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Weighted weak type $(1,1)$ estimates for singular integrals and Littlewood–Paley functions

by

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Abstract. We prove some weighted weak type $(1,1)$ inequalities for certain singular integrals and Littlewood–Paley functions.

1. Introduction. Let K be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ which satisfies

$$\int_{a<|x|<b} K(x)\,dx = 0 \quad \text{for all } a,b \text{ such that } 0<a<b. \tag{1.1}$$

We assume that $n \ge 2$. We consider a singular integral operator which can be defined by

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)\,dy = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x-y)f(y)\,dy,$$

where $f \in C_0^\infty(\mathbb{R}^n)$ (the space of infinitely differentiable functions with compact support). Define

$$V(\theta) = \sup_{R>0} V_R(\theta), \quad \text{where} \quad V_R(\theta) = \int_R^{2R} |K(r\theta)|r^{n-1}\,dr.$$

Also put for $t \in (0,1]$, $q>0$,

$$\omega_q(t) = \sup_{|s|<tR/2} \Big(R^{-n} \int_{S^{n-1}} \int_R^{2R} |R^n[K((r-s)\theta) - K(r\theta)]|^q r^{n-1}\,dr\,d\sigma(\theta)\Big)^{1/q},$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{n-1} of $\mathbb{R}^n$ and the supremum is taken over all s and R such that $|s|<tR/2$. When $q=\infty$, we can define $\omega_\infty(t)$ by the usual modification.

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Let $L\log L(S^{n-1})$ denote the space of all those measurable functions Ω on S^{n-1} which satisfy

$$\int\limits_{S^{n-1}} |\Omega(\theta)| \log^+ |\Omega(\theta)| \, d\sigma(\theta) < \infty,$$

where $\log^+ x = \max(\log x, 0)$ $(x > 0)$, $\log^+ 0 = 0$. The following is known:

Theorem A. *Suppose $V \in L\log L(S^{n-1})$ and $\int_0^1 \omega_1(t)\, dt/t < \infty$. Suppose T is bounded on $L^2(\mathbb{R}^n)$. Then T is of weak type* $(1,1)$.

This is due to Seeger [10] (see also Tao [13], Seeger–Tao [11] for further developments). When $n \le 5$ and $K(x) = \Omega(x')/|x|^n$ $(x' = x/|x|)$, $\Omega \in L\log L(S^{n-1})$, the result was previously proved by Christ–Rubio de Francia [2] (see also Christ [1]). Hofmann [6] proved the result when $n = 2$ and $K(x) = \Omega(x')/|x|^n$ with $\Omega \in L^q(S^{n-1})$ for some $q > 1$.

For a non-negative function Ω on S^{n-1}, we define a maximal function

$$M_\Omega(f)(x) = \sup_{r>0} r^{-n} \int\limits_{|y|<r} |f(x-y)|\Omega(y')\, dy.$$

Put $M^s(f) = [M(|f|^s)]^{1/s}$, for $s > 0$, where M denotes the Hardy–Littlewood maximal operator, and $M^s_\Omega(f) = [M_\Omega(|f|^s)]^{1/s}$. Note that $M^s_\Omega(f) \le (\|\Omega\|_1/n)^{1/s-1/t} M^t_\Omega(f)$ if $s < t$.

Let w be a measurable, almost everywhere positive function on $\mathbb{R}^n$. We call such w a *weight function*. We denote by $L^p(w)$ $(p > 0)$ the space of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{L^p(w)} = (\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx)^{1/p} < \infty$, and by $L^{1,\infty}(w)$ the weak $L^1(w)$ space of all those functions f which satisfy

$$\|f\|_{L^{1,\infty}(w)} = \sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty,$$

where $w(E) = \int_E w(x)\, dx$. In [14], Vargas proved the following when $n = 2$ (see also [7]):

Theorem B. *Let $q, \beta > 1$. Suppose $\Omega \in L^q(S^1)$, $\int_{S^1} \Omega(\theta)\, d\sigma(\theta) = 0$ and $K(x) = \Omega(x')/|x|^2$. Put*

$$W(x) = \|\Omega\|_q M^\beta(w)(x) + \|\Omega\|_q^{1/\beta'} M^\beta M^\beta_{|\widetilde{\Omega}|} M^\beta(w)(x),$$

where $\widetilde{\Omega}(\theta) = \Omega(-\theta)$ and $1/\beta + 1/\beta' = 1$. Then T is bounded from $L^1(W)$ to $L^{1,\infty}(w)$; more precisely, there exists a constant $C = C(\beta, q)$ such that

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^2 : |T(f)(x)| > \lambda\}) \le C \int\limits_{\mathbb{R}^2} |f(x)| W(x)\, dx.$$

When $K(x) = \Omega(x')/|x|^n$, $\Omega \in L^\infty(S^{n-1})$ and $w \in A_1$, it is noted in Fan–Sato [5] that T is bounded from $L^1(w)$ to $L^{1,\infty}(w)$, which follows from Theorem B when $n = 2$. Here A_p denotes the weight class of Muckenhoupt.

In this note we generalize Theorem B to the case of general convolution kernels (not necessarily homogeneous) on $\mathbb{R}^n$ for all $n \geq 2$.

THEOREM 1. *Suppose the following*:

(1) $V \in L^r(S^{n-1})$ *for some* $1 < r \leq \infty$,
(2) $c_\omega := \int_0^1 \omega_q(t)\,dt/t < \infty$ *for some* $1 < q \leq \infty$,
(3) $w \in A_2$.

Let $s, t, u > 1$, $\varepsilon \in (0, r-1)$. *Then there exists a constant* C *depending only on* n, r, q, s, t, u, ε *and the* A_2 *constant for* w *such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}) \leq C\|f\|_{L^1(W)},$$

where

$$W = \|V\|_r^{1/s'} M^s M^s_{\widetilde{V}}(w) + c_\omega M^t M^{q'}(w) + \|V\|_r M^u(w) + \|V\|_r^{-\varepsilon} M M_{\widetilde{V}^{1+\varepsilon}}(w).$$

Recall that $w \in A_p$ $(1 < p < \infty)$ satisfies

$$\sup_Q \Big(|Q|^{-1} \int_Q w(x)\,dx\Big)\Big(|Q|^{-1} \int_Q w(x)^{-1/(p-1)}dx\Big)^{p-1} < \infty\,,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and this supremum is called the A_p *constant* for w. When $q < \infty$ in Theorem 1, we can replace $M^t M^{q'}(w)$ by $M^{q'}(w)$. Since $M^s_\Omega(f) \leq c\|\Omega\|_q^{1/s} M^{sq'}(f)$ for $q > 1$, by Theorem 1 we have the following:

COROLLARY 1. *Suppose* $V \in L^q(S^{n-1})$, $\int_0^1 \omega_q(t)\,dt/t < \infty$ *and* $w^{q'} \in A_1$ *for some* $1 < q \leq \infty$. *Then* T *is bounded from* $L^1(w)$ *to* $L^{1,\infty}(w)$.

Put

$$V^*(\theta) = \sup_{r>0} r^n |K(r\theta)|.$$

THEOREM 2. *Suppose the following*:

(1) $V \in L^r(S^{n-1})$ *for some* $1 < r \leq \infty$,
(2) $N := V^*[\log^+(V^*/\|V\|_r)]^{1+\varepsilon} \in L^1(S^{n-1})$ *for some* $\varepsilon > 0$,
(3) $\omega_1(t) \leq c_0 t^\alpha$ *for some* $\alpha \in (0,1]$,
(4) $w \in A_2$.

Let $s, t, u > 1$. *Then there exists a constant* C *depending only on* n, r, s, t, u, ε *and the* A_2 *constant for* w *such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}) \leq C\|f\|_{L^1(W)},$$

where

$$\begin{aligned} W = {} & \|V\|_r^{1/s'} M^s M^s_{\widetilde{V}}(w) + \alpha^{-1}(\|V\|_r + c_0) M^u(w) \\ & + M M_{\widetilde{N}}(w) + \alpha^{-1} M^t M_{\widetilde{V}^*}(w). \end{aligned}$$

We observe that a homogeneous kernel $K(x) = \Omega(x')/|x|^n$ with $\Omega \in L^1$ satisfies condition (3) of Theorem 2 with $c_0 = c\|\Omega\|_1$ and $\alpha = 1$.

REMARK 1. For any $q, \beta > 1$, let $p = [(\beta-1)q+1]/\beta$. Then $p > 1$. Since $t(\log^+ t)^{1+\varepsilon} \le ct^p$, by Hölder's inequality we have, if $V \in L^r$ and $V^* \in L^q$,

$$M_{\widetilde{V}^*[\log^+(\widetilde{V}^*/\|V\|_r)]^{1+\varepsilon}}(w) \le c\|V\|_r^{(1-q)/\beta'}\|V^*\|_q^{q/\beta'} M^\beta_{\widetilde{V}^*}(w).$$

Therefore, if $w(x) = |x|^\gamma$, $-n + (n-1)/q < \gamma \le 0$, in Theorem 2 and if we further assume $V^* \in L^q$, then by taking s, t, u and β sufficiently close to 1 we see that $W(x) \le c|x|^\gamma$ (see Muckenhoupt–Wheeden [8]). Thus $T : L^1(|x|^\gamma) \to L^{1,\infty}(|x|^\gamma)$.

REMARK 2. If the kernel has the form $K(x) = \Omega(x')/|x|^n$, with $\Omega \in L^q(S^{n-1})$, $q > 1$, then by Theorem 2 and Remark 1 we have

$$\|T(f)\|_{L^{1,\infty}(w)} \le C \int_{\mathbb{R}^n} |f|[\|\Omega\|_q^{1/s'} M^s M^s_{|\widetilde{\Omega}|}(w) + \|\Omega\|_q M^u(w)]\, dx$$

for $1 < s, u < \infty$ and $w \in A_2$. For any weight function v, put $w = M^\beta(v)$ for $1 < \beta < \infty$. If $M^\beta(v)$ is finite a.e., then $w \in A_1$ and so $w \in A_2$. (We shall also use this fact in what follows.) Moreover, we have $v \le w$ a.e. Thus

$$\|T(f)\|_{L^{1,\infty}(v)} \le C \int_{\mathbb{R}^n} |f|[\|\Omega\|_q^{1/s'} M^s M^s_{|\widetilde{\Omega}|} M^\beta(v) + \|\Omega\|_q M^u M^\beta(v)]\, dx.$$

Since $M^u M^\beta(v) \le cM^\beta(v)$ if $u < \beta$, Theorem B follows from this when $n = 2$.

To prove Theorems 1 and 2, we use the following L^2-estimates:

THEOREM 3. *Suppose the following*:

(1) $V \in L^r(S^{n-1})$ *for some* $1 < r \le \infty$,
(2) $c_\omega := \int_0^1 \omega_q(t)\, dt/t < \infty$ *for some* $1 < q \le \infty$,
(3) $w \in A_2$.

Let $s, t > 1$. *Then there exists a constant* C *depending only on* n, r, q, s, t *and the* A_2 *constant for* w *such that*

$$\|T(f)\|_{L^2(w)} \le C\|f\|_{L^2(W)},$$

where

$$W = \|V\|_r^{2-1/s} M^s M^s_{\widetilde{V}}(w) + c_\omega^2 M^t M^{q'}(w).$$

THEOREM 4. *Suppose the following*:

(1) $V \in L^r(S^{n-1})$ *for some* $1 < r \le \infty$,
(2) $V^* \in L^1(S^{n-1})$,
(3) $d_\omega := \int_0^1 [\omega_1(t)]^{1/2}\, dt/t < \infty$,
(4) $w \in A_2$.

Let $s,t>1$. Then there exists a constant C depending only on n, r, s, t and the A_2 constant for w such that

$$\|T(f)\|_{L^2(w)} \le C\|f\|_{L^2(W)},$$

where

$$W = \|V\|_r^{2-1/s} M^s M^s_{\widetilde{V}}(w) + d_\omega^2 M^t M_{\widetilde{V}^*}(w).$$

We can also prove similar results for certain Littlewood–Paley functions. Let $\psi \in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. We define the *Littlewood–Paley function* by

$$g(f)(x) = g_\psi(f)(x) = \Big(\int_0^\infty |\psi_t * f(x)|^2 \,\frac{dt}{t}\Big)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$. Suppose ψ is supported in $\{1 \le |x| \le 2\}$. For $t \in (0,1]$, $q>0$, put

$$\widetilde{\omega}_q(t) = \sup_{|s|<t/2} \Big(\int_{S^{n-1}} \Big[\int_0^\infty |\psi((r-s)\theta) - \psi(r\theta)|^2\,dr/r\Big]^{q/2} d\sigma(\theta)\Big)^{1/q}.$$

When $q=\infty$, we can define $\widetilde{\omega}_\infty(t)$ by the usual modification. Let

$$V(\theta) = \Big(\int_0^\infty |\psi(r\theta)|^2\,dr/r\Big)^{1/2} \qquad (\theta \in S^{n-1}). \tag{1.2}$$

Then we have the following:

THEOREM 5. *Suppose the following*:

(1) *$V \in L^1(S^{n-1})$ and $N := V[\log^+(V/\|V\|_1)]^{1+\varepsilon} \in L^1(S^{n-1})$ for some $\varepsilon > 0$,*

(2) *$\psi \in L^r(\mathbb{R}^n)$ for some $1 < r \le \infty$,*

(3) *$c_{\widetilde{\omega}} := \int_0^1 \widetilde{\omega}_q(t)\,dt/t < \infty$ for some $1 < q \le \infty$.*

Let $s,u>1$. Then there exists a constant C depending only on n, r, q, s, u and ε such that

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : g_\psi(f)(x) > \lambda\}) \le C\|f\|_{L^1(W)},$$

where

$$W = \|V\|_1^{-1/s'} \|\psi\|_r^{2/s'} M^s M^s_{\widetilde{V}}(w) + c_{\widetilde{\omega}} M M^{q'}(w) + \|V\|_1 M^u(w) + M M_{\widetilde{N}}(w).$$

As Theorem 1 implies Corollary 1, we easily see that Theorem 5 implies the following:

COROLLARY 2. *Suppose $V \in L^q(S^{n-1})$, $\int_0^1 \widetilde{\omega}_q(t)\,dt/t < \infty$ and $w^{q'} \in A_1$ for some $1 < q \le \infty$. Then g_ψ is bounded from $L^1(w)$ to $L^{1,\infty}(w)$.*

THEOREM 6. *Suppose the following*:

(1) *$V \in L^1(S^{n-1})$ and $N := V[\log^+(V/\|V\|_1)]^{1+\varepsilon} \in L^1(S^{n-1})$ for some $\varepsilon > 0$,*

(2) $\psi \in L^r(\mathbb{R}^n)$ *for some* $1 < r \le \infty$,
(3) $\widetilde{\omega}_1(t) \le c_0 t^\alpha$ *for some* $\alpha \in (0,1]$.

Let $s, u > 1$. *Then there exists a constant* C *depending only on* n, r, s, u *and* ε *such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : g_\psi(f)(x) > \lambda\}) \le C\|f\|_{L^1(W)},$$

where

$$W = \|V\|_1^{-1/s'} \|\psi\|_r^{2/s'} M^s M_{\widetilde{V}}^{s}(w) + \alpha^{-1}(\|V\|_1 + c_0)M^u(w) + MM_{\widetilde{N}}(w).$$

In Theorems 5 and 6, the assumption $w \in A_2$ is not needed, unlike in Theorems 1 and 2.

Remark 3. In Theorem 6, if we further assume that $V \in L^p$ for some $p > 1$, then as in Remark 1 we see that $g_\psi : L^1(|x|^\gamma) \to L^{1,\infty}(|x|^\gamma)$ for $-n + (n-1)/p < \gamma \le 0$.

Remark 4. From the proofs of Theorems 5 and 6 below, we can see that if $V \in L \log L$ and $\int_0^1 \widetilde{\omega}_1(t)\, dt/t < \infty$, then g_ψ is of weak type $(1,1)$ (the case when $w \equiv 1$).

To prove Theorems 5 and 6, we use the following L^2-estimates:

Theorem 7. *Suppose the following*:

(1) $V \in L^1(S^{n-1})$,
(2) $\psi \in L^r(\mathbb{R}^n)$ *for some* $1 < r \le \infty$.

Let $s > 1$. *Then there exists a constant* C *depending only on* n, s *and* r *such that*

$$\|g_\psi(f)\|_{L^2(w)} \le C\|V\|_1^{1/(2s)} \|\psi\|_r^{1-1/s} \|f\|_{L^2(M^s M_{\widetilde{V}}^{s}(w))}.$$

Suppose Ψ satisfies either the hypotheses of Theorem 5 or those of Theorem 6 for ψ. Let

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k \Psi_{2^k}(x),$$

where $\mathbb{Z}$ denotes the set of integers and $\{c_k\}$ is a sequence of non-negative numbers such that $\sum_k c_k < \infty$. Then we see that

$$g_\psi(f) \le \sum_k c_k g_{\Psi_{2^k}}(f) = \Big(\sum_k c_k\Big) g_\Psi(f).$$

This implies that g_ψ satisfies the estimates similar to those for g_Ψ.

Let

$$\psi(x) = |x|^{-n+\varrho}\Omega(x')\chi_{(0,1]}(|x|) \quad (\varrho > 0),$$

where $\Omega \in L^1(S^{n-1})$ satisfies $\int \Omega(\theta)\, d\sigma(\theta) = 0$ and χ_E denotes the characteristic function of a set E. Put $\mu_\varrho(f) = g_\psi(f)$. Then $\mu_\varrho(f)$ is known as the *Marcinkiewicz integral* (see Stein [12]). Take $\Psi(x) = |x|^{-n+\varrho}\Omega(x')\chi_{(1,2]}(|x|)$,

$c_k = 2^{k\varrho}$ for $k < 0$ and $c_k = 0$ for $k \geq 0$. Then $\psi = \sum_{k\in\mathbb{Z}} c_k \Psi_{2^k}$. Therefore, if $\Omega \in L^r(S^{n-1})$, $1 < r \leq \infty$, it is easy to see that we can apply Theorem 6 to get results for $\mu_\varrho(f)$. We refer to Ding–Fan–Pan [3] and Fan–Sato [5] for recent results on Marcinkiewicz integrals. In particular, in [5] the weak $(1,1)$ boundedness of μ_ϱ is proved under the assumption $\Omega \in L\log L$.

We shall give the proofs of Theorems 3 and 4 in Section 2 by applying the method of Duoandikoetxea–Rubio de Francia [4]. The proofs of Theorems 1 and 2 will be given in Sections 3 and 4, respectively. The principal part of the proofs of Theorems 1 and 2 is based on the estimates obtained by Seeger [10], which are crucial for the proof of Theorem A. Also we use a variant of the interpolation method given by Vargas [14]. An interpolation with change of measures between the $L^{p,1}(v_i\,d\nu)$-$L^p(w_i\,d\mu)$ estimates $(i = 1,2)$ was used in [14]. To prove Theorems 1 and 2 by using Seeger's results, we apply an interpolation with change of measures between the $L^{p_i,1}(v_i\,d\nu)$-$L^{p_i,\infty}(w_i\,d\mu)$ estimates, where $1 < p_i < \infty$. This method has already been used in Fan–Sato [5]; see [5] for more details. Theorem 7 will be proved in Section 5. Finally, we shall prove Theorems 5 and 6 in Section 6 by using the results given in Fan–Sato [5].

2. Proofs of Theorems 3 and 4. We consider a kernel of the form $K(x) = t^{-n}h(t)\Omega(t,\theta)$ with $x = t\theta$, $t > 0$, $\theta \in S^{n-1}$. Put $K_k(x) = K(x)\chi_{[2^k,2^{k+1})}(|x|)$. For $a > 0$ let

$$I_a(h) = \sup_{j\in\mathbb{Z}} \Bigg(\int_{2^j}^{2^{j+1}} |h(t)|^a \,\frac{dt}{t} \Bigg)^{1/a}.$$

We fix θ and write $\Omega(t,\theta) = \Omega_\theta(t)$. We assume that Ω_θ is of bounded variation on each interval $[2^k, 2^{k+1}]$ and put

$$V_k(\Omega)(\theta) = V(\Omega_\theta, [2^k, 2^{k+1}]),$$

where $V(H,I)$ denotes the total variation of a function H over an interval I. Let

$$\Omega_k^*(\theta) = \sup_{t\in[2^k,2^{k+1}]} |\Omega(t,\theta)|.$$

For $q,s \geq 1$, put

$$E(\Omega; q,s) = \sup_k \|\Omega_k^*\|_q + \sup_k \|\Omega_k^*\|_q^{1/s} \|V_k(\Omega)\|_q^{1/s'}.$$

We denote by $\widehat{f}$ the Fourier transform of f.

Lemma 1. *Suppose that $E(\Omega; q,s) < \infty$ and $I_s(h) < \infty$ for some $q,s \in (1,2]$. Then*

$$|\widehat{K}_k(\xi)| \leq cE(\Omega; q,s) I_s(h) |2^k\xi|^{-(q-1)(s-1)/(2qs)}, \tag{2.1}$$

where c depends only on the dimension n.

To prove this, we apply the method of [4]. We give the proof for the sake of completeness.

Proof of Lemma 1. We may assume that $E(\Omega;q,s)=1$ and $I_s(h)=1$ by homogeneity. Define a measure τ_t concentrated on $\{|x|=t\}$ by

$$\langle \tau_t, f\rangle = \int\limits_{\mathbb{S}^{n-1}} f(t\theta)\Omega_t(\theta)\,d\sigma(\theta) \quad \text{for } f\in C_0^\infty(\mathbb{R}^n),$$

where we write $\Omega_t(\theta)=\Omega(t,\theta)$. Then we see that

$$\begin{aligned}
(2.2)\qquad |\widehat{K}_k(\xi)| &= \bigg|\int\limits_{2^k}^{2^{k+1}} h(t)\Big(\int\limits_{\mathbb{S}^{n-1}} \Omega_t(\theta)\exp(-2\pi i t\xi\theta)\,d\sigma(\theta)\Big)\frac{dt}{t}\bigg|\\
&= \bigg|\int\limits_{2^k}^{2^{k+1}} h(t)\widehat{\tau}_t(\xi)\,\frac{dt}{t}\bigg|\\
&\le \Big(\int\limits_{2^k}^{2^{k+1}} |h(t)|^s\,\frac{dt}{t}\Big)^{1/s}\Big(\int\limits_{2^k}^{2^{k+1}} |\widehat{\tau}_t(\xi)|^{s'}\,\frac{dt}{t}\Big)^{1/s'}\\
&\le \|\Omega_k^*\|_1^{(s'-2)/s'}\Big(\int\limits_{2^k}^{2^{k+1}} |\widehat{\tau}_t(\xi)|^2\,\frac{dt}{t}\Big)^{1/s'},
\end{aligned}$$

where the first inequality follows from Hölder's inequality.

Let $A=1/(2q')$ and $\xi'=\xi/|\xi|$. Put

$$\Theta_k(\theta,\omega)=\Omega_k^*(\theta)\Omega_k^*(\omega)+\Omega_k^*(\theta)V_k(\Omega)(\omega)+\Omega_k^*(\omega)V_k(\Omega)(\theta).$$

Then, since

$$V(\Omega_\theta\bar{\Omega}_\omega,[2^k,2^{k+1}])\le \Omega_k^*(\theta)V_k(\Omega)(\omega)+\Omega_k^*(\omega)V_k(\Omega)(\theta),$$

applying integration by parts and Hölder's inequality, we have

$$\begin{aligned}
(2.3)\qquad &\int\limits_{2^k}^{2^{k+1}} |\widehat{\tau}_t(\xi)|^2\,\frac{dt}{t}\\
&= \iint\Big(\int\limits_{2^k}^{2^{k+1}} \Omega_\theta(t)\bar{\Omega}_\omega(t)\exp(-2\pi i t\xi(\theta-\omega))\,\frac{dt}{t}\Big)\,d\sigma(\theta)\,d\sigma(\omega)\\
&\le c\iint \Theta_k(\theta,\omega)\min(1,|2^k\xi(\theta-\omega)|^{-1})\,d\sigma(\theta)\,d\sigma(\omega)\\
&\le c\iint \Theta_k(\theta,\omega)|2^k\xi|^{-A}|\xi'(\theta-\omega)|^{-A}\,d\sigma(\theta)\,d\sigma(\omega)\\
&\le c|2^k\xi|^{-A}(\|\Omega_k^*\|_q^2+\|\Omega_k^*\|_q\|V_k(\Omega)\|_q)\Big(\iint|\xi'(\theta-\omega)|^{-1/2}\,d\sigma(\theta)\,d\sigma(\omega)\Big)^{1/q'}.
\end{aligned}$$

Combining (2.2) and (2.3), we get (2.1).

Now we give the proof of Theorem 3. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(x) \ge 0$, $\operatorname{supp}(\phi) \subset \{|x| < 2^{-10}\}$ and $\int \phi(x)\,dx = 1$. Let $\gamma \in C^\infty(\mathbb{R})$ be such that $\operatorname{supp}(\gamma) \subset \{1/2 \le t \le 2\}$ and

$$\sum_{j=-\infty}^{\infty} \gamma(2^j t) = 1 \quad \text{for all } t \ne 0.$$

For $k \in \mathbb{Z}$ and $\delta > 0$, put

$$K_j^k(t\theta) = \gamma(2^{-j}t) \int K(\varrho\theta) 2^{-j+\delta|k|} \phi(2^{-j+\delta|k|}(t-\varrho))\,d\varrho.$$

Then

$$|K_j^k(t\theta)| \le c 2^{\delta|k|} 2^{-jn} V(\theta) \chi_{[2^{j-1}, 2^{j+1}]}(t).$$

Also we see that

$$\begin{aligned} |(\partial/\partial t)(t^n K_j^k(t\theta))| &\le |n t^{n-1} K_j^k(t\theta)| + |t^n (\partial/\partial t) K_j^k(t\theta)| \\ &\le c 2^{\delta|k|} 2^{-j} V(\theta) \chi_{[2^{j-1}, 2^{j+1}]}(t) \\ &\quad + c 2^{2\delta|k|} 2^{-j} V(\theta) \chi_{[2^{j-1}, 2^{j+1}]}(t), \end{aligned}$$

and hence $V(H, [2^j, 2^{j+1}]) \le c 2^{2\delta|k|} V(\theta)$, where $H(t) = t^n K_j^k(t\theta)$. Thus by Lemma 1 we have

$$|\widehat{K_j^k}(\xi)| \le c 2^{2\delta|k|} \|V\|_r |2^j \xi|^{-(r-1)/(4r)}.$$

Moreover, $\|K_j^k\|_1 \le c 2^{\delta|k|} \|V\|_1$, and so, by (1.1),

$$|\widehat{K_j^k}(\xi)| \le c 2^{\delta|k|} \|V\|_1 \min(1, |2^j \xi|) \le c 2^{\delta|k|} \|V\|_1 |2^j \xi|^{(r-1)/(4r)}.$$

Combining these results, we obtain

$$|\widehat{K_j^k}(\xi)| \le c 2^{2\delta|k|} \|V\|_r \min(|2^j \xi|, |2^j \xi|^{-1})^{(r-1)/(4r)}. \tag{2.4}$$

Define Δ_k by the Fourier transform

$$\widehat{\Delta_k(f)}(\xi) = \gamma(2^k |\xi|) \widehat{f}(\xi).$$

Now decompose

$$Tf = \sum_k \sum_j K_j^k * \Delta_{j+k} f + \sum_k \sum_j S_j^k * \Delta_{j+k} f,$$

where $S_j^k = K_j - K_j^k$, $K_j(x) = \gamma(2^{-j}|x|) K(x)$. If $w \in A_2$, by the Littlewood–Paley inequality we have

$$\Big\| \sum_j K_j^k * \Delta_{j+k} f \Big\|_{L^2(w)}^2 \le c_w \sum_j \|K_j^k * \Delta_{j+k} f\|_{L^2(w)}^2. \tag{2.5}$$

By (2.4) we have

$$\|K_j^k * \Delta_{j+k} f\|_2 \le c \|V\|_r 2^{2\delta|k|} 2^{-\varepsilon|k|} \|f\|_2 \quad (\varepsilon = (r-1)/(4r)). \tag{2.6}$$

On the other hand, for a weight function w we have

$$\|K_j^k * \Delta_{j+k} f\|_{L^2(w)} \leq c\|V\|_1^{1/2} 2^{\delta|k|} \|f\|_{L^2(MM_{\widetilde{V}}(w))}. \tag{2.7}$$

Interpolating between (2.6) and (2.7), we obtain

$$\|K_j^k * \Delta_{j+k} f\|_{L^2(w^\theta)} \leq c\|V\|_r^{1-\theta/2} 2^{2\delta|k|} 2^{-(1-\theta)\varepsilon|k|} \|f\|_{L^2(MM_{\widetilde{V}}(w)^\theta)} \tag{2.8}$$

for all $\theta \in (0,1)$. Thus

$$\begin{aligned} \sum_j \|K_j^k * \Delta_{j+k} f\|^2_{L^2(w^\theta)} &\leq c\|V\|_r^{2-\theta} 2^{4\delta|k|} 2^{-2(1-\theta)\varepsilon|k|} \sum_j \|\widetilde{\Delta}_{j+k} f\|^2_{L^2(MM_{\widetilde{V}}(w)^\theta)} \\ &\leq c\|V\|_r^{2-\theta} 2^{4\delta|k|} 2^{-2(1-\theta)\varepsilon|k|} \|f\|^2_{L^2(MM_{\widetilde{V}}(w)^\theta)}, \end{aligned} \tag{2.9}$$

where $\widetilde{\Delta}_j$ is another decomposition operator such that

$$\widehat{\widetilde{\Delta}_j f}(\xi) = \widetilde{\gamma}(2^j \xi) \widehat{f}(\xi)$$

with $\widetilde{\gamma} \in C^\infty(\mathbb{R}^n)$ satisfying $\widetilde{\gamma}(\xi) = 1$ for $1/2 \leq |\xi| \leq 2$, $\mathrm{supp}(\widetilde{\gamma}) \subset \{1/4 \leq |\xi| \leq 4\}$. For any $\theta \in (0,1)$, choose δ small enough to satisfy $2\delta - (1-\theta)\varepsilon < 0$. Then, for a weight function w such that $w^\theta \in A_2$, by (2.5) and (2.9) we have

$$\begin{aligned} \Big\| \sum_k \sum_j K_j^k * \Delta_{j+k} f \Big\|_{L^2(w^\theta)} &\leq \sum_k \Big\| \sum_j K_j^k * \Delta_{j+k} f \Big\|_{L^2(w^\theta)} \\ &\leq c\|V\|_r^{1-\theta/2} \sum_k 2^{2\delta|k|} 2^{-(1-\theta)\varepsilon|k|} \|f\|_{L^2(MM_{\widetilde{V}}(w)^\theta)} \\ &\leq c\|V\|_r^{1-\theta/2} \|f\|_{L^2(MM_{\widetilde{V}}(w)^\theta)}. \end{aligned} \tag{2.10}$$

For $w \in A_2$, substituting $w^{1/\theta}$ for w in (2.10), we see that, for all $\theta \in (0,1)$,

$$\Big\| \sum_k \sum_j K_j^k * \Delta_{j+k} f \Big\|_{L^2(w)} \leq c\|V\|_r^{1-\theta/2} \|f\|_{L^2(MM_{\widetilde{V}}(w^{1/\theta})^\theta)}. \tag{2.11}$$

Similarly, if $w \in A_2$, then

$$\Big\| \sum_j S_j^k * \Delta_{j+k} f \Big\|^2_{L^2(w)} \leq c_w \sum_j \|S_j^k * \Delta_{j+k} f\|^2_{L^2(w)}.$$

Using Hölder's inequality, for $1 < t < \infty$ we see that

$$\begin{aligned} \sum_j \|S_j^k * \Delta_{j+k} f\|^2_{L^2(w)} &\leq c\omega_q (2^{-\delta|k|-1})^2 \sum_j \|\Delta_{j+k} f\|^2_{L^2(M^{q'}(w))} \\ &\leq c\omega_q (2^{-\delta|k|-1})^2 \|f\|^2_{L^2(M^t M^{q'}(w))}. \end{aligned}$$

Thus

$$(2.12)\quad \Big\|\sum_k\sum_j S_j^k * \Delta_{j+k}f\Big\|_{L^2(w)} \le c\Big[\sum_k \omega_q(2^{-\delta|k|-1})\Big]\|f\|_{L^2(M^tM^{q'}(w))}.$$

Combining (2.11) and (2.12), we get the conclusion of Theorem 3.

We turn to the proof of Theorem 4. We note that

$$|K_j^k(t\theta)| \le c2^{-jn}V^*(\theta)\chi_{[2^{j-1},2^{j+1}]}(t).$$

Therefore,

$$\|S_j^k * \Delta_{j+k}f\|^2_{L^2(w)} \le c\omega_1(2^{-\delta|k|-1})\|\Delta_{j+k}f\|^2_{L^2(M_{\widetilde{V}^*}(w))},$$

and hence, for $1 < t < \infty$,

$$\sum_j \|S_j^k * \Delta_{j+k}f\|^2_{L^2(w)} \le c\omega_1(2^{-\delta|k|-1})\|f\|^2_{L^2(M^tM_{\widetilde{V}^*}(w))}.$$

Thus, using the assumption $w \in A_2$, we have

$$\Big\|\sum_k\sum_j S_j^k * \Delta_{j+k}f\Big\|_{L^2(w)} \le c\Big[\sum_k \omega_1(2^{-\delta|k|-1})^{1/2}\Big]\|f\|_{L^2(M^tM_{\widetilde{V}^*}(w))}.$$

We can handle $\sum_k\sum_j K_j^k * \Delta_{j+k}f$ as in the proof of Theorem 3. This completes the proof of Theorem 4.

3. Proof of Theorem 1. By Calderón–Zygmund decomposition at height $\mu = \lambda/A$ with $A = \|V\|_r + c_\omega$, we have a collection $\{Q\}$ of non-overlapping closed dyadic cubes and functions g, b such that

$$f = g + b,\qquad \mu \le |Q|^{-1}\int_Q |f| \le c\mu,\qquad v\Big(\bigcup Q\Big) \le c\|f\|_{L^1(M(v))}/\mu,$$

$$\|g\|_\infty \le c\mu,\qquad \|g\|_{L^1(v)} \le c\|f\|_{L^1(M(v))},$$

$$b = \sum_Q b_Q,\quad \operatorname{supp}(b_Q) \subset Q,\quad \int b_Q = 0,\qquad \|b_Q\|_1 \le c\mu|Q|,$$

where v is any weight function. Put $B_j = \sum_{\ell(Q)=2^j} b_Q$ for $j \in \mathbb{Z}$, where $\ell(Q)$ denotes the sidelength of Q.

Let the functions γ and ϕ be as in Section 2. Put $K_j(x) = \gamma(2^{-j}|x|)K(x)$ as before. For a positive integer s and $\delta, \eta > 0$, define

$$H_j^s(t\theta) = \chi_{D_s^\eta}(\theta)\gamma(2^{-j}t)\int K(\varrho\theta)2^{-j+\delta s}\phi(2^{-j+\delta s}(t-\varrho))\,d\varrho,$$

where

$$D_s^\eta = \{\theta \in S^{n-1} : V(\theta) \le 2^{\eta s}\|V\|_r\}.$$

Put $E_s^\eta = S^{n-1} \setminus D_s^\eta$ and

$$R_j^s(t\theta) = \chi_{E_s^\eta}(\theta)\gamma(2^{-j}t)\int K(\varrho\theta)2^{-j+\delta s}\phi(2^{-j+\delta s}(t-\varrho))\,d\varrho.$$

Decompose $K_j(t\theta) = H_j^s(t\theta) + R_j^s(t\theta) + S_j^s(t\theta)$, where

$$S_j^s(t\theta) = \gamma(2^{-j}t)\int [K(t\theta) - K((t-\varrho)\theta)]2^{-j+\delta s}\phi(2^{-j+\delta s}\varrho)\,d\varrho.$$

By Seeger [10], for some $\varepsilon_0 > 0$ we have

$$\Big|\Big\{x \in \mathbb{R}^n : \Big|\sum_j H_j^s * B_{j-s}(x)\Big| > \lambda\Big\}\Big| \le c\|V\|_r 2^{c(\delta+\eta)s}2^{-\varepsilon_0 s}\lambda^{-1}\sum \|b_Q\|_1. \tag{3.1}$$

On the other hand, $|H_j^s(t\theta)| \le c\|V\|_r 2^{\eta s}2^{\delta s}2^{-jn}\chi_{[2^{j-1},2^{j+1}]}(t)$, so we have

$$\sum_j \|H_j^s * B_{j-s}\|_{L^1(w)} \le c\|V\|_r 2^{\eta s}2^{\delta s}\sum_Q \inf_Q M(w)\|b_Q\|_1, \tag{3.2}$$

where $\inf_Q M(w) = \inf_{x\in Q} M(w)(x)$.

For $t > 0$, put

$$F_t^s = \Big\{x \in \mathbb{R}^n : \Big|\sum_j H_j^s * B_{j-s}(x)\Big| > t\Big\}.$$

Taking η and δ small enough and interpolating between (3.1) and (3.2) by a variant of the method of Vargas [14], for any $\theta \in (0,1)$ we obtain

$$w(F^s_{c_\tau 2^{-\tau s}\lambda}) \le c\|V\|_r\lambda^{-1}2^{-\zeta s}\|f\|_{L^1(M(w^{1/\theta})^\theta)},$$

where $\tau, \zeta > 0$ depend on θ and c_τ satisfies $c_\tau\sum_s 2^{-\tau s} = 1$. (See Fan–Sato [5] for more details about the proof of this estimate.) Thus

$$w\Big(\Big\{x \in \mathbb{R}^n : \Big|\sum_{s=1}^{\infty}\sum_{j=-\infty}^{\infty} H_j^s * B_{j-s}(x)\Big| > \lambda\Big\}\Big) \le c_\theta\|V\|_r\lambda^{-1}\|f\|_{L^1(M(w^{1/\theta})^\theta)}. \tag{3.3}$$

Next we note that

$$|R_j^s(t\theta)| \le c2^{\delta s}2^{-jn}V_s(\theta)\chi_{[2^{j-1},2^{j+1}]}(t),$$

where $V_s(\theta) = V(\theta)\chi_{E_s^\eta}(\theta)$. Therefore

$$\|R_j^s * B_{j-s}\|_{L^1(w)} \le c2^{\delta s}\|B_{j-s}\|_{L^1(M_{\widetilde{V}_s}(w))},$$

and hence, if $\eta\varepsilon > \delta$,

$$\begin{aligned} \sum_{s\ge 1}\sum_j \|R_j^s * B_{j-s}\|_{L^1(w)} &\le c\sum_s 2^{\delta s}\|f\|_{L^1(MM_{\widetilde{V}_s}(w))} \\ &\le c\sum_s 2^{\delta s}2^{-\eta\varepsilon s}\|V\|_r^{-\varepsilon}\|f\|_{L^1(MM_{\widetilde{V}_s^{1+\varepsilon}}(w))} \\ &\le c\|V\|_r^{-\varepsilon}\|f\|_{L^1(MM_{\widetilde{V}^{1+\varepsilon}}(w))}. \end{aligned} \tag{3.4}$$

Also by Hölder's inequality we have

$$\Big\|\sum_j S_j^s * B_{j-s}\Big\|_{L^1(w)} \le c\omega_q(2^{-\delta s})\|f\|_{L^1(MM^{q'}(w))}.$$

Thus

$$\Big\|\sum_s\sum_j S_j^s * B_{j-s}\Big\|_{L^1(w)} \le c\Big[\sum_s \omega_q(2^{-\delta s})\Big]\|f\|_{L^1(MM^{q'}(w))}. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we have

$$\begin{aligned} &w(\{x\in\mathbb{R}^n\setminus E : |T(b)(x)|>\lambda\}) \\ &\quad \le c\lambda^{-1}\int |f|[\|V\|_r M(w^{1/\theta})^\theta + \|V\|_r^{-\varepsilon} MM_{\widetilde{V}^{1+\varepsilon}}(w) + c_\omega MM^{q'}(w)]\,dx, \end{aligned} \tag{3.6}$$

where $E=\bigcup Q^*$ with Q^* denoting a suitable concentric enlargement of Q (see [1], [5], [10], e.g., for more details about this argument). We can handle $T(g)$ by Theorem 3 as follows:

$$\begin{aligned} &w(\{x\in\mathbb{R}^n : |T(g)(x)|>\lambda\}) \le \lambda^{-2}\|Tg\|_{L^2(w)}^2 \\ &\quad \le c\lambda^{-2}\|V\|_r^{2-1/s}\|g\|^2_{L^2(M^sM_{\widetilde{V}}^{\frac{s}{}}(w))} + c\lambda^{-2}c_\omega^2\|g\|^2_{L^2(M^tM^{q'}(w))} \\ &\quad \le c\lambda^{-1}\|V\|_r^{2-1/s}A^{-1}\|f\|_{L^1(M^sM_{\widetilde{V}}^{\frac{s}{}}(w))} + c\lambda^{-1}c_\omega^2A^{-1}\|f\|_{L^1(M^tM^{q'}(w))} \\ &\quad \le c\lambda^{-1}\|V\|_r^{1-1/s}\|f\|_{L^1(M^sM_{\widetilde{V}}^{\frac{s}{}}(w))} + c\lambda^{-1}c_\omega\|f\|_{L^1(M^tM^{q'}(w))}. \end{aligned} \tag{3.7}$$

Also we note that $w(E)\le cA\lambda^{-1}\|f\|_{L^1(M(w))}$. Combining this estimate with (3.6) and (3.7), we get the conclusion of Theorem 1.

4. Proof of Theorem 2. We use notation similar to that of Section 3. We apply the Calderón–Zygmund decomposition with $\mu=\lambda/A$, $A=c_0\alpha^{-1}+\|V\|_r$. Let

$$H_j^s(t\theta)=\chi_{D_s^\eta}(\theta)\gamma(2^{-j}t)\int K(\varrho\theta)2^{-j+\delta s}\phi(2^{-j+\delta s}(t-\varrho))\,d\varrho,$$

where $D_s^\eta=\{\theta\in S^{n-1}: V^*(\theta)\le 2^{\eta s}\|V\|_r\}$. Put $E_s^\eta=S^{n-1}\setminus D_s^\eta$ and

$$R_j^s(t\theta)=\chi_{E_s^\eta}(\theta)K_j(t\theta).$$

Decompose $K_j(t\theta)=H_j^s(t\theta)+R_j^s(t\theta)+S_j^s(t\theta)$, where

$$S_j^s(t\theta)=\chi_{D_s^\eta}(\theta)\gamma(2^{-j}t)\int[K(t\theta)-K((t-\varrho)\theta)]2^{-j+\delta s}\phi(2^{-j+\delta s}\varrho)\,d\varrho.$$

Since $|S_j^s(t\theta)|\le c\|V\|_r2^{\eta s}2^{-jn}\chi_{[2^{j-1},2^{j+1}]}(t)$, we have

$$\sum_j\|S_j^s * B_{j-s}\|_{L^1(w)} \le c\|V\|_r2^{\eta s}\sum_Q\inf_Q M(w)\|b_Q\|_1. \tag{4.1}$$

On the other hand,

$$\sum_j \|S_j^s * B_{j-s}\|_1 \le \sum_j \|S_j^s\|_1 \|B_{j-s}\|_1 \le cc_0 2^{-\alpha\delta s} \sum_Q \|b_Q\|_1. \tag{4.2}$$

Interpolating between (4.1) and (4.2) as in Section 3, by taking η small enough, for any $\theta \in (0,1)$ we obtain

$$\sum_j \|S_j^s * B_{j-s}\|_{L^1(w)} \le c\|V\|_r^\theta c_0^{1-\theta} 2^{-\tau s} \|f\|_{L^1(M(w^{1/\theta})^\theta)} \tag{4.3}$$

with some $\tau > 0$ depending on θ and α.

Now we note that

$$|R_j^s(t\theta)| \le c2^{-jn} V_s^*(\theta) \chi_{[2^{j-1}, 2^{j+1}]}(t),$$

where $V_s(\theta) = V^*(\theta)\chi_{E_s^\eta}(\theta)$. Therefore

$$\|R_j^s * B_{j-s}\|_{L^1(w)} \le c\|B_{j-s}\|_{L^1(M_{\widetilde{V}_s^*}(w))}.$$

Thus for any $\varepsilon > 0$,

$$\begin{aligned}\sum_{s\ge 1}\sum_j \|R_j^s * B_{j-s}\|_{L^1(w)} &\le c\sum_s \|f\|_{L^1(MM_{\widetilde{V}_s^*}(w))} \\ &\le c\eta^{-(1+\varepsilon)} \sum_s s^{-(1+\varepsilon)} \int |f| MM_{\widetilde{V}_s^*[\log^+(\widetilde{V}_s^*/\|V\|_r)]^{1+\varepsilon}}(w)\, dx \\ &\le c\int |f| MM_{\widetilde{V}^*[\log^+(\widetilde{V}^*/\|V\|_r)]^{1+\varepsilon}}(w)\, dx.\end{aligned} \tag{4.4}$$

We can handle $\sum_s \sum_j H_j^s * B_{j-s}$ just as in the proof of Theorem 1. So we have

$$\begin{aligned} w\Big(\Big\{x \in \mathbb{R}^n : \Big|\sum_s \sum_j H_j^s * B_{j-s}(x)\Big| > \lambda\Big\}\Big) \\ \le c_\theta \|V\|_r \lambda^{-1} \|f\|_{L^1(M(w^{1/\theta})^\theta)}.\end{aligned} \tag{4.5}$$

By (4.3)–(4.5) we can treat $T(b)$ as in the proof of Theorem 1 by choosing a suitable exceptional set E. For the estimation of $T(g)$ we use Theorem 4 in the same way as we used Theorem 3 in the proof of Theorem 1. This completes the proof.

5. Proof of Theorem 7. Suppose ψ is supported in $\{1 \le |x| \le 2\}$ and let V be as in (1.2). We write

$$\psi_t(u\theta) = \sum_{j=-\infty}^{\infty} t^{-n}\psi(t^{-1}u\theta)\chi_{(1,2]}(2^{-j}t) = \sum_{j=-\infty}^{\infty} L_j(u\theta, t)$$

for $u \ge 0$, $\theta \in S^{n-1}$. Note that

$$|L_j(u\theta, t)|_{\mathcal{H}} \le c2^{-jn} V(\theta)\chi_{[1,4]}(2^{-j}u), \tag{5.1}$$

where $|L_j(u\theta,t)|_{\mathcal{H}} = (\int_0^\infty |L_j(u\theta,t)|^2\,dt/t)^{1/2}$. Now, we decompose

$$f*\psi_t(x) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \Delta_{j+k}(f*\psi_t)(x)\chi_{(2^k,2^{k+1}]}(t) = \sum_j F_j(x,t),$$

say, where Δ_j is as in Section 2. Set

$$T_j(f)(x) = \Big(\int_0^\infty |F_j(x,t)|^2\,\frac{dt}{t}\Big)^{1/2} = \Big(\sum_k |[L_k]_t * \Delta_{j+k}(f)(x)|_{\mathcal{H}}^2\Big)^{1/2},$$

where we write $[L_k]_t(x) = L_k(x,t)$.

Let

$$U_{j,k}(f)(x) = |[L_k]_t * \Delta_{j+k}(f)(x)|_{\mathcal{H}}.$$

Then by Hölder's inequality and (5.1), we have

$$\|U_{j,k}(f)\|_{L^2(w)}^2 \le c\|V\|_1\|f\|_{L^2(MM_{\widetilde{V}}(w))}^2. \tag{5.2}$$

On the other hand, by Plancherel's theorem

$$\|U_{j,k}(f)\|_2^2 = \int_{\mathbb{R}^n}\Big(\int_1^2 |\widehat{\psi}(2^k t\xi)|^2\,\frac{dt}{t}\Big)|\widehat{f}(\xi)\gamma(2^{k+j}|\xi|)|^2\,d\xi.$$

It is known that

$$\Big(\int_1^2 |\widehat{\psi}(2^k t\xi)|^2\,\frac{dt}{t}\Big) \le c\|\psi\|_r^2 \min(|2^k\xi|, |2^k\xi|^{-1})^{\varepsilon} \quad \text{for } \varepsilon\in(0,1/r')$$

(see Sato [9]). Therefore

$$\|U_{j,k}(f)\|_2^2 \le c\|\psi\|_r^2 2^{-\varepsilon|j|}\int |\widehat{f}(\xi)\gamma(2^{k+j}|\xi|)|^2\,d\xi \le c\|\psi\|_r^2 2^{-\varepsilon|j|}\|f\|_2^2. \tag{5.3}$$

Interpolating between (5.2) and (5.3), we get

$$\|U_{j,k}(f)\|_{L^2(w^\theta)}^2 \le c\|V\|_1^\theta \|\psi\|_r^{2(1-\theta)} 2^{-(1-\theta)\varepsilon|j|}\|f\|_{L^2(MM_{\widetilde{V}}(w)^\theta)}^2$$

for all $\theta\in(0,1)$. Substituting $w^{1/\theta}$ for w and writing $s=1/\theta$, we have

$$\|U_{j,k}(f)\|_{L^2(w)}^2 \le c\|V\|_1^\theta \|\psi\|_r^{2(1-\theta)} 2^{-(1-\theta)\varepsilon|j|}\|f\|_{L^2(M^s M_{\widetilde{V}}^s(w))}^2.$$

Let $\widetilde{\Delta}_j$ be as in Section 2. Then

$$\begin{aligned}\|T_j(f)\|_{L^2(w)}^2 &= \sum_k \|U_{j,k}(f)\|_{L^2(w)}^2 = \sum_k \|U_{j,k}(\widetilde{\Delta}_{j+k}f)\|_{L^2(w)}^2\\ &\le c\|V\|_1^\theta \|\psi\|_r^{2(1-\theta)} 2^{-(1-\theta)\varepsilon|j|}\sum_k \|\widetilde{\Delta}_{j+k}f\|_{L^2(M^s M_{\widetilde{V}}^s(w))}^2\\ &\le c\|V\|_1^\theta \|\psi\|_r^{2(1-\theta)} 2^{-(1-\theta)\varepsilon|j|}\|f\|_{L^2(M^s M_{\widetilde{V}}^s(w))}^2.\end{aligned}$$

From this we get the conclusion of Theorem 7, since $g_\psi(f) \le \sum_j T_j(f)$.

6. Proofs of Theorems 5 and 6. Let ϕ be as in Section 2. First we prove Theorem 5. For $\beta > 0$, let

$$D_s = \{\theta \in S^{n-1} : V(\theta) > 2^{\beta s}\|V\|_1\},$$

where s is a positive integer. We write $x = u\theta$, $u \geq 0$, $\theta \in S^{n-1}$ and $\psi(u\theta) = \Omega(u,\theta)$. Put $\Omega_s(u,\theta) = \Omega(u,\theta)\chi_{D_s}(\theta)$ and $\Omega^s = \Omega - \Omega_s$. As in the proof of Theorem 7, we decompose $\psi_t(x) = \sum_j L_j(x,t)$. Split L_j as

$$L_j(x,t) = t^{-n}\Omega(t^{-1}|x|,\theta)\chi_{(1,2]}(2^{-j}t) = K_j^s(x,t) + R_j^s(x,t) + S_j^s(x,t),$$

where

$$\begin{aligned} K_j^s(x,t) &= t^{-n}\Omega^s(\cdot,\theta) * \phi_{2^{-\beta s}}(t^{-1}u)\chi_{(1,2]}(2^{-j}t),\\ R_j^s(x,t) &= t^{-n}\Omega_s(\cdot,\theta) * \phi_{2^{-\beta s}}(t^{-1}u)\chi_{(1,2]}(2^{-j}t),\\ S_j^s(x,t) &= t^{-n}\Omega(t^{-1}u,\theta)\chi_{(1,2]}(2^{-j}t)\\ &\qquad - t^{-n}\Omega(\cdot,\theta) * \phi_{2^{-\beta s}}(t^{-1}u)\chi_{(1,2]}(2^{-j}t). \end{aligned}$$

We use the Calderón–Zygmund decomposition with $\mu = \lambda/A$, $A = \|V\|_1$. We note

$$\sup_{0\leq m\leq \ell} \|(\partial/\partial u)^m \Omega^s(\cdot,\theta) * \phi_{2^{-\beta s}}(u)\|_{L^2(du)} \leq c2^{\beta s(\ell+1)}\|V\|_1,$$

uniformly in $\theta \in S^{n-1}$. Thus, taking β small enough, as in Fan–Sato [5] we have

$$\Big|\Big\{x \in \mathbb{R}^n : \Big|\sum_{j\in\mathbb{Z}} [K_j^s]_t * B_{j-s}(x)\Big|_{\mathcal{H}} > \lambda\Big\}\Big| \leq c2^{-\varepsilon_0 s}\sum_Q |Q| \tag{6.1}$$

for some $\varepsilon_0 > 0$, where $|\cdot|_{\mathcal{H}}$ is as in Section 5. On the other hand,

$$|K_j^s(u\theta,t)|_{\mathcal{H}} \leq c2^{\beta s}\|V\|_1 u^{-n}\chi_{[1/4,8]}(2^{-j}u),$$

so that

$$\begin{aligned} \Big\|\sum_j [K_j^s]_t * B_{j-s}\Big\|_{L^1_w(\mathcal{H})} &= \Big\|\Big(\int_0^\infty \Big|\sum_j [K_j^s]_t * B_{j-s}\Big|^2 dt/t\Big)^{1/2}\Big\|_{L^1(w)} \\ &\leq c2^{\beta s}\|V\|_1 \sum_Q \inf_Q M(w)\|b_Q\|_1. \end{aligned} \tag{6.2}$$

Using the estimates obtained by interpolating between (6.1) and (6.2), taking β small enough, as in the proof of Theorem 1 we have, for any $\theta \in (0,1)$,

$$\begin{aligned} w\Big(\Big\{x \in \mathbb{R}^n : \Big|\sum_s\sum_j [K_j^s]_t * B_{j-s}(x)\Big|_{\mathcal{H}} > \lambda\Big\}\Big)& \\ \leq cA\lambda^{-1}\|f\|_{L^1(M(w^{1/\theta})^\theta)}&. \end{aligned} \tag{6.3}$$

Next, note that

$$|R_j^s(u\theta,t)|_{\mathcal{H}} \leq cV_s(\theta)u^{-n}\chi_{[1/4,8]}(2^{-j}u),$$

where $V_s(\theta) = V(\theta)\chi_{D_s}(\theta)$. Thus

$$\Big\|\sum_j [R_j^s]_t * B_{j-s}\Big\|_{L^1_w(\mathcal{H})} \le c\sum_j \|B_{j-s}\|_{L^1(M_{\widetilde{V}_s}(w))} \le c\|f\|_{L^1(MM_{\widetilde{V}_s}(w))}.$$

Therefore, for any $\varepsilon > 0$,

$$\sum_{s\ge 1}\Big\|\sum_j [R_j^s]_t * B_{j-s}\Big\|_{L^1_w(\mathcal{H})} \le c\beta^{-1-\varepsilon}\sum_s s^{-1-\varepsilon}\int |f| MM_{\widetilde{V}_s[\log^+(\widetilde{V}_s/\|V\|_1)]^{1+\varepsilon}}(w)\,dx \le c\int |f| MM_{\widetilde{V}[\log^+(\widetilde{V}/\|V\|_1)]^{1+\varepsilon}}(w)\,dx. \tag{6.4}$$

By Hölder's inequality, we see that

$$\|[S_j^s]_t * B_{j-s}\|_{L^1_w(\mathcal{H})} \le c\widetilde{\omega}_q(2^{-\delta s})\|B_{j-s}\|_{L^1(M^{q'}(w))}.$$

Thus

$$\sum_s\sum_j \|[S_j^s]_t * B_{j-s}\|_{L^1_w(\mathcal{H})} \le c\Big[\sum_s \widetilde{\omega}_q(2^{-\delta s})\Big]\|f\|_{L^1(MM^{q'}(w))}. \tag{6.5}$$

By (6.3)–(6.5) we can treat $g_\psi(b)$ as in the proof of Theorem 1 by choosing a suitable exceptional set E. To estimate $g_\psi(g)$ we apply Theorem 7:

$$w(\{x\in\mathbb{R}^n : |g_\psi(g)(x)| > \lambda\}) \le \lambda^{-2}\|g_\psi(g)\|^2_{L^2(w)} \le c\lambda^{-1}\|V\|_1^{1/s-1}\|\psi\|_r^{2-2/s}\|f\|_{L^1(M^sM^{s}_{\widetilde{V}}(w))}.$$

This completes the proof of Theorem 5.

Next we turn to the proof of Theorem 6. Decompose

$$L_j(x,t) = t^{-n}\Omega(t^{-1}|x|,\theta)\chi_{(1,2]}(2^{-j}t) = K_j^s(x,t) + R_j^s(x,t) + S_j^s(x,t),$$

where

$$\begin{aligned} K_j^s(x,t) &= t^{-n}\Omega^s(\cdot,\theta) * \phi_{2^{-\beta s}}(t^{-1}u)\chi_{(1,2]}(2^{-j}t),\\ R_j^s(x,t) &= t^{-n}\Omega_s(t^{-1}u,\theta)\chi_{(1,2]}(2^{-j}t),\\ S_j^s(x,t) &= t^{-n}\Omega^s(t^{-1}u,\theta)\chi_{(1,2]}(2^{-j}t)\\ &\qquad - t^{-n}\Omega^s(\cdot,\theta) * \phi_{2^{-\beta s}}(t^{-1}u)\chi_{(1,2]}(2^{-j}t). \end{aligned}$$

Here Ω^s, Ω_s are as above. We use the Calderón–Zygmund decomposition also with $\mu = \lambda/A$, $A = \|V\|_1$.

Since $\widetilde{\omega}_1(t) \le c_0 t^\alpha$, we have

$$\sum_j \|[S_j^s]_t * B_{j-s}\|_{L^1(\mathcal{H})} \le cc_0 2^{-\beta\alpha s}\sum_j \|B_{j-s}\|_1. \tag{6.6}$$

On the other hand, $|S_j^s(u\theta,t)|_{\mathcal{H}} \le c2^{\beta s}\|V\|_1 u^{-n}\chi_{[1/4,8]}(2^{-j}u)$, and hence

$$\Big\|\sum_j [S_j^s]_t * B_{j-s}\Big\|_{L^1_w(\mathcal{H})} \le c2^{\beta s}\|V\|_1\sum_Q \inf_Q M(w)\|b_Q\|_1. \tag{6.7}$$

Using the estimates obtained by interpolating between (6.6) and (6.7), taking β small enough, we have, for any $\theta \in (0,1)$,

$$w\Big(\Big\{x \in \mathbb{R}^n : \Big|\sum_s \sum_j [S_j^s]_t * B_{j-s}(x)\Big|_{\mathcal{H}} > \lambda\Big\}\Big) \\ \leq c\alpha^{-1}(\|V\|_1 + c_0)\lambda^{-1}\|f\|_{L^1(M(w^{1/\theta})^\theta)}.$$

The rest of the proof is similar to the case of Theorem 5. This completes the proof.

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