of Hardy–Littlewood [HL], we obtain the following inequality in the case of small indices.

LEMMA 3.3. *Let $0 < p \le 1$ and $b_j \ge 0$, $c_j > 0$, $\forall j = 1, \ldots, n$, and let $g : I^n \to [0, \infty)$ be increasing in each variable. Then there exists $K = K(p, b)$ such that*

$$\left\{ \int_{I^n} \frac{\prod_{j=1}^n (1-t_j)^{c_j-1}}{\prod_{j=1}^n (1-r_jt_j)^{b_j}}\, g(t)\, dt \right\}^p \le K \int_{I^n} \frac{\prod_{j=1}^n (1-t_j)^{pc_j-1}}{\prod_{j=1}^n (1-r_jt_j)^{pb_j}}\, g^p(t)\, dt. \tag{3.1}$$

*Proof.* Let $\lambda_{k_j} = 1 - 2^{-k_j}$ and $I_k = \prod_{j=1}^n [\lambda_{k_j-1}, \lambda_{k_j})$. Then in $I_k$ we have

$$1-\lambda_{k_j} \le 1-t_j \le 2(1-\lambda_{k_j}), \quad 1-\varrho\lambda_{k_j} \le 1-\varrho t_j \le 2(1-\varrho\lambda_{k_j}),$$
$$g(t) \le g(\lambda_k), \quad |I_k| = \prod_{j=1}^n (1-\lambda_{k_j}),$$

where $\lambda_k = (\lambda_{k_1}, \ldots, \lambda_{k_n})$, $\varrho \in I$ and $t = (t_1, \ldots, t_n) \in I_k$.

We claim that

$$\left\{ \int_{I_k} \frac{\prod_{j=1}^n (1-t_j)^{c_j-1}}{\prod_{j=1}^n (1-r_jt_j)^{b_j}}\, g(t)\, dt \right\}^p \le K \int_{I_{k+1}} \frac{\prod_{j=1}^n (1-t_j)^{pc_j-1}}{\prod_{j=1}^n (1-r_jt_j)^{pb_j}}\, g^p(t)\, dt. \tag{3.2}$$

Indeed, the integrand on the left side of (3.2) is enlarged if we replace $t_j$ by the constant $\lambda_{k_j}$ and $g(t)$ by the constant $g(\lambda_k)$, up to a constant independent of $k$. Then we can calculate the resulting integral and its $p$th power. The result is further enlarged if we replace $\lambda_k$ by $t \in I_{k+1}$, so (3.2) holds.

Now we write the integral over $I^n$ as the sum of integrals over $I_k$, and then apply the inequality $(a+b)^p \le a^p + b^p$ for $0 < p < 1$. The desired conclusion follows from (3.2) by summing over $k$. ■

Now we can prove the generalized Hardy inequality.

*Proof of Proposition 3.1.* (i) First assume that $1 < k < \infty$. Set

$$J = \int_{I^n} \frac{\prod_{j=1}^n (1-r_j)^{c_j-1}}{\prod_{j=1}^n (1-r_j\varrho)^{b_j}}\, g(r)\, dr.$$

Rewrite the integrand of $J$ as the product of

$$\prod_{j=1}^n (1-r_j)^{b_j-a_j-\varepsilon-1/k'} \prod_{j=1}^n (1-\varrho r_j)^{a_j-b_j}$$

and

$$\prod_{j=1}^n (1-r_j)^{c_j-b_j+a_j+\varepsilon-1/k} \prod_{j=1}^n (1-\varrho r_j)^{-a_j} g(r).$$

Here $\varepsilon$ is a sufficiently small positive number and $k'$ is the conjugate index of $k$. Applying Hölder's inequality and Lemma 3.2(ii), we have

$$J^k \le \frac{K}{\prod_{j=1}^n (1-\varrho)^{k\varepsilon}} \int_{I^n} \frac{\prod_{j=1}^n (1-r_j)^{k(c_j-b_j+a_j+\varepsilon)-1}}{\prod_{j=1}^n (1-\varrho r_j)^{ka_j}}\, g^k(r)\, dr. \tag{3.3}$$

Note that here we used the assumption $b_j > a_j$.

For $0 < k \le 1$, Lemma 3.3 shows that

$$J^k \le K \int_{I^n} \frac{\prod_{j=1}^n (1-r_j)^{kc_j-1}}{\prod_{j=1}^n (1-\varrho r_j)^{kb_j}}\, g^k(r)\, dr.$$

Consequently, for any $0 < k < \infty$, $J^k$ can be estimated by the integral over $I^n$, so that Fubini's theorem and Lemma 3.2(i) yield the desired result.

(ii) Since the expression in brackets in (ii) is again $J$, and $J^k$ can be estimated by the integral over $I^n$, the desired result follows from Fubini's theorem.

(iii) Set

$$\widetilde{J} = \int_I \frac{(1-\varrho)^{\delta-1}}{\prod_{j=1}^n (1-r_j\varrho)^{b_j}}\, g(\varrho)\, d\varrho.$$

For $1 < k < \infty$, we take $\varepsilon > 0$ sufficiently small and rewrite the above integrand as the product of

$$(1-\varrho)^{|b|-|a|-n\varepsilon-1/k'} \prod_{j=1}^n (1-r_j\varrho)^{a_j-b_j}$$

and

$$(1-\varrho)^{\delta-|b|+|a|+n\varepsilon-1/k}\prod_{j=1}^{n}(1-r_j\varrho)^{-a_j}g(\varrho).$$

Then from Hölder's inequality and Lemma 3.2(i),

$$\widetilde{J}^k \le K\frac{1}{\prod_{j=1}^n(1-r_j)^{k\varepsilon}}\int\limits_I \frac{(1-\varrho)^{k(\delta-|b|+|a|+n\varepsilon)-1}}{\prod_{j=1}^n(1-r_j\varrho)^{ka_j}}\,g^k(\varrho)\,d\varrho.$$

If $0<k<1$, Lemma 3.3 shows that $\widetilde{J}^k$ can be estimated by the integral over $I$. The desired result now follows from Fubini's theorem and Lemma 3.2(ii). ∎

**4. Weighted Bergman projections.** In this section, we consider the boundedness of weighted Bergman operators on $L^{p,q,\alpha}(U^n)$. We refer to [FR], [SW] for boundedness properties of weighted Bergman operators on $L^p$ spaces in the unit ball of $\mathbb{C}^n$.

THEOREM 4.1. *Under Hypothesis* 2.1,

(i) $\mathcal{T}_\beta : L^{p,q,\alpha}(U^n)\to H^{p,q,\alpha}(U^n)$ *is bounded provided* $1\le p<\infty$ *and* $1\le q\le\infty$;

(ii) $\mathcal{T}_\beta : H^{p,q,\alpha}(U^n)\to H^{p,q,\alpha}(U^n)$ *is always bounded.*

Let $\delta_{jk}$ be the Kronecker delta, i.e., $\delta_{jk}=1$ if $j=k$, and 0 otherwise.

LEMMA 4.2. *Let* $u=(u_1,\dots,u_n)\in U^n$ *and* $r_j=|u_j|$, $j=1,\dots,n$. *If* $\alpha_j>1$, *then for any* $\varrho\in I$ *and* $t\in I^n$,

(i) $$\int\limits_T\frac{d\sigma_1(\zeta)}{\prod_{j=1}^n|1-\varrho\zeta\overline{u}_j|^{\alpha_j}}\le K(\alpha)\,\frac{1}{\prod_{j=1}^n(1-\varrho r_j)^{\alpha_j-\delta_{j,1}}};$$

(ii) $$\int\limits_{T^n}\frac{d\sigma_n(\zeta)}{\prod_{j=1}^n|1-t_j\zeta_j\overline{u}_j|^{\alpha_j}}\le K(\alpha)\,\frac{1}{\prod_{j=1}^n(1-t_jr_j)^{\alpha_j-1}}.$$

*Proof.* The case of $n=1$ is well known (see [D]). Assertion (i) then follows from the inequality

$$\prod_{j=2}^n|1-\varrho\zeta\overline{u}_j|^{-\alpha_j}\le\prod_{j=2}^n(1-\varrho r_j)^{-\alpha_j}.$$

Assertion (ii) is obvious since it can be reduced to the case $n=1$. ∎

*Proof of Theorem 4.1.* For any function $F$ in $U^n$ and

$$\beta_j>\frac{\alpha_j+1}{p}+\frac{1}{\min\{q,1\}}-2,\qquad \alpha_j>-1,\qquad \forall j=1,\dots,n,$$

let

$$G_v(u) = F(u)\prod_{j=1}^{n}(1-\overline{v}_j u_j)^{-(\beta_j+2)} \tag{4.1}$$

for any $v, u \in U^n$. By (2.1) and the polar coordinates formula,

$$|\mathcal{T}_\beta F(v)| \le K \int_{I^n} \prod_{j=1}^{n}(1-r_j)^{\beta_j} M_1(r, G_v)\, dr. \tag{4.2}$$

Assume that $1 \le q \le \infty$. Minkowski's inequality shows that

$$M_q(t, \mathcal{T}_\beta F) \le K \int_{I^n} \prod_{j=1}^{n}(1-r_j)^{\beta_j} M_q(t, M_1(r, G_v))\, dr. \tag{4.3}$$

Let $v, u \in U^n$ and $t_j = |v_j|$, $r_j = |u_j|$. We claim that

$$M_q(t, M_1(r, G_v)) \le K M_q(r, F) \prod_{j=1}^{n}(1-t_j r_j)^{-(\beta_j+1)}. \tag{4.4}$$

From this claim together with (4.3), we find

$$M_q(t, \mathcal{T}_\beta F) \le K \int_{I^n} \frac{\prod_{j=1}^{n}(1-r_j)^{\beta_j}}{\prod_{j=1}^{n}(1-t_j r_j)^{\beta_j+1}} M_q(r, F)\, dr. \tag{4.5}$$

Therefore, applying Proposition 3.1(ii) with $b_j = c_j = \beta_j + 1$, $a_j = (\alpha_j+1)/p$, $k = p$ and $g(r) = M_q(r, F)$ we get

$$\int_{I^n} \prod_{j=1}^{n}(1-t_j^2)^{\alpha_j} M_q^p(t, \mathcal{T}_\beta F)\, dt \le K \int_{I^n} \prod_{j=1}^{n}(1-r_j^2)^{\alpha_j} M_q^p(r, F)\, dr.$$

Here we used Hypothesis (2.1)(iii), which assures $b_j > a_j$.

To prove the case $1 \le q \le \infty$, it remains to prove claim (4.4). Rewrite $G_v(u) = G_v^{(1)}(u) G_v^{(2)}(u)$, where

$$G_v^{(1)}(u) = F(u)\prod_{j=1}^{n}(1-\overline{v}_j u_j)^{-(\beta_j+2)/q},$$

$$G_v^{(2)}(u) = \prod_{j=1}^{n}(1-\overline{v}_j u_j)^{-(\beta_j+2)/q'}.$$

From Hölder's inequality we have

$$\begin{aligned} M_1(r, G_v) &\le M_q(r, G_v^{(1)}) M_{q'}(r, G_v^{(2)}) \\ &\le K(q,\beta) M_q(r, G_v^{(1)}) \prod_{j=1}^{n}(1-t_j r_j)^{-(\beta_j+1)/q'}. \end{aligned} \tag{4.6}$$

If $q=\infty$, then $G_v^{(1)}=F$ and $q'=1$, so that (4.4) follows from (4.6). If $1\le q<\infty$, notice that by (1.1), the definition of $G_v^{(1)}$ and Lemma 4.2(ii),

$$M_q^q(t,M_q(r,G_v^{(1)}))=\int\limits_{T^n}\int\limits_{T^n}|G_{t\eta}^{(1)}(r\zeta)|^q\,d\sigma_n(\zeta)\,d\sigma_n(\eta)$$
$$\le KM_q^q(r,F)\prod_{j=1}^n(1-t_jr_j)^{-(\beta_j+1)},$$

and the claim also follows from (4.6).

Assume now that $0<q<1$ and that $F$ is holomorphic in $U^n$. Then $G_v$ is holomorphic in $U^n$ for any given $v\in U^n$, so that $M_1(r,G_v)$ is increasing in each $r_j$, $j=1,\dots,n$. Thus from (4.2) and Lemma 3.3, we obtain

$$|\mathcal{T}_\beta F(v)|^q\le K\int\limits_{I^n}\prod_{j=1}^n(1-r_j)^{q(\beta_j+1)-1}M_1^q(r,G_v)\,dr.$$

Combining this with

$$M_1^q(r,G_v)\le K(q)\prod_{j=1}^n(1-r_j)^{q-1}M_q^q(r,G_v),$$

which holds since $G_v$ is holomorphic (see [Fr]), we deduce that

$$|\mathcal{T}_\beta F(v)|^q\le K\int\limits_{U^n}\frac{\prod_{j=1}^n(1-|u_j|^2)^{q(\beta_j+2)-2}}{\prod_{j=1}^n|1-v_j\overline{u}_j|^{q(\beta_j+2)}}\,|F(u)|^q\,dm_n(u).$$

Now, integrating over $T^n$ and changing the order of integration, from Lemma 4.2(ii) we have

$$M_q^q(t,\mathcal{T}_\beta F)\le K\int\limits_{I^n}\frac{\prod_{j=1}^n(1-r_j)^{q(\beta_j+2)-2}}{\prod_{j=1}^n(1-t_jr_j)^{q(\beta_j+2)-1}}\,M_q^q(r,F)\,dr.$$

By applying Proposition 3.1(ii), we obtain

$$\int\limits_{I^n}\prod_{j=1}^n(1-t_j^2)^{\alpha_j}M_q^p(t,\mathcal{T}_\beta F)\,dt\le K\int\limits_{I^n}\prod_{j=1}^n(1-r_j^2)^{\alpha_j}M_q^p(r,F)\,dr.$$

This completes the proof. ∎

**5. Proof of Theorem 2.2.** This section is devoted to proving Theorem 2.2.

*Proof of Theorem 2.2(i).* By applying the method of [Sh2 (I), Proposition 2.3], every function in $L_a^{p,q,\alpha}(U^n)$ can be approximated by its slice functions, so $A(U^n)$ is dense in $H^{p,q,\alpha}(U^n)$. We claim that

$$\mathcal{T}_\beta|_{A(U^n)}=\mathrm{Id}.$$

In fact the case $n = 1$ is well known (see [FR]) and the general case follows from this special case by iteration. More precisely, let $F \in A(U^n)$ and rewrite (2.1) as

$$\mathcal{T}_\beta F(v) = \prod_{k=2}^{n} (\beta_k + 1) \int\limits_{U^{n-1}} \frac{\prod_{k=2}^{n} (1 - |u_k|^2)^{\beta_k}}{\prod_{k=2}^{n} (1 - v_k \overline{u}_k)^{\beta_j + 2}} \, dm_1(u_2) \cdots dm_1(u_n)$$
$$\cdot \int\limits_{U} (\beta_1 + 1) \frac{(1 - |u_1|^2)^{\beta_1}}{(1 - v_1 \overline{u}_1)^{\beta_1 + 2}} \, F(u_1, \ldots, u_n) \, dm_1(u_1).$$

Note that the second integral is equal to $F(v_1, u_2, \ldots, u_n)$. By continuing this procedure, we finally have

$$\mathcal{T}_\beta F(v_1, \ldots, v_n) = F(v_1, \ldots, v_n),$$

as desired. Thus the boundedness of $\mathcal{T}_\beta$ ensured by Theorem 4.1(ii) implies that

$$\mathcal{T}_\beta |_{H^{p,q,\alpha}(U^n)} = \mathrm{Id}. \quad \blacksquare \tag{5.1}$$

*Proof of Theorem 2.2(ii).* Let $z \in U$, $u \in U^n$ and $\varrho = |z|$, $r_j = |u_j|$. Let

$$G_z(u) = F(u) \prod_{j=1}^{n} (1 - \overline{z} u_j)^{-(\beta_j + 2)}. \tag{5.2}$$

By (2.2) and the polar coordinates formula,

$$|\mathcal{D}_\beta F(z)| \le K \int\limits_{I^n} \prod_{j=1}^{n} (1 - r_j)^{\beta_j} M_1(r, G_z) \, dr. \tag{5.3}$$

Assume that $0 < q < 1$ and that $F$ is holomorphic on $U^n$. Then $G_z$ is holomorphic on $U^n$ for any given $z \in U$. By Lemma 3.3,

$$|\mathcal{D}_\beta F(z)|^q \le K \int\limits_{I^n} \prod_{j=1}^{n} (1 - r_j)^{q(\beta_j + 1) - 1} M_1^q(r, G_z) \, dr.$$

Since $M_1^q(r, G_z) \le K(q) \prod_{j=1}^{n} (1 - r_j)^{q-1} M_q^q(r, G_z)$ (see [Fr]), we have

$$|\mathcal{D}_\beta F(z)|^q \le K \int\limits_{U^n} \frac{\prod_{j=1}^{n} (1 - |u_j|^2)^{q(\beta_j + 2) - 2}}{\prod_{j=1}^{n} |1 - z \overline{u}_j|^{q(\beta_j + 2)}} \, |F(u)|^q \, dm_n(u).$$

Consequently, Lemma 4.2(i) shows that

$$M_q^q(\varrho, \mathcal{D}_\beta F) \le K \int\limits_{I^n} \frac{\prod_{j=1}^{n} (1 - r_j)^{q(\beta_j + 2) - 2}}{\prod_{j=1}^{n} (1 - \varrho r_j)^{q(\beta_j + 2) - \delta_{j1}}} \, M_q^q(r, F) \, dr.$$

Applying Proposition 3.1(i), we obtain

$$\int\limits_{I} (1 - \varrho)^{|\alpha| + (p/q + 1)(n-1)} M_q^p(\varrho, \mathcal{D}_\beta F) \, d\varrho \le K \int\limits_{I^n} \prod_{j=1}^{n} (1 - r_j)^{\alpha_j} M_q^p(r, F) \, dr.$$

This proves the case $0 < q < 1$.

Let $1 \le q \le \infty$. By (5.3), Minkowski's inequality shows that

$$M_q(\varrho, \mathcal{D}_\beta F) \le K \int_{I^n} \prod_{j=1}^{n} (1-r_j)^{\beta_j} M_q(\varrho, M_1(r, G_z))\, dr.$$

We claim that

$$M_q(\varrho, M_1(r, G_z)) \le K M_q(r, F) \prod_{j=1}^{n} (1 - r_j \varrho)^{-(\beta_j+1+(1-\delta_{j1})/q)}. \tag{5.4}$$

From (5.4), we have

$$M_q(\varrho, \mathcal{D}_\beta F) \le K \int_{I^n} \frac{\prod_{j=1}^{n}(1-r_j)^{\beta_j}}{\prod_{j=1}^{n}(1-\varrho r_j)^{\beta_j+1+(1-\delta_{j1})/q}} M_q(r, F)\, dr, \tag{5.5}$$

and apply Proposition 3.1(i) to obtain the desired result.

It remains to prove (5.4). We rewrite $G_z(u) = G_z^{(1)}(u) G_z^{(2)}(u)$, where

$$G_z^{(1)}(u) = F(u) \prod_{j=1}^{n} (1 - \overline{z} u_j)^{-(\beta_j+2)/q},$$

$$G_z^{(2)}(u) = \prod_{j=1}^{n} (1 - \overline{z} u_j)^{-(\beta_j+2)/q'}.$$

From Hölder's inequality and Lemma 4.2 we have

$$\begin{aligned} M_1(r, G_z) &\le M_q(r, G_z^{(1)}) M_{q'}(r, G_z^{(2)}) \\ &\le K(q, \beta) M_q(r, G_z^{(1)}) \prod_{j=1}^{n} (1 - |z| r_j)^{-(\beta_j+1)/q'}. \end{aligned} \tag{5.6}$$

If $q = \infty$, then $G_z^{(1)} = F$ and $q' = 1$, so that claim (5.4) follows from (5.6). If $1 \le q < \infty$, notice that

$$\begin{aligned} M_q^q(\varrho, M_q(r, G_z^{(1)})) &= \int_T \int_{T^n} |G_{\varrho\eta}^{(1)}(r\zeta)|^q \, d\sigma_n(\zeta)\, d\sigma_1(\eta) \\ &\le K M_q^q(r, F) \prod_{j=1}^{n} (1 - r_j \varrho)^{-(\beta_j+2-\delta_{j1})}. \end{aligned}$$

Hence claim (5.4) also follows from (5.6). This ends the proof. ∎

*Proof of Theorem 2.2(iii).* Let $f$ be holomorphic on $U$ and let

$$G_u(z) = f(z) \prod_{j=1}^{n} (1 - z \overline{u}_j)^{-(\beta_j+2)} \tag{5.7}$$

for any $u \in U^n$ and $z \in U$. From (2.3),

$$|\mathcal{E}_\beta f(u)| \le K \int_I (1-\varrho)^{|\beta|+2n-2} M_1(\varrho, G_u)\, d\varrho. \tag{5.8}$$

First assume that $0 < q < 1$. Since $G_u$ is holomorphic on $U$ for any given $u \in U^n$, it follows from Lemma 3.3 with $n = 1$ that

$$|\mathcal{E}_\beta f(u)|^q \leq K \int_I (1-\varrho)^{q(|\beta|+2n-1)-1} M_1^q(\varrho, G_u)\, d\varrho.$$

Notice that $M_1^q(\varrho, G_u) \leq K(q)(1-\varrho)^{q-1} M_q^q(\varrho, G_u)$. Recalling the definition of $G_u$ in (5.7) we have

$$|\mathcal{E}_\beta f(u)|^q \leq K \int_U \frac{(1-|z|^2)^{q(|\beta|+2n)-2}}{\prod_{j=1}^n |1-z\overline{u}_j|^{q(\beta_j+2)}} |f(z)|^q \, dm_1(z),$$

so that Lemma 4.2(ii) gives

$$M_q^q(r, \mathcal{E}_\beta f) \leq K \int_I \frac{(1-\varrho)^{q(|\beta|+2n)-2}}{\prod_{j=1}^n (1-\varrho r_j)^{q(\beta_j+2)-1}} M_q^q(\varrho, f)\, d\varrho,$$

and Proposition 3.1(iii) implies

$$\int_{I^n} \prod_{j=1}^n (1-r_j)^{\alpha_j} M_q^p(r, \mathcal{E}_\beta f) dr \leq K \int_I (1-\varrho)^{|\alpha|+(p/q+1)(n-1)} M_q^p(\varrho, f)\, d\varrho.$$

Now we assume that $1 \leq q \leq \infty$. Then, from (5.8), Minkowski's inequality shows that

$$M_q(r, \mathcal{E}_\beta f) \leq K \int_I (1-\varrho)^{|\beta|+2n-2} M_q(r, M_1(\varrho, G_u))\, d\varrho. \tag{5.9}$$

We claim that, for $r \in I^n$ defined by $r_j = |u_j|$ and $\varrho = |z| \in I$,

$$M_q(r, M_1(\varrho, G_u)) \leq K M_q(\varrho, f) \prod_{j=1}^n (1-r_j\varrho)^{-(\beta_j+1+(1-\delta_{j1})/q')}. \tag{5.10}$$

From this claim and (5.9), we find

$$M_q(r, \mathcal{E}_\beta f) \leq K \int_I \frac{(1-\varrho)^{|\beta|+2n-2}}{\prod_{j=1}^n (1-\varrho r_j)^{\beta_j+1+(1-\delta_{j1})/q'}} M_q(\varrho, f)\, d\varrho. \tag{5.11}$$

Therefore, applying Proposition 3.1(iii) we get the desired result:

$$\int_{I^n} \prod_{j=1}^n (1-r_j)^{\alpha_j} M_q^p(r, \mathcal{E}_\beta f)\, dr \leq K \int_I (1-\varrho)^{|\alpha|+(p/q+1)(n-1)} M_q^p(\varrho, f)\, d\varrho.$$

It remains to prove the claim in (5.10). To this end, write $G_u(z) = G_u^{(1)}(z) G_u^{(2)}(z)$, where

$$G_u^{(1)}(z) = f(z) \prod_{j=1}^n (1-z\overline{u}_j)^{-(\beta_j+2)/q},$$

$$G_u^{(2)}(z) = \prod_{j=1}^n (1-z\overline{u}_j)^{-(\beta_j+2)/q'}.$$

From Hölder's inequality and Lemma 4.2(i) we have, for any $\beta_j > -1$,

$$\begin{aligned} (5.12)\quad M_1(\varrho, G_u) &\le M_q(\varrho, G_u^{(1)}) M_{q'}(\varrho, G_u^{(2)}) \\ &\le K(q,\beta) M_q(\varrho, G_u^{(1)}) \prod_{j=1}^{n} (1 - |u_j|\varrho)^{-(\beta_j + 2 - \delta_{j1})/q'}. \end{aligned}$$

If $q = \infty$, then $G_u^{(1)} = f$ and $q' = 1$ so that (5.10) follows directly from (5.12). If $1 \le q < \infty$, notice that in virtue of (1.1), Fubini's theorem and Lemma 4.2,

$$\begin{aligned} M_q^q(r, M_q(\varrho, G_u^{(1)})) &= \int_{T^n} \int_{T} |G_{r\eta}^{(1)}(\varrho\zeta)|^q \, d\sigma_1(\zeta)\, d\sigma_n(\eta) \\ &\le K M_q^q(\varrho, f) \prod_{j=1}^{n} (1 - r_j \varrho)^{-(\beta_j+1)}. \end{aligned}$$

Then (5.10) follows from (5.12) and Lemma 3.2(i). This completes the proof. ∎

## References


[AB] P. Ahern and J. Bruna, *Maximal and area integral characterizations of Hardy–Sobolev spaces in the unit ball of* $\mathbb{C}^n$, Rev. Mat. Iberoamericana 4 (1988), 123–153.

[AJ] P. Ahern and M. Jevtić, *Duality and multipliers for mixed norm spaces*, Michigan Math. J. 30 (1983), 53–64.

[BP] A. Benedek and R. Panzone, *The space* $L^P$*, with mixed norm*, Duke Math. J. 28 (1961), 301–324.

[CM] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[Det] J. Detraz, *Restriction à la diagonale des classes de Hardy du bidisque*, C. R. Acad. Sci. Paris 287 (1978), 997–999.

[DjS] A. E. Djrbashian and F. A. Shamoian, *Topics in the Theory of* $A^p_\alpha$ *Spaces*, Leipzig, Teubner, 1988.

[D] P. L. Duren, *Theory of* $H^p$ *Spaces*, Academic Press, New York, 1970.

[DS] P. L. Duren and A. L. Shields, *Restrictions of* $H^p$ *functions to the diagonal of the polydisc*, Duke Math. J. 42 (1975), 751–753.

[Fl] T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38 (1972), 746–765.

[FR] F. Forelli and W. Rudin, *Projections on spaces of holomorphic functions in the ball*, Indiana Univ. Math. J. 24 (1974), 593–602.

[Fr] A. P. Frazier, *The dual space of* $H^p$ *of the polydisc for* $0 < p < 1$, Duke Math. J. 39 (1972), 369–379.

[HL] G. H. Hardy and J. E. Littlewood, *Some properties of functional integrals II*, Math. Z. 34 (1932), 403–439.

[HLP] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1934.

[HO] C. Horowitz and D. Oberlin, *Restriction of* $H^p$ *functions to the diagonal of* $U^n$, Indiana Univ. Math. J. 24 (1975), 767–772.

[J] M. Jevtić, *Projection theorems, fractional derivatives and inclusion theorems for mixed-norm spaces on the ball*, Analysis 9 (1989), 83–105.

[L] D. H. Luecking, *Embedding theorems for spaces of analytic functions via Khinchine's inequality*, Michigan Math. J. 40 (1993), 333–358.

[MR] B. Moulin et J. P. Rosay, *Sur la restriction des fonctions pluri-sous-harmoniques à la diagonale du polydisque*, Indiana Univ. Math. J. 26 (1977), 869–973.

[Pa] M. Pavlović, *Decompositions of $L^p$ and Hardy spaces of polyharmonic functions*, J. Math. Anal. Appl. 216 (1997), 499–509.

[RL] G. B. Ren and C. W. Liu, *Diagonal mappings on bounded symmetric domains*, Chinese J. Contemp. Math. 22 (2001), 11–22.

[Ru1] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.

[Ru2] —, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer, Berlin, 1980.

[Sh] F. A. Shamoian, *Diagonal mapping and problems of representation in anisotropic spaces of functions that are holomorphic in a polydisc*, Siberian Math. J. 31 (1990), 350–365.

[Sha] J. H. Shapiro, *Mackey topologies, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces*, Duke Math. J. 43 (1976), 187–202.

[Sh1] J. H. Shi, *Inequalities for the integral means of holomorphic functions and their derivatives in the ball of $\mathbb{C}^n$*, Trans. Amer. Math. Soc. 328 (1991), 619–637.

[Sh2] —, *Duality and multipliers for mixed norm spaces in the ball (I), (II)*, Complex Variables 25 (1994), 119–130; 131–157.

[SR] J. H. Shi and G. B. Ren, *Boundedness of the Cesàro operator on mixed norm spaces*, Proc. Amer. Math. Soc. 126 (1998), 3553–3560.

[SW] A. L. Shields and D. L. Williams, *Bounded projections, duality and multipliers in spaces of harmonic functions*, J. Reine Angew. Math. 299/300 (1978), 256–279.



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*Received August 13, 2002*
*Revised version December 4, 2003* (5017)