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Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces

by

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Abstract. The problem of boundedness of the Hardy–Littewood maximal operator in local and global Morrey-type spaces is reduced to the problem of boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions. This allows obtaining sufficient conditions for boundedness for all admissible values of the parameters. Moreover, in case of local Morrey-type spaces, for some values of the parameters, these sufficient conditions are also necessary.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centred at x of radius r.

Definition 1. Let $0 < p, \theta \leq \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w}$ and $GM_{p\theta,w}$ the *local* and *global Morrey-type spaces* respectively, defined to be the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p\theta,w}} \equiv \|f\|_{LM_{p\theta,w}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)},$$
$$\|f\|_{GM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w}}$$

respectively.

Lemma 1. *Let* $0 < p, \theta \leq \infty$ *and let* w *be a non-negative measurable function on* $(0,\infty)$.

(i) *If*

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty \quad \textit{for all } t > 0, \tag{1}$$

then $LM_{p\theta,w} = GM_{p\theta,w} = \Theta$, *where* Θ *is the set of all functions equivalent to* 0 *on* $\mathbb{R}^n$.

(ii) *If*

$$\|w(r)r^{n/p}\|_{L_\theta(0,t)} = \infty \quad \textit{for all } t > 0, \tag{2}$$

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then $f(0) = 0$ *for all* $f \in LM_{p\theta,w}$ *continuous at* 0, *and* $GM_{p\theta,w} = \Theta$ *for* $0 < p < \infty$.

Proof. (i) Let (1) be satisfied and f be not equivalent to zero. Then $A = \|f\|_{L_p(B(0,t_0))} > 0$ for some $t_0 > 0$. Hence

$$\|f\|_{GM_{p\theta,w}} \geq \|f\|_{LM_{p\theta,w}} \geq \|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(t_0,\infty)} \geq A\|w(r)\|_{L_\theta(t_0,\infty)}.$$

Therefore $\|f\|_{GM_{p\theta,w}} = \|f\|_{LM_{p\theta,w}} = \infty$.

(ii) Let (2) be satisfied. If $f \in LM_{p\theta,w}$ and there exists

$$\lim_{r\to 0} |B(0,r)|^{-1/p}\|f\|_{L_p(B(0,r))} = B, \tag{3}$$

where $|B(0,r)|$ is the volume of $B(0,r)$, then $B = 0$. Indeed, assume $B > 0$. Then there exists $t_0 > 0$ such that

$$|B(0,r)|^{-1/p}\|f\|_{L_p(B(0,r))} \geq B/2 \tag{4}$$

for all $0 < r \leq t_0$. Consequently,

$$\|f\|_{LM_{p\theta,w}} \geq \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,t_0)} \geq \frac{B}{2}\, v_n^{1/p}\|w(r)r^{n/p}\|_{L_\theta(0,t_0)},$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. Hence $\|f\|_{LM_{p\theta,w}} = \infty$, so $f \notin LM_{p\theta,w}$ and we have arrived at a contradiction.

If $f \in LM_{p\theta,w}$ and it is continuous at 0, then (3) holds with $B = |f(0)|$. Hence $f(0) = 0$.

Next let $0 < p < \infty$ and let $f \in GM_{p\theta,w}$. Then by the Lebesgue theorem on differentiation of integrals, for almost all $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} |B(x,r)|^{-1/p}\|f\|_{L_p(B(x,r))} = |f(x)|.$$

By the above argument, $f(x) = 0$ for all those x. Hence f is equivalent to zero. ∎

DEFINITION 2. Let $0 < p,\theta \leq \infty$. We denote by $\Omega_{p,\theta}$ the set of all non-negative measurable functions w on $(0,\infty)$ such that for some $t_1, t_2 > 0$,

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r)r^{n/p}\|_{L_\theta(0,t_2)} < \infty. \tag{5}$$

In what follows, keeping in mind Lemma 1, we always assume that $w \in \Omega_{p,\theta}$.

The spaces $GM_{p\infty,r^{-\lambda/p}}$, where $0 < \lambda < n$, are the classical Morrey spaces introduced in [6] and applied to studying the local behaviour of solutions of second order elliptic partial differential equations. The interpolation properties of the spaces $GM_{p\infty,w}$ were investigated by S. Spanne in [11]. The spaces $GM_{p\theta,r^{-\lambda}}$ were used by G. Lu [4] for studying embedding theorems for vector fields of Hörmander type. T. Mizuhara [5] and E. Nakai [7] studied the boundedness of various integral operators in the spaces $GM_{p\infty,w}$.

For a measurable set $\Omega \subset \mathbb{R}^n$ and a non-negative measurable function v on Ω, let $L_{p,v}(\Omega)$ be the weighted L_p-space of all measurable functions f

on Ω for which

$$\|f\|_{L_{p,v}(\Omega)} = \|vf\|_{L_p(\Omega)} < \infty.$$

If $0 < p \le \theta \le \infty$, then

$$\|f\|_{LM_{p\theta,w}} \le \|f\|_{L_{p,W}(\mathbb{R}^n)}, \tag{6}$$

and if $0 < \theta \le p \le \infty$, then

$$\|f\|_{L_{p,W}(\mathbb{R}^n)} \le \|f\|_{LM_{p\theta,w}}, \tag{7}$$

where

$$W(x) = \|w\|_{L_\theta(|x|,\infty)} \quad \text{for } x \in \mathbb{R}^n.$$

These inequalities are particular cases of general inequalities for Lebesgue spaces with mixed quasinorms (see, for example, [8, Section 3.37]). In particular, for $0 < p \le \infty$,

$$\|f\|_{LM_{pp,w}} = \|f\|_{L_{p,W_p}(\mathbb{R}^n)}.$$

where $W_p(x) = \|w\|_{L_p(|x|,\infty)}$ for $x \in \mathbb{R}^n$.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The *Hardy–Littlewood maximal operator* M is defined by

$$(Mf)(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy.$$

F. Chiarenza and M. Frasca [2] have proved the boundedness of M in the classical Morrey spaces. T. Mizuhara [5] and E. Nakai [7] have obtained sufficient conditions on w for the boundedness of M in $GM_{p\infty,w}$. In this paper we improve, in particular, the results obtained in [5, 7]. Moreover, for some values of parameters we obtain necessary and sufficient conditions for the operator M to be bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$.

An application of the known necessary and sufficient conditions for the boundedness of the maximal operator from one weighted Lebesgue space to another [10] immediately implies the following result for local Morrey-type spaces. Here and throughout, χ_E denotes the characteristic function of the set E.

THEOREM 1. *Let* $1 < p_1 \le p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{p_1,\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$.

- *If* $p_1 \ge \theta_1$ *and* $p_2 \le \theta_2$, *and for some* $c_1 > 0$ *and all balls* $B \subset \mathbb{R}^n$,

$$\|M(\chi_B W_1^{p_1/(1-p_1)})\|_{L_{p_2,W_2}(B)} \le c_1 \|W_1^{1/(1-p_1)}\|_{L_{p_1}(B)}, \tag{8}$$

where

$$W_1(x) = \|w_1\|_{L_{\theta_1}(|x|,\infty)}, \qquad W_2(x) = \|w_2\|_{L_{\theta_2}(|x|,\infty)}, \qquad x \in \mathbb{R}^n,$$

then the operator M *is bounded from* $LM_{p_1\theta_1,w_1}$ *to* $LM_{p_2\theta_2,w_2}$ *and from* $GM_{p_1\theta_1,w_1}$ *to* $GM_{p_2\theta_2,w_2}$.

• *If* $p_1 \le \theta_1$ *and* $p_2 \ge \theta_2$, *then condition* (8) *is necessary for the boundedness of* M *from* $LM_{p_1\theta_1,w_1}$ *to* $LM_{p_2\theta_2,w_2}$.

• *In particular*, *if* $\theta_1 = p_1$ *and* $\theta_2 = p_2$, *then condition* (8) *is necessary and sufficient for the boundedness of* M *from* $LM_{p_1p_1,w_1}$ *to* $LM_{p_2p_2,w_2}$.

Proof. Let $p_1 \ge \theta_1$ and $p_2 \le \theta_2$. By applying (6), the sufficiency of (8) for the boundedness of M and (7) we get

$$\|Mf\|_{LM_{p_2\theta_2,w_2}} \le \|Mf\|_{L_{p_2,W_2}(\mathbb{R}^n)} \le c_2\|f\|_{L_{p_1,W_1}(\mathbb{R}^n)} \le c_2\|f\|_{LM_{p_1\theta_1,w_1}}, \tag{9}$$

where $c_2 > 0$ is independent of f.

Conversely, if $p_1 \le \theta_1$, $p_2 \ge \theta_2$, and

$$\|Mf\|_{LM_{p_2\theta_2,w_2}} \le c_3\|f\|_{LM_{p_1\theta_1,w_1}},$$

where $c_3 > 0$ is independent of f, then by (6),

$$\|Mf\|_{L_{p_2,W_2}(\mathbb{R}^n)} \le c_3\|f\|_{L_{p_1,W_1}(\mathbb{R}^n)}, \tag{10}$$

which is known to imply (8).

Also (9) implies that

$$\|Mf\|_{GM_{p_2\theta_2,w_2}} \le c_2\|f\|_{GM_{p_1\theta_1,w_1}}. \blacksquare$$

In order to obtain conditions on w_1 and w_2 ensuring the boundedness of M for other values of the parameters and to obtain simpler conditions for $p = \theta_1 = \theta_2$ we shall reduce the problem of boundedness of M in local Morrey-type spaces to the problem of boundedness of Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions.

As in [2, 5, 7], we start with the inequality

$$\int_{B(0,r)} (Mf)(x)^p\,dx \le c_4 \int_{\mathbb{R}^n} |f(x)|^p (M\chi_{B(0,r)})(x)\,dx, \tag{11}$$

where $1 < p < \infty$ and $c_4 > 0$ is independent of $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and r. This is a particular case of a more general inequality established by C. L. Fefferman and E. Stein [3]:

$$\int_{\mathbb{R}^n} (Mf)(x)^p |\varphi(x)|\,dx \le c_5 \int_{\mathbb{R}^n} |f(x)|^p (M\varphi)(x)\,dx,$$

where $1 < p < \infty$ and $c_5 > 0$ is independent of $f, \varphi \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 2. *For all* $r > 0$ *and* $x \in \mathbb{R}^n$,

$$\left(\frac{r}{|x|+r}\right)^n \le (M\chi_{B(0,r)})(x) \le 4^n\left(\frac{r}{|x|+r}\right)^n. \tag{12}$$

This statement is known, at least with some constant on the right-hand side. For the sake of completeness we give the proof.

Proof. If $n=1$, then $M\chi_{(-r,r)}=1$ for $|x|<r$ and $M\chi_{(-r,r)}=r/(|x|+r)$ if $|x|\geq r$, and (12) is trivial.

Assume that $n\geq 2$. If $|x|<r$, then $M\chi_{B(0,r)}=1$ and (12) is trivial again.

Assume that $|x|\geq r$. If $\varrho\geq|x|$, then

$$G(x,r,\varrho)=\frac{1}{|B(x,\varrho)|}\int\limits_{B(x,\varrho)}\chi_{B(0,r)}(y)\,dy=\frac{|B(x,\varrho)\cap B(0,r)|}{|B(x,\varrho)|}\leq\left(\frac{r}{\varrho}\right)^n\leq\left(\frac{r}{|x|}\right)^n.$$

If $0<\varrho\leq|x|-r$, then $G(x,r,\varrho)=0$.

Assume that $|x|-r\leq\varrho\leq|x|$. In order to estimate $G(x,r,\varrho)$ from above note that

$$B(x,\varrho)\cap B(0,r)\subset B_{n-1}(0,h)\times(|x|-\varrho,r),$$

where $B_{n-1}(0,h)$ is the ball centered at the origin of radius h in $\mathbb{R}^{n-1}$, and h is the height in the triangle with side lengths r, ϱ and $|x|$, perpendicular to the side of length $|x|$. Since

$$h=\frac{\sqrt{\varrho^2-(|x|-r)^2}\sqrt{(|x|+r)^2-\varrho^2}}{2|x|}\leq\sqrt{(\varrho^2-(|x|-r)^2)\frac{r}{|x|}},$$

it follows that

$$\begin{aligned}G(x,r,\varrho)&\leq\frac{v_{n-1}}{v_n}\,\frac{((\varrho^2-(|x|-r)^2)r/|x|)^{(n-1)/2}(\varrho-(|x|-r))}{\varrho^n}\\&=\frac{v_{n-1}}{v_n}\left(\left(1-\left(\frac{|x|-r}{\varrho}\right)^2\right)\frac{r}{|x|}\right)^{(n-1)/2}\left(1-\frac{|x|-r}{\varrho}\right)\\&\leq\frac{v_{n-1}}{v_n}\left(\left(1-\left(\frac{|x|-r}{|x|}\right)^2\right)\frac{r}{|x|}\right)^{(n-1)/2}\left(1-\frac{|x|-r}{|x|}\right)\\&\leq\frac{v_{n-1}}{v_n}\,2^{(n-1)/2}\left(\frac{r}{|x|}\right)^n.\end{aligned}$$

Since $v_{n-1}/v_n\leq n/2\leq 2^{n/2}$, for $|x|\geq r$ and for all $\varrho\geq|x|-r$ we have

$$G(x,r,\varrho)\leq 2^n\left(\frac{r}{|x|}\right)^n\leq 4^n\left(\frac{r}{|x|+r}\right)^n.$$

Since

$$M\chi_{B(0,r)}(x)=\sup_{\varrho\geq|x|-r}G(x,r,\varrho),$$

the upper bound in (12) follows. The lower bound is much simpler:

$$M\chi_{B(0,r)}(x)\geq G(x,r,|x|+r)=\left(\frac{r}{|x|+r}\right)^n. \qquad\blacksquare$$

Corollary 1. *Let $1 < p < \infty$. Then there exists $c_6 > 0$ such that*

$$\|Mf\|_{L_p(B(0,r))} \le c_6 \left(r^n \int\limits_{\mathbb{R}^n} \frac{|f(x)|^p}{(|x| + r)^n}\, dx \right)^{1/p},$$

for all $r > 0$ and all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

In order to estimate the right-hand side integral we first prove the following equality.

Lemma 3. *Let ϕ be a non-negative measurable function on $\mathbb{R}^n$. Then for all $r \ge 0$,*

$$\int\limits_{|x| \ge r} \frac{\phi(x)\, dx}{|x|^n} = n \int\limits_r^\infty \Big(\int\limits_{r \le |x| \le t} \phi(x)\, dx \Big) \frac{dt}{t^{n+1}}.$$

Proof. By applying spherical coordinates we get

$$\begin{aligned} \int\limits_r^\infty \Big(\int\limits_{r \le |x| \le t} \phi(x)\, dx \Big) \frac{dt}{t^{n+1}} &= \int\limits_r^\infty \Big(\int\limits_{S^{n-1}} \Big(\int\limits_r^t \phi(\varrho\sigma) \varrho^{n-1} d\varrho \Big)\, d\sigma \Big) \frac{dt}{t^{n+1}} \\ &= \int\limits_{S^{n-1}} \Big(\int\limits_r^\infty \Big(\int\limits_{\max\{\varrho, r\}}^\infty \frac{dt}{t^{n+1}} \Big) \phi(\varrho\sigma) \varrho^{n-1} d\varrho \Big)\, d\sigma \\ &= \frac{1}{n} \int\limits_{S^{n-1}} \int\limits_r^\infty \frac{\phi(\varrho\sigma)\varrho^{n-1}}{\varrho^n}\, d\varrho\, d\sigma = \frac{1}{n} \int\limits_{|x| \ge r} \frac{\phi(x)\, dx}{|x|^n}. \ \blacksquare \end{aligned}$$

Lemma 4. *Let φ be a non-negative measurable function on $\mathbb{R}^n$. Then for all $r \ge 0$,*

$$n\, 2^{-n} \int\limits_r^\infty \Big(\int\limits_{B(0,t)} \varphi(x)\, dx \Big) \frac{dt}{t^{n+1}} \le \int\limits_{\mathbb{R}^n} \frac{\varphi(x)\, dx}{(|x| + r)^n} \le n \int\limits_r^\infty \Big(\int\limits_{B(0,t)} \varphi(x)\, dx \Big) \frac{dt}{t^{n+1}}.$$

Proof. Since

$$\int\limits_{\mathbb{R}^n} \frac{\varphi(x)\, dx}{(|x| + r)^n} \le r^{-n} \int\limits_{|x| \le r} \varphi(x)\, dx + \int\limits_{|x| > r} \frac{\varphi(x)}{|x|^n}\, dx$$

and

$$\int\limits_{\mathbb{R}^n} \frac{\varphi(x) dx}{(|x| + r)^n} \ge 2^{-n} \Big(r^{-n} \int\limits_{|x| \le r} \varphi(x)\, dx + \int\limits_{|x| > r} \frac{\varphi(x)}{|x|^n}\, dx \Big),$$

the statement follows by Lemma 3 because

$$\begin{aligned} \int\limits_{|x| \ge r} \frac{\varphi(x)}{|x|^n}\, dx &= n \int\limits_r^\infty \Big(\int\limits_{|x| \le t} \varphi(x)\, dx - \int\limits_{|x| \le r} \varphi(x)\, dx \Big) \frac{dt}{t^{n+1}} \\ &= n \int\limits_r^\infty \Big(\int\limits_{|x| \le t} \varphi(x)\, dx \Big) \frac{dt}{t^{n+1}} - r^{-n} \int\limits_{|x| \le r} \varphi(x)\, dx. \ \blacksquare \end{aligned} \tag{13}$$

COROLLARY 2. *Let $1 < p < \infty$. Then there exists $c_7 > 0$ such that*

$$\|Mf\|_{L_p(B(0,r))} \le c_7 \Big(r^n \int_r^\infty \Big(\int_{B(0,t)} |f(x)|^p \, dx \Big) \frac{dt}{t^{n+1}} \Big)^{1/p} \tag{14}$$

for all $r > 0$ and all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Let H be the *Hardy operator*:

$$(Hg)(r) = \int_0^r g(t)\, dt, \qquad 0 < r < \infty.$$

LEMMA 5. *Let $1 < p < \infty$, $0 < \theta \le \infty$, $w \in \Omega_{p,\theta}$. Then there exists $c_8 > 0$ such that*

$$\|Mf\|_{LM_{p\theta,w}} \le c_8 \|Hg\|^{1/p}_{L_{\theta/p,v}(0,\infty)}$$

for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, where

$$g(t) = \int_{B(0,t^{-1/n})} |f(y)|^p \, dy, \tag{15}$$

$$v(r) = w(r^{-1/n})^p r^{-1-(1+1/n)p/\theta}. \tag{16}$$

Proof. By Corollary 2, for $\theta < \infty$ we have

$$\begin{aligned}
\|Mf\|_{LM_{p\theta,w}} &= \big\| w(r) \|Mf\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)} \\
&\le c_7^{1/p} \bigg\| w(r) r^{n/p} \Big(\int_r^\infty \Big(\int_{B(0,t)} |f(x)|^p \, dx \Big) \frac{dt}{t^{n+1}} \Big)^{1/p} \bigg\|_{L_\theta(0,\infty)} \\
&= c_7^{1/p} n^{-1/p} \bigg\| w(r) r^{n/p} \Big(\int_0^{r^{-n}} \Big(\int_{B(0,\tau^{-1/n})} |f(x)|^p \, dx \Big) d\tau \Big)^{1/p} \bigg\|_{L_\theta(0,\infty)}
\end{aligned} \tag{17}$$

$$= c_7^{1/p} n^{-1/p} \Big(\int_0^\infty (w(r) r^{n/p})^\theta \Big(\int_0^{r^{-n}} g(\tau)\, d\tau \Big)^{\theta/p} dr \Big)^{1/\theta}$$

$$= c_7^{1/p} n^{-1/p-1/\theta} \Big(\int_0^\infty (w(\varrho^{-1/n}) \varrho^{-1/p})^\theta \varrho^{-1/n-1} \Big(\int_0^\varrho g(\tau)\, d\tau \Big)^{\theta/p} d\varrho \Big)^{1/\theta}$$

$$= c_8 \|Hg\|^{1/p}_{L_{\theta/p,v}(0,\infty)}.$$

If $\theta = \infty$, then by a similar argument

$$\begin{aligned}
\|Mf\|_{LM_{p\infty,w}} &= \operatorname*{ess\,sup}_{0<r<\infty} w(r) \|Mf\|_{L_p(B(0,r))} \\
&\le c_8' \operatorname*{ess\,sup}_{0<\varrho<\infty} w(\varrho^{-1/n}) \varrho^{-1/p} Hg^{1/p}(\varrho), \qquad c_8' = c_7^{-1/p} n^{-1/p}. \ \blacksquare
\end{aligned}$$

COROLLARY 3. *Let $1 < p < \infty$, $0 < \theta \le \infty$, $w \in \Omega_{p,\theta}$. Then*

$$\|Mf\|_{GM_{p\theta,w}} \le c_8 \sup_{x\in\mathbb{R}^n} \|H(g(x,\cdot))\|^{1/p}_{L_{\theta/p,v}(0,\infty)}$$

for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$, where v is given by (16) *and*

$$g(x,t) = \int\limits_{B(x,t^{-1/n})} |f(y)|^p\,dy = \int\limits_{B(0,t^{-1/n})} |f(x+y)|^p\,dy. \tag{18}$$

Proof. Since for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\|(Mf)(x+\cdot)\|_{L_p(B(0,r))} \\
&\quad = \bigg(\int\limits_{B(0,r)} \bigg(\sup_{r>0} \frac{1}{|B(x+z,r)|} \int\limits_{B(x+z,r)} |f(y)|\,dy\bigg)^p dz\bigg)^{1/p} \\
&\quad = \bigg(\int\limits_{B(0,r)} \bigg(\sup_{r>0} \frac{1}{|B(z,r)|} \int\limits_{B(z,r)} |f(x+u)|\,du\bigg)^p dz\bigg)^{1/p} \\
&\quad = \|M(f(x+\cdot))\|_{L_p(B(0,r))},
\end{aligned} \tag{19}$$

we have

$$\begin{aligned}
\|Mf\|_{GM_{p\theta,w}} &= \sup_{x\in\mathbb{R}^n} \|w(r)\|(Mf)(x+\cdot)\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)} \\
&= \sup_{x\in\mathbb{R}^n} \|w(r)\|M(f(x+\cdot))\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)} \\
&= \sup_{x\in\mathbb{R}^n} \|M(f(x+\cdot))\|_{LM_{p\theta,w}} \\
&\le c_8 \sup_{x\in\mathbb{R}^n} \bigg\|H\bigg(\int\limits_{B(0,t^{-1/n})} f(x+y)|^p\,dy\bigg)\bigg\|^{1/p}_{L_{\theta/p,v}(0,\infty)} \\
&= c_8 \sup_{x\in\mathbb{R}^n} \|H(g(x,\cdot))\|^{1/p}_{L_{\theta/p,v}(0,\infty)}. \quad \blacksquare
\end{aligned} \tag{20}$$

THEOREM 2. *Let $0 < p_2 \le p_1 < \infty$, $p_1 > 1$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{p_1,\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$. Assume that, for some $q > 1$ satisfying $p_2 \le q \le p_1$, the operator H is bounded from $L_{\theta_1/q,v_1}(0,\infty)$ to $L_{\theta_2/q,v_2}(0,\infty)$ on the cone of all non-negative functions φ non-increasing on $(0,\infty)$ and satisfying $\lim_{t\to\infty}\varphi(t) = 0$, where*

$$v_1(r) = (w_1(r^{-1/n})r^{1/q-1/p_1-(1+1/n)/\theta_1})^q, \tag{21}$$

$$v_2(r) = (w_2(r^{-1/n})r^{-1/p_2-(1+1/n)/\theta_2})^q. \tag{22}$$

Then the operator M is bounded from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ and from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$.

Proof. 1. By Hölder's inequality with exponent $q/p_2 \geq 1$,

$$\begin{aligned} \|Mf\|_{LM_{p_2\theta_2,w_2}} &= \big\|w_2(r)\|Mf\|_{L_{p_2}(B(0,r))}\big\|_{L_{\theta_2}(0,\infty)} \\ &\leq \big\|w_2(r)|B(0,r)|^{1/p_2-1/q}\|Mf\|_{L_q(B(0,r))}\big\|_{L_{\theta_2}(0,\infty)} \\ &= v_n^{1/p_2-1/q}\big\|u_2(r)\|Mf\|_{L_q(B(0,r))}\big\|_{L_{\theta_2}(0,\infty)} \\ &= v_n^{1/p_2-1/q}\|Mf\|_{LM_{q\theta_2,u_2}}, \end{aligned} \tag{23}$$

where

$$u_2(r) = w_2(r) r^{n(1/p_2-1/q)}.$$

Since $q > 1$, by Lemma 5 applied to $LM_{q\theta_2,u_2}$ we have

$$\|Mf\|_{LM_{q\theta_2,u_2}} \leq c_9 \|Hg\|^{1/q}_{L_{\theta_2/q,v_2}(0,\infty)},$$

where

$$g(t) = \int\limits_{B(0,t^{-1/n})} |f(y)|^q\, dy, \tag{24}$$

and $c_9 > 0$ is independent of f, because $u_2(r^{-1/n})^q r^{-1-(1+1/n)q/\theta_2} = v_2(r)$.

Since g is non-negative, non-increasing on $(0,\infty)$ and $\lim_{t\to+\infty} g(t) = 0$, and H is bounded from $L_{\theta_1/q,v_1}(0,\infty)$ to $L_{\theta_2/q,v_2}(0,\infty)$ on the cone of functions containing g, we have

$$\|Mf\|_{LM_{p\theta_2,w_2}} \leq c_{10} \|g\|^{1/q}_{L_{\theta_1/q,v_1}(0,\infty)},$$

where $c_{10} > 0$ is independent of f.

Finally, by Hölder's inequality with exponent $p_1/q \geq 1$,

$$g(t) \leq |B(0,t^{-1/n})|^{1-q/p_1} \|f\|^q_{L_{p_1}(B(0,t^{-1/n}))}.$$

Hence

$$\begin{aligned} \|Mf\|_{LM_{p_2\theta_2,w_2}} &\leq c_{11}\Big(\int\limits_0^\infty v_1(t)^{\theta_1/q} t^{-(1-q/p_1)\theta_1/q} \|f\|^{\theta_1}_{L_{p_1}(B(0,t^{-1/n}))}\, dt\Big)^{1/\theta_1} \\ &= c_{11} n^{1/\theta_1}\Big(\int\limits_0^\infty v_1(r^{-n})^{\theta_1/q} r^{n(\theta_1/q-\theta_1/p_1)-n-1} \|f\|^{\theta_1}_{L_{p_1}(B(0,r))}\, dr\Big)^{1/\theta_1} \\ &= c_{11} n^{1/\theta_1}\Big(\int\limits_0^\infty (w_1(r)\|f\|_{L_{p_1}(B(0,r))})^{\theta_1}\, dr\Big)^{1/\theta_1} = c_{11} n^{1/\theta_1} \|f\|_{LM_{p_1\theta_1,w_1}}, \end{aligned} \tag{25}$$

where $c_{11} > 0$ is independent of f.

2. To prove the boundedness of M from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$ we apply Corollary 3. Since for all $x \in \mathbb{R}^n$ the function $g(x,t)$ defined by (18) is non-negative and non-increasing on $(0,\infty)$ and $\lim_{t\to\infty} g(x,t) = 0$, as in

Step 1 it follows that

$$\|Mf\|_{GM_{p_2\theta_2,w_2}} \le c_{10} \sup_{x\in\mathbb{R}^n} \|g(x,\cdot)\|^{1/q}_{L_{\theta_1/q,v_1}(0,\infty)}.$$

Since by Hölder's inequality

$$g(x,t) \le |B(0,t^{-1/n})|^{1-q/p_1} \|f(x+\cdot)\|^q_{L_{p_1}(B(0,t^{-1/n}))},$$

as in the second part of Step 1 we obtain

$$\begin{aligned}\|Mf\|_{GM_{p_2\theta_2,w_2}} &\le c_{11} n^{1/\theta_1} \sup_{x\in\mathbb{R}^n} \Big(\int_0^\infty w_1(r)\|f(x+\cdot)\|^{\theta_1}_{L_{p_1}(B(0,r))}\,dr\Big)^{1/\theta_1} \\ &= c_{11} n^{1/\theta_1} \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p_1\theta_1,w_1}} \\ &= c_{11} n^{1/\theta_1} \|f\|_{GM_{p_1\theta_1,w_1}}. \ \blacksquare\end{aligned}$$

In order to obtain explicit sufficient conditions on the weight functions ensuring the boundedness of M, we first apply the following simple statement.

LEMMA 6. *Let* $0 < \theta \le \infty$, *and let* w *and* v *be positive measurable functions on* $(0,\infty)$. *Then the inequality*

$$\|Hg\|_{L_{\theta,w}(0,\infty)} \le c_{12}\|g\|_{L_{\infty,v}(0,\infty)} \tag{26}$$

is satisfied for all non-negative functions g *with* $c_{12} > 0$ *independent of* g *if*, *and only if*,

$$B \equiv \Big\| w(r)\int_0^r \frac{dt}{v(t)}\Big\|_{L_\theta(0,\infty)} < \infty. \tag{27}$$

Moreover, *the minimal value of* c_{12} *is equal to* B.

Proof. Necessity. Taking $g = 1/v$ in (26) we obtain (27).

Sufficiency. It suffices to note that

$$\begin{aligned}\|Hg\|_{L_{\theta,w}(0,\infty)} &= \Big\| w(r)\int_0^r g(t)\,dt\Big\|_{L_\theta(0,\infty)} \\ &= \Big\| w(r)\int_0^r \frac{g(t)}{v(t)}\,v(t)\,dt\Big\|_{L_\theta(0,\infty)} \le B\|g\|_{L_{\infty,v}(0,\infty)}. \ \blacksquare\end{aligned} \tag{28}$$

Applying Lemma 6 to Theorem 2 we obtain the following sufficient condition for the boundedness of the maximal operator in local and global Morrey-type spaces.

THEOREM 3. *Let* $1 < p < \infty$, $0 < \theta_2 \le \infty$, $w_1 \in \Omega_{p,\infty}$, $w_2 \in \Omega_{p,\theta_2}$ *and suppose that*

$$\Big\| w_2^p(r) r^{n+(n+1)p/\theta_2} \int_r^\infty \frac{1}{w_1^p(t)t^{n+1}}\,dt\Big\|_{L_{\theta_2/p}(0,\infty)} < \infty. \tag{29}$$

Then the operator M is bounded from $LM_{p\infty,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\infty,w_1}$ to $GM_{p\theta_2,w_2}$.

A similar result was obtained by E. Nakai [7] for $w_1 = w_2 = w^{-1/p}(r)$ and $\theta_2 = \infty$ with an extra condition: there exists $c_{13} > 0$ such that $r \le t \le 2r \Rightarrow 1/c_{13} \le w(t)/w(r) \le c_{13}$.

Necessary and sufficient conditions for the validity of

$$\|H\varphi\|_{L_{\theta_2/q,v_2}(0,\infty)} \le c_{14}\|\varphi\|_{L_{\theta_1/q,v_1}(0,\infty)}, \tag{30}$$

where $c_{14} > 0$ is independent of φ, for all non-negative non-increasing functions φ are known for most of the cases. For detailed information see [12], [13]. Application of any of those conditions gives sufficient conditions for the boundedness of the maximal operator from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ and from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$. However, since the reverse of inequality (11) does not hold (take $f \equiv 1$), there is no guarantee that an application of necessary and sufficient conditions on v_1 and v_2 ensuring the validity of (30) will imply necessary and sufficient conditions for the boundedness of M from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$.

Fortunately for certain values of the parameters this is indeed the case, namely for $1 < p < \infty$, $0 < \theta_1 \le \theta_2 < \infty$, $\theta_1 \le p$.

Note that in this case necessary conditions (which are also sufficient) for the validity of (30) for non-negative non-increasing functions are obtained by taking $\varphi = \chi_{(0,t)}$ with an arbitrary $t > 0$.

Since in the proof of Theorem 2 inequality (30) is applied to the function $\varphi = g$, where g is given by (24), it is natural to choose, as test functions, functions f_t, $t > 0$, for which $\int_{B(0,u^{-1/n})} |f_t(y)|^q\,dy$ is equal or close to $A(t)\chi_{(0,t)}(u)$, $u > 0$, where $A(t)$ is independent of u. The simplest choice is

$$f_t(y) = \chi_{B(0,2t)\setminus B(0,t)}(y), \qquad y \in \mathbb{R}^n,\ t > 0. \tag{31}$$

Lemma 7. *Let $r, t > 0$, and $0 < p \le \infty$. Then*

$$\|f_t\|_{L_p(B(0,r))} = 0, \quad 0 < r \le t, \qquad \|f_t\|_{L_p(B(0,r))} \le c_{15}t^{n/p}, \quad t < r < \infty,$$

where $c_{15} > 0$ depends only on n and p.

Proof. The statement follows since for measurable $G, \Omega \subset \mathbb{R}^n$,

$$\|\chi_G\|_{L_p(\Omega)} = |G \cap \Omega|^{1/p}.$$

Hence $\|f_t\|_{L_p(B(0,r))} = |(B(0,2t)\setminus B(0,t)) \cap B(0,r)|^{1/p}$. ∎

Lemma 8. *For all $t > 0$ and $x \in \mathbb{R}^n$,*

$$\frac{1}{2}\left(\frac{t}{|x|+t}\right)^n \le (Mf_t)(x) \le 8^n\left(\frac{t}{|x|+t}\right)^n. \tag{32}$$

Proof. By setting $r = |x|+2t$ and noting $B(x, |x|+2t) \supset B(0, 2t)\backslash B(0, t)$ we get

$$(Mf_t)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \chi_{B(0,2t)\backslash B(0,t)}(z)\, dz$$
$$\geq \frac{1}{B(x, |x| + 2t)} \int_{B(x,|x|+2t)} \chi_{B(0,2t)\backslash B(0,t)}\, dz$$
$$= \frac{1}{v_n} \frac{1}{(|x| + 2t)^n} \int_{t<|z|<2t} dz$$
$$= (2^n - 1) \frac{t^n}{(|x| + 2t)^n} \geq (1 - 2^{-n}) \frac{t^n}{(|x| + t)^n} \geq \frac{1}{2} \frac{t^n}{(|x| + t)^n}.$$

Since $|f_t| \leq \chi_{B(0,2t)}$, by Lemma 2 we obtain

$$(Mf_t)(x) \leq (M\chi_{B(0,2t)})(x) \leq 4^n\left(\frac{2t}{|x| + 2t}\right)^n \leq 8^n\left(\frac{t}{|x| + t}\right)^n. \blacksquare$$

For functions F, G defined on $(0, \infty) \times (0, \infty)$ we shall write $F \asymp G$ if there exist $c, c' > 0$ such that $cF(r, t) \leq G \leq c'F(r, t)$ for all $r, t \in (0, \infty)$.

LEMMA 9. *For all $r, t > 0$ and $0 < p \leq \infty$,*

$$\|Mf_t\|_{L_p(B(0,r))} \asymp r^{n/p} \begin{cases} \min\{1, (t/r)^n\}, & 0 < p < 1, \\ \min\{1, (t/r)^n \ln(e + r/t)\}, & p = 1, \\ \min\{1, (t/r)^{n/p}\}, & 1 < p \leq \infty. \end{cases} \tag{33}$$

Proof. By Lemma 8 we get

$$\left(\frac{1}{2}\right)^p t^{np} \int_{B(0,r)} \frac{1}{(|y| + t)^{np}}\, dy$$
$$\leq \int_{B(0,r)} (Mf_t)^p(y)\, dy \leq 8^{np} t^{np} \int_{B(0,r)} \frac{1}{(|y| + t)^{np}}\, dy.$$

Furthermore

$$\int_{B(0,r)} \frac{1}{(|y| + t)^{np}}\, dy = nv_n \int_0^r \frac{\tau^{n-1}}{(\tau + t)^{np}}\, d\tau.$$

If $0 < r \leq t$, then

$$\frac{(2t)^{-np} r^n}{n} = (2t)^{-np} \int_0^r \tau^{n-1}\, d\tau < \int_0^r \frac{\tau^{n-1}}{(\tau + t)^{np}}\, d\tau \tag{34}$$
$$\leq t^{-np} \int_0^r \tau^{n-1}\, d\tau = \frac{t^{-np} r^n}{n}.$$

Hence

$$2^{-p(n+1)}r^n \le \frac{1}{v_n} \int\limits_{B(0,r)} ((Mf_t)(y))^p \, dy \le 8^{np} r^n.$$

If $r > t$ then we consider separately three cases.

1. If $0 < p < 1$, then by applying (34) with $r = t$ we get

$$\frac{2^{-np}}{n} r^{n-np} \le \frac{2^{-np}}{n} \left(t^{n-np} + r^{n-np} - t^n r^{-np}\right) = \frac{2^{-np}}{n} t^{n-np} + (2r)^{-np} \int\limits_t^r \tau^{n-1} \, d\tau$$

$$\le \int\limits_0^r \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau = \int\limits_0^t \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau + \int\limits_t^r \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau$$

$$\le \frac{t^{n-np}}{n} + \int\limits_t^r \tau^{n-1-np} \, d\tau = \frac{t^{n-np}}{n} + \frac{r^{n-np} - t^{n-np}}{n(1-p)} \le \frac{r^{n-np}}{n(1-p)}.$$

Hence

$$2^{-p(n+1)} r^{n-np} t^{np} \le \frac{1}{v_n} \int\limits_{B(0,r)} ((Mf_t)(y))^p \, dy \le \frac{8^{np}}{1-p} r^{n-np} t^{np}.$$

2. If $p = 1$, then

$$2^{-n}\left(\frac{1}{n} + \ln\frac{r}{t}\right) = (2t)^{-n} \int\limits_0^t \tau^{n-1} \, d\tau + 2^{-n} \int\limits_t^r \frac{d\tau}{\tau} \le \int\limits_0^r \frac{\tau^{n-1}}{(\tau+t)^n} \, d\tau$$

$$= \int\limits_0^t \frac{\tau^{n-1}}{(\tau+t)^n} \, d\tau + \int\limits_t^r \frac{\tau^{n-1}}{(\tau+t)^n} \, d\tau \le t^{-n} \int\limits_0^t \tau^{n-1} d\tau + \int\limits_t^r \frac{d\tau}{\tau} = \frac{1}{n} + \ln\frac{r}{t}.$$

Hence

$$2^{-(n+p)}\left(1 + n\ln\frac{r}{t}\right) t^n \le \frac{1}{v_n} \int\limits_{B(0,r)} (Mf_t)(y) \, dy \le 8^{np}\left(1 + n\ln\frac{r}{t}\right) t^n.$$

3. Finally, if $1 < p < \infty$, then

$$2^{-np}\frac{t^{n-np}}{n} \le \int\limits_0^t \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau \le \int\limits_0^r \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau$$

$$= \int\limits_0^t \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau + \int\limits_t^r \frac{\tau^{n-1}}{(\tau+t)^{np}} \, d\tau$$

$$\le \frac{t^{n-np}}{n} + \int\limits_t^\infty \tau^{n-1-np} \, d\tau = \frac{p}{p-1} \, \frac{t^{n-np}}{n}.$$

Hence

$$2^{-p(n+1)}t^n \le \frac{1}{v_n} \int\limits_{B(0,r)} ((Mf_t)(y))^p\,dy \le 8^{np}\,\frac{p}{p-1}\,t^n.$$

These estimates imply the statement of the lemma. ∎

COROLLARY 4. *For* $0 < p \le \infty$, $p \ne 1$,

$$\|Mf_t\|_{L_p(B(0,r))} \asymp \left(\frac{t}{r+t}\right)^{n\min\{1,1/p\}} r^{n/p}.$$

THEOREM 4. *Let* $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{p,\theta_1}$, $w_2 \in \Omega_{p,\theta_2}$.

• *If* $\theta_1 \le \theta_2$, *and* $\theta_1 \le p$ *and for some* $c_{16} > 0$ *and all* $t > 0$,

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{n/p}\right\|_{L_{\theta_2}(0,\infty)} \le c_{16}\|w_1\|_{L_{\theta_1}(t,\infty)}, \tag{35}$$

then M *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$.

• *For any* $0 < \theta_1, \theta_2 \le \infty$ *condition* (35) *is necessary for the boundedness of* M *from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$.

• *In particular*, *if* $\theta_1 \le \theta_2$, $\theta_1 \le p$, *then condition* (35) *is necessary and sufficient for the boundedness of* M *from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$.

Proof. Sufficiency. It is known [13] that a necessary and sufficient condition for the validity of (30) with $q = p$ for all non-negative decreasing functions φ on $(0,\infty)$ has the form: for some $c_{17} > 0$ and all $t > 0$,

$$\|v_2(r)\min\{t,r\}\|_{L_{\theta_2/p}(0,\infty)} \le c_{17}\|v_1(r)\|_{L_{\theta_1/p}(0,t)}. \tag{36}$$

Applying this condition to the functions v_1 and v_2 given by (21) and (22) we obtain (35).

Indeed, taking into account (21) and (22) and replacing $r^{-1/n}$ by ϱ and $t^{-1/n}$ by τ, we get

$$\|w_2(\varrho)\varrho^{n/p}\min\{\tau^{-n/p}, \varrho^{-n/p}\}\|_{L_{\theta_2}(0,\infty)} \le c_{18}\|w_1\|_{L_{\theta_1}(\tau,\infty)},$$

where $c_{18} > 0$ is independent of $\tau > 0$. Hence (35) follows since

$$\varrho^{n/p}\min\{\tau^{-n/p}, \varrho^{-n/p}\} \asymp \left(\frac{\varrho}{\varrho+\tau}\right)^{n/p}.$$

Necessity. Assume that, for some $c_{19} > 0$ and all $f \in LM_{p\theta_1,w_1}$,

$$\|Mf\|_{LM_{p\theta_2,w_2}} \le c_{19}\|f\|_{LM_{p\theta_1,w_1}}. \tag{37}$$

Take $f = f_t$, where f_t is defined by (31). Then by Lemma 7 the right-hand side of (37) does not exceed

$$c_{15}t^{n/p}\|w_1\|_{L_{\theta_1}(t,\infty)},$$

where $c_{15} > 0$ is independent of $t > 0$. Furthermore by Corollary 4 (case $p > 1$) the left-hand side of (37) is equivalent to

$$\left\| w_2(r) \left(\frac{rt}{t+r} \right)^{n/p} \right\|_{L_{\theta_2}(0,\infty)}.$$

Hence (35) follows. ■

Remark 1. It is unclear whether for $1 < p < \infty$, $\theta_1 \le \theta_2$, $\theta_1 \le p$ condition (35) is necessary for the boundedness of M from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$. (If we take $f = f_t$ in (37), with LM replaced by GM, then (35) does not follow.)

Remark 2. If $p = 1$, $0 < \theta_1, \theta_2 \le \infty$, then a similar argument shows that the condition: there exists $c_{20} > 0$ such that for all $t > 0$,

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{n} \ln \left(e + \frac{r}{t} \right) \right\|_{L_{\theta_2}(0,\infty)} \le c_{20} \|w_1\|_{L_{\theta_1}(t,\infty)},$$

is necessary for the boundedness of M from $LM_{1\theta_1,w_1}$ to $LM_{1\theta_2,w_2}$.

Remark 3. Under the assumptions of Theorem 4 the boundedness of the maximal operator from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ is equivalent to the boundedness of the Hardy operator from $L_{\theta_1/p,v_1}(0,\infty)$ to $L_{\theta_2/p,v_2}(0,\infty)$ where $v_1(r) = (w_1(r^{-1/n}) r^{-(1+1/n)1/\theta_1})^p$, $v_2(r) = (w_2(r^{-1/n}) r^{-1/p-(1+1/n)1/\theta_2})^p$ on the cone of non-negative non-increasing functions. This is proved by finding necessary and sufficient conditions on w_1 and w_2, namely (35), for the boundedness of both operators. It may be of interest to find a direct proof of this equivalence. (One of the implications is established in Theorem 2.)

Next we consider the local and global weak Morrey-type spaces and study the boundedness of the maximal operator M in these spaces.

Definition 3. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. Denote by $LWM_{p\theta,w}$ and $GWM_{p\theta,w}$ the *local* and *global weak Morrey-type spaces* respectively, defined to be the spaces of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LWM_{p\theta,w}} \equiv \|f\|_{LWM_{p\theta,w}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{WL_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$
$$\|f\|_{GWM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LWM_{p\theta,w}},$$

respectively, where for $p < \infty$,

$$\|f\|_{WL_p(B(0,r))} = \sup_{t>0} t (\mathrm{meas}\, \{x \in B(0,r) : |f(x)| > t\})^{1/p}.$$

If $p = \infty$, then $WL_\infty \equiv L_\infty$ and $LWM_{\infty\theta,w} \equiv LM_{\infty\theta,w}$, $GWM_{\infty\theta,w} \equiv GM_{\infty\theta,w}$.

Note that for any $0 < p, \theta \le \infty$,

$$\|f\|_{LWM_{p\theta,w}} \le \|f\|_{LM_{p\theta,w}}, \qquad \|f\|_{GWM_{p\theta,w}} \le \|f\|_{GM_{p\theta,w}}$$

for all $f \in LM_{p\theta,w}$ and $f \in GM_{p\theta,w}$ respectively.

As in [2], [5] and [7] the proof of the boundedness of the maximal operator for $p = 1$ is based on the inequality

$$\text{meas}\,\{x \in B(0,r) : (Mf)(x) > t\} \le \frac{c_{21}}{t} \int\limits_{\mathbb{R}^n} |f(x)|(M\chi_{B(0,r)})(x)\,dx, \tag{38}$$

where $c_{21} > 0$ is independent of $f \in L_1^{\text{loc}}(\mathbb{R}^n)$, t and r. This is a particular case of a more general inequality established by C. L. Fefferman and E. Stein [3]:

$$\int\limits_{\{x\in\mathbb{R}^n : (Mf)(x)>t\}} |\varphi(x)|\,dx \le \frac{c_{22}}{t} \int\limits_{\mathbb{R}^n} |f(x)|(M\varphi)(x)\,dx,$$

where $c_{22} > 0$ is independent of $f, \varphi \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Using inequality (38), the relation

$$\|Mf\|_{WL_p(\mathbb{R}^n)} \le \|Mf\|_{L_p(\mathbb{R}^n)}, \qquad 0 < p \le \infty,$$

and the properties of the maximal operator in local Morrey-type spaces established in the first part of the paper, we get the following corresponding properties of the maximal operator in local weak Morrey-type spaces:

Lemma 10. *Let $1 \le p < \infty$. Then there exists $c_{23} > 0$ such that*

$$\|Mf\|_{WL_p(B(0,r))} \le c_{23}\left(r^n \int\limits_r^\infty \left(\int\limits_{B(0,t)} |f(x)|^p\,dx\right)\frac{dt}{t^{n+1}}\right)^{1/p} \tag{39}$$

for all $r > 0$ and all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Lemma 11. *Let $1 \le p < \infty$, $0 < \theta \le \infty$. Then there exists $c_{24} > 0$ such that*

$$\|Mf\|_{LWM_{p\theta,w}} \le c_{24}\|Hg\|^{1/p}_{L_{\theta/p,v}(0,\infty)}$$

for all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$, where g and v are given by (15), (16) *respectively.*

Corollary 5. *Let $1 \le p < \infty$, $0 < \theta \le \infty$. Then*

$$\|Mf\|_{GWM_{p\theta,w}} \le c_{24} \sup_{x\in\mathbb{R}^n} \|H(g(x,\cdot))\|^{1/p}_{L_{\theta/p,v}(0,\infty)}$$

for all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$, where $g(x,\cdot)$ is given by (18).

Proof. We consider two cases:

1. If $1 < p < \infty$, the assertion follows by the proof of Corollary 3.
2. If $p = 1$, then for all $x \in \mathbb{R}^n$,

$$(40)\qquad \|(Mf)(x+\cdot)\|_{WL_1(B(0,r))}$$
$$= \sup_{t>0} t\left|\left\{z\in B(0,r):\ \sup_{r>0}\frac{1}{|B(x+z,r)|}\int\limits_{B(x+z,r)}|f(y)|\,dy>t\right\}\right|$$
$$= \sup_{t>0} t\left|\left\{z\in B(0,r):\sup_{r>0}\frac{1}{|B(z,r)|}\int\limits_{B(z,r)}|f(x+y)|\,dy>t\right\}\right|$$
$$= \|M(f(x+\cdot))\|_{WL_1(B(0,r))},$$

hence we have

$$(41)\quad \|Mf\|_{GWM_{1\theta,w}} = \sup_{x\in\mathbb{R}^n}\big\|w(r)\|(Mf)(x+\cdot)\|_{WL_1(B(0,r))}\big\|_{L_\theta(0,\infty)}$$
$$= \sup_{x\in\mathbb{R}^n}\big\|w(r)\|M(f(x+\cdot))\|_{WL_1(B(0,r))}\big\|_{L_\theta(0,\infty)}$$
$$= \sup_{x\in\mathbb{R}^n}\|M(f(x+\cdot))\|_{LWM_{1\theta,w}}$$
$$\le c_{24}\sup_{x\in\mathbb{R}^n}\Big\|H\Big(\int\limits_{B(0,t^{-1/n})}|f(x+y)|\,dy\Big)\Big\|_{L_{\theta,v}(0,\infty)}$$
$$= c_{24}\sup_{x\in\mathbb{R}^n}\|H(g(x,\cdot))\|_{L_{\theta,v}(0,\infty)}.\ \blacksquare$$

THEOREM 5. *Let* $0<p_2\le p_1<\infty$, $p_1>1$, $0<\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{p_1,\theta_1}$, $w_2\in\Omega_{p_2,\theta_2}$. *Assume that, for some* $q>1$ *satisfying* $p_2\le q\le p_1$, *the operator* H *is bounded from* $L_{\theta_1/q,v_1}(0,\infty)$ *to* $L_{\theta_2/q,v_2}(0,\infty)$ *on the cone of all non-negative functions* φ *non-increasing on* $(0,\infty)$ *and satisfying* $\lim_{t\to\infty}\varphi(t)=0$, *where* v_1,v_2 *are defined by* (21), (22) *respectively. Then the operator* M *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GWM_{p\theta_2,w_2}$.

THEOREM 6. *Let* $1\le p<\infty$, $0<\theta_2\le\infty$, $w_1\in\Omega_{p,\infty}$, $w_2\in\Omega_{p,\theta_2}$. *Let also condition* (29) *be satisfied. Then the operator* M *is bounded from* $LM_{p\infty,w_1}$ *to* $LWM_{p\theta_2,w_2}$ *and from* $GM_{p\infty,w_1}$ *to* $GWM_{p\theta_2,w_2}$.

LEMMA 12. *Let* $r,t>0$, *and* $0<p\le\infty$. *Then*

$$\|f_t\|_{WL_p(B(0,r))}=0,\quad 0<r<t,\qquad \|f_t\|_{WL_p(B(0,r))}\le c_{15}t^{n/p},\quad r\ge t.$$

Proof. The statement follows from Lemma 7 since for all measurable $G,\Omega\subset\mathbb{R}^n$,

$$\|\chi_G\|_{WL_p(\Omega)}=|G\cap\Omega|^{1/p}=\|\chi_G\|_{L_p(\Omega)},$$

hence $\|f_t\|_{WL_p(B(0,r))}=\|f_t\|_{L_p(B(0,r))}$. ■

LEMMA 13. *For all* $0<p\le\infty$,

$$(42)\qquad \|Mf_t\|_{WL_p(B(0,r))}\asymp\left(\frac{t}{r+t}\right)^{n\min\{1/p,1\}}r^{n/p}.$$

Proof. By Lemma 8 we have

$$\|Mf_t\|_{WL_p(B(0,r))} \asymp t^n \Big\| \Big(\frac{1}{|x|+t}\Big)^n \Big\|_{WL_p(B(0,r))}.$$

Furthermore,

$$\begin{aligned}\Big\| \Big(\frac{1}{|x|+t}\Big)^n \Big\|_{WL_p(B(0,r))} &= \sup_{\tau>0} \tau \,\mathrm{meas} \Big\{ x \in B(0,r) : \frac{1}{(|x|+t)^n} > \tau \Big\}^{1/p} \\ &= \sup_{\tau>0} \tau |B(0,r) \cap B(0, \tau^{-1/n} - t)|^{1/p} \\ &= v_n \sup_{0<\tau<t^{-n}} \tau (\min\{r, \tau^{-1/n} - t\})^{n/p} \\ &= v_n \max\{ \sup_{0<\tau\le(t+r)^{-n}} \tau r^{n/p}, \sup_{(t+r)^{-n}<\tau<t^{-n}} \tau(\tau^{-1/n}-t)^{n/p}\} \\ &= v_n \max\{(t+r)^{-n} r^{n/p}, \sup_{(t+r)^{-n}<\tau<t^{-n}} \tau(\tau^{-1/n}-t)^{n/p}\} \\ &= \sup_{(t+r)^{-n}\le\tau<t^{-n}} \tau(\tau^{-1/n}-t)^{n/p}.\end{aligned}$$

If $0 < p \le 1$, then the function $\phi(\tau) = \tau(\tau^{-1/n} - t)^{n/p}$ decreases on $[(t+r)^{-n}, t^{-n})$, therefore

$$\sup_{(t+r)^{-n}\le\tau<t^{-n}} \tau(\tau^{-1/n}-t)^{n/p} = \frac{r^{n/p}}{(t+r)^n}.$$

If $p > 1$, then for $t \ge (p-1)r$, φ also decreases on $[(t+r)^{-n}, t^{-n})$ and for $t < (p-1)r$ the supremum is attained at $\tau = \big(\frac{p-1}{pt}\big)^n$. Hence

$$\begin{aligned}\sup_{(t+r)^{-n}\le\tau<t^{-n}} \tau(\tau^{-1/n}-t)^{n/p} &= c_{25} \begin{cases} \dfrac{r^{n/p}}{(t+r)^n}, & t \ge (p-1)r, \\ t^{n/p-n}, & t < (p-1)r, \end{cases} \\ &\asymp \Big(\frac{rt}{t+r}\Big)^{n/p} t^{-n},\end{aligned}$$

where $c_{25} > 0$ depends only on p and n. Therefore the statement follows. ■

THEOREM 7. *Let* $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{p,\theta_1}$, $w_2 \in \Omega_{p,\theta_2}$.

- *If* $\theta_1 \le \theta_2$, $\theta_1 \le p$ *and inequality* (35) *is satisfied, then* M *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GWM_{p\theta_2,w_2}$.
- *For any* $0 < \theta_1, \theta_2 \le \infty$ *condition* (35) *is necessary for the boundedness of* M *from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$.
- *In particular, if* $\theta_1 \le \theta_2$, $\theta_1 \le p$, *then condition* (35) *is necessary and sufficient for the boundedness of* M *from* $LM_{p\theta_1,w_1}$ *to* $LWM_{p\theta_2,w_2}$.

Proof. Sufficiency follows from Theorem 5 as in the proof of Theorem 4. The proof of necessity is also essentially the same as in the proof of Theorem 4, with Lemma 9 replaced by Lemma 13. ∎

REMARK 4. When defining global Morrey-type spaces, one might consider a weight function w depending not only on $r>0$, but also on $x\in\mathbb{R}^n$, and consider the space of all functions $f\in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\Big\|\big\|w(x,r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)}\Big\|_{L_\infty(\mathbb{R}^n)}<\infty.$$

For $\theta=\infty$ such quasinorms were considered in [7]. Moreover, it is also reasonable to replace $L_\infty(\mathbb{R}^n)$ by $L_\eta(\mathbb{R}^n)$, where $0<\eta\le\infty$, thus assuming that

$$\|f\|_{GM_{p\theta\eta,w}}=\Big\|\big\|w(x,r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)}\Big\|_{L_\eta(\mathbb{R}^n)}<\infty.$$

If in Theorem 2 formulas (21) and (22) are replaced by

$$v_1(x,r)=(w_1(x,r^{-1/n})r^{1/q-1/p_1-(1+1/n)1/\theta_1})^q,$$
$$v_2(x,r)=(w_2(x,r^{-1/n})r^{-1/p_2-(1+1/n)1/\theta_2})^q$$

and

$$\sup_{x\in\mathbb{R}^n}\|H\|_{L_{\theta_1/q,v_1(x,r)}(0,\infty)\cap C\to L_{\theta_2/q,v_2(x,r)}(0,\infty)\cap C}<\infty,$$

where C is the cone of all non-negative functions φ non-increasing on $(0,\infty)$ and satisfying $\lim_{t\to\infty}\varphi(t)=0$, then the maximal operator M is also bounded from $GM_{p_1\theta_1\eta,w_1}$ to $GM_{p_2\theta_2\eta,w_2}$. Similar remarks refer to all other inequalities of the paper involving global Morrey-type spaces or global weak Morrey-type spaces.

A brief exposition of the results of this paper, without proofs, is given in [1].

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